

# A theory of plots

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# 1 Plots on a set

We denote by  $\mathit{Set}$  the category of sets and maps. For a category  $\mathcal{C}$  and an object  $X$  of  $\mathcal{C}$ , we denote by  $h_X$  the presheaf on  $\mathcal{C}$  represented by  $X$ , that is,  $h_X : \mathcal{C}^{op} \rightarrow \mathit{Set}$  is a functor defined by  $h_X(U) = \mathcal{C}(U, X)$  and  $h_X(f : U \rightarrow V) = (f^* : \mathcal{C}(U, X) \rightarrow \mathcal{C}(V, X))$ . For a morphism  $\varphi : X \rightarrow Y$  in  $\mathcal{C}$ , let  $h_\varphi : h_X \rightarrow h_Y$  be a natural transformation defined by  $(h_\varphi)_U = \varphi_* : \mathcal{C}(U, X) \rightarrow \mathcal{C}(U, Y)$ .

**Definition 1.1** Let  $\mathcal{C}$  be a category,  $F : \mathcal{C} \rightarrow \mathit{Set}$  a functor and  $X$  a set  $X$ . Define a presheaf  $F_X$  on  $\mathcal{C}$  to be a composition  $\mathcal{C}^{op} \xrightarrow{F^{op}} \mathit{Set}^{op} \xrightarrow{h_X} \mathit{Set}$ . Here  $F^{op} : \mathcal{C}^{op} \rightarrow \mathit{Set}^{op}$  is a functor defined by  $F^{op}(U) = F(U)$  for  $U \in \mathit{Ob}\mathcal{C}$  and  $F^{op}(f) = F(f)$  for  $f \in \mathit{Mor}\mathcal{C}$ . An element of  $\prod_{U \in \mathit{Ob}\mathcal{C}} F_X(U)$  is called an  $F$ -parametrization of  $X$ .

**Definition 1.2** Let  $(\mathcal{C}, J)$  be a site,  $X$  a set and  $F : \mathcal{C} \rightarrow \mathit{Set}$  a functor. Assume that  $\mathcal{C}$  has a terminal object  $1_{\mathcal{C}}$  and that  $F(1_{\mathcal{C}})$  consists of a single element  $*$ . If a subset  $\mathcal{D}$  of  $\prod_{U \in \mathit{Ob}\mathcal{C}} F_X(U)$  satisfies the following conditions, we call  $\mathcal{D}$  a the-ology on  $X$  with respect to  $F$  and  $(\mathcal{C}, J)$  or just a the-ology on  $X$  for short and call a pair  $(X, \mathcal{D})$  a the-ological object. An element of  $\mathcal{D}$  is called an  $F$ -plot of  $(X, \mathcal{D})$ .

- (i)  $\mathcal{D} \supset F_X(1_{\mathcal{C}})$
- (ii) For a morphism  $f : U \rightarrow V$  in  $\mathcal{C}$ ,  $F_X(f) : F_X(V) \rightarrow F_X(U)$  maps  $\mathcal{D} \cap F_X(V)$  into  $\mathcal{D} \cap F_X(U)$ .
- (iii) For an object  $U$  of  $\mathcal{C}$ , an element  $x$  of  $F_X(U)$  belongs to  $\mathcal{D} \cap F_X(U)$  if there exists a covering  $(U_i \xrightarrow{f_i} U)_{i \in I}$  such that  $F_X(f_i) : F_X(U) \rightarrow F_X(U_i)$  maps  $x$  into  $\mathcal{D} \cap F_X(U_i)$  for any  $i \in I$ .

**Remark 1.3** For a subset  $\mathcal{D}$  of  $\prod_{U \in \mathit{Ob}\mathcal{C}} F_X(U)$  and  $U \in \mathit{Ob}\mathcal{C}$ , we put  $F_{\mathcal{D}}(U) = \mathcal{D} \cap F_X(U)$ .

- (1)  $\mathcal{D}$  satisfies condition (i) of (1.2) if and only if  $F_{\mathcal{D}}(1_{\mathcal{C}}) = F_X(1_{\mathcal{C}})$ .
- (2)  $\mathcal{D}$  satisfies condition (ii) of (1.2) if and only if a correspondence  $U \mapsto F_{\mathcal{D}}(U)$  defines a subpresheaf  $F_{\mathcal{D}}$  of  $F_X$ .

**Proposition 1.4** Condition (iii) of (1.2) is equivalent to the following condition if we assume condition (ii).

- (iii') For an object  $U$  of  $\mathcal{C}$ , an element  $x$  of  $F_X(U)$  belongs to  $\mathcal{D} \cap F_X(U)$  if there exists  $R \in J(U)$  such that  $F_X(f) : F_X(U) \rightarrow F_X(\mathit{dom}(f))$  maps  $x$  into  $\mathcal{D} \cap F_X(\mathit{dom}(f))$  for any  $f \in R$ .

*Proof.* It is clear that (iii') implies (iii) since  $R \in J(U)$  is a covering of  $U$ . Assume that (iii) is satisfied and that  $(U_i \xrightarrow{f_i} U)_{i \in I}$  is a covering of  $U$  such that  $F_X(f_i) : F_X(U) \rightarrow F_X(U_i)$  maps  $x \in F_X(U)$  into  $\mathcal{D} \cap F_X(U_i)$  for any  $i \in I$ . Let  $R$  be a sieve generated by  $(U_i \xrightarrow{f_i} U)_{i \in I}$ , which is given by

$$R(V) = \{f \in h_U(V) \mid f = f_i g \text{ for some } i \in I \text{ and } g \in \mathcal{C}(V, U_i)\}.$$

Then, for  $f \in R$ , there exist  $i \in I$  and  $g : \mathit{dom}(f) \rightarrow U_i$  such that  $f = f_i g$ . Since  $F_X(f_i)(x) \in \mathcal{D} \cap F_X(U_i)$  implies  $F_X(f)(x) = F_X(g)F_X(f_i)(x) \in \mathcal{D} \cap F_X(\mathit{dom}(f))$  by (ii), it follows from (iii') that  $x \in \mathcal{D} \cap F_X(U)$ .  $\square$

**Definition 1.5** Let  $(\mathcal{C}, J)$  be a site and  $F : \mathcal{C} \rightarrow \mathit{Set}$  a functor. If  $(\mathcal{C}, J)$  and  $F$  satisfies the following condition,  $(\mathcal{C}, J)$  is called an  $F$ -preconcrete site. Moreover, if  $F : \mathcal{C} \rightarrow \mathit{Set}$  is faithful,  $(\mathcal{C}, J)$  is called an  $F$ -concrete site.

(PCS) For every covering  $(U_i \xrightarrow{f_i} U)_{i \in I}$ ,  $(F(U_i) \xrightarrow{F(f_i)} F(U))_{i \in I}$  is an epimorphic family in  $\mathit{Set}$ .

**Remark 1.6** (1) Let  $X$  be a set and  $(\mathcal{C}, J)$  an  $F$ -preconcrete site. For a covering  $(U_i \xrightarrow{f_i} U)_{i \in I}$  in  $(\mathcal{C}, J)$ , since  $(F(U_i) \xrightarrow{F(f_i)} F(U))_{i \in I}$  is an epimorphic family in  $\mathit{Set}$ , the map  $(F_X(f_i))_{i \in I} : F_X(U) \rightarrow \prod_{i \in I} F_X(U_i)$  induced by  $F_X(f_i) = F(f_i)^* : F_X(U) \rightarrow F_X(U_i)$ 's is injective. Hence  $F_X$  is a separated presheaf on  $\mathcal{C}$  and  $F_{\mathcal{D}}$  is also a separated presheaf for a the-ology  $\mathcal{D}$  on  $X$ .

(2) Assume that  $\mathcal{C}$  has a terminal object  $1_{\mathcal{C}}$ . If  $F : \mathcal{C} \rightarrow \mathit{Set}$  is a functor defined by  $F(U) = \mathcal{C}(1_{\mathcal{C}}, U)$  and  $F(f : U \rightarrow V) = (f_* : \mathcal{C}(1_{\mathcal{C}}, U) \rightarrow \mathcal{C}(1_{\mathcal{C}}, V))$ , an  $F$ -preconcrete site is called a preconcrete site and an  $F$ -concrete site is called a concrete site.

For a map  $\varphi : X \rightarrow Y$  and a functor  $F : \mathcal{C} \rightarrow \mathit{Set}$ , we define a morphism  $F_\varphi : F_X \rightarrow F_Y$  of presheaves by  $(F_\varphi)_U = \varphi_* : F_X(U) = \mathit{Set}(F(U), X) \rightarrow \mathit{Set}(F(U), Y) = F_Y(U)$ .

**Definition 1.7** Let  $(\mathcal{C}, J)$  be a site and  $F : \mathcal{C} \rightarrow \text{Set}$  a functor.

(1) Let  $(X, \mathcal{D})$  and  $(Y, \mathcal{E})$  be the-ological objects. If the map  $(F_\varphi)_U : F_X(U) \rightarrow F_Y(U)$  induced by a map  $\varphi : X \rightarrow Y$  maps  $\mathcal{D} \cap F_X(U)$  into  $\mathcal{E} \cap F_Y(U)$  for each  $U \in \text{Ob } \mathcal{C}$ , we call  $\varphi$  a morphism of the-ological objects. We denote this by  $\varphi : (X, \mathcal{D}) \rightarrow (Y, \mathcal{E})$ .

(2) We define a category  $\mathcal{P}_F(\mathcal{C}, J)$  of the-ological objects as follows. Objects of  $\mathcal{P}_F(\mathcal{C}, J)$  are the-ological objects and morphisms of  $\mathcal{P}_F(\mathcal{C}, J)$  are morphism of the-ological objects.

**Remark 1.8** Let  $\varphi : (X, \mathcal{D}) \rightarrow (Y, \mathcal{E})$  be a morphism of the-ological objects. It follows from the definition of a morphism of the-ological objects that  $(F_\varphi)_U : F_X(U) \rightarrow F_Y(U)$  defines a map  $(F_\varphi)_U : F_{\mathcal{D}}(U) \rightarrow F_{\mathcal{E}}(U)$  which is natural in  $U \in \text{Ob } \mathcal{C}$ . Thus we have a morphism  $F_\varphi : F_{\mathcal{D}} \rightarrow F_{\mathcal{E}}$  of presheaves.

**Definition 1.9** For the-ologies  $\mathcal{D}$  and  $\mathcal{E}$  on  $X$ , we say that  $\mathcal{D}$  is finer than  $\mathcal{E}$  and that  $\mathcal{E}$  is coarser than  $\mathcal{D}$  if  $\mathcal{D} \subset \mathcal{E}$ .

**Remark 1.10** We put  $\mathcal{D}_{\text{coarse}, X} = \coprod_{U \in \text{Ob } \mathcal{C}} F_X(U)$ . It is clear that  $\mathcal{D}_{\text{coarse}, X}$  is the coarsest the-ology on  $X$ . For a map  $f : Y \rightarrow X$  and a the-ology  $\mathcal{E}$  on  $Y$ ,  $f : (Y, \mathcal{E}) \rightarrow (X, \mathcal{D}_{\text{coarse}, X})$  is a morphism of the-ologies.

**Proposition 1.11** Let  $(\mathcal{D}_i)_{i \in I}$  be a family of the-ologies on a set  $X$ . Then,  $\bigcap_{i \in I} \mathcal{D}_i$  is a the-ology on  $X$  that is the finest the-ology among the-ologies on  $X$  which are coarser than  $\mathcal{D}_i$  for any  $i \in I$ .

*Proof.* Put  $\mathcal{E} = \bigcap_{i \in I} \mathcal{D}_i$ . Since  $\mathcal{D}_i \supset F_X(1_{\mathcal{C}})$  for any  $i \in I$ ,  $\mathcal{E} \supset F_X(1_{\mathcal{C}})$  holds. For a morphism  $f : U \rightarrow V$  of  $\mathcal{C}$ , since  $F_X(f) : F_X(V) \rightarrow F_X(U)$  maps  $\mathcal{D}_i \cap F_X(V)$  to  $\mathcal{D}_i \cap F_X(U)$  for any  $i \in I$ ,  $F_X(f)$  maps  $\mathcal{E} \cap F_X(V)$  to  $\mathcal{E} \cap F_X(U)$ . Suppose that there exists a covering  $(U_j \xrightarrow{f_j} U)_{j \in J}$  such that  $F_X(f_j) : F_X(U) \rightarrow F_X(U_j)$  maps  $x \in F_X(U)$  into  $\mathcal{E} \cap F_X(U_j)$  for any  $j \in J$ . Hence  $F_X(f_j)$  maps  $x$  into  $\mathcal{D}_i \cap F_X(U_j)$  for any  $j \in J$  which implies  $x \in \mathcal{D}_i \cap F_X(U)$ . Thus we have  $x \in \mathcal{E} \cap F_X(U)$ .  $\square$

For a set  $X$ , we denote by  $\mathcal{P}_F(\mathcal{C}, J)_X$  a subcategory of  $\mathcal{P}_F(\mathcal{C}, J)$  consisting of objects of the form  $(X, \mathcal{D})$  and morphisms of the form  $id_X : (X, \mathcal{D}) \rightarrow (X, \mathcal{E})$ . Then,  $\mathcal{P}_F(\mathcal{C}, J)_X$  is regarded as an ordered set of the-ologies on  $X$ . We often denote by  $\mathcal{D}$  an object  $(X, \mathcal{D})$  of  $\mathcal{P}_F(\mathcal{C}, J)_X$  for short. It follows from (1.10) that  $\mathcal{D}_{\text{coarse}, X}$  is the maximum (terminal) object of  $\mathcal{P}_F(\mathcal{C}, J)_X$ .

**Corollary 1.12**  $\mathcal{P}_F(\mathcal{C}, J)_X$  is complete as an ordered set.

*Proof.* Let  $\Sigma$  be a non-empty subset of  $\mathcal{P}_F(\mathcal{C}, J)_X$ . Then,  $\inf \Sigma = \bigcap_{\mathcal{D} \in \Sigma} \mathcal{D}$  by (1.11). We denote by  $\hat{\Sigma}$  a subset of  $\mathcal{P}_F(\mathcal{C}, J)_X$  consisting of elements which contain every elements of  $\Sigma$ . Then it follows from (1.11) that  $\bigcap_{\mathcal{E} \in \hat{\Sigma}} \mathcal{E}$  is an element of  $\mathcal{P}_F(\mathcal{C}, J)_X$ . Thus we see  $\sup \Sigma = \bigcap_{\mathcal{E} \in \hat{\Sigma}} \mathcal{E}$ .  $\square$

**Proposition 1.13** Let  $\mathcal{S}$  be a subset of  $\coprod_{U \in \text{Ob } \mathcal{C}} F_X(U)$  which contains  $F_X(1_{\mathcal{C}})$ . For  $f \in \text{Mor } \mathcal{C}$ , define a subset  $\mathcal{S}_f$  of  $F_X(\text{dom}(f))$  by  $\mathcal{S}_f = F_X(f)(\mathcal{S} \cap F_X(\text{codom}(f)))$ . For  $U \in \text{Ob } \mathcal{C}$ , we define a subset  $\mathcal{S}(U)$  of  $F_X(U)$  by

$$\mathcal{S}(U) = \left\{ x \in F_X(U) \mid \text{There exists } R \in J(U) \text{ such that } F_X(g)(x) \in \bigcup_{f \in \text{Mor } \mathcal{C}} \mathcal{S}_f \text{ for all } g \in R. \right\}.$$

If we put  $\mathcal{G}(\mathcal{S}) = \coprod_{U \in \text{Ob } \mathcal{C}} \mathcal{S}(U)$  and  $\Sigma = \{ \mathcal{D} \in \mathcal{P}_F(\mathcal{C}, J)_X \mid \mathcal{D} \supset \mathcal{S} \}$ , then  $\mathcal{G}(\mathcal{S}) = \inf \Sigma \in \mathcal{P}_F(\mathcal{C}, J)_X$ .

*Proof.* Since  $\mathcal{S}_{id_U} = \mathcal{S} \cap F_X(U)$ ,  $\mathcal{S} \subset \bigcup_{f \in \text{Mor } \mathcal{C}} \mathcal{S}_f$  holds. For  $x \in \left( \bigcup_{f \in \text{Mor } \mathcal{C}} \mathcal{S}_f \right) \cap F_X(U)$ , there exists  $f \in \text{Mor } \mathcal{C}$  such that  $\text{dom}(f) = U$  and  $x \in \mathcal{S}_f \cap F_X(U)$ . Then, we have  $x = \alpha F(f)$  for some  $\alpha \in \mathcal{S} \cap F_X(\text{codom}(f))$ . For  $g \in h_U$ , since  $F_X(g)(x) = F_X(g)(\alpha F(f)) = \alpha F(fg) = F_X(fg)(\alpha) \in F_X(fg)(\mathcal{S} \cap F_X(\text{codom}(f))) = \mathcal{S}_{fg}$  and  $h_U \in J(R)$ , it follows that  $x \in \mathcal{S}(U)$ . Hence we have  $\left( \bigcup_{f \in \text{Mor } \mathcal{C}} \mathcal{S}_f \right) \cap F_X(U) \subset \mathcal{S}(U)$  and  $\mathcal{G}(\mathcal{S}) \supset \bigcup_{f \in \text{Mor } \mathcal{C}} \mathcal{S}_f \supset \mathcal{S} \supset F_X(1_{\mathcal{C}})$ .

Let  $f : U \rightarrow V$  be a morphism in  $\mathcal{C}$ . For  $x \in \mathcal{G}(\mathcal{S}) \cap F_X(V) = \mathcal{S}(V)$ , there exists  $R \in J(V)$  such that  $F_X(g)(x) \in \bigcup_{f \in \text{Mor } \mathcal{C}} \mathcal{S}_f$  for all  $g \in R$ . Hence there exists  $s_g \in \text{Mor } \mathcal{C}$  for each  $g \in R$  such that  $F_X(g)(x) \in \mathcal{S}_{s_g}$ . It follows that there exists  $x_g \in \mathcal{S} \cap F_X(\text{codom}(s_g))$  which satisfies  $F_X(s_g)(x_g) = F_X(g)(x)$  for each  $g \in R$ .

Define a sieve  $h_f^{-1}(R)$  on  $U$  by  $h_f^{-1}(R) = \{j \in \text{Mor } \mathcal{C} \mid \text{codom}(j) = U, fj \in R\}$ . Then, for  $j \in h_f^{-1}(R)$ , since  $F_X(j)(F_X(f)(x)) = F_X(fj)(x) = F_X(s_{fj})(x_{fj}) \in F_X(s_{fj})(\mathcal{S} \cap F_X(\text{codom}(s_{fj}))) = \mathcal{S}_{fj}$  and  $h_f^{-1}(R) \in J(U)$  hold, we have  $F_X(f)(x) \in \mathcal{G}(\mathcal{S}) \cap F_X(U) = \mathcal{S}(U)$ . Thus  $F_X(f) : F_X(V) \rightarrow F_X(U)$  maps  $\mathcal{G}(\mathcal{S}) \cap F_X(V)$  into  $\mathcal{G}(\mathcal{S}) \cap F_X(U)$ .

For  $U \in \text{Ob } \mathcal{C}$  and  $x \in F_X(U)$ , suppose that there exists  $R \in J(U)$  such that  $F_X(f) : F_X(U) \rightarrow F_X(\text{dom}(f))$  maps  $x$  into  $\mathcal{G}(\mathcal{S}) \cap F_X(\text{dom}(f)) = \mathcal{S}(\text{dom}(f))$  for any  $f \in R$ . Then, there exists  $S_f \in J(\text{dom}(f))$  such that

$$F_X(fg)(x) = F_X(g)(F_X(f)(x)) \in \bigcup_{j \in \text{Mor } \mathcal{C}} \mathcal{S}_j \cdots (*)$$

holds for any  $g \in S_f$ . Put  $T = \{fg \mid f \in R, g \in S_f\}$ . Since  $T \in J(U)$ ,  $(*)$  implies  $x \in \mathcal{S}(U) = \mathcal{G}(\mathcal{S}) \cap F_X(U)$ . Hence we conclude that  $\mathcal{G}(\mathcal{S})$  is a the-ology on  $X$ .

Suppose that a the-ology  $\mathcal{D}$  on  $X$  contains  $\mathcal{S}$ . For  $f \in \text{Mor } \mathcal{C}$ , since

$$\mathcal{S}_f = F_X(f)(\mathcal{S} \cap F_X(\text{codom}(f))) \subset F_X(f)(\mathcal{D} \cap F_X(\text{codom}(f))) \subset \mathcal{D} \cap F_X(\text{dom}(f)),$$

We have  $\bigcup_{f \in \text{Mor } \mathcal{C}} \mathcal{S}_f \subset \mathcal{D}$  which implies  $\mathcal{S}(U) \subset \mathcal{D}$  for any  $U \in \text{Ob } \mathcal{C}$  by (1.4). Hence  $\mathcal{G}(\mathcal{S}) \subset \mathcal{D}$  holds.  $\square$

**Remark 1.14** (1) For  $U \in \text{Ob } \mathcal{C}$ , the subset  $\mathcal{S}(U)$  of  $F_X(U)$  defined in (1.13) coincides with

$$\left\{ x \in F_X(U) \mid \text{There exists a covering } (U_i \xrightarrow{g_i} U)_{i \in I} \text{ such that } F_X(g_i)(x) \in \bigcup_{f \in \text{Mor } \mathcal{C}} \mathcal{S}_f \text{ for all } i \in I. \right\}.$$

In fact, since  $R \in J(U)$  is a covering of  $U$ ,  $\mathcal{S}(U)$  is contained in the above set. Suppose that, for  $x \in F_X(U)$ , there exists a covering  $(U_i \xrightarrow{g_i} U)_{i \in I}$  such that  $F_X(g_i)(x) \in \bigcup_{f \in \text{Mor } \mathcal{C}} \mathcal{S}_f$  for any  $i \in I$ . We choose  $f_i \in \text{Mor } \mathcal{C}$

which satisfies  $F_X(g_i)(x) \in \mathcal{S}_{f_i}$  for each  $i \in I$ . Let  $R$  be a sieve on  $U$  generated by  $(U_i \xrightarrow{g_i} U)_{i \in I}$ . For  $j \in R$ , there exist  $i \in I$  and  $k \in \mathcal{C}(\text{dom}(j), U_i)$  such that  $j = g_i k$ . Then we have  $F_X(j)(x) = F_X(k)(F_X(g_i)(x))$ , which belongs to  $F_X(k)(\mathcal{S}_{f_i}) = F_X(f_i k)(\mathcal{S} \cap F_X(\text{codom}(f_i))) = \mathcal{S}_{f_i k}$ . Therefore we have  $x \in \mathcal{S}(U)$  and the above set is contained in  $\mathcal{S}(U)$ .

(2) Let  $\Sigma$  be a non-empty subset of  $\mathcal{P}_F(\mathcal{C}, J)_X$  and put  $\mathcal{S}(\Sigma) = \bigcup_{\mathcal{D} \in \Sigma} \mathcal{D}$ . For  $f \in \text{Mor } \mathcal{C}$  and  $x \in \mathcal{S}(\Sigma)_f$ , there exist  $\mathcal{D} \in \Sigma$  and  $y \in \mathcal{D} \cap F_X(\text{codom}(f))$  such that  $x = F_X(f)(y)$  which belongs to  $\mathcal{D} \cap F_X(\text{dom}(f))$ . It follows that  $\bigcup_{f \in \text{Mor } \mathcal{C}} \mathcal{S}(\Sigma)_f \subset \mathcal{S}(\Sigma)$  holds. Since  $\mathcal{S}(\Sigma) \subset \bigcup_{f \in \text{Mor } \mathcal{C}} \mathcal{S}(\Sigma)_f$ , we have  $\mathcal{S}(\Sigma) = \bigcup_{f \in \text{Mor } \mathcal{C}} \mathcal{S}(\Sigma)_f$ . Thus, for  $U \in \text{Ob } \mathcal{C}$ , the following equality holds.

$$\mathcal{S}(\Sigma)(U) = \left\{ x \in F_X(U) \mid \text{There exists a covering } (U_i \xrightarrow{g_i} U)_{i \in I} \text{ such that } F_X(g_i)(x) \in \bigcup_{\mathcal{D} \in \Sigma} \mathcal{D} \text{ for all } i \in I. \right\}$$

Hence  $\sup \Sigma = \mathcal{G}(\mathcal{S}(\Sigma)) = \bigcup_{U \in \mathcal{C}} \mathcal{S}(\Sigma)(U)$ .

**Definition 1.15** For a subset  $\mathcal{S}$  of  $\prod_{U \in \text{Ob } \mathcal{C}} F_X(U)$  containing  $F_X(1_{\mathcal{C}})$ , we call  $\mathcal{G}(\mathcal{S})$  defined in (1.13) the the-ology generated by  $\mathcal{S}$ .

**Definition 1.16** Let  $(\mathcal{C}, J)$  be a site and  $X$  a set. We put  $\mathcal{D}_{\text{disc}, X} = \bigcap_{\mathcal{D} \in \text{Ob } \mathcal{P}_F(\mathcal{C}, J)_X} \mathcal{D}$  and call this the discrete the-ology on  $X$ .  $\mathcal{D}_{\text{disc}, X}$  is the finest the-ology on  $X$ .

**Remark 1.17** (1) For any map  $f : X \rightarrow Y$  and a the-ology  $\mathcal{E}$  on  $Y$ ,  $f : (X, \mathcal{D}_{\text{disc}, X}) \rightarrow (Y, \mathcal{E})$  is a morphism of the-ologies. In particular,  $(X, \mathcal{D}_{\text{disc}, X})$  is the minimum (initial) object of  $\mathcal{P}_F(\mathcal{C}, J)_X$ .

(2) Since  $\mathcal{D}_{\text{disc}, X} \supset F_X(1_{\mathcal{C}})$ ,  $\mathcal{D}_{\text{disc}, X}$  contains the image of the map  $F_X(o_U) : F_X(1_{\mathcal{C}}) \rightarrow F_X(U)$  induced by the unique map  $o_U : U \rightarrow 1_{\mathcal{C}}$  for any  $U \in \text{Ob } \mathcal{C}$ . Hence every constant map in  $F_X(U)$  belongs to  $\mathcal{D}_{\text{disc}, X}$ .

(3) Let  $\mathcal{S}_{\text{const}}$  be the set of all constant maps in  $\prod_{U \in \text{Ob } \mathcal{C}} F_X(U)$ . Then  $\mathcal{S}_{\text{const}} = \bigcup_{f \in \text{Mor } \mathcal{C}} (\mathcal{S}_{\text{const}})_f$ . Hence

$\mathcal{D}_{\text{disc}, X} \cap F_X(U) = \mathcal{D}(\mathcal{S}_{\text{const}}) \cap F_X(U)$  coincides with the following set.

$$\left\{ x \in F_X(U) \mid \text{There exists a covering } (U_i \xrightarrow{g_i} U)_{i \in I} \text{ such that } F_X(g_i)(x) \text{ is a constant map for all } i \in I. \right\}$$

## 2 Category of the-ology

For a map  $f : X \rightarrow Y$  and  $(Y, \mathcal{E}) \in \text{Ob } \mathcal{P}_F(\mathcal{C}, J)$ , we define a the-ology  $\mathcal{E}^f$  on  $X$  to be the coarsest the-ology such that  $f : (X, \mathcal{E}^f) \rightarrow (Y, \mathcal{E})$  is a morphism of the-ologies.

**Proposition 2.1** For a map  $f : X \rightarrow Y$  and  $(Y, \mathcal{E}) \in \text{Ob } \mathcal{P}_F(\mathcal{C}, J)$ ,  $\mathcal{E}^f$  is given by

$$\mathcal{E}^f = \coprod_{U \in \text{Ob } \mathcal{C}} (F_f)^{-1}(\mathcal{E} \cap F_Y(U)) = \coprod_{U \in \text{Ob } \mathcal{C}} \{\varphi \in F_X(U) \mid f\varphi \in \mathcal{E} \cap F_Y(U)\}.$$

*Proof.* We put  $\bar{\mathcal{E}} = \coprod_{U \in \text{Ob } \mathcal{C}} \{\varphi \in F_X(U) \mid f\varphi \in \mathcal{E} \cap F_Y(U)\}$ . Since  $\mathcal{E} \supset F_Y(1_{\mathcal{C}})$ ,  $\bar{\mathcal{E}} \supset F_X(1_{\mathcal{C}})$  holds.

For a morphism  $\rho : U \rightarrow V$  of  $\mathcal{C}$  and  $\psi \in \bar{\mathcal{E}} \cap F_X(V)$ , then  $f\psi \in \mathcal{E} \cap F_Y(V)$  implies that  $fF_X(\rho)(\psi) = f\psi\rho_* = F_Y(\rho)(f\psi)$  is contained in  $\mathcal{E} \cap F_Y(U)$ , which shows that  $F_X(\rho)(\psi)$  is contained in  $\bar{\mathcal{E}} \cap F_X(U)$ . Thus  $F_X(\rho) : F_X(V) \rightarrow F_X(U)$  maps  $\bar{\mathcal{E}} \cap F_X(V)$  to  $\bar{\mathcal{E}} \cap F_X(U)$ .

For  $\varphi \in F_X(U)$ , assume that there exists a covering  $(U_i \xrightarrow{\rho_i} U)_{i \in I}$  such that  $F_X(\rho_i) : F_X(U) \rightarrow F_X(U_i)$  maps  $\varphi$  into  $\bar{\mathcal{E}} \cap F_X(U_i)$  for any  $i \in I$ . Then,  $F_Y(\rho_i)(f\varphi) = f\varphi\rho_{i*} = fF_X(\rho_i)(\varphi) \in \mathcal{E} \cap F_Y(U_i)$  for any  $i \in I$ . Hence  $f\varphi \in \mathcal{E} \cap F_Y(U)$  which implies  $\varphi \in \bar{\mathcal{E}} \cap F_X(U)$ . Therefore  $\bar{\mathcal{E}}$  is a the-ology on  $X$ .

Suppose that  $\mathcal{D}$  is a the-ology on  $X$  such that  $f : (X, \mathcal{D}) \rightarrow (Y, \mathcal{E})$  is a morphism of the-ologies. Then,  $(F_f)_U : F_X(U) \rightarrow F_Y(U)$  maps  $\mathcal{D} \cap F_X(U)$  into  $\mathcal{E} \cap F_Y(U)$  for each  $U \in \text{Ob } \mathcal{C}$ . Then  $\mathcal{D} \cap F_X(U)$  is contained in  $\{\varphi \in F_X(U) \mid f\varphi \in \mathcal{E} \cap F_Y(U)\}$ . Hence we have  $\mathcal{D} \subset \bar{\mathcal{E}}$  which shows  $\bar{\mathcal{E}} = \mathcal{E}^f$ .  $\square$

The following result is straightforward from the definition of  $\mathcal{E}^f$ .

**Proposition 2.2** Let  $(\mathcal{E}_i)_{i \in I}$  a family of the-ologies on a set  $Y$ , For a map  $f : X \rightarrow Y$ ,  $\left(\bigcap_{i \in I} \mathcal{E}_i\right)^f = \bigcap_{i \in I} \mathcal{E}_i^f$  holds.

Let us define a forgetful functor  $\Gamma : \mathcal{P}_F(\mathcal{C}, J) \rightarrow \text{Set}$  by  $\Gamma(X, \mathcal{D}) = X$  for an object  $(X, \mathcal{D})$  of  $\mathcal{P}_F(\mathcal{C}, J)$  and  $\Gamma(\varphi : (X, \mathcal{D}) \rightarrow (Y, \mathcal{E})) = (\varphi : X \rightarrow Y)$  for a morphism  $\varphi : (X, \mathcal{D}) \rightarrow (Y, \mathcal{E})$  in  $\mathcal{P}_F(\mathcal{C}, J)$ .

It is clear that  $\Gamma$  is faithful. In other words, if we put

$$\mathcal{P}_F(\mathcal{C}, J)_f((X, \mathcal{D}), (Y, \mathcal{E})) = \Gamma^{-1}(f) \cap \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))$$

for a map  $f : X \rightarrow Y$  and  $(X, \mathcal{D}), (Y, \mathcal{E}) \in \text{Ob } \mathcal{P}_F(\mathcal{C}, J)$ ,  $\mathcal{P}_F(\mathcal{C}, J)_f((X, \mathcal{D}), (Y, \mathcal{E}))$  has at most one element. We see that  $\mathcal{P}_F(\mathcal{C}, J)_f((X, \mathcal{D}), (Y, \mathcal{E}))$  is not empty if and only if  $\mathcal{D} \subset \mathcal{E}^f$  which is equivalent that  $\mathcal{P}_F(\mathcal{C}, J)_X((X, \mathcal{D}), (X, \mathcal{E}^f))$  is not empty.

**Proposition 2.3** For maps  $f : X \rightarrow Y$ ,  $g : W \rightarrow X$  and an object  $(Y, \mathcal{E})$  of  $\mathcal{P}_F(\mathcal{C}, J)_Y$ ,  $\mathcal{E}^{fg} = (\mathcal{E}^f)^g$  holds and  $\Gamma : \mathcal{P}_F(\mathcal{C}, J) \rightarrow \text{Set}$  is a fibered category.

*Proof.* For  $U \in \text{Ob } \mathcal{C}$ ,  $\varphi \in \mathcal{E}^{fg} \cap F_W(U)$  holds if and only if  $fg\varphi \in \mathcal{E} \cap F_Y(U)$  which is equivalent to  $g\varphi \in \mathcal{E}^f \cap F_X(U)$ . Moreover  $g\varphi \in \mathcal{E}^f \cap F_X(U)$  holds if and only if  $\varphi \in (\mathcal{E}^f)^g \cap F_W(U)$ . Thus we have  $\mathcal{E}^{fg} = (\mathcal{E}^f)^g$ . We put  $f^*(Y, \mathcal{E}) = (X, \mathcal{E}^f)$  and let  $\alpha_f(Y, \mathcal{E}) : f^*(Y, \mathcal{E}) = (X, \mathcal{E}^f) \rightarrow (Y, \mathcal{E})$  be the unique morphism in  $\mathcal{P}_F(\mathcal{C}, J)$  that satisfies  $\Gamma(\alpha_f(Y, \mathcal{E})) = f$ . For an object  $(X, \mathcal{D})$  of  $\mathcal{P}_F(\mathcal{C}, J)_X$ , a map

$$\mathcal{P}_F(\mathcal{C}, J)_X((X, \mathcal{D}), (X, \mathcal{E}^f)) \rightarrow \mathcal{P}_F(\mathcal{C}, J)_f((X, \mathcal{D}), (Y, \mathcal{E}))$$

which maps  $\varphi$  to  $\alpha_f(Y, \mathcal{E})\varphi$  is bijective, namely  $\alpha_f(Y, \mathcal{E})$  is a cartesian morphism. The equality  $\mathcal{E}^{fg} = (\mathcal{E}^f)^g$  implies that the following composition coincides with  $\alpha_{fg}(Y, \mathcal{E})$ .

$$(W, \mathcal{E}^{fg}) = (W, (\mathcal{E}^f)^g) \xrightarrow{\alpha_g(X, \mathcal{E}^f)} (X, \mathcal{E}^f) \xrightarrow{\alpha_f(Y, \mathcal{E})} (Y, \mathcal{E})$$

Therefore  $\Gamma : \mathcal{P}_F(\mathcal{C}, J) \rightarrow \text{Set}$  is a fibered category.  $\square$

For a map  $f : X \rightarrow Y$  and  $(X, \mathcal{D}) \in \text{Ob } \mathcal{P}_F(\mathcal{C}, J)$ , we define a the-ology  $\mathcal{D}_f$  on  $Y$  to be the finest the-ology such that  $f : (X, \mathcal{D}) \rightarrow (Y, \mathcal{D}_f)$  is a morphism of the-ologies, that is,  $\mathcal{D}_f = \bigcap_{\mathcal{E} \in \Sigma} \mathcal{E}$ , where

$$\Sigma = \left\{ \mathcal{E} \in \text{Ob } \mathcal{P}_F(\mathcal{C}, J)_Y \mid \mathcal{E} \supset \coprod_{U \in \text{Ob } \mathcal{C}} (F_f)_U(\mathcal{D} \cap F_X(U)) \right\}.$$

**Remark 2.4** We can also describe  $\mathcal{D}_f$  by using (1.13) as follows. Consider a subset  $\mathcal{S}$  of  $\coprod_{U \in \text{Ob } \mathcal{C}} F_Y(U)$  given

by  $\mathcal{S} = F_Y(1_{\mathcal{C}}) \coprod \left( \coprod_{U \in \text{Ob } \mathcal{C}, U \neq 1_{\mathcal{C}}} (F_f)_U(\mathcal{D} \cap F_X(U)) \right)$ . Then, if  $U \neq 1_{\mathcal{C}}$ , we have  $\mathcal{S} \cap F_Y(U) = (F_f)_U(\mathcal{D} \cap F_X(U))$

and the subset  $\mathcal{S}_g = F_Y(g)(\mathcal{S} \cap F_Y(\text{codom}(g)))$  of  $F_Y(\text{dom}(g))$  for  $g \in \text{Mor } \mathcal{C}$  is given by

$$\mathcal{S}_g = F_Y(g)((F_f)_{\text{codom}(g)}(\mathcal{D} \cap F_X(\text{codom}(g)))) = (F_f)_{\text{dom}(g)}(F_X(g)(\mathcal{D} \cap F_X(\text{codom}(g))))$$

if  $\text{codom}(g) \neq 1_{\mathcal{C}}$ . Since  $F_X(g) : F_X(\text{codom}(g)) \rightarrow F_X(\text{dom}(g))$  maps  $\mathcal{D} \cap F_X(\text{codom}(g))$  into  $\mathcal{D} \cap F_X(\text{dom}(g))$ , the above equality implies  $\mathcal{S}_g \subset (F_f)_{\text{dom}(g)}(\mathcal{D} \cap F_X(\text{dom}(g))) = \mathcal{S}_{id_{\text{dom}(g)}}$ . If  $\text{codom}(g) = 1_{\mathcal{C}}$ ,  $g$  is the unique morphism  $o_V : V \rightarrow 1_{\mathcal{C}}$ . Hence we have  $\bigcup_{g \in \text{Mor } \mathcal{C}} \mathcal{S}_g = \bigcup_{V \in \text{Ob } \mathcal{C}, V \neq 1_{\mathcal{C}}} \mathcal{S}_{id_V} \cup \bigcup_{V \in \text{Ob } \mathcal{C}} \mathcal{S}_{o_V}$ . It follows that the following equality holds for  $V \in \text{Ob } \mathcal{C}$ .

$$\left( \bigcup_{g \in \text{Mor } \mathcal{C}} \mathcal{S}_g \right) \cap F_Y(V) = \mathcal{S}_{id_V} \cup \mathcal{S}_{o_V} = (F_f)_V(\mathcal{D} \cap F_X(V)) \cup F_Y(o_V)(F_Y(1_{\mathcal{C}}))$$

For  $U \in \text{Ob } \mathcal{C}$ , the subset  $\mathcal{S}(U)$  of  $F_Y(U)$  defined in (1.13) is the set of elements  $x$  of  $F_Y(U)$  which satisfy the following condition (\*).

(\*) There exists  $R \in J(U)$  such that, for each  $h \in R$ ,  $F_Y(h)(x) : F(\text{dom}(h)) \rightarrow Y$  is a constant map or there exists  $y \in \mathcal{D} \cap F_X(\text{dom}(h))$  which satisfies  $F_Y(h)(x) = (F_f)_{\text{dom}(h)}(y)$ .

We remark that if  $f : X \rightarrow Y$  is surjective, we can replace the above condition by the following condition.

(\*)' There exists  $R \in J(U)$  such that, for each  $h \in R$ , there exists  $y \in \mathcal{D} \cap F_X(\text{dom}(h))$  which satisfies  $F_Y(h)(x) = (F_f)_{\text{dom}(h)}(y)$ .

If we put  $\mathcal{G}(\mathcal{S}) = \prod_{U \in \text{Ob } \mathcal{C}} \mathcal{S}(U)$ , we have  $\mathcal{D}_f = \mathcal{G}(\mathcal{S})$ .

**Proposition 2.5**  $\Gamma : \mathcal{P}_F(\mathcal{C}, J) \rightarrow \text{Set}$  is a bifibered category.

*Proof.* For a map  $f : X \rightarrow Y$ , we define a functor  $f_* : \mathcal{P}_F(\mathcal{C}, J)_X \rightarrow \mathcal{P}_F(\mathcal{C}, J)_Y$  as follows. For an object  $(X, \mathcal{D})$  of  $\mathcal{P}_F(\mathcal{C}, J)_X$ , we put  $f_*(X, \mathcal{D}) = (Y, \mathcal{D}_f)$ . If  $\mathcal{D}$  and  $\mathcal{D}'$  are the-ologies on  $X$  such that  $\mathcal{D} \subset \mathcal{D}'$ , then  $\mathcal{D}_f \subset \mathcal{D}'_f$ . Hence we can put  $f_*(id_X : (X, \mathcal{D}) \rightarrow (X, \mathcal{D}')) = (id_Y : (Y, \mathcal{D}_f) \rightarrow (Y, \mathcal{D}'_f))$ .

For an object  $(X, \mathcal{D})$  of  $\mathcal{P}_F(\mathcal{C}, J)_X$  and an object  $(Y, \mathcal{E})$  of  $\mathcal{P}_F(\mathcal{C}, J)_Y$ ,  $\mathcal{D}_f \subset \mathcal{E}$  holds if and only if  $(F_f)_U(\mathcal{D} \cap F_X(U)) \subset \mathcal{E}$  for any  $U \in \text{Ob } \mathcal{C}$ , which is equivalent to  $\mathcal{D} \subset \mathcal{E}^f$ . Thus  $\mathcal{P}_F(\mathcal{C}, J)_Y(f_*(X, \mathcal{D}), (Y, \mathcal{E}))$  is not empty if and only if  $\mathcal{P}_F(\mathcal{C}, J)_X((X, \mathcal{D}), f^*(Y, \mathcal{E}))$  is not empty. It follows that  $f_*$  is a left adjoint of  $f^*$  and that  $\Gamma : \mathcal{P}_F(\mathcal{C}, J) \rightarrow \text{Set}$  is a bifibered category.  $\square$

**Proposition 2.6** Let  $p : \mathcal{F} \rightarrow \mathcal{E}$  be a prefibered category. If  $\mathcal{F}_X$  has an initial object for any object  $X$  of  $\mathcal{E}$ , then  $p$  has a left adjoint.

*Proof.* We denote by  $0_X$  an initial object of  $\mathcal{F}_X$  and define a functor  $L : \mathcal{E} \rightarrow \mathcal{F}$  as follows. We put  $L(X) = 0_X$  for an object  $X$  of  $\mathcal{E}$ . For a morphism  $f : X \rightarrow Y$  in  $\mathcal{E}$  and an object  $N$  of  $\mathcal{F}_Y$ , we denote by  $i_f : 0_X \rightarrow f^*(0_Y)$  unique morphism in  $\mathcal{F}_X$  and by  $\alpha_f(N) : f^*(N) \rightarrow N$  the cartesian morphism that is mapped to  $f$  by  $p$ . Put  $L(f) = \alpha_f(0_Y)i_f$ . Since the identity morphism of  $0_X$  is unique morphism in  $\mathcal{E}_X$  from  $0_X$  to  $0_X$ ,  $L(id_X)$  is the identity morphism of  $0_X$  if  $X = Y$ . For composable morphisms  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  in  $\mathcal{E}$ , let  $f^*(i_g) : f^*(0_Y) \rightarrow f^*(g^*(0_Z))$  and  $c_{g,f}(0_Z) : f^*(g^*(0_Z)) \rightarrow (gf)^*(0_Z)$  be unique morphisms in  $\mathcal{F}_X$  that make the upper and the lower rectangles of the following diagram commutative, respectively.

$$\begin{array}{ccccc} & & f^*(0_Y) & \xrightarrow{\alpha_f(0_Y)} & 0_Y \\ & \nearrow i_f & \downarrow f^*(i_g) & & \downarrow i_g \\ 0_X & & f^*(g^*(0_Z)) & \xrightarrow{\alpha_f(g^*(0_Z))} & g^*(0_Z) \\ & \searrow i_{gf} & \downarrow c_{g,f}(0_Z) & & \downarrow \alpha_g(0_Z) \\ & & (gf)^*(0_Z) & \xrightarrow{\alpha_{gf}(0_Z)} & 0_Z \end{array}$$

Since  $i_f$ ,  $f^*(i_g)$ ,  $c_{g,f}(0_Z)$  and  $i_{gf}$  are morphisms in  $\mathcal{F}_X$ , the left triangle of the above diagram is commutative. Hence  $L(gf) = L(g)L(f)$  holds, which shows that  $L$  is a functor.  $pL$  is the identity functor of  $\mathcal{E}$  since  $p(i_f) = id_X$  and  $p(\alpha_f(0_Y)) = f$  hold for any morphism  $f : X \rightarrow Y$  in  $\mathcal{E}$ . We denote by  $\eta : id_{\mathcal{E}} \rightarrow pL$  the identity natural transformation. For an object  $M$  of  $\mathcal{F}$ , let  $\varepsilon_M : Lp(M) = 0_{p(M)} \rightarrow M$  be unique morphism in  $\mathcal{F}_{p(M)}$ . For a morphism  $\varphi : M \rightarrow N$  in  $\mathcal{F}$ , there exists unique morphism  $\tilde{\varphi} : M \rightarrow p(\varphi)^*(N)$  in  $\mathcal{F}_{p(M)}$  that makes the right triangle of the following diagram commute. The right triangle of the following diagram commutes by the definition of  $L$  and the lower trapezoid of the following diagram commutes by the definition of  $p(\varphi)^*(\varepsilon_N)$ . Since  $\varepsilon_M$ ,  $\tilde{\varphi}$ ,  $i_{p(\varphi)}$ ,  $\alpha_{p(\varphi)}(0_{p(N)})$  are morphisms in  $\mathcal{F}_{p(M)}$  and  $0_{p(M)}$  is an initial object of  $\mathcal{F}_{p(M)}$ , the upper trapezoid of the following diagram is also commutative.

$$\begin{array}{ccc}
0_{p(M)} & \xrightarrow{\varepsilon_M} & M \\
\downarrow Lp(\varphi) & \searrow i_{p(\varphi)} & \swarrow \tilde{\varphi} \\
& p(\varphi)^*(0_{p(M)}) & \xrightarrow{p(\varphi)^*(\varepsilon_N)} & p(\varphi)^*(N) \\
& \swarrow \alpha_{p(\varphi)}(0_{p(N)}) & & \searrow \alpha_{p(\varphi)}(N) \\
0_{p(N)} & \xrightarrow{\varepsilon_N} & N \\
& & \downarrow \varphi
\end{array}$$

Thus we have a natural transformation  $\varepsilon : Lp \rightarrow id_{\mathcal{F}}$ . For an object  $M$  of  $\mathcal{F}$ , since  $p(\varepsilon_M)$  is the identity morphism of  $p(M)$ , a composition  $p(M) \xrightarrow{\eta_{p(M)}} p(M) = pLp(M) \xrightarrow{p(\varepsilon_M)} p(M)$  is also the identity morphism of  $M$ . For an object  $X$  of  $\mathcal{E}$ , since  $\varepsilon_{L(X)} : LpL(X) = 0_X \rightarrow 0_X = L(X)$  is the identity morphism of  $0_X$ , a composition  $L(X) \xrightarrow{L(\eta_X)} LpL(X) \xrightarrow{\varepsilon_{L(X)}} L(X)$  is the identity morphism of  $L(X) = 0_X$ . Therefore  $L$  is a left adjoint of  $p$ .  $\square$

**Corollary 2.7** *Let  $p : \mathcal{F} \rightarrow \mathcal{E}$  be a bifibered category. If  $\mathcal{F}_X$  has a terminal object for any object  $X$  of  $\mathcal{E}$ , then  $p$  has a right adjoint.*

*Proof.* Since  $p : \mathcal{F} \rightarrow \mathcal{E}$  is a cofibered category,  $p^{op} : \mathcal{F}^{op} \rightarrow \mathcal{E}^{op}$  is a fibered category. By the assumption,  $\mathcal{F}_X^{op}$  has an initial object and it follows from (2.6) that  $p^{op}$  has a left adjoint  $L : \mathcal{E}^{op} \rightarrow \mathcal{F}^{op}$  of  $p^{op}$ . Hence  $L^{op} : \mathcal{E} \rightarrow \mathcal{F}$  is a right adjoint of  $p$ .  $\square$

**Remark 2.8** *Under the assumption of the above corollary, a right adjoint  $R : \mathcal{E} \rightarrow \mathcal{F}$  of  $p$  is given as follows. For an object  $X$  of  $\mathcal{E}$ , we denote by  $1_X$  a terminal object of  $\mathcal{F}_X$  and put  $R(X) = 1_X$ . For each morphism  $f : X \rightarrow Y$  of  $\mathcal{E}$  and an object  $M$  of  $\mathcal{F}_X$ , we choose a right adjoint  $f_* : \mathcal{F}_X \rightarrow \mathcal{F}_Y$  of the inverse image functor  $f^* : \mathcal{F}_Y \rightarrow \mathcal{F}_X$  and a cocartesian morphism  $\alpha^f(M) : M \rightarrow f_*(M)$  which is mapped to  $f$  by  $p$ . We define  $R(f) : 1_X \rightarrow 1_Y$  to be a composition  $1_X \xrightarrow{\alpha^f(M)} f_*(1_X) \xrightarrow{o_Y} 1_Y$ , where  $o_Y$  is the unique morphism in  $\mathcal{F}_Y$ .*

By (2.5) and (2.7), we deduce the following result.

**Corollary 2.9**  $\Gamma : \mathcal{P}_F(\mathcal{C}, J) \rightarrow \text{Set}$  has left and right adjoints.

**Remark 2.10** *A left adjoint  $\mathcal{L} : \text{Set} \rightarrow \mathcal{P}_F(\mathcal{C}, J)$  and the right adjoint  $\mathcal{R} : \text{Set} \rightarrow \mathcal{P}_F(\mathcal{C}, J)$  of  $\Gamma$  are given by  $\mathcal{L}(X) = (X, \mathcal{D}_{disc, X})$ ,  $\mathcal{L}(\varphi : X \rightarrow Y) = (\varphi : (X, \mathcal{D}_{disc, X}) \rightarrow (Y, \mathcal{D}_{disc, Y}))$  and  $\mathcal{R}(X) = (X, \mathcal{D}_{coarse, X})$ ,  $\mathcal{R}(\varphi : X \rightarrow Y) = (\varphi : (X, \mathcal{D}_{coarse, X}) \rightarrow (Y, \mathcal{D}_{coarse, Y}))$ .*

Let  $\{(X_i, \mathcal{D}_i)\}_{i \in I}$  be a family of objects of  $\mathcal{P}_F(\mathcal{C}, J)$ . We denote by  $\text{pr}_i : \prod_{j \in I} X_j \rightarrow X_i$  the projection to the  $i$ -th component and  $\iota_i : X_i \rightarrow \prod_{j \in I} X_j$  the inclusion to the  $i$ -th summand. Put  $\mathcal{D}^I = \bigcap_{j \in I} \mathcal{D}_i^{\text{pr}_i}$ . Then,  $\mathcal{D}^I$  is the finest the-ology such that  $\text{pr}_i : \left( \prod_{j \in I} X_j, \mathcal{D}^I \right) \rightarrow (X_i, \mathcal{D}_i)$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$  for any  $i \in I$ .

Let  $\mathcal{D}_I$  be the coarsest the-ology on  $\prod_{j \in I} X_j$  such that  $\iota_i : (X_i, \mathcal{D}_i) \rightarrow \left( \prod_{j \in I} X_j, \mathcal{D}_I \right)$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$  for any  $i \in I$ . If we put  $\mathcal{S}_I = \left\{ \mathcal{E} \in \text{Ob } \mathcal{P}_F(\mathcal{C}, J) \mid \prod_{j \in I} X_j \Big| \mathcal{E} \supset \bigcup_{j \in I} (\mathcal{D}_j)_{\iota_j} \right\}$ , then  $\mathcal{D}_I = \bigcap_{\mathcal{E} \in \mathcal{S}_I} \mathcal{E}$ .

**Proposition 2.11** (1)  $\left( \left( \prod_{j \in I} X_j, \mathcal{D}^I \right) \xrightarrow{\text{pr}_i} (X_i, \mathcal{D}_i) \right)_{i \in I}$  is a product of  $\{(X_i, \mathcal{D}_i)\}_{i \in I}$ .

(2)  $\left( (X_i, \mathcal{D}_i) \xrightarrow{\iota_i} \left( \prod_{j \in I} X_j, \mathcal{D}_I \right) \right)_{i \in I}$  is a coproduct of  $\{(X_i, \mathcal{D}_i)\}_{i \in I}$ .

*Proof.* (1) Let  $\{\varphi_i : (Y, \mathcal{E}) \rightarrow (X_i, \mathcal{D}_i)\}_{i \in I}$  be a family of morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ . Let  $\varphi : Y \rightarrow \prod_{j \in I} X_j$  be the unique map that satisfies  $\text{pr}_i \varphi = \varphi_i$  for any  $i \in I$ . For  $U \in \text{Ob } \mathcal{C}$ ,  $x \in \mathcal{E} \cap F_Y(U)$  and  $i \in I$ , it follows that  $\text{pr}_i(F_\varphi)_U(x) = (F_{\text{pr}_i})_U(F_\varphi)_U(x) = (F_{\varphi_i})_U(x) \in \mathcal{D}_i \cap F_{X_i}(U)$  which shows  $(F_\varphi)_U(x) \in \mathcal{D}_i^{\text{pr}_i}$ . Thus  $(F_\varphi)_U(x) \in \bigcap_{i \in I} \mathcal{D}_i^{\text{pr}_i} = \mathcal{D}^I$  and  $\varphi : (Y, \mathcal{E}) \rightarrow \left( \prod_{j \in I} X_j, \mathcal{D}^I \right)$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

(2) Let  $\{\psi_i : (X_i, \mathcal{D}_i) \rightarrow (Y, \mathcal{E})\}_{i \in I}$  be a family of morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ . Let  $\psi : \prod_{j \in I} X_j \rightarrow Y$  be the unique map that satisfies  $\psi \iota_i = \psi_i$  for any  $i \in I$ . We claim that  $\mathcal{E}^\psi \supset \bigcup_{j \in I} (\mathcal{D}_j)_{\iota_j}$  which holds if and only if  $\mathcal{E}^\psi \supset (F_{\iota_j})_U(\mathcal{D}_j \cap F_{X_j}(U))$  for any  $j \in I$  and  $U \in \text{Ob}\mathcal{C}$ . In fact, for  $x \in \mathcal{D}_j \cap F_{X_j}(U)$ , since we have  $\psi(F_{\iota_j})_U(x) = (F_{\psi_{\iota_j}})_U(x) = (F_{\psi_j})_U(x) \in \mathcal{E} \cap F_Y(U)$ ,  $(F_{\iota_j})_U(x)$  belongs to  $\mathcal{E}^\psi \cap F_{\prod_{j \in I} X_j}(U)$ . It follows that  $\mathcal{E}^\psi$  contains  $\mathcal{D}_I$  which implies that  $\psi : \left(\prod_{j \in I} X_j, \mathcal{D}_I\right) \rightarrow (Y, \mathcal{E})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .  $\square$

**Definition 2.12** We call  $\left(\prod_{j \in I} X_j, \mathcal{D}_I\right)$  the product the-ology on  $\prod_{j \in I} X_j$  and denote this by  $\prod_{j \in I} (X_j, \mathcal{D}_j)$ . Similarly, we call  $\left(\prod_{j \in I} X_j, \mathcal{D}^I\right)$  the sum the-ology on  $\prod_{j \in I} X_j$  and denote this by  $\prod_{j \in I} (X_j, \mathcal{D}_j)$ .

**Remark 2.13** Let  $(X, \mathcal{D})$  and  $(Y, \mathcal{E})$  be objects of  $\mathcal{P}_F(\mathcal{C}, J)$ . We denote by  $\text{pr}_X : X \times Y \rightarrow X$ ,  $\text{pr}_Y : X \times Y \rightarrow Y$  the projections and by  $\iota_y : X \times \{y\} \rightarrow X \times Y$  the inclusion map for  $y \in Y$ . Since  $\text{pr}_Y \iota_y : X \times \{y\} \rightarrow Y$  is a constant map, we have  $\mathcal{E}^{\text{pr}_Y \iota_y} = \mathcal{D}_{\text{coarse}, X \times \{y\}}$ . Hence  $(\mathcal{D}^{\text{pr}_X} \cap \mathcal{E}^{\text{pr}_Y})^{\iota_y} = \mathcal{D}^{\text{pr}_X \iota_y} \cap \mathcal{E}^{\text{pr}_Y \iota_y} = \mathcal{D}^{\text{pr}_X \iota_y}$  holds by (2.2) and (2.3). Let  $j_y : X \rightarrow X \times \{y\}$  be a map defined by  $j_y(x) = (x, y)$ . Then  $\text{pr}_X \iota_y$  is the inverse of  $j_y$  and  $j_y : (X, \mathcal{D}) \rightarrow (X \times \{y\}, (\mathcal{D}^{\text{pr}_X} \cap \mathcal{E}^{\text{pr}_Y})^{\iota_y})$  is an isomorphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

**Proposition 2.14** Let  $f, g : (X, \mathcal{D}) \rightarrow (Y, \mathcal{E})$  be morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ . Then, equalizers and coequalizers of  $f$  and  $g$  exist.

*Proof.* Put  $Z = \{x \in X \mid f(x) = g(x)\}$  and let  $i : Z \rightarrow X$  be the inclusion map. Suppose that a morphism  $h : (V, \mathcal{F}) \rightarrow (X, \mathcal{D})$  in  $\mathcal{P}_F(\mathcal{C}, J)$  satisfies  $fh = gh$ . Let  $\tilde{h} : V \rightarrow Z$  be the unique map that satisfies  $i\tilde{h} = h$ . For  $U \in \text{Ob}\mathcal{C}$  and  $\varphi \in \mathcal{F} \cap F_V(U)$ , we have  $i(F_{\tilde{h}})_U(\varphi) = (F_{i\tilde{h}})_U(\varphi) = (F_h)_U(\varphi) \in \mathcal{D} \cap F_X(U)$ , which shows  $(F_{\tilde{h}})_U(\varphi) \in \mathcal{D}^i \cap F_Z(U)$ . Therefore  $\tilde{h} : (V, \mathcal{F}) \rightarrow (Z, \mathcal{D}^i)$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$  and  $i : (Z, \mathcal{D}^i) \rightarrow (X, \mathcal{D})$  is an equalizer of  $f$  and  $g$ .

Let  $W$  be the quotient set of  $Y$  by an equivalence relation on  $Y$  generated by  $f(x) \sim g(x)$  for  $x \in X$ . We denote by  $q : Y \rightarrow W$  the quotient map. Suppose that a morphism  $h : (Y, \mathcal{E}) \rightarrow (V, \mathcal{F})$  in  $\mathcal{P}_F(\mathcal{C}, J)$  satisfies  $hf = hg$ . Let  $\tilde{h} : W \rightarrow V$  be the unique map that satisfies  $\tilde{h}q = h$ . For  $U \in \text{Ob}\mathcal{C}$  and  $\psi \in \mathcal{E} \cap F_Y(U)$ , since  $\tilde{h}(F_q)_U(\psi) = (F_{\tilde{h}q})_U(\psi) = (F_h)_U(\psi) \in \mathcal{F} \cap F_V(U)$  holds, we have  $(F_q)_U(\psi) \in \mathcal{F}^{\tilde{h}}$ . Hence  $\mathcal{F}^{\tilde{h}}$  contains  $(F_q)_U(\mathcal{E} \cap F_Y(U))$  for any  $U \in \text{Ob}\mathcal{C}$  which implies that  $\mathcal{F}^{\tilde{h}} \supset \mathcal{E}_q$  holds and  $\tilde{h} : (W, \mathcal{E}_q) \rightarrow (V, \mathcal{F})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . Thus we see that  $q : (Y, \mathcal{E}) \rightarrow (W, \mathcal{E}_q)$  is a coequalizer of  $f$  and  $g$ .  $\square$

**Remark 2.15** Suppose that  $X$  is a set which has only one element and  $\mathcal{D}$  is a the-ology on  $X$ . Since  $F_X(U)$  is also a set which has only one element for any  $U \in \text{Ob}\mathcal{C}$ , the map  $F_X(o_U) : F_X(1_{\mathcal{C}}) \rightarrow F_X(U)$  induced by the unique morphism  $o_U : U \rightarrow 1_{\mathcal{C}}$  surjective. Since  $F_X(1_{\mathcal{C}}) \subset \mathcal{D}$ , the condition (ii) of (1.3) implies  $F_X(U) \subset \mathcal{D}$ . Thus  $\mathcal{D} = \prod_{U \in \text{Ob}\mathcal{C}} F_X(U)$  holds, namely  $\mathcal{D}_{\text{coarse}, \{1\}}$  is the only the-ology on  $\{1\}$ . We also remark that  $(\{1\}, \mathcal{D}_{\text{coarse}, \{1\}})$  is a terminal object of  $\mathcal{P}_F(\mathcal{C}, J)$ .

**Proposition 2.16** Let  $f : (X, \mathcal{D}) \rightarrow (Y, \mathcal{E})$  and  $g : (Z, \mathcal{F}) \rightarrow (Y, \mathcal{E})$  be morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ . We consider the following cartesian square in  $\text{Set}$ .

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{\tilde{g}} & X \\ \downarrow \tilde{f} & & \downarrow f \\ Z & \xrightarrow{g} & Y \end{array}$$

Then,  $(Z, \mathcal{F}) \xleftarrow{\tilde{f}} (X \times_Y Z, \mathcal{D}^{\tilde{g}} \cap \mathcal{E}^{\tilde{f}}) \xrightarrow{\tilde{g}} (X, \mathcal{D})$  is a limit of a diagram  $(X, \mathcal{D}) \xrightarrow{f} (Y, \mathcal{E}) \xleftarrow{g} (Z, \mathcal{F})$  in  $\mathcal{P}_F(\mathcal{C}, J)$ .

*Proof.* We denote by  $\text{pr}_X : X \times Z \rightarrow X$  and  $\text{pr}_Z : X \times Z \rightarrow Z$  the projections. Let  $j : X \times_Y Z \rightarrow X \times Z$  be the inclusion map. Then,  $j$  is an equalizer of maps  $f\text{pr}_X, g\text{pr}_Z : X \times Z \rightarrow Y$  in  $\text{Set}$ . It follows from (2.14) that

$$j : (X \times_Y Z, (\mathcal{D}^{\text{pr}_X} \cap \mathcal{F}^{\text{pr}_Z})^j) \rightarrow (X \times Z, \mathcal{D}^{\text{pr}_X} \cap \mathcal{F}^{\text{pr}_Z})$$

is an equalizer of morphisms  $f\text{pr}_X, g\text{pr}_Z : (X \times Z, \mathcal{D}^{\text{pr}_X} \cap \mathcal{F}^{\text{pr}_Z}) \rightarrow (Y, \mathcal{E})$  in  $\mathcal{P}_F(\mathcal{C}, J)$ . Now the assertion follows from an equality  $(\mathcal{D}^{\text{pr}_X} \cap \mathcal{F}^{\text{pr}_Z})^j = (\mathcal{D}^{\text{pr}_X})^j \cap (\mathcal{F}^{\text{pr}_Z})^j = \mathcal{D}^{\text{pr}_X j} \cap \mathcal{F}^{\text{pr}_Z j} = \mathcal{D}^{\tilde{g}} \cap \mathcal{E}^{\tilde{f}}$  obtained from (2.2) and (2.3).  $\square$



For objects  $(X, \mathcal{D}), (Y, \mathcal{E})$  of  $\mathcal{P}_F(\mathcal{C}, J)$ , we define a map  $\text{ev} : X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})) \rightarrow Y$  by  $\text{ev}(x, f) = f(x)$  and also define a set  $\Sigma_{\mathcal{D}, \mathcal{E}}$  of the-ologies on  $\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))$  by

$$\Sigma_{\mathcal{D}, \mathcal{E}} = \{\mathcal{F} \in \mathcal{P}_F(\mathcal{C}, J)_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))} \mid \mathcal{E}^{\text{ev}} \supset \mathcal{D}^{\text{pr}_1} \cap \mathcal{F}^{\text{pr}_2}\}.$$

Here  $\text{pr}_1 : X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})) \rightarrow X$  and  $\text{pr}_2 : X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})) \rightarrow \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))$  are the projections. Then,  $\Sigma_{\mathcal{D}, \mathcal{E}}$  is the set of the-ology  $\mathcal{F}$  on  $\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))$  such that

$$\text{ev} : (X, \mathcal{D}) \times (\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})), \mathcal{F}) \rightarrow (Y, \mathcal{E})$$

is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

**Lemma 2.17**  $\Sigma_{\mathcal{D}, \mathcal{E}}$  is not empty.

*Proof.* It suffices to show that the discrete the-ology  $\mathcal{D}_{\text{disc}, \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}$  on  $\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))$  belongs to  $\Sigma_{\mathcal{D}, \mathcal{E}}$ . For  $U \in \text{Ob } \mathcal{C}$  and  $f \in \mathcal{D}_{\text{disc}, \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))} \cap F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(U)$ , there exists a covering  $(U_i \xrightarrow{g_i} U)_{i \in I}$  such that  $F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(g_i)(f)$  is a constant map for every  $i \in I$  by (1.17). We also take  $x \in \mathcal{D} \cap F_X(U)$ . Then,  $(x, f) : F(U) \rightarrow X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))$  is regarded as an element of  $F_{X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(U)$  which is mapped by

$$F_{X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(g_i) : F_{X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(U) \rightarrow F_{X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(U_i)$$

to a map  $(F_X(g_i)(x), F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(g_i)(f)) = (xF(g_i), fF(g_i)) : F(U_i) \rightarrow X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))$ . It follows from the commutativity of a diagram

$$\begin{array}{ccc} F_{X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(U) & \xrightarrow{(F_{\text{ev}})_U} & F_Y(U) \\ \downarrow F_{X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(g_i) & & \downarrow F_Y(g_i) \\ F_{X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(U_i) & \xrightarrow{(F_{\text{ev}})_{U_i}} & F_Y(U_i) \end{array}$$

that  $F_Y(g_i)(F_{\text{ev}})_U$  maps  $(x, f)$  to  $(F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(g_i)(f))(F_X(g_i)(x)) = (fF(g_i))(xF(g_i)) \in F_Y(U_i)$ . By the assumption on  $(U_i \xrightarrow{g_i} U)_{i \in I}$ ,  $F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(g_i)(f) = fF(g_i) : F(U_i) \rightarrow \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))$  is a constant map. Hence if we denote the image of this map by  $c$ ,  $(F_c)_{U_i}$  maps  $\mathcal{D} \cap F_X(U_i)$  to  $\mathcal{E} \cap F_Y(U_i)$  and we have  $(F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(g_i)(f))(F_X(g_i)(x)) = c(xF(g_i)) \in \mathcal{E} \cap F_Y(U_i)$  since  $xF(g_i) \in \mathcal{D} \cap F_X(U_i)$ . Therefore  $F_Y(g_i)(F_{\text{ev}})_U(x, f) \in \mathcal{E} \cap F_Y(U_i)$  for any  $i \in I$ , which shows  $(F_{\text{ev}})_U(x, f)$  belongs to  $\mathcal{E} \cap F_Y(U)$ . Thus  $\text{ev} : (X, \mathcal{D}) \times (\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})), \mathcal{D}_{\text{disc}, \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}) \rightarrow (Y, \mathcal{E})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .  $\square$

For  $U \in \text{Ob } \mathcal{C}$ , we consider the following condition (E) on an element  $\varphi$  of  $F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(U)$ .

(E) For any  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$ ,  $g \in \mathcal{C}(W, V)$  and  $\psi \in \mathcal{D} \cap F_X(V)$ , the following composition belongs to  $\mathcal{E} \cap F_Y(W)$ .

$$F(W) \xrightarrow{(F(g), F(f))} F(V) \times F(U) \xrightarrow{\psi \times \varphi} X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})) \xrightarrow{\text{ev}} Y$$

Define a set  $\mathcal{E}^{\mathcal{D}}$  of  $F$ -parametrizations of a set  $\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))$  so that  $\mathcal{E}^{\mathcal{D}} \cap F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(U)$  is a subset of  $F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(U)$  consisting of elements which satisfy the above condition (E).

**Proposition 2.18**  $\mathcal{E}^{\mathcal{D}}$  is a the-ology on  $\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))$ .

*Proof.* For  $\varphi \in F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(1_{\mathcal{C}})$ ,  $V, W \in \text{Ob } \mathcal{C}$ ,  $g \in \mathcal{C}(W, V)$  and  $\psi \in \mathcal{D} \cap F_X(V)$ , a composition

$$F(W) \xrightarrow{(F(g), F(o_W))} F(V) \times F(1_{\mathcal{C}}) \xrightarrow{\psi \times \varphi} X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})) \xrightarrow{\text{ev}} Y$$

coincides with  $(F_{\varphi(*)})_W(F_X(g)(\psi))$ . Here  $o_W : W \rightarrow 1_{\mathcal{C}}$  denotes the unique morphism and  $*$  is unique element of  $F(1_{\mathcal{C}})$ . Since  $(F_{\varphi(*)})_W : F_X(W) \rightarrow F_Y(W)$  maps  $\mathcal{D} \cap F_X(W)$  to  $\mathcal{E} \cap F_Y(W)$  and  $F_X(g)(\psi)$  belongs to  $\mathcal{D} \cap F_X(W)$ ,  $(F_{\varphi(*)})_W(F_X(g)(\psi))$  is an element of  $\mathcal{E} \cap F_Y(W)$ . Hence  $\mathcal{E}^{\mathcal{D}}$  contains  $F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(1_{\mathcal{C}})$ .

Let  $j : Z \rightarrow U$  be a morphism in  $\mathcal{C}$ . For  $\varphi \in \mathcal{E}^{\mathcal{D}} \cap F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(U)$ ,  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, Z)$ ,  $g \in \mathcal{C}(W, V)$  and  $\psi \in \mathcal{D} \cap F_X(V)$ , since a composition

$$F(W) \xrightarrow{(F(g), F(f))} F(V) \times F(Z) \xrightarrow{\psi \times F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(j)(\varphi)} X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})) \xrightarrow{\text{ev}} Y$$

coincides with  $F(W) \xrightarrow{(F(g), F(f))} F(V) \times F(U) \xrightarrow{\psi \times \varphi} X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})) \xrightarrow{\text{ev}} Y$  and the latter composition belongs to  $\mathcal{E} \cap F_Y(W)$  which shows  $F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(j)(\varphi) \in \mathcal{E}^{\mathcal{D}} \cap F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(Z)$ .

Assume that, for  $\varphi \in F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(U)$ , there exists  $R \in J(U)$  such that  $F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(j)(\varphi)$  belongs to  $\mathcal{E}^{\mathcal{D}} \cap F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(\text{dom}(j))$  for any  $j \in R$ . We take  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$ ,  $g \in \mathcal{C}(W, V)$  and  $\psi \in \mathcal{D} \cap F_X(V)$  and put  $h_f^{-1}(R) = \{i \in \text{Mor } \mathcal{C} \mid \text{codom}(i) = W, fi \in R\}$ . Then,  $h_f^{-1}(R) \in J(W)$ . For any  $i \in h_f^{-1}(R)$ , a composition

$$F(\text{dom}(i)) \xrightarrow{F(i)} F(W) \xrightarrow{(F(g), F(f))} F(V) \times F(U) \xrightarrow{\psi \times \varphi} X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})) \xrightarrow{\text{ev}} Y$$

coincides with a composition

$$F(\text{dom}(i)) \xrightarrow{(F(gi), F(id_{\text{dom}(i)}))} F(V) \times F(\text{dom}(i)) \xrightarrow{\psi \times F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(fi)(\varphi)} X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})) \xrightarrow{\text{ev}} Y$$

which belongs to  $\mathcal{E} \cap F_Y(\text{dom}(i))$  since  $F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(fi)(\varphi) \in \mathcal{E}^{\mathcal{D}} \cap F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(\text{dom}(fi))$ . Hence we have  $F_Y(i)(\text{ev}(\psi \times \varphi)(F(g), F(f))) \in \mathcal{E} \cap F_Y(\text{dom}(i))$  for any  $i \in h_f^{-1}(R)$  and this shows that  $\text{ev}(\psi \times \varphi)(F(g), F(f))$  belongs to  $\mathcal{E} \cap F_Y(W)$ . Hence  $\varphi \in \mathcal{E}^{\mathcal{D}} \cap F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(U)$  follows from the definition of  $\mathcal{E}^{\mathcal{D}}$ .  $\square$

We denote by  $(Y, \mathcal{E})^{(X, \mathcal{D})}$  an object  $(\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})), \mathcal{E}^{\mathcal{D}})$  of  $\mathcal{P}_F(\mathcal{C}, J)$ .

**Proposition 2.19**  $\mathcal{E}^{\mathcal{D}}$  is maximum element of  $\Sigma_{\mathcal{D}, \mathcal{E}}$ .

*Proof.* For  $U \in \text{Ob } \mathcal{C}$  and  $\xi \in \mathcal{D}^{\text{pr}_1} \cap (\mathcal{E}^{\mathcal{D}})^{\text{pr}_2} \cap F_{X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(U)$ , it follows from  $\text{pr}_1 \xi \in \mathcal{D} \cap F_X(U)$  and  $\text{pr}_2 \xi \in \mathcal{E}^{\mathcal{D}} \cap F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(U)$  that the following composition belongs to  $\mathcal{E} \cap F_Y(U)$ .

$$F(U) \xrightarrow{(F(id_U), F(id_U))} F(U) \times F(U) \xrightarrow{\text{pr}_1 \xi \times \text{pr}_2 \xi} X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})) \xrightarrow{\text{ev}} Y$$

Since this composition coincides with  $\text{ev} \xi$ , we see that  $\xi \in \mathcal{E}^{\text{ev}}$  holds. Hence we have  $\mathcal{E}^{\text{ev}} \supset \mathcal{D}^{\text{pr}_1} \cap (\mathcal{E}^{\mathcal{D}})^{\text{pr}_2}$  and  $\mathcal{E}^{\mathcal{D}}$  is an element of  $\Sigma_{\mathcal{D}, \mathcal{E}}$ .

For  $\mathcal{F} \in \Sigma_{\mathcal{D}, \mathcal{E}}$  and  $W \in \text{Ob } \mathcal{C}$ , since  $\text{ev} : (X, \mathcal{D}) \times (\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})), \mathcal{F}) \rightarrow (Y, \mathcal{E})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ ,  $(F_{\text{ev}})_W : F_{X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(W) \rightarrow F_Y(W)$  maps  $\mathcal{D}^{\text{pr}_1} \cap \mathcal{F}^{\text{pr}_2} \cap F_{X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(W)$  into  $\mathcal{E} \cap F_Y(W)$ . For  $\varphi \in \mathcal{F} \cap F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(U)$ , we take  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$ ,  $g \in \mathcal{C}(W, V)$  and  $\psi \in \mathcal{D} \cap F_X(V)$ . Then, we have  $\varphi F(f) = F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(f)(\varphi) \in \mathcal{F} \cap F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(W)$  and  $\psi F(g) = F_X(g)(\psi) \in \mathcal{D} \cap F_X(W)$  which implies  $(\psi F(g), \varphi F(f)) \in \mathcal{D}^{\text{pr}_1} \cap \mathcal{F}^{\text{pr}_2} \cap F_{X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(W)$ . It follows that a composition  $F(W) \xrightarrow{(F(g), F(f))} F(V) \times F(U) \xrightarrow{\psi \times \varphi} X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})) \xrightarrow{\text{ev}} Y$  belongs to  $\mathcal{E} \cap F_Y(W)$ . Therefore  $\varphi \in \mathcal{E}^{\mathcal{D}}$  holds and this shows  $\mathcal{F} \subset \mathcal{E}^{\mathcal{D}}$ . Thus  $\mathcal{E}^{\mathcal{D}}$  is maximum element of  $\Sigma_{\mathcal{D}, \mathcal{E}}$ .  $\square$

**Lemma 2.20** Let  $(X, \mathcal{D})$  be an object of  $\mathcal{P}_F(\mathcal{C}, J)$  and  $\xi : (Y, \mathcal{E}) \rightarrow (Z, \mathcal{F})$  a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

(1)  $id_X \times \xi : X \times Y \rightarrow X \times Z$  defines a morphism  $id_X \times \xi : (X, \mathcal{D}) \times (Y, \mathcal{E}) \rightarrow (X, \mathcal{D}) \times (Z, \mathcal{F})$  in  $\mathcal{P}_F(\mathcal{C}, J)$ .

(2) A map  $\xi_* : \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})) \rightarrow \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Z, \mathcal{F}))$  defined by  $\xi_*(\alpha) = \xi \alpha$  defines a morphism  $\xi_* : (\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})), \mathcal{E}^{\mathcal{D}}) \rightarrow (\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Z, \mathcal{F})), \mathcal{F}^{\mathcal{D}})$  in  $\mathcal{P}_F(\mathcal{C}, J)$ .

(3) A map  $\xi^* : \mathcal{P}_F(\mathcal{C}, J)((Z, \mathcal{F}), (X, \mathcal{D})) \rightarrow \mathcal{P}_F(\mathcal{C}, J)((Y, \mathcal{E}), (X, \mathcal{D}))$  defined by  $\xi^*(\alpha) = \alpha \xi$  defines a morphism  $\xi^* : (\mathcal{P}_F(\mathcal{C}, J)((Z, \mathcal{F}), (X, \mathcal{D})), \mathcal{D}^{\mathcal{F}}) \rightarrow (\mathcal{P}_F(\mathcal{C}, J)((Y, \mathcal{E}), (X, \mathcal{D})), \mathcal{D}^{\mathcal{E}})$  in  $\mathcal{P}_F(\mathcal{C}, J)$ .

*Proof.* (1) We denote by  $\text{pr}'_X : X \times Z \rightarrow X$  and  $\text{pr}'_Z : X \times Z \rightarrow Z$  the projections. Since  $\text{pr}'_X(id_X \times \xi) = \text{pr}_X$  and  $\text{pr}'_Z(id_X \times \xi) = \xi \text{pr}_Y$ , the following equalities hold for  $U \in \text{Ob } \mathcal{C}$  and  $\varphi \in \mathcal{D}^{\text{pr}_X} \cap \mathcal{E}^{\text{pr}_Y} \cap F_{X \times Y}(U)$ .

$$(F_{\text{pr}'_X})_U(F_{id_X \times \xi})_U(\varphi) = (F_{\text{pr}_X})_U(\varphi) \in \mathcal{D} \cap F_X(U), \quad (F_{\text{pr}'_Z})_U(F_{id_X \times \xi})_U(\varphi) = (F_{\xi})_U(F_{\text{pr}_Y})_U(\varphi) \in \mathcal{F} \cap F_Z(U)$$

Hence  $(F_{id_X \times \xi})_U : F_{X \times Y}(U) \rightarrow F_{X \times Z}(U)$  maps  $\mathcal{D}^{\text{pr}_X} \cap \mathcal{E}^{\text{pr}_Y} \cap F_{X \times Y}(U)$  into  $\mathcal{D}^{\text{pr}'_X} \cap \mathcal{F}^{\text{pr}'_Z} \cap F_{X \times Z}(U)$ . Thus  $id_X \times \xi : (X, \mathcal{D}) \times (Y, \mathcal{E}) = (X \times Y, \mathcal{D}^{\text{pr}_X} \cap \mathcal{E}^{\text{pr}_Y}) \rightarrow (X \times Z, \mathcal{D}^{\text{pr}'_X} \cap \mathcal{F}^{\text{pr}'_Z}) = (X, \mathcal{D}) \times (Z, \mathcal{F})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

(2) For  $U \in \text{Ob } \mathcal{C}$  and  $\varphi \in \mathcal{E}^{\mathcal{D}} \cap F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(U)$ , we take  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$ ,  $g \in \mathcal{C}(W, V)$  and  $\psi \in \mathcal{D} \cap F_X(V)$ . Since a composition  $F(W) \xrightarrow{(F(g), F(f))} F(V) \times F(U) \xrightarrow{\psi \times \varphi} X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})) \xrightarrow{\text{ev}} Y$  belongs to  $\mathcal{E} \cap F_Y(W)$ , and  $\xi$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ , the composition of the upper row of the following diagram belongs to  $\mathcal{F} \cap F_Z(W)$  by the commutativity of the diagram.

$$\begin{array}{ccccc} F(W) & \xrightarrow{(F(g), F(f))} & F(V) \times F(U) & \xrightarrow{\psi \times (F_{\xi_*})_U(\varphi)} & X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Z, \mathcal{F})) & \xrightarrow{\text{ev}} & Z \\ & & \downarrow \psi \times \varphi & & & \nearrow \xi & \\ & & X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})) & \xrightarrow{\text{ev}} & Y & & \end{array}$$

Hence  $(F_{\xi_*})_U : F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(U) \rightarrow F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Z, \mathcal{F}))}(U)$  maps  $\mathcal{E}^{\mathcal{D}} \cap F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))}(U)$  into  $\mathcal{F}^{\mathcal{D}} \cap F_{\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Z, \mathcal{F}))}(U)$ . Thus  $\xi_* : (\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})), \mathcal{E}^{\mathcal{D}}) \rightarrow (\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Z, \mathcal{F})), \mathcal{F}^{\mathcal{D}})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

(3) For  $U \in \text{Ob } \mathcal{C}$  and  $\varphi \in \mathcal{D}^{\mathcal{F}} \cap F_{\mathcal{P}_F(\mathcal{C}, J)((Z, \mathcal{F}), (X, \mathcal{D}))}(U)$ , we take  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$ ,  $g \in \mathcal{C}(W, V)$  and  $\psi \in \mathcal{E} \cap F_Y(V)$ . Since  $\xi$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ , we have  $(F_{\xi})_V(\psi) \in \mathcal{F} \cap F_Z(V)$  and this implies that a composition  $F(W) \xrightarrow{(F(g), F(f))} F(V) \times F(U) \xrightarrow{(F_{\xi})_V(\psi) \times \varphi} Z \times \mathcal{P}_F(\mathcal{C}, J)((Z, \mathcal{F}), (X, \mathcal{D})) \xrightarrow{\text{ev}} X$  belongs to  $\mathcal{D} \cap F_X(W)$ . Thus the composition of the upper row of the following diagram belongs to  $\mathcal{D} \cap F_X(W)$  by the commutativity of the diagram.

$$\begin{array}{ccccc} F(W) & \xrightarrow{(F(g), F(f))} & F(V) \times F(U) & \xrightarrow{\psi \times (F_{\xi_*})_U(\varphi)} & Y \times \mathcal{P}_F(\mathcal{C}, J)((Y, \mathcal{E}), (X, \mathcal{D})) & \xrightarrow{\text{ev}} & X \\ & & & \searrow (F_{\xi})_V(\psi) \times \varphi & & \nearrow \text{ev} & \\ & & & & Y \times \mathcal{P}_F(\mathcal{C}, J)((Z, \mathcal{F}), (X, \mathcal{D})) & & \end{array}$$

Hence  $(F_{\xi_*})_U : F_{\mathcal{P}_F(\mathcal{C}, J)((Z, \mathcal{F}), (X, \mathcal{D}))}(U) \rightarrow F_{\mathcal{P}_F(\mathcal{C}, J)((Y, \mathcal{E}), (X, \mathcal{D}))}(U)$  maps  $\mathcal{D}^{\mathcal{F}} \cap F_{\mathcal{P}_F(\mathcal{C}, J)((Z, \mathcal{F}), (X, \mathcal{D}))}(U)$  into  $\mathcal{E}^{\mathcal{E}} \cap F_{\mathcal{P}_F(\mathcal{C}, J)((Y, \mathcal{E}), (X, \mathcal{D}))}(U)$ . Thus  $\xi^* : (\mathcal{P}_F(\mathcal{C}, J)((Z, \mathcal{F}), (X, \mathcal{D})), \mathcal{D}^{\mathcal{F}}) \rightarrow (\mathcal{P}_F(\mathcal{C}, J)((Y, \mathcal{E}), (X, \mathcal{D})), \mathcal{E}^{\mathcal{E}})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .  $\square$

For objects  $(X, \mathcal{D})$ ,  $(Y, \mathcal{E})$  of  $\mathcal{P}_F(\mathcal{C}, J)$  and  $y \in Y$ , we define a map  $\iota_y : X \rightarrow X \times Y$  by  $\iota_y(x) = (x, y)$ . We denote by  $\text{pr}_X : X \times Y \rightarrow X$  and  $\text{pr}_Y : X \times Y \rightarrow Y$  the projections. Since  $\text{pr}_X \iota_y$  is the identity map of  $X$  and  $\text{pr}_Y \iota_y$  is the constant map whose image is  $\{y\}$ ,  $(F_{\text{pr}_X})_U(F_{\iota_y})_U : F_X(U) \rightarrow F_X(U)$  maps  $\mathcal{D} \cap F_X(U)$  to  $\mathcal{D} \cap F_X(U)$  and  $(F_{\text{pr}_Y})_U(F_{\iota_y})_U : F_X(U) \rightarrow F_Y(U)$  maps  $\mathcal{D} \cap F_X(U)$  to  $\mathcal{E} \cap F_Y(U)$  for any  $U \in \text{Ob } \mathcal{C}$ . Therefore  $(F_{\iota_y})_U : F_X(U) \rightarrow F_{X \times Y}(U)$  maps  $\mathcal{D} \cap F_X(U)$  to  $\mathcal{D}^{\text{pr}_X} \cap \mathcal{E}^{\text{pr}_Y} \cap F_{X \times Y}(U)$ , that is,  $\iota_y$  belongs to  $\mathcal{P}_F(\mathcal{C}, J)((Y, \mathcal{E}), (X \times Y, \mathcal{D}^{\text{pr}_X} \cap \mathcal{E}^{\text{pr}_Y}))$ . Thus a map  $\eta : Y \rightarrow \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (X \times Y, \mathcal{D}^{\text{pr}_X} \cap \mathcal{E}^{\text{pr}_Y}))$  is defined by  $\eta(y) = \iota_y$ .

**Lemma 2.21** *The map  $\eta : Y \rightarrow \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (X \times Y, \mathcal{D}^{\text{pr}_X} \cap \mathcal{E}^{\text{pr}_Y}))$  defined above defines a morphism  $\eta : (Y, \mathcal{E}) \rightarrow (X \times Y, \mathcal{D}^{\text{pr}_X} \cap \mathcal{E}^{\text{pr}_Y})^{(X, \mathcal{D})} = ((X, \mathcal{D}) \times (Y, \mathcal{E}))^{(X, \mathcal{D})}$  in  $\mathcal{P}_F(\mathcal{C}, J)$ .*

*Proof.* It suffices to verify that  $(F_{\eta})_U(\varphi) \in (\mathcal{D}^{\text{pr}_X} \cap \mathcal{E}^{\text{pr}_Y})^{\mathcal{D}}$  holds for any  $U \in \text{Ob } \mathcal{C}$  and  $\varphi \in \mathcal{E} \cap F_Y(U)$ . We take  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$ ,  $g \in \mathcal{C}(W, V)$  and  $\psi \in \mathcal{D} \cap F_X(V)$ . The image of  $u \in F(W)$  by the following composition is  $\text{ev}(\psi(gu), \iota_{\varphi}(fu)) = (\psi(gu), \varphi(fu)) = (F_X(g)(\psi)(u), F_Y(f)(\varphi)(u))$ .

$$F(W) \xrightarrow{(F(g), F(f))} F(V) \times F(U) \xrightarrow{\psi \times (F_{\eta})_U(\varphi)} X \times \mathcal{P}_F(\mathcal{C}, J)((Y, \mathcal{E}), (X \times Y, \mathcal{D}^{\text{pr}_X} \cap \mathcal{E}^{\text{pr}_Y})) \xrightarrow{\text{ev}} X \times Y$$

Hence the following diagram is commutative.

$$\begin{array}{ccccc} & & X & \xleftarrow{\text{pr}_X} & X \times Y \\ & \nearrow F_X(f)(\varphi) & & \searrow & \\ F(W) & \xrightarrow{(F(g), F(f))} & F(V) \times F(U) & \xrightarrow{\psi \times (F_{\eta})_U(\varphi)} & X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (X \times Y, \mathcal{D}^{\text{pr}_X} \cap \mathcal{E}^{\text{pr}_Y})) & \xrightarrow{\text{ev}} & X \times Y \\ & \searrow F_Y(g)(\psi) & & \nearrow & \\ & & Y & \xleftarrow{\text{pr}_Y} & X \times Y \end{array}$$

Since  $F_X(f)(\varphi) \in \mathcal{D} \cap F_X(W)$  and  $F_Y(g)(\psi) \in \mathcal{E} \cap F_Y(W)$ , the composition of the middle row of the above map belongs to  $\mathcal{D}^{\text{pr}_X} \cap \mathcal{E}^{\text{pr}_Y} \cap F_{X \times Y}(W)$ .  $\square$

For an object  $(X, \mathcal{D})$ , we define functors  $P_{(X, \mathcal{D})}, E_{(X, \mathcal{D})} : \mathcal{P}_F(\mathcal{C}, J) \rightarrow \mathcal{P}_F(\mathcal{C}, J)$  as follows. We put

$$\begin{aligned} P_{(X, \mathcal{D})}(Y, \mathcal{E}) &= (X, \mathcal{D}) \times (Y, \mathcal{E}) = (X \times Y, \mathcal{D}^{\text{pr}_X} \cap \mathcal{E}^{\text{pr}_Y}) & P_{(X, \mathcal{D})}(\xi) &= id_X \times \xi \\ E_{(X, \mathcal{D})}(Y, \mathcal{E}) &= (Y, \mathcal{E})^{(X, \mathcal{D})} = (\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})), \mathcal{E}^{\mathcal{D}}) & E_{(X, \mathcal{D})}(\xi) &= \xi_* \end{aligned}$$

for an object  $(Y, \mathcal{E})$  of  $\mathcal{P}_F(\mathcal{C}, J)$  and a morphism  $\xi : (Y, \mathcal{E}) \rightarrow (Z, \mathcal{F})$  in  $\mathcal{P}_F(\mathcal{C}, J)$ . Then, the following maps define natural transformations  $\text{ev}_{(X, \mathcal{D})} : P_{(X, \mathcal{D})}E_{(X, \mathcal{D})} \rightarrow id_{\mathcal{P}_F(\mathcal{C}, J)}$  and  $\eta_{(X, \mathcal{D})} : id_{\mathcal{P}_F(\mathcal{C}, J)} \rightarrow E_{(X, \mathcal{D})}P_{(X, \mathcal{D})}$ .

$$\begin{aligned} \text{ev} &= (\text{ev}_{(X, \mathcal{D})})_{(Y, \mathcal{E})} : P_{(X, \mathcal{D})}E_{(X, \mathcal{D})}(Y, \mathcal{E}) = (X, \mathcal{D}) \times (Y, \mathcal{E})^{(X, \mathcal{D})} \rightarrow (Y, \mathcal{E}) \\ \eta &= (\eta_{(X, \mathcal{D})})_{(Y, \mathcal{E})} : (Y, \mathcal{E}) \rightarrow ((X, \mathcal{D}) \times (Y, \mathcal{E}))^{(X, \mathcal{D})} = E_{(X, \mathcal{D})}P_{(X, \mathcal{D})}(Y, \mathcal{E}) \end{aligned}$$

**Proposition 2.22**  $\mathcal{P}_F(\mathcal{C}, J)$  is cartesian closed.

*Proof.* Let  $(X, \mathcal{D})$  and  $(Y, \mathcal{E})$  be objects of  $\mathcal{P}_F(\mathcal{C}, J)$ . It is easy to verify that the following composition is the identity map of  $X \times Y$ .

$$P_{(X, \mathcal{D})}(Y, \mathcal{E}) \xrightarrow{P_{(X, \mathcal{D})}((\eta_{(X, \mathcal{D})})_{(Y, \mathcal{E})})} P_{(X, \mathcal{D})}E_{(X, \mathcal{D})}P_{(X, \mathcal{D})}(Y, \mathcal{E}) \xrightarrow{(\text{ev}_{(X, \mathcal{D})})_{P_{(X, \mathcal{D})}(Y, \mathcal{E})}} P_{(X, \mathcal{D})}(Y, \mathcal{E})$$

Let  $\text{pr}_1 : X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})) \rightarrow X$  and  $\text{pr}_2 : X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})) \rightarrow \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E}))$  be the projections. Then, the underlying set of  $E_{(X, \mathcal{D})}P_{(X, \mathcal{D})}E_{(X, \mathcal{D})}(Y, \mathcal{E})$  is

$$\mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (X \times \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{D}), (Y, \mathcal{E})), \mathcal{D}^{\text{pr}_1} \cap (\mathcal{E}^{\mathcal{D}})^{\text{pr}_2}).$$

For  $\varphi \in E_{(X, \mathcal{D})}(Y, \mathcal{E})$ , since  $(\text{ev}_{(X, \mathcal{D})})_{(Y, \mathcal{E})} \iota_\varphi : X \rightarrow Y$  maps  $x \in X$  to  $\varphi(x)$ , we have  $(\text{ev}_{(X, \mathcal{D})})_{(Y, \mathcal{E})} \iota_\varphi = \varphi$ , which implies that the following composition is the identity map of  $E_{(X, \mathcal{D})}(Y, \mathcal{E})$ .

$$E_{(X, \mathcal{D})}(Y, \mathcal{E}) \xrightarrow{(\eta_{(X, \mathcal{D})})_{E_{(X, \mathcal{D})}(Y, \mathcal{E})}} E_{(X, \mathcal{D})}P_{(X, \mathcal{D})}E_{(X, \mathcal{D})}(Y, \mathcal{E}) \xrightarrow{E_{(X, \mathcal{D})}((\text{ev}_{(X, \mathcal{D})})_{(Y, \mathcal{E})})} E_{(X, \mathcal{D})}(Y, \mathcal{E})$$

Therefore,  $E_{(X, \mathcal{D})}$  is a right adjoint of  $P_{(X, \mathcal{D})}$  with unit  $\eta_{(X, \mathcal{D})}$  and counit  $\text{ev}_{(X, \mathcal{D})}$ .  $\square$

### 3 Locally cartesian closedness

For a category  $\mathcal{E}$ , let  $\mathcal{E}^{(2)}$  be the category of morphisms in  $\mathcal{E}$  defined as follows. Put  $\text{Ob } \mathcal{E}^{(2)} = \text{Mor } \mathcal{E}$  and a morphism from  $\mathbf{E} = (E \xrightarrow{\pi} X)$  to  $\mathbf{F} = (F \xrightarrow{\rho} Y)$  is a pair  $\langle \xi : E \rightarrow F, f : X \rightarrow Y \rangle$  of morphisms in  $\mathcal{E}$  which satisfies  $\rho\xi = f\pi$ . The composition of morphisms  $\langle \xi, f \rangle : \mathbf{E} \rightarrow \mathbf{F}$  and  $\langle \zeta, g \rangle : \mathbf{F} \rightarrow \mathbf{G}$  is defined to be  $\langle \zeta\xi, gf \rangle : \mathbf{E} \rightarrow \mathbf{G}$ . We define a functor  $\wp : \mathcal{E}^{(2)} \rightarrow \mathcal{E}$  by  $\wp(E \xrightarrow{\pi} X) = X$  and  $\wp(\langle \xi, f \rangle) = f$ . For an object  $X$  of  $\mathcal{E}$ , we denote by  $\mathcal{E}_X^{(2)}$  a subcategory of  $\mathcal{E}^{(2)}$  given as follows. We mention that  $\mathcal{E}_X^{(2)}$  is often denoted by  $\mathcal{E}/X$  in literatures.

$$\text{Ob } \mathcal{E}_X^{(2)} = \{\mathbf{E} \in \text{Ob } \mathcal{E}^{(2)} \mid \wp(\mathbf{E}) = X\}, \quad \text{Mor } \mathcal{E}_X^{(2)} = \{\xi \in \text{Mor } \mathcal{E}^{(2)} \mid \wp(\xi) = \text{id}_X\}$$

For a morphism  $f : X \rightarrow Y$  in  $\mathcal{E}$ , an object  $\mathbf{E}$  of  $\mathcal{E}_X^{(2)}$  and an object  $\mathbf{F}$  of  $\mathcal{E}_Y^{(2)}$ , we denote by  $\mathcal{E}_f^{(2)}(\mathbf{E}, \mathbf{F})$  the set of all morphisms  $\xi : \mathbf{E} \rightarrow \mathbf{F}$  in  $\mathcal{E}^{(2)}$  such that  $\wp(\xi) = f$ .

If  $\mathcal{E}$  has finite limits,  $\wp : \mathcal{E}^{(2)} \rightarrow \mathcal{E}$  is a fibered category as we explain below. For a morphism  $f : X \rightarrow Y$  in  $\mathcal{E}$  and an object  $\mathbf{F} = (F \xrightarrow{\rho} Y)$  of  $\mathcal{E}_Y^{(2)}$ , consider the following cartesian square in  $\mathcal{E}$ .

$$\begin{array}{ccc} F \times_Y X & \xrightarrow{f_\rho} & F \\ \downarrow \rho_f & & \downarrow \rho \\ X & \xrightarrow{f} & Y \end{array}$$

We put  $f^*(\mathbf{F}) = (F \times_Y X \xrightarrow{\rho_f} X)$  and  $\alpha_f(\mathbf{F}) = \langle f_\rho, f \rangle : f^*(\mathbf{F}) \rightarrow \mathbf{F}$ . The following result is straightforward from the definition of cartesian square.

**Proposition 3.1**  $\alpha_f(\mathbf{F})$  is a cartesian morphism, that is, for any object  $\mathbf{G}$  of  $\mathcal{E}_X^{(2)}$  the map

$$\alpha_f(\mathbf{F})_* : \mathcal{E}_X^{(2)}(\mathbf{G}, f^*(\mathbf{F})) \rightarrow \mathcal{E}_f^{(2)}(\mathbf{G}, \mathbf{F})$$

defined by  $\alpha_f(\mathbf{F})_*(\xi) = \alpha_f(\mathbf{F})\xi$  is bijective.

**Remark 3.2** For the identity morphism  $\text{id}_X$  of  $X \in \text{Ob } \mathcal{E}$  and  $\mathbf{F} \in \text{Ob } \mathcal{E}_X^{(2)}$ , the identity morphism  $\text{id}_{\mathbf{F}}$  of  $\mathbf{F}$  is obviously cartesian. In this case, we can regard  $F$  as  $F \times_X X$  and identify  $\text{id}_X^*(\mathbf{F})$  with  $\mathbf{F}$ . Hence  $\alpha_{\text{id}_X}(\mathbf{F}) : \text{id}_X^*(\mathbf{F}) \rightarrow \mathbf{F}$  is the identity morphism of  $\mathbf{F}$ .

For objects  $\mathbf{E}, \mathbf{F}$  of  $\mathcal{E}_Y^{(2)}$  and a morphism  $\varphi : \mathbf{E} \rightarrow \mathbf{F}$  in  $\mathcal{E}_Y^{(2)}$ , let  $f^*(\varphi) : f^*(\mathbf{E}) \rightarrow f^*(\mathbf{F})$  be the unique morphism in  $\mathcal{E}_X^{(2)}$  that is mapped to a composition  $f^*(\mathbf{E}) \xrightarrow{\alpha_f(\mathbf{E})} \mathbf{E} \xrightarrow{\varphi} \mathbf{F}$  by the following bijection given in (3.1).

$$\alpha_f(\mathbf{F})_* : \mathcal{E}_X^{(2)}(f^*(\mathbf{E}), f^*(\mathbf{F})) \rightarrow \mathcal{E}_f^{(2)}(f^*(\mathbf{E}), \mathbf{F})$$

Thus we have the inverse image functor  $f^* : \mathcal{E}_Y^{(2)} \rightarrow \mathcal{E}_X^{(2)}$  associated with a morphism  $f : X \rightarrow Y$  in  $\mathcal{E}$ . It follows from the definition of  $f^*$  that the bijection in (3.1) is natural in  $\mathbf{F}$ .

For morphisms  $f : X \rightarrow Y$ ,  $g : Z \rightarrow X$  in  $\mathcal{E}$  and an object  $\mathbf{E}$  of  $\mathcal{E}_Y^{(2)}$ , let  $\mathbf{c}_{f,g}(\mathbf{E}) : g^*(f^*(\mathbf{E})) \rightarrow (fg)^*(\mathbf{E})$  be the unique morphism in  $\mathcal{E}_Z^{(2)}$  that is mapped to a composition  $g^*(f^*(\mathbf{E})) \xrightarrow{\alpha_g(f^*(\mathbf{E}))} f^*(\mathbf{E}) \xrightarrow{\alpha_f(\mathbf{E})} \mathbf{E}$  by the following bijection given in (3.1).

$$\alpha_{fg}(\mathbf{E})_* : \mathcal{E}_Z^{(2)}(g^*(f^*(\mathbf{E})), (fg)^*(\mathbf{E})) \rightarrow \mathcal{E}_{fg}^{(2)}(g^*(f^*(\mathbf{E})), \mathbf{E})$$

**Proposition 3.3**  $\mathbf{c}_{f,g}(\mathbf{E})$  is an isomorphism in  $\mathcal{E}_Z^{(2)}$ . Hence  $\wp : \mathcal{E}^{(2)} \rightarrow \mathcal{E}$  is a fibered category.

*Proof.* We consider the following diagrams in  $\mathcal{E}$  such that the left and right rectangles of the left diagram (i) and the right diagram (ii) are cartesian.

$$(i) \quad \begin{array}{ccccc} (E \times_Y X) \times_X Z & \xrightarrow{g\pi_f} & E \times_Y X & \xrightarrow{f\pi} & E \\ \downarrow (\pi_f)_g & & \downarrow \pi_f & & \downarrow \pi \\ Z & \xrightarrow{g} & X & \xrightarrow{f} & Y \end{array} \quad (ii) \quad \begin{array}{ccc} E \times_Y Z & \xrightarrow{(fg)\pi} & E \\ \downarrow \pi_{fg} & & \downarrow \pi \\ Z & \xrightarrow{fg} & Y \end{array}$$

Hence there exists unique morphism  $\mathbf{c}_{f,g}(\mathbf{E}) : (E \times_Y X) \times_X Z \rightarrow E \times_Y Z$  that makes the following diagram commute.

$$\begin{array}{ccc} (E \times_Y X) \times_X Z & \xrightarrow{g\pi_f} & E \times_Y X \\ \searrow \mathbf{c}_{f,g}(\mathbf{E}) & & \downarrow f\pi \\ & & E \times_Y Z \xrightarrow{(fg)\pi} E \\ \searrow (\pi_f)_g & & \downarrow \pi \\ & & Z \xrightarrow{fg} Y \end{array}$$

Since the outer rectangle of diagram (i) is also cartesian, it follows that  $\mathbf{c}_{f,g}(\mathbf{E})$  is an isomorphism. Since  $\alpha_f(\mathbf{E})\alpha_g(f^*(\mathbf{E})) = \langle f\pi g\pi_f, fg \rangle$  and  $\alpha_{fg}(\mathbf{E}) = \langle (fg)\pi, fg \rangle$ ,  $\alpha_{fg}(\mathbf{E})_*$  maps  $\langle \mathbf{c}_{f,g}(\mathbf{E}), id_Z \rangle$  to  $\alpha_f(\mathbf{E})\alpha_g(f^*(\mathbf{E}))$  by the commutativity of the above diagram. Thus we have  $\mathbf{c}_{f,g}(\mathbf{E}) = \langle \mathbf{c}_{f,g}(\mathbf{E}), id_Z \rangle$  which is an isomorphism.  $\square$

**Remark 3.4** (1) It follows from the definition of  $\mathbf{c}_{f,g}(\mathbf{E})$ , the following diagram is commutative.

$$\begin{array}{ccc} g^*f^*(\mathbf{E}) & \xrightarrow{\alpha_g(f^*(\mathbf{E}))} & f^*(\mathbf{E}) \\ \downarrow \mathbf{c}_{f,g}(\mathbf{E}) & & \downarrow \alpha_f(\mathbf{E}) \\ (fg)^*(\mathbf{E}) & \xrightarrow{\alpha_{fg}(\mathbf{E})} & \mathbf{E} \end{array}$$

Hence we have  $\mathbf{c}_{f,id_X}(\mathbf{E}) = \mathbf{c}_{id_Y,f}(\mathbf{E}) = id_{f^*(\mathbf{E})}$  by (3.2) and the uniqueness of  $\mathbf{c}_{f,g}(\mathbf{E})$ .

(2) There exists unique morphisms  $id_{E \times_Y g} : E \times_Y Z \rightarrow E \times_Y X$  and  $\mathbf{c}_{f,g}(\mathbf{E})^{-1} : E \times_Y Z \rightarrow (E \times_Y X) \times_X Z$  in  $\mathcal{E}$  that makes the following diagram commute. The inverse  $\mathbf{c}_{f,g}(\mathbf{E})^{-1} : (fg)^*(\mathbf{E}) \rightarrow g^*(f^*(\mathbf{E}))$  of  $\mathbf{c}_{f,g}(\mathbf{E})$  is given by  $\mathbf{c}_{f,g}(\mathbf{E})^{-1} = \langle \mathbf{c}_{f,g}(\mathbf{E})^{-1}, id_Z \rangle$ .

$$\begin{array}{ccccc} E \times_Y Z & \xrightarrow{id_{E \times_Y g}} & (E \times_Y X) \times_X Z & \xrightarrow{g\pi_f} & E \times_Y X \xrightarrow{f\pi} E \\ \downarrow \pi_{fg} & \searrow \mathbf{c}_{f,g}(\mathbf{E})^{-1} & \downarrow (\pi_f)_g & \downarrow \pi_f & \downarrow \pi \\ Z & \xrightarrow{fg} & X & \xrightarrow{f} & Y \end{array}$$

The following results are easily verified. In fact, these facts hold for the case that  $\wp$  is a general fibered category ([3]).

**Proposition 3.5** For composable morphisms  $f : X \rightarrow Y$ ,  $g : Z \rightarrow X$  in  $\mathcal{E}$  and a morphism  $\varphi : \mathbf{E} \rightarrow \mathbf{F}$  in  $\mathcal{E}_Y^{(2)}$ , the following diagram commutes. In other words,  $\mathbf{c}_{f,g}$  gives a natural transformation  $g^*f^* \rightarrow (fg)^*$  of functors from  $\mathcal{E}_Y^{(2)}$  to  $\mathcal{E}_Z^{(2)}$ .

$$\begin{array}{ccc}
g^*f^*(\mathbf{E}) & \xrightarrow{c_{f,g}(\mathbf{E})} & (fg)^*(\mathbf{E}) \\
\downarrow g^*f^*(\varphi) & & \downarrow (fg)^*(\varphi) \\
g^*f^*(\mathbf{F}) & \xrightarrow{c_{f,g}(\mathbf{F})} & (fg)^*(\mathbf{F})
\end{array}$$

**Proposition 3.6** ([3], p.172 Proposition 7.4.) For a diagram  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$  in  $\mathcal{E}$  and an object  $\mathbf{E}$  of  $\mathcal{E}_W^{(2)}$ , the following diagram commutes.

$$\begin{array}{ccccc}
f^*(g^*h^*)(\mathbf{E}) & \xrightarrow{f^*(c_{h,g}(\mathbf{E}))} & f^*(hg)^*(\mathbf{E}) & \xrightarrow{c_{hg,f}(\mathbf{E})} & ((hg)f)^*(\mathbf{E}) \\
\parallel & & & & \parallel \\
(f^*g^*)h^*(\mathbf{E}) & \xrightarrow{c_{g,f}(h^*(\mathbf{E}))} & (gf)^*h^*(\mathbf{E}) & \xrightarrow{c_{h,gf}(\mathbf{E})} & (h(gf))^*(\mathbf{E})
\end{array}$$

For a morphism  $f : X \rightarrow Y$  in  $\mathcal{E}$ , define a functor  $f_* : \mathcal{E}_X^{(2)} \rightarrow \mathcal{E}_Y^{(2)}$  as follows. We put  $f_*(\mathbf{E}) = (E \xrightarrow{f\rho} Y)$  for an object  $\mathbf{E} = (E \xrightarrow{\rho} X)$  of  $\mathcal{E}_X^{(2)}$ . We put  $f_*(\langle \xi, id_X \rangle) = \langle \xi, id_Y \rangle : f_*(\mathbf{E}) \rightarrow f_*(\mathbf{F})$  for a morphism  $\langle \xi, id_X \rangle : \mathbf{E} \rightarrow \mathbf{F}$  in  $\mathcal{E}_X^{(2)}$ .

**Proposition 3.7**  $f_* : \mathcal{E}_X^{(2)} \rightarrow \mathcal{E}_Y^{(2)}$  is a left adjoint of  $f^* : \mathcal{E}_Y^{(2)} \rightarrow \mathcal{E}_X^{(2)}$ . Hence  $\varphi : \mathcal{E}^{(2)} \rightarrow \mathcal{E}$  is a bifibered category.

*Proof.* For an object  $\mathbf{E}$  of  $\mathcal{E}_X^{(2)}$  and an object  $\mathbf{F}$  of  $\mathcal{E}_Y^{(2)}$ , we define a map  $\Phi_{\mathbf{E},\mathbf{F}} : \mathcal{E}_f^{(2)}(\mathbf{E}, \mathbf{F}) \rightarrow \mathcal{E}_f^{(2)}(f_*(\mathbf{E}), \mathbf{F})$  by  $\Phi_{\mathbf{E},\mathbf{F}}(\langle \xi, f \rangle) = \langle \xi, id_Y \rangle$ . It is clear that  $\Phi_{\mathbf{E},\mathbf{F}}$  is bijective and natural in  $\mathbf{E}$  and  $\mathbf{F}$ . It follows from (3.1) that we have a bijection  $\Phi_{\mathbf{E},\mathbf{F}}\alpha_f(\mathbf{F})_* : \mathcal{E}_X^{(2)}(\mathbf{E}, f^*(\mathbf{F})) \rightarrow \mathcal{E}_Y^{(2)}(f_*(\mathbf{E}), \mathbf{F})$  which is natural in  $\mathbf{E}$  and  $\mathbf{F}$ .  $\square$

**Remark 3.8** The unit  $\eta : id_{\mathcal{E}_X^{(2)}} \rightarrow f^*f_*$  and the counit  $\varepsilon : f_*f^* \rightarrow id_{\mathcal{E}_Y^{(2)}}$  of the adjunction  $f_* \dashv f^*$  are given as follows. For an object  $\mathbf{E}$  of  $\mathcal{E}_X^{(2)}$ , there exists unique morphism  $\eta_{\mathbf{E}} : \mathbf{E} \rightarrow f^*(f_*(\mathbf{E}))$  in  $\mathcal{E}_X^{(2)}$  such that  $\alpha_f(f_*(\mathbf{E}))_* : \mathcal{E}_X^{(2)}(\mathbf{E}, f^*(f_*(\mathbf{E}))) \rightarrow \mathcal{E}_f^{(2)}(\mathbf{E}, f_*(\mathbf{E}))$  maps  $\eta_{\mathbf{E}}$  to  $(\langle id_{\mathbf{E}}, f \rangle : \mathbf{E} \rightarrow f_*(\mathbf{E})) \in \mathcal{E}_f^{(2)}(\mathbf{E}, f_*(\mathbf{E}))$  by (3.1). It is easy to verify that  $\eta_{\mathbf{E}}$  is natural in  $\mathbf{E}$ . For an object  $\mathbf{F} = (F \xrightarrow{\pi} Y)$  of  $\mathcal{E}_Y^{(2)}$ , consider the following cartesian square.

$$\begin{array}{ccc}
F \times_Y X & \xrightarrow{f\pi} & F \\
\downarrow \pi_f & & \downarrow \pi \\
X & \xrightarrow{f} & Y
\end{array}$$

Then, we have  $f_*(f^*(\mathbf{F})) = (F \times_Y X \xrightarrow{f\pi_f} Y)$  and define  $\varepsilon_{\mathbf{F}} : f_*(f^*(\mathbf{F})) \rightarrow \mathbf{F}$  by  $\varepsilon_{\mathbf{F}} = \langle f\pi, id_Y \rangle$ .

$\mathcal{P}_F(\mathcal{C}, J)$  is complete and cocomplete by (2.11) and (2.14), in particular  $\mathcal{P}_F(\mathcal{C}, J)$  has finite limits. Hence we can consider the fibered category  $\varphi : \mathcal{P}_F(\mathcal{C}, J)^{(2)} \rightarrow \mathcal{P}_F(\mathcal{C}, J)$  of morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$  by (3.3). It follows from (3.7) that the inverse image functors of this fibered category have left adjoints. We show that the inverse image functors also have right adjoints below.

Let  $\varphi : (X, \mathcal{D}) \rightarrow (Y, \mathcal{F})$  be a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$  and  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (X, \mathcal{D}))$  an object of  $\mathcal{P}_F(\mathcal{C}, J)^{(2)}$ . For  $y \in Y$ , we denote by  $\iota_y : \varphi^{-1}(y) \rightarrow X$  the inclusion map and consider a the-ology  $\mathcal{D}^{\iota_y}$  on  $\varphi^{-1}(y)$ . We define a subset  $E(\varphi; y)$  of  $\mathcal{P}_F(\mathcal{C}, J)((\varphi^{-1}(y), \mathcal{D}^{\iota_y}), (E, \mathcal{E}))$  by  $E(\varphi; y) = \emptyset$  if  $\varphi^{-1}(y) = \emptyset$  and

$$E(\varphi; y) = \{\alpha \in \mathcal{P}_F(\mathcal{C}, J)((\varphi^{-1}(y), \mathcal{D}^{\iota_y}), (E, \mathcal{E})) \mid \pi\alpha = \iota_y\}$$

if  $\varphi^{-1}(y) \neq \emptyset$ . Put  $E(\varphi) = \coprod_{y \in Y} E(\varphi; y)$  and define map  $\varphi_{!E} : E(\varphi) \rightarrow Y$  by  $\varphi_{!E}(\alpha) = y$  if  $\alpha \in E(\varphi; y)$ . Note that the image of  $\varphi_{!E}$  coincides with the image of  $\varphi$ . We consider the following cartesian square (\*) in  $\mathbf{Set}$ .

$$(*) \quad \begin{array}{ccc}
E(\varphi) \times_Y X & \xrightarrow{\tilde{\varphi}_{\mathbf{E}}} & E(\varphi) \\
\downarrow \tilde{\varphi}_{!E} & & \downarrow \varphi_{!E} \\
X & \xrightarrow{\varphi} & Y
\end{array}$$

Define a map  $\varepsilon_{\mathbf{E}}^{\varphi} : E(\varphi) \times_Y X \rightarrow E$  by  $\varepsilon_{\mathbf{E}}^{\varphi}(\alpha, x) = \alpha(x)$  if  $\alpha \in E(\varphi; y)$  and  $x \in \varphi^{-1}(y)$  for  $y \in Y$ . Then,  $\varepsilon_{\mathbf{E}}^{\varphi}$  makes the following diagram commute.

$$\begin{array}{ccc}
E(\varphi) \times_Y X & \xrightarrow{\varepsilon_E^\varphi} & E \\
& \searrow \widetilde{\varphi_{!E}} & \downarrow \pi \\
& & X
\end{array}$$

Let  $\Sigma_{E,\varphi}$  the set of all the-ology  $\mathcal{L}$  on  $E(\varphi)$  such that  $\mathcal{L} \subset \mathcal{F}^{\varphi_{!E}}$  and  $\mathcal{D}^{\widetilde{\varphi_{!E}}} \cap \mathcal{L}^{\widetilde{\varphi_E}} \subset \mathcal{E}^{\varepsilon_E^\varphi}$  hold. We note that  $\mathcal{L} \in \Sigma_{E,\varphi}$  if and only if both  $\varphi_{!E} : (E(\varphi), \mathcal{L}) \rightarrow (Y, \mathcal{F})$  and  $\varepsilon_E^\varphi : (E(\varphi) \times_Y X, \mathcal{D}^{\widetilde{\varphi_{!E}}} \cap \mathcal{L}^{\widetilde{\varphi_E}}) \rightarrow (E, \mathcal{E})$  are morphisms in  $\mathcal{P}_F(\mathcal{C}, \mathcal{J})$ .

**Proposition 3.9**  $\Sigma_{E,\varphi}$  is not empty.

*Proof.* It suffices to show that the discrete the-ology  $\mathcal{D}_{disc,E(\varphi)}$  on  $E(\varphi)$  belongs to  $\Sigma_{E,\varphi}$ . It follows from (1.17) that  $\mathcal{D}_{disc,E(\varphi)} \subset \mathcal{F}^{\varphi_{!E}}$  holds. For  $U \in \text{Ob}\mathcal{C}$ , suppose that  $\psi \in \mathcal{D}^{\widetilde{\varphi_{!E}}} \cap \mathcal{D}_{disc,E(\varphi)}^{\widetilde{\varphi_E}} \cap F_{E(\varphi) \times_Y X}(U)$ . Then, we have  $\widetilde{\varphi_{!E}}\psi \in \mathcal{D} \cap F_X(U)$  and  $\widetilde{\varphi_E}\psi \in \mathcal{D}_{disc,E(\varphi)} \cap F_{E(\varphi)}(U)$ . Hence there exists a covering  $(U_i \xrightarrow{g_i} U)_{i \in I}$  such that  $F_{E(\varphi)}(g_i)(\widetilde{\varphi_E}\psi) : F_{E(\varphi)}(U_i) \rightarrow E(\varphi)$  is a constant map for every  $i \in I$  by (1.17). We denote by  $\alpha_i \in E(\varphi)$  the image of  $F_{E(\varphi)}(g_i)(\widetilde{\varphi_E}\psi)$  and put  $y_i = \varphi_{!E}(\alpha_i)$ . Then we have  $\alpha_i \in E(\varphi; y_i)$  and the image of  $F_X(g_i)(\widetilde{\varphi_{!E}}\psi) = \widetilde{\varphi_{!E}}\psi F(g_i) : F(U_i) \rightarrow X$  is contained in  $\varphi^{-1}(y_i)$ . Hence we have a map  $\xi_i : F(U_i) \rightarrow \varphi^{-1}(y_i)$  satisfying  $\iota_{y_i}\xi_i = F_X(g_i)(\widetilde{\varphi_{!E}}\psi) \in \mathcal{D} \cap F_X(U_i)$ , which shows  $\xi_i \in \mathcal{D}^{\iota_{y_i}} \cap F_{\varphi^{-1}(y_i)}(U_i)$ . Since we have an equality  $F_{E(\varphi) \times_Y X}(g_i)(\psi) = (F_{E(\varphi)}(g_i)(\widetilde{\varphi_E}\psi), \iota_{y_i}\xi_i) : F(U_i) \rightarrow E(\varphi) \times_Y X$ , it follows that the following equality holds.

$$F_E(g_i)(F_{\varepsilon_E^\varphi}(\psi)) = F_{\varepsilon_E^\varphi}(F_{E(\varphi) \times_Y X}(g_i)(\psi)) = \alpha_i \xi_i = F_{\alpha_i}(\xi_i)$$

Since  $\alpha_i : (\varphi^{-1}(y_i), \mathcal{D}^{\iota_{y_i}}) \rightarrow (E, \mathcal{E})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, \mathcal{J})$ , we have  $F_{\alpha_i}(\xi_i) \in \mathcal{E} \cap F_E(U_i)$  for any  $i \in I$ . Therefore  $F_{\varepsilon_E^\varphi}(\psi) \in \mathcal{E} \cap F_E(U)$  holds and we see that  $\mathcal{D}^{\widetilde{\varphi_{!E}}} \cap \mathcal{D}_{disc,E(\varphi)}^{\widetilde{\varphi_E}} \subset \mathcal{E}^{\varepsilon_E^\varphi}$  holds.  $\square$

For  $U \in \text{Ob}\mathcal{C}$ , we consider the following condition (LE) on an element  $\gamma$  of  $F_{E(\varphi)}(U)$ .

(LE) If  $V, W \in \text{Ob}\mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$ ,  $g \in \mathcal{C}(W, V)$  and  $\psi \in \mathcal{D} \cap F_X(V)$  satisfy  $\varphi\psi F(g) = \varphi_{!E}\gamma F(f)$ , a composition  $F(W) \xrightarrow{(\gamma F(f), \psi F(g))} E(\varphi) \times_Y X \xrightarrow{\varepsilon_E^\varphi} E$  belongs to  $\mathcal{E} \cap F_E(W)$  and a composition  $F(U) \xrightarrow{\gamma} E(\varphi) \xrightarrow{\varphi_{!E}} Y$  belongs to  $\mathcal{F} \cap F_Y(U)$ .

Define a set  $\mathcal{D}_{E,\varphi}$  of  $F$ -parametrizations of a set  $E(\varphi)$  so that  $\mathcal{D}_{E,\varphi} \cap F_{E(\varphi)}(U)$  is a subset of  $F_{E(\varphi)}(U)$  consisting of elements which satisfy the above condition (LE) for any  $U \in \text{Ob}\mathcal{C}$ .

**Proposition 3.10**  $\mathcal{D}_{E,\varphi}$  is a the-ology on  $E(\varphi)$ .

*Proof.* Suppose that  $\gamma \in F_{E(\varphi)}(1_{\mathcal{C}})$ ,  $V, W \in \text{Ob}\mathcal{C}$ ,  $g \in \mathcal{C}(W, V)$  and  $\psi \in \mathcal{D} \cap F_X(V)$  satisfy  $\varphi\psi F(g) = \varphi_{!E}\gamma F(o_W)$ . Put  $y_\varphi = \varphi_{!E}(\gamma(*))$ . Then,  $\gamma(*) \in E(\varphi; y_\varphi)$  and  $\gamma(*) : (\varphi^{-1}(y_\varphi), \mathcal{D}^{\iota_{y_\varphi}}) \rightarrow (E, \mathcal{E})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, \mathcal{J})$  and  $\pi\gamma(*) = \iota_{y_\varphi}$  holds. There exists unique map  $\bar{\psi} : F(W) \rightarrow \varphi^{-1}(y_\varphi)$  that satisfies  $\iota_{y_\varphi}\bar{\psi} = \psi F(g) = F_X(g)(\psi)$ . Since  $F_X(g)(\psi) \in \mathcal{D} \cap F_X(W)$ , we have  $\bar{\psi} \in \mathcal{D}^{\iota_{y_\varphi}} \cap F_{\varphi^{-1}(y_\varphi)}(W)$ . This implies  $(F_{\gamma(*)})_W(\bar{\psi}) \in \mathcal{E} \cap F_E(W)$ . On the other hand, a composition  $F(W) \xrightarrow{(\gamma F(o_W), \psi F(g))} E(\varphi) \times_Y X \xrightarrow{\varepsilon_E^\varphi} E$  coincides with  $\gamma(*)\bar{\psi} = (F_{\gamma(*)})_W(\bar{\psi})$  which belongs to  $\mathcal{E} \cap F_E(W)$ . Moreover we have  $\varphi_{!E}\gamma \in F_Y(1_{\mathcal{C}}) \subset \mathcal{F}$ . Hence  $\mathcal{D}_{E,\varphi}$  contains  $F_{E(\varphi)}(1_{\mathcal{C}})$ .

Let  $j : Z \rightarrow U$  be a morphism in  $\mathcal{C}$ . For  $\gamma \in \mathcal{D}_{E,\varphi} \cap F_{E(\varphi)}(U)$ ,  $V, W \in \text{Ob}\mathcal{C}$ ,  $f \in \mathcal{C}(W, Z)$ ,  $g \in \mathcal{C}(W, V)$  and  $\psi \in \mathcal{D} \cap F_X(V)$ , assume that  $\varphi\psi F(g) = \varphi_{!E}F_{E(\varphi)}(j)(\gamma)F(f)$  holds. Since a composition

$$F(W) \xrightarrow{(F_{E(\varphi)}(j)(\gamma)F(f), \psi F(g))} E(\varphi) \times_Y X \xrightarrow{\varepsilon_E^\varphi} E$$

coincides with  $F(W) \xrightarrow{(\gamma F(jf), \psi F(g))} E(\varphi) \times_Y X \xrightarrow{\varepsilon_E^\varphi} E$  which belongs to  $\mathcal{E} \cap F_E(W)$  since  $\gamma \in \mathcal{D}_{E,\varphi} \cap F_{E(\varphi)}(U)$ . Since  $\varphi_{!E}\gamma \in \mathcal{F} \cap F_Y(U)$ ,  $\varphi_{!E}F_{E(\varphi)}(j)(\gamma) = F_Y(j)(\varphi_{!E}\gamma) \in \mathcal{F} \cap F_Y(Z)$  holds. Thus  $F_{E(\varphi)}(j)(\gamma)$  belongs to  $\mathcal{D}_{E,\varphi} \cap F_{E(\varphi)}(Z)$ .

Assume that, for  $\gamma \in F_{E(\varphi)}(U)$ , there exists  $R \in J(U)$  such that  $F_{E(\varphi)}(j)(\gamma)$  belongs to  $\mathcal{D}_{E,\varphi} \cap F_{E(\varphi)}(\text{dom}(j))$  for any  $j \in R$ . Suppose that  $\varphi\psi F(g) = \varphi_{!E}\gamma F(f)$  holds for  $V, W \in \text{Ob}\mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$ ,  $g \in \mathcal{C}(W, V)$  and  $\psi \in \mathcal{D} \cap F_X(V)$ . If we put  $h_f^{-1}(R) = \{i \in \text{Mor}\mathcal{C} \mid \text{codom}(i) = W, fi \in R\}$ , then we have  $h_f^{-1}(R) \in J(W)$  and  $F_{E(\varphi)}(fi)(\gamma) \in \mathcal{D}_{E,\varphi} \cap F_{E(\varphi)}(\text{dom}(i))$  for any  $i \in h_f^{-1}(R)$ . Hence the following composition belongs to  $\mathcal{E} \cap F_E(\text{dom}(i))$  for any  $i \in h_f^{-1}(R)$ .

$$F(\text{dom}(i)) \xrightarrow{(F_{E(\varphi)}(fi)(\gamma), \psi F(gi))} E(\varphi) \times_Y X \xrightarrow{\varepsilon_E^\varphi} E$$

Since the above composition coincides with a composition  $F(\text{dom}(i)) \xrightarrow{F(i)} F(W) \xrightarrow{(\gamma F(f), \psi F(g))} X \times_Y E(\varphi) \xrightarrow{\varepsilon_E^\varphi} E$ , it follows that a composition  $F(W) \xrightarrow{(\gamma F(f), \psi F(g))} X \times_Y E(\varphi) \xrightarrow{\varepsilon_E^\varphi} E$  belongs to  $\mathcal{E} \cap F_E(W)$ . Since  $F_{E(\varphi)}(j)(\gamma)$  belongs to  $\mathcal{D}_{\mathbf{E}, \varphi} \cap F_{E(\varphi)}(\text{dom}(j))$ , we have  $F_Y(j)(\varphi!_E \gamma) = \varphi!_E F_{E(\varphi)}(j)(\gamma) \in \mathcal{F} \cap F_Y(\text{dom}(j))$  for any  $j \in R$ . It follows that  $\varphi!_E \gamma \in \mathcal{F} \cap F_Y(U)$ . Thus we have  $\gamma \in \mathcal{D}_{\mathbf{E}, \varphi} \cap F_{E(\varphi)}(U)$ .  $\square$

**Proposition 3.11**  $\mathcal{D}_{\mathbf{E}, \varphi}$  is maximum element of  $\Sigma_{\mathbf{E}, \varphi}$ .

*Proof.* For  $U \in \text{Ob } \mathcal{C}$  and  $\xi \in \mathcal{D}^{\widetilde{\varphi!_E}} \cap \mathcal{D}_{\mathbf{E}, \varphi}^{\widetilde{\varphi!_E}} \cap F_{E(\varphi) \times_Y X}(U)$ ,  $\varphi \widetilde{\varphi!_E} \xi = \varphi!_E \widetilde{\varphi!_E} \xi$  holds and it follows from  $\widetilde{\varphi!_E} \xi \in \mathcal{D} \cap F_X(U)$  and  $\widetilde{\varphi!_E} \xi \in \mathcal{D}_{\mathbf{E}, \varphi} \cap F_{E(\varphi)}(U)$  that a composition  $F(U) \xrightarrow{(\widetilde{\varphi!_E} \xi, \widetilde{\varphi!_E} \xi)} E(\varphi) \times_Y X \xrightarrow{\varepsilon_E^\varphi} Y$  belongs to  $\mathcal{E} \cap F_Y(U)$ . Since this composition coincides with  $\varepsilon_E^\varphi \xi$ , we see that  $\xi \in \mathcal{E}^{\varepsilon_E^\varphi}$  holds. Hence  $\mathcal{D}^{\widetilde{\varphi!_E}} \cap \mathcal{D}_{\mathbf{E}, \varphi}^{\widetilde{\varphi!_E}}$  is contained in  $\mathcal{E}^{\varepsilon_E^\varphi}$ . It is clear from the definition of  $\mathcal{D}_{\mathbf{E}, \varphi}$  that  $\mathcal{D}_{\mathbf{E}, \varphi}$  is contained in  $\mathcal{F}^{\varphi!_E}$ . Thus  $\mathcal{D}_{\mathbf{E}, \varphi}$  is an element of  $\Sigma_{\mathbf{E}, \varphi}$ .

For  $\mathcal{L} \in \Sigma_{\mathbf{E}, \varphi}$  and  $U \in \text{Ob } \mathcal{C}$ , suppose that  $\gamma \in \mathcal{L} \cap F_{E(\varphi)}(U)$ ,  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$ ,  $g \in \mathcal{C}(W, V)$  and that  $\psi \in \mathcal{D} \cap F_X(V)$  satisfies  $\varphi \psi F(g) = \varphi!_E \gamma F(f)$ . Since  $\mathcal{L} \subset \mathcal{F}^{\varphi!_E}$ , a composition  $F(U) \xrightarrow{\gamma} E(\varphi) \xrightarrow{\varphi!_E} Y$  belongs to  $\mathcal{F} \cap F_Y(U)$ . On the other hand, since  $\widetilde{\varphi!_E}(\gamma F(f), \psi F(g)) = F_X(g)(\psi) \in \mathcal{D} \cap F_X(W)$  and  $\widetilde{\varphi!_E}(\gamma F(f), \psi F(g)) = F_{E(\varphi)}(\gamma) \in \mathcal{L} \cap F_{E(\varphi)}(W)$  hold, we have  $(\gamma F(f), \psi F(g)) \in \mathcal{D}^{\widetilde{\varphi!_E}} \cap \mathcal{L}^{\widetilde{\varphi!_E}} \subset \mathcal{E}^{\varepsilon_E^\varphi}$ . It follows that a composition  $F(W) \xrightarrow{(\gamma F(f), \psi F(g))} E(\varphi) \times_Y X \xrightarrow{\varepsilon_E^\varphi} E$  belongs to  $\mathcal{E} \cap F_E(W)$ . Therefore  $\gamma \in \mathcal{D}_{\mathbf{E}, \varphi}$  holds and this shows  $\mathcal{L} \subset \mathcal{D}_{\mathbf{E}, \varphi}$ . Since  $\mathcal{D}_{\mathbf{E}, \varphi}$  is an element of  $\Sigma_{\mathbf{E}, \varphi}$  by (2.18),  $\mathcal{D}_{\mathbf{E}, \varphi}$  is maximum element of  $\Sigma_{\mathbf{E}, \varphi}$ .  $\square$

Let  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (X, \mathcal{D}))$ ,  $\mathbf{G} = ((G, \mathcal{G}) \xrightarrow{\rho} (X, \mathcal{D}))$  be objects of  $\mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{D})}^{(2)}$  and  $\varphi : (X, \mathcal{D}) \rightarrow (Y, \mathcal{F})$  a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . Let  $\langle \xi, id_X \rangle : \mathbf{E} \rightarrow \mathbf{G}$  be a morphism in  $\mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{D})}^{(2)}$ . If  $\alpha \in E(\varphi; y)$  for  $y \in Y$ , we have  $\rho \xi \alpha = \pi \alpha = \iota_y$ , hence  $\xi \alpha \in G(\varphi; y)$ . Thus we can define a map  $\xi_\varphi : E(\varphi) \rightarrow G(\varphi)$  by  $\xi_\varphi(\alpha) = \xi \alpha$ . We consider the following diagram whose outer trapezoid and lower rectangle are cartesian.

$$\begin{array}{ccccc}
E(\varphi) \times_Y X & \xrightarrow{\quad \widetilde{\varphi!_E} \quad} & & & E(\varphi) \\
& \searrow^{\xi_\varphi \times_Y id_X} & & & \swarrow_{\xi_\varphi} \\
& & G(\varphi) \times_Y X & \xrightarrow{\quad \widetilde{\varphi!_G} \quad} & G(\varphi) \\
& \searrow^{\widetilde{\varphi!_E}} & \downarrow \widetilde{\varphi!_G} & & \downarrow \varphi!_G \\
& & X & \xrightarrow{\quad \varphi \quad} & Y
\end{array}$$

Since the right triangle of the above diagram is commutative, there exists unique map

$$\xi_\varphi \times_Y id_X : E(\varphi) \times_Y X \rightarrow G(\varphi) \times_Y X$$

that makes the above diagram commutative.

**Proposition 3.12**  $\xi_\varphi : (E(\varphi), \mathcal{D}_{\mathbf{E}, \varphi}) \rightarrow (G(\varphi), \mathcal{D}_{\mathbf{G}, \varphi})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$  and the following diagram is commutative.

$$\begin{array}{ccc}
E(\varphi) \times_Y X & \xrightarrow{\varepsilon_E^\varphi} & E \\
\downarrow \xi_\varphi \times_Y id_X & & \downarrow \xi \\
G(\varphi) \times_Y X & \xrightarrow{\varepsilon_G^\varphi} & G
\end{array}$$

*Proof.* It is clear from the definitions of  $\varepsilon_E^\varphi$ ,  $\varepsilon_G^\varphi$  and  $\xi_\varphi$  that the above diagram is commutative. For  $U \in \text{Ob } \mathcal{C}$  and  $\gamma \in \mathcal{D}_{\mathbf{E}, \varphi} \cap F_{E(\varphi)}(U)$ , we take  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$ ,  $g \in \mathcal{C}(W, V)$  and  $\psi \in \mathcal{D} \cap F_X(V)$  satisfy  $\varphi \psi F(g) = \varphi!_E \gamma F(f)$ . Since  $\varphi!_G F_{\xi_\varphi}(\gamma) = F_{\varphi!_G \xi_\varphi}(\gamma) = F_{\varphi!_E}(\gamma) = \varphi!_E \gamma$ ,  $\varphi \psi F(g) = \varphi!_E \gamma F(f)$  holds. It follows from the assumption  $\gamma \in \mathcal{D}_{\mathbf{E}, \varphi} \cap F_{E(\varphi)}(U)$  that a composition  $F(U) \xrightarrow{F_{\xi_\varphi}(\gamma)} G(\varphi) \xrightarrow{\varphi!_G} Y$  belongs to  $\mathcal{F} \cap F_Y(U)$  and that a composition  $F(W) \xrightarrow{(\gamma F(f), \psi F(g))} E(\varphi) \times_Y X \xrightarrow{\varepsilon_E^\varphi} E$  belongs to  $\mathcal{E} \cap F_E(W)$ . We note that the following diagram is commutative.

$$\begin{array}{ccccc}
F(W) & \xrightarrow{(\gamma F(f), \psi F(g))} & E(\varphi) \times_Y X & \xrightarrow{\varepsilon_E^\varphi} & E \\
& \searrow^{(F_{\xi_\varphi}(\gamma) F(f), \psi F(g))} & \downarrow \xi_\varphi \times_Y id_X & & \downarrow \xi \\
& & G(\varphi) \times_Y X & \xrightarrow{\varepsilon_G^\varphi} & G
\end{array}$$



Since  $\xi : (E, \mathcal{E}) \rightarrow (G, \mathcal{G})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ , a composition  $F(W) \xrightarrow{(F_{\xi\varphi}(\gamma)F(f), \psi F(g))} G(\varphi) \times_Y X \xrightarrow{\varepsilon_G^{\mathcal{E}}} E$  belongs to  $\mathcal{E} \cap F_G(W)$  by the commutativity of the above diagram.  $\square$

**Remark 3.13** We note that  $\mathbf{X} = ((X, \mathcal{D}) \xrightarrow{id_X} (X, \mathcal{D}))$  is a terminal object of  $\mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{D})}^{(2)}$ . For  $y \in Y$ , since  $X(\varphi; y) = \{\iota_y\}$  if  $\varphi^{-1}(y)$  is not empty,  $X(\varphi)$  is identified with the image  $\varphi(X)$  of  $\varphi$  and  $\varphi_{!X} : X(\varphi) \rightarrow Y$  is identified with the inclusion map  $\varphi(X) \rightarrow Y$ . For an object  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (X, \mathcal{D}))$  of  $\mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{D})}^{(2)}$ , the map  $\pi_\varphi : E(\varphi) \rightarrow X(\varphi)$  induced by the unique morphism  $\langle \pi, id_X \rangle : \mathbf{E} \rightarrow \mathbf{X}$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{D})}^{(2)}$  maps  $E(\varphi; y)$  to  $\{\iota_y\}$  if  $\varphi^{-1}(y)$  is not empty.

**Remark 3.14** Let  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (X, \mathcal{D}))$ ,  $\mathbf{G} = ((G, \mathcal{G}) \xrightarrow{\rho} (X, \mathcal{D}))$ ,  $\mathbf{H} = ((H, \mathcal{H}) \xrightarrow{\chi} (X, \mathcal{D}))$  be objects of  $\mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{D})}^{(2)}$  and  $\langle \xi, id_X \rangle : \mathbf{E} \rightarrow \mathbf{G}$ ,  $\langle \zeta, id_X \rangle : \mathbf{G} \rightarrow \mathbf{H}$  be morphisms in  $\mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{D})}^{(2)}$ . For a morphism  $\varphi : (X, \mathcal{D}) \rightarrow (Y, \mathcal{F})$ , it follows from the definition of  $\xi_\varphi$  that  $(\zeta\xi)_\varphi : E(\varphi) \rightarrow H(\varphi)$  coincides with a composition  $E(\varphi) \xrightarrow{\xi_\varphi} G(\varphi) \xrightarrow{\zeta_\varphi} H(\varphi)$ . We also note that  $(id_E)_\varphi$  coincides with the identity map of  $E(\varphi)$ .

We define a functor  $\varphi_! : \mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{D})}^{(2)} \rightarrow \mathcal{P}_F(\mathcal{C}, J)_{(Y, \mathcal{F})}^{(2)}$  by putting  $\varphi_!(\mathbf{E}) = ((E(\varphi), \mathcal{D}_{E, \varphi}) \xrightarrow{\varphi_{!E}} (Y, \mathcal{F}))$  for an object  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (X, \mathcal{D}))$  of  $\mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{D})}^{(2)}$  and  $\varphi_!(\langle \xi, id_X \rangle) = \langle \xi_\varphi, id_Y \rangle : \varphi_!(\mathbf{E}) \rightarrow \varphi_!(\mathbf{G})$  for a morphism  $\langle \xi, id_X \rangle : \mathbf{E} \rightarrow \mathbf{G}$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{D})}^{(2)}$ . It follows from (3.11) and (3.12) that we have a natural transformation  $\varepsilon^\varphi : \varphi^* \varphi_! \rightarrow id_{\mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{D})}^{(2)}}$  defined by

$$\varepsilon_{\mathbf{E}}^\varphi = \langle \varepsilon_{\mathbf{E}}^\varphi, id_X \rangle : ((E(\varphi) \times_Y X, \mathcal{D}_{E, \varphi}^{\mathbf{E}} \cap \mathcal{D}_{\widetilde{\varphi_{!E}}}^{\mathbf{E}}) \xrightarrow{\widetilde{\varphi_{!E}}} (X, \mathcal{D})) \rightarrow ((E, \mathcal{E}) \xrightarrow{\pi} (X, \mathcal{D})).$$

For an object  $\mathbf{G} = ((G, \mathcal{G}) \xrightarrow{\rho} (Y, \mathcal{F}))$  of  $\mathcal{P}_F(\mathcal{C}, J)_{(Y, \mathcal{F})}^{(2)}$ , we consider the following cartesian square in  $\mathcal{P}_F(\mathcal{C}, J)$ .

$$\begin{array}{ccc} (G \times_Y X, \mathcal{G}^{\varphi\rho} \cap \mathcal{D}^{\rho\varphi}) & \xrightarrow{\varphi\rho} & (G, \mathcal{G}) \\ \downarrow \rho_\varphi & & \downarrow \rho \\ (X, \mathcal{D}) & \xrightarrow{\varphi} & (Y, \mathcal{F}) \end{array}$$

Then,  $\varphi^*(\mathbf{G}) = ((G \times_Y X, \mathcal{G}^{\varphi\rho} \cap \mathcal{D}^{\rho\varphi}) \xrightarrow{\rho_\varphi} (X, \mathcal{D}))$  and  $(G \times_Y X)(\varphi)$  is described as a set as follows.

$$\begin{aligned} (G \times_Y X)(\varphi) &= \coprod_{y \in Y} (G \times_Y X)(\varphi; y) = \coprod_{y \in Y} \{ \alpha \in \mathcal{P}_F(\mathcal{C}, J)((\varphi^{-1}(y), \mathcal{D}^{\iota_y}), (G \times_Y X, \mathcal{D}^{\rho\varphi} \cap \mathcal{G}^{\varphi\rho})) \mid \rho_\varphi \alpha = \iota_y \} \\ &= \coprod_{y \in Y} \{ (\lambda, \iota_y) \in \mathcal{P}_F(\mathcal{C}, J)((\varphi^{-1}(y), \mathcal{D}^{\iota_y}), (G \times_Y X, \mathcal{D}^{\rho\varphi} \cap \mathcal{G}^{\varphi\rho})) \mid \lambda : \varphi^{-1}(y) \rightarrow G \text{ satisfies } \rho\lambda = \varphi\iota_y \} \\ &= \coprod_{y \in Y} \{ (\lambda, \iota_y) \in \mathcal{P}_F(\mathcal{C}, J)((\varphi^{-1}(y), \mathcal{D}^{\iota_y}), (G \times_Y X, \mathcal{D}^{\rho\varphi} \cap \mathcal{G}^{\varphi\rho})) \mid \lambda : \varphi^{-1}(y) \rightarrow G \text{ satisfies } \lambda(\varphi^{-1}(y)) \subset \rho^{-1}(\iota_y) \} \end{aligned}$$

For  $v \in G$ , let us denote by  $c_v : \varphi^{-1}(\rho(v)) \rightarrow G$  the constant map whose image is  $\{v\}$ . Then we have  $c_v(\varphi^{-1}(\rho(v))) = \{v\} \subset \rho^{-1}(\rho(v))$  which implies  $(c_v, \iota_{\rho(v)}) \in (G \times_Y X)(\varphi)$ . Define a map  $\eta_{\mathbf{G}}^\varphi : G \rightarrow (G \times_Y X)(\varphi)$  by  $\eta_{\mathbf{G}}^\varphi(v) = (c_v, \iota_{\rho(v)})$ . Then,  $\eta_{\mathbf{G}}^\varphi$  makes the following diagram commute.

$$\begin{array}{ccc} G & \xrightarrow{\eta_{\mathbf{G}}^\varphi} & (G \times_Y X)(\varphi) \\ & \searrow \rho & \downarrow \varphi_{! \varphi^*(\mathbf{G})} \\ & & Y \end{array}$$

**Proposition 3.15**  $\eta_{\mathbf{G}}^\varphi : (G, \mathcal{G}) \rightarrow ((G \times_Y X)(\varphi), \mathcal{D}_{\varphi^*(\mathbf{G}), \varphi})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

*Proof.* For  $U \in \text{Ob } \mathcal{C}$  and  $\gamma \in \mathcal{G} \cap F_G(U)$ , we take  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$ ,  $g \in \mathcal{C}(W, V)$  and  $\psi \in \mathcal{D} \cap F_X(V)$  such that  $\varphi\psi F(g) = \varphi_{! \varphi^*(\mathbf{G})} F_{\eta_{\mathbf{G}}^\varphi}(\gamma) F(f)$  holds. Since  $F_{\eta_{\mathbf{G}}^\varphi}(\gamma) = \eta_{\mathbf{G}}^\varphi \gamma$ , a composition

$$F(U) \xrightarrow{F_{\eta_{\mathbf{G}}^\varphi}(\gamma)} (G \times_Y X)(\varphi) \xrightarrow{\varphi_{! \varphi^*(\mathbf{G})}} Y$$

coincides with  $\rho\gamma = F_\rho(\gamma)$  which belongs to  $\mathcal{F} \cap F_Y(U)$ . On the other hand, it follows from the definitions of

$\varepsilon_{\varphi^*(\mathbf{G})}^\varphi$  and  $\eta_{\mathbf{G}}^\varphi$  that the following composition coincides with a map  $(\gamma F(f), \psi F(g)) : F(W) \rightarrow G \times_Y X$ .

$$F(W) \xrightarrow{(F_{\eta_{\mathbf{G}}^\varphi}(\gamma)F(f), \psi F(g))} (G \times_Y X)(\varphi) \times_Y X \xrightarrow{\varepsilon_{\varphi^*(\mathbf{G})}^\varphi} G \times_Y X$$

Since  $\gamma \in \mathcal{G} \cap F_G(U)$  and  $\psi \in \mathcal{D} \cap F_X(V)$ ,  $(\gamma F(f), \psi F(g)) = (F_G(f)(\gamma), F_X(g)(\psi)) \in \mathcal{G}^{\varphi\rho} \cap \mathcal{D}^{\rho\varphi} \cap F_{G \times_Y X}(W)$  holds. It follows that  $F_{\eta_{\mathbf{G}}^\varphi}(\gamma)$  belongs to  $\mathcal{D}_{\varphi^*(\mathbf{G}), \varphi} \cap F_{(G \times_Y X)(\varphi)}(U)$ .  $\square$

For objects  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (Y, \mathcal{F}))$ ,  $\mathbf{G} = ((G, \mathcal{G}) \xrightarrow{\rho} (Y, \mathcal{F}))$  of  $\mathcal{P}_F(\mathcal{C}, J)_{(Y, \mathcal{F})}^{(2)}$  and a morphism  $\varphi : (X, \mathcal{D}) \rightarrow (Y, \mathcal{F})$  in  $\mathcal{P}_F(\mathcal{C}, J)$ , we consider the following cartesian squares in  $\mathcal{P}_F(\mathcal{C}, J)$ .

$$\begin{array}{ccc} (E \times_Y X, \mathcal{E}^{\varphi\pi} \cap \mathcal{D}^{\pi\varphi}) & \xrightarrow{\varphi\pi} & (E, \mathcal{E}) & & (G \times_Y X, \mathcal{G}^{\varphi\rho} \cap \mathcal{D}^{\rho\varphi}) & \xrightarrow{\varphi\rho} & (G, \mathcal{G}) \\ \downarrow \pi_\varphi & & \downarrow \pi & & \downarrow \rho_\varphi & & \downarrow \rho \\ (X, \mathcal{D}) & \xrightarrow{\varphi} & (Y, \mathcal{F}) & & (X, \mathcal{D}) & \xrightarrow{\varphi} & (Y, \mathcal{F}) \end{array}$$

Let  $\langle \zeta, id_Y \rangle : \mathbf{E} \rightarrow \mathbf{G}$  be a morphism in  $\mathcal{P}_F(\mathcal{C}, J)_{(Y, \mathcal{F})}^{(2)}$ . Since  $\rho\zeta = \pi$  holds, there exists unique morphism  $\zeta \times_Y id_X : (E \times_Y X, \mathcal{E}^{\varphi\pi} \cap \mathcal{D}^{\pi\varphi}) \rightarrow (G \times_Y X, \mathcal{G}^{\varphi\rho} \cap \mathcal{D}^{\rho\varphi})$  in  $\mathcal{P}_F(\mathcal{C}, J)$  that makes the following diagram commutative.

$$\begin{array}{ccccc} E \times_Y X & \xrightarrow{\varphi\pi} & E & & \\ \downarrow \pi_\varphi & \searrow \zeta \times_Y id_X & \downarrow \zeta & & \\ G \times_Y X & \xrightarrow{\varphi\rho} & G & & \\ \downarrow \rho_\varphi & & \downarrow \rho & & \\ X & \xrightarrow{\varphi} & Y & & \end{array}$$

The following result is easily verified from the definitions of  $\eta_{\mathbf{E}}^\varphi$ ,  $\eta_{\mathbf{G}}^\varphi$  and  $(\zeta \times_Y id_X)_\varphi$ .

**Proposition 3.16** For a morphism  $\langle \zeta, id_Y \rangle : ((E, \mathcal{E}) \xrightarrow{\pi} (Y, \mathcal{F})) \rightarrow ((G, \mathcal{G}) \xrightarrow{\rho} (Y, \mathcal{F}))$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(Y, \mathcal{F})}^{(2)}$ , the following diagram is commutative.

$$\begin{array}{ccc} E & \xrightarrow{\eta_{\mathbf{E}}^\varphi} & (E \times_Y X)(\varphi) \\ \downarrow \zeta & & \downarrow (\zeta \times_Y id_X)_\varphi \\ G & \xrightarrow{\eta_{\mathbf{G}}^\varphi} & (G \times_Y X)(\varphi) \end{array}$$

It follows from (3.15) and (3.16) that there is a natural transformation  $\eta^\varphi : id_{\mathcal{P}_F(\mathcal{C}, J)_{(Y, \mathcal{F})}^{(2)}} \rightarrow \varphi! \varphi^*$  defined by

$$\eta_{\mathbf{G}}^\varphi = \langle \eta_{\mathbf{G}}^\varphi, id_Y \rangle : ((G, \mathcal{G}) \xrightarrow{\rho} (Y, \mathcal{F})) \rightarrow ((G \times_Y X)(\varphi) \xrightarrow{\varphi! \varphi^*(\mathbf{G})} (Y, \mathcal{F}))$$

for an object  $\mathbf{G} = ((G, \mathcal{G}) \xrightarrow{\rho} (Y, \mathcal{F}))$  of  $\mathcal{P}_F(\mathcal{C}, J)_{(Y, \mathcal{F})}^{(2)}$ .

Consider the following diagram, where the outer trapezoid and the lower rectangle are cartesian.

$$\begin{array}{ccccc} G \times_Y X & \xrightarrow{\varphi\rho} & G & & \\ \downarrow \rho_\varphi & \searrow \eta_{\mathbf{G}}^\varphi \times_Y id_X & \downarrow \eta_{\mathbf{G}}^\varphi & & \\ (G \times_Y X)(\varphi) \times_Y X & \xrightarrow{\varphi! \varphi^*(\mathbf{G})} & (G \times_Y X)(\varphi) & & \\ \downarrow (\varphi! \varphi^*(\mathbf{G}))_\varphi & & \downarrow \varphi! \varphi^*(\mathbf{G}) & & \\ X & \xrightarrow{\varphi} & Y & & \end{array}$$

Since the right triangle of the above diagram is commutative, there exists unique map  $\eta_{\mathbf{G}}^\varphi \times_Y id_X : G \times_Y X \rightarrow (G \times_Y X)(\varphi) \times_Y X$  that makes the above diagram commute.

**Lemma 3.17** For an objects  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (X, \mathcal{D}))$ ,  $\mathbf{G} = ((G, \mathcal{G}) \xrightarrow{\rho} (Y, \mathcal{F}))$  of  $\mathcal{P}_F(\mathcal{C}, J)_{(Y, \mathcal{F})}^{(2)}$  and a morphism  $\varphi : (X, \mathcal{D}) \rightarrow (Y, \mathcal{F})$  in  $\mathcal{P}_F(\mathcal{C}, J)$ , the following compositions are both identity maps.

$$E(\varphi) \xrightarrow{\eta_{\varphi_1(\mathbf{E})}^\varphi} (E(\varphi) \times_Y X)(\varphi) \xrightarrow{(\varepsilon_{\mathbf{E}}^\varphi)_\varphi} E(\varphi), \quad G \times_Y X \xrightarrow{\eta_{\mathbf{G}}^\varphi \times_Y id_X} (G \times_Y X)(\varphi) \times_Y X \xrightarrow{\varepsilon_{\varphi^*(\mathbf{G})}^\varphi} G \times_Y X$$

*Proof.* For  $\alpha \in E(\varphi)$ , suppose  $\alpha \in E(\varphi; y)$  for  $y \in Y$ , then the following equality holds for  $x \in \varphi^{-1}(y)$ .

$$((\varepsilon_{\mathbf{E}}^\varphi)_\varphi \eta_{\varphi_1(\mathbf{E})}^\varphi(\alpha))(x) = ((\varepsilon_{\mathbf{E}}^\varphi)_\varphi(c_\alpha, \iota_y))(x) = \varepsilon_{\mathbf{E}}^\varphi(\alpha, x) = \alpha(x)$$

For  $(v, x) \in G \times_Y X$ , then we have  $\rho(v) = \varphi(x)$  and  $v \in \rho^{-1}(\varphi(x))$ . Hence we have the following equality.

$$\varepsilon_{\varphi^*(\mathbf{G})}^\varphi(\eta_{\mathbf{G}}^\varphi \times_Y id_X)(v, x) = \varepsilon_{\varphi^*(\mathbf{G})}^\varphi((c_v, \iota_y), x) = (c_v, \iota_y)(x) = (v, x)$$

Thus the assertion follows.  $\square$

For an object  $\mathbf{G} = ((G, \mathcal{G}) \xrightarrow{\rho} (Y, \mathcal{F}))$  of  $\mathcal{P}_F(\mathcal{C}, J)_{(Y, \mathcal{F})}^{(2)}$  and an object  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (X, \mathcal{D}))$  of  $\mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{D})}^{(2)}$ , since compositions

$$\varphi_1(\mathbf{E}) \xrightarrow{\eta_{\varphi_1(\mathbf{E})}^\varphi} \varphi_1 \varphi^* \varphi_1(\mathbf{E}) \xrightarrow{\varphi_1(\varepsilon_{\mathbf{E}}^\varphi)} \varphi_1(\mathbf{E}), \quad \varphi^*(\mathbf{G}) \xrightarrow{\varphi^*(\eta_{\mathbf{G}}^\varphi)} \varphi^* \varphi_1 \varphi^*(\mathbf{G}) \xrightarrow{\varepsilon_{\varphi^*(\mathbf{G})}^\varphi} \varphi^*(\mathbf{G})$$

are both identity morphisms by (3.17), we have the following result.

**Proposition 3.18**  $\varphi_1$  is a right adjoint of  $\varphi^*$ . Hence  $\mathcal{P}_F(\mathcal{C}, J)$  is locally cartesian closed.

**Remark 3.19** Let  $\mathbf{E} = ((Y, \mathcal{E}) \xrightarrow{\pi} (X, \mathcal{D}))$ ,  $\mathbf{F} = ((Z, \mathcal{F}) \xrightarrow{\rho} (X, \mathcal{D}))$  and  $\mathbf{G} = ((W, \mathcal{G}) \xrightarrow{\chi} (X, \mathcal{D}))$  be objects of  $\mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{D})}^{(2)}$ . It follows from (2.7) and (3.18) that there exist natural bijections

$$\begin{aligned} \mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{D})}^{(2)}(\rho_* \rho^*(\mathbf{E}), \mathbf{G}) &\rightarrow \mathcal{P}_F(\mathcal{C}, J)_{(Z, \mathcal{F})}^{(2)}(\rho^*(\mathbf{E}), \rho^*(\mathbf{G})), \\ \mathcal{P}_F(\mathcal{C}, J)_{(Z, \mathcal{F})}^{(2)}(\rho^*(\mathbf{E}), \rho^*(\mathbf{G})) &\rightarrow \mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{D})}^{(2)}(\mathbf{E}, \rho_! \rho^*(\mathbf{G})). \end{aligned}$$

We note that the product  $\mathbf{E} \times \mathbf{F}$  of  $\mathbf{E}$  and  $\mathbf{F}$  is given by  $\mathbf{E} \times \mathbf{F} = \rho_* \rho^*(\mathbf{E})$ . Hence if we put  $\mathbf{G}^F = \rho_! \rho^*(\mathbf{G})$ , we have a natural bijection

$$\mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{D})}^{(2)}(\mathbf{E} \times \mathbf{F}, \mathbf{G}) \rightarrow \mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{D})}^{(2)}(\mathbf{E}, \mathbf{G}^F).$$

This shows that  $\mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{D})}^{(2)}$  is cartesian closed.

## 4 Strong subobject classifier

**Definition 4.1** Let  $\mathcal{E}$  be a category.

(1) Two morphisms  $p : X \rightarrow Y$  and  $i : Z \rightarrow W$  in  $\mathcal{E}$  are said to be orthogonal if the following left diagram is commutative, there exists unique morphism  $s : Y \rightarrow Z$  that makes the following right diagram commute.

$$\begin{array}{ccc} X & \xrightarrow{u} & Z \\ \downarrow p & & \downarrow i \\ Y & \xrightarrow{v} & W \end{array} \quad \begin{array}{ccc} X & \xrightarrow{u} & Z \\ \downarrow p & \dashrightarrow s & \downarrow i \\ Y & \xrightarrow{v} & W \end{array}$$

If  $p$  and  $i$  are orthogonal, we denote this by  $p \perp i$ .

(2) For a class  $C$  of morphisms in  $\mathcal{E}$ , we put

$$C^\perp = \{i \in \text{Mor } \mathcal{E} \mid p \perp i \text{ if } p \in C\}, \quad {}^\perp C = \{p \in \text{Mor } \mathcal{E} \mid p \perp i \text{ if } i \in C\}.$$

(3) Let  $E$  be the class of all epimorphisms in  $\mathcal{E}$ . A monomorphism  $i : Z \rightarrow W$  in  $\mathcal{E}$  is called a strong monomorphism if  $i$  belongs to  $E^\perp$ .

(4) Let  $M$  be the class of all monomorphisms in  $\mathcal{E}$ . An epimorphism  $p : X \rightarrow Y$  in  $\mathcal{E}$  is called a strong epimorphism if  $p$  belongs to  ${}^\perp M$ .

**Proposition 4.2** Let  $C$  be a class of morphisms in  $\mathcal{E}$ .

(1) If  $D$  is a class of morphisms in  $\mathcal{E}$  which contains  $C$ , then  $C^\perp \supset D^\perp$  and  ${}^\perp C \supset {}^\perp D$ .

(2)  $C \subset {}^\perp(C^\perp)$  and  $C \subset ({}^\perp C)^\perp$  hold.

(3)  $({}^\perp(C^\perp))^\perp = C^\perp$  and  ${}^\perp({}^\perp C)^\perp = {}^\perp C$  hold.

*Proof.* (1) Since  $f \in C$  implies  $f \in D$ , the assertion is straightforward from the definition (4.1).

(2) For  $p \in C$ , we have  $p \perp j$  for any  $j \in C^\perp$ , which shows  $p \in {}^\perp(C^\perp)$ . Thus we have  $C \subset {}^\perp(C^\perp)$ . For  $i \in C$ , we have  $p \perp i$  for any  $p \in {}^\perp C$ , which shows  $i \in ({}^\perp C)^\perp$ . Thus we have  $C \subset ({}^\perp C)^\perp$ .

(3) It follows from (1) and (2) that we have  $({}^\perp(C^\perp))^\perp \subset C^\perp$  and  ${}^\perp({}^\perp C)^\perp \subset {}^\perp C$ . Suppose that  $i \in C^\perp$  and  $p \in {}^\perp(C^\perp)$ . Then,  $p \perp j$  for any  $j \in C^\perp$  in particular, we have  $p \perp i$ . Hence  $p \perp i$  holds for any  $p \in {}^\perp(C^\perp)$ ,

which implies  $i \in (\perp(C^\perp))^\perp$ . Thus we have  $C^\perp \subset (\perp(C^\perp))^\perp$ . Suppose that  $i \in \perp C$  and  $p \in (\perp C)^\perp$ . Then,  $p \perp i$  for any  $j \in \perp C$  in particular, we have  $p \perp i$ . Hence  $p \perp i$  holds for any  $p \in (\perp C)^\perp$ , which implies  $i \in ((\perp C)^\perp)^\perp$ . Thus we have  $\perp C \subset ((\perp C)^\perp)^\perp$ .  $\square$

**Proposition 4.3** (1) *If  $i : Z \rightarrow W$  is an equalizer of  $f, g : W \rightarrow V$ , then  $i$  is a strong monomorphism.*

(2) *If  $p : X \rightarrow Y$  is a coequalizer of  $f, g : U \rightarrow X$ , then  $p$  is a strong epimorphism.*

*Proof.* (1) Suppose that the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{u} & Z \\ \downarrow p & & \downarrow i \\ Y & \xrightarrow{v} & W \end{array}$$

Then, we have  $fv p = f i u = g i u = g v p$ . Hence if  $p$  is an epimorphism, it follows that  $fv = gv$ . Since  $i$  is an equalizer of  $f, g : W \rightarrow V$ , there exists unique  $s : Y \rightarrow Z$  that satisfies  $v = is$ . Then,  $isp = vp = iu$  which implies  $sp = u$  since  $i$  is a monomorphism.

(2) Suppose that the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{u} & Z \\ \downarrow p & & \downarrow i \\ Y & \xrightarrow{v} & W \end{array}$$

Then, we have  $iu f = v p f = v p g = i u g$ . Hence if  $i$  is a monomorphism, it follows that  $u f = u g$ . Since  $p$  is a coequalizer of  $f, g : U \rightarrow X$ , there exists unique  $s : Y \rightarrow Z$  that satisfies  $u = sp$ . Then,  $isp = iu = vp$  which implies  $is = v$  since  $p$  is an epimorphism.  $\square$

**Definition 4.4** *Let  $\mathcal{E}$  be a category with a terminal object  $1_{\mathcal{E}}$ . If a morphism  $t : 1_{\mathcal{E}} \rightarrow \Omega$  satisfies the following condition, we call  $t$  a strong subobject classifier of  $\mathcal{E}$ .*

(\*) *For each strong monomorphism  $\sigma : Y \rightarrow X$  in  $\mathcal{E}$ , there exists unique morphism  $\phi_\sigma : X \rightarrow \Omega$  that makes the following square cartesian.*

$$\begin{array}{ccc} Y & \xrightarrow{o_Y} & 1_{\mathcal{E}} \\ \downarrow \sigma & & \downarrow t \\ X & \xrightarrow{\phi_\sigma} & \Omega \end{array}$$

**Remark 4.5** *Assume that the outer rectangle of the following left diagram is cartesian. If  $h : V \rightarrow X$  satisfies  $fh = gsh$ , then there exists unique morphism  $k : V \rightarrow Y$  that satisfies  $\sigma k = h$  by the assumption.*

$$\begin{array}{ccc} Y & \xrightarrow{s\sigma} & W \\ \downarrow \sigma & \nearrow s & \downarrow g \\ X & \xrightarrow{f} & Z \end{array} \quad \begin{array}{ccccc} & & V & & \\ & & \swarrow & \searrow & \\ & & k & & sh \\ & & \swarrow & \searrow & \\ & & Y & \xrightarrow{s\sigma} & W \\ & & \downarrow \sigma & \nearrow s & \downarrow g \\ & & X & \xrightarrow{f} & Z \end{array}$$

Hence if  $\sigma : Y \rightarrow X$  is a monomorphism,  $\sigma$  is an equalizer of  $f, gs : X \rightarrow Z$ . It follows that if  $\mathcal{E}$  has a strong subobject classifier, each strong monomorphism in  $\mathcal{E}$  is an equalizer of a certain pair of morphisms.

**Proposition 4.6** *A morphism  $i : (Y, \mathcal{E}) \rightarrow (X, \mathcal{D})$  in  $\mathcal{P}_F(\mathcal{C}, J)$  is a monomorphism if and only if  $i : Y \rightarrow X$  is injective.*

*Proof.* It is clear that  $i : (Y, \mathcal{E}) \rightarrow (X, \mathcal{D})$  in  $\mathcal{P}_F(\mathcal{C}, J)$  is a monomorphism if  $i : Y \rightarrow X$  is injective. Suppose that  $i : (Y, \mathcal{E}) \rightarrow (X, \mathcal{D})$  is a monomorphism in  $\mathcal{P}_F(\mathcal{C}, J)$  and that  $i(a) = i(b)$  holds for  $a, b \in Y$ . Define maps  $f, g : \{1\} \rightarrow Y$  by  $f(1) = a$  and  $g(1) = b$ . Then  $f, g : (\{1\}, \mathcal{D}_{disc, \{1\}}) \rightarrow (Y, \mathcal{E})$  are morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$  which satisfy  $if = ig$ . Thus we have  $f = g$  which implies  $a = b$ .  $\square$

**Proposition 4.7** Let  $\sigma : (Y, \mathcal{F}) \rightarrow (X, \mathcal{D})$  be a strong monomorphism in  $\mathcal{P}_F(\mathcal{C}, J)$  and denote by  $i : \sigma(Y) \rightarrow X$  the inclusion map. Then there is a surjection  $\tilde{\sigma} : Y \rightarrow \sigma(Y)$  which satisfies  $i\tilde{\sigma} = \sigma$ . This map gives an isomorphism  $\tilde{\sigma} : (Y, \mathcal{F}) \rightarrow (\sigma(Y), \mathcal{D}^i)$  in  $\mathcal{P}_F(\mathcal{C}, J)$ .

*Proof.* Since  $\sigma : Y \rightarrow X$  is injective by (4.6),  $\tilde{\sigma}$  is bijective. Since  $(F_\sigma)_U = (F_i)_U(F_{\tilde{\sigma}})_U : F_Y(U) \rightarrow F_X(U)$  maps  $\mathcal{F} \cap F_Y(U)$  into  $\mathcal{D} \cap F_X(U)$ ,  $(F_{\tilde{\sigma}})_U : F_Y(U) \rightarrow F_X(U)$  maps  $\mathcal{F} \cap F_Y(U)$  into  $(F_i)_U^{-1}(\mathcal{D} \cap F_X(U)) = \mathcal{D}^i \cap F_{\sigma(Y)}(U)$  for  $U \in \text{Ob}\mathcal{C}$ . Hence  $\tilde{\sigma} : (Y, \mathcal{F}) \rightarrow (\sigma(Y), \mathcal{D}^i)$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . Consider the following left commutative diagram.

$$\begin{array}{ccc} (Y, \mathcal{F}) & \xrightarrow{id_Y} & (Y, \mathcal{F}) \\ \downarrow \tilde{\sigma} & & \downarrow \sigma \\ (S(Y), \mathcal{D}^i) & \xrightarrow{i} & (X, \mathcal{D}) \end{array} \quad \begin{array}{ccc} (Y, \mathcal{F}) & \xrightarrow{id_Y} & (Y, \mathcal{F}) \\ \downarrow \tilde{\sigma} & \nearrow s & \downarrow \sigma \\ (S(Y), \mathcal{D}^i) & \xrightarrow{i} & (X, \mathcal{D}) \end{array}$$

Since  $\tilde{\sigma} : (Y, \mathcal{F}) \rightarrow (\sigma(Y), \mathcal{D}^i)$  is an epimorphism in  $\mathcal{P}_F(\mathcal{C}, J)$  and  $\sigma : (Y, \mathcal{F}) \rightarrow (X, \mathcal{D})$  is a strong monomorphism in  $\mathcal{P}_F(\mathcal{C}, J)$ , there exists a morphism  $s : (S(Y), \mathcal{D}^i) \rightarrow (Y, \mathcal{F})$  in  $\mathcal{P}_F(\mathcal{C}, J)$  which makes the above right diagram commute. Hence we have  $s\tilde{\sigma} = id_Y$  and  $i\tilde{\sigma}s = \sigma s = i$ . Since  $i$  is a monomorphism, the latter equality implies  $\tilde{\sigma}s = id_{s(Y)}$ . Therefore  $\tilde{\sigma} : (Y, \mathcal{F}) \rightarrow (\sigma(Y), \mathcal{D}^i)$  is an isomorphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .  $\square$

Let  $t : \{1\} \rightarrow \{0, 1\}$  be an inclusion map. Then,  $t : (\{1\}, \mathcal{D}_{coarse, \{1\}}) \rightarrow (\{0, 1\}, \mathcal{D}_{coarse, \{0, 1\}})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

**Proposition 4.8** Let  $(X, \mathcal{D})$  be an object of  $\mathcal{P}_F(\mathcal{C}, J)$  and  $Y$  a subset of  $X$ . We denote by  $\sigma : Y \rightarrow X$  the inclusion map and define a map  $\phi_\sigma : X \rightarrow \{0, 1\}$  by  $\phi_\sigma(x) = \begin{cases} 1 & x \in Y \\ 0 & x \notin Y \end{cases}$ . Then, the following diagram is a cartesian square in  $\mathcal{P}_F(\mathcal{C}, J)$ .

$$\begin{array}{ccc} (Y, \mathcal{D}^\sigma) & \xrightarrow{o_Y} & (\{1\}, \mathcal{D}_{coarse, \{1\}}) \\ \downarrow \sigma & & \downarrow t \\ (X, \mathcal{D}) & \xrightarrow{\phi_\sigma} & (\{0, 1\}, \mathcal{D}_{coarse, \{0, 1\}}) \end{array}$$

*Proof.* Let  $f : (W, \mathcal{F}) \rightarrow (X, \mathcal{D})$  be a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$  which satisfies  $\phi_\sigma f = t o_W$ . Then, we have  $\phi_\sigma f(W) \subset \{1\}$  which shows  $f(W) \subset Y$ . Hence there is unique map  $\tilde{f} : W \rightarrow Y$  which satisfies  $\sigma \tilde{f} = f$ . For each  $U \in \text{Ob}\mathcal{C}$ , since  $(F_\sigma)_U(F_{\tilde{f}})_U = (F_f)_U : F_W(U) \rightarrow F_X(U)$  maps  $\mathcal{F} \cap F_W(U)$  into  $\mathcal{D} \cap F_X(U)$ , it follows that  $(F_{\tilde{f}})_U : F_W(U) \rightarrow F_Y(U)$  maps  $\mathcal{F} \cap F_W(U)$  into  $(F_\sigma)_U^{-1}(\mathcal{D} \cap F_X(U)) = \mathcal{D}^\sigma \cap F_Y(U)$ . Thus  $\tilde{f} : (W, \mathcal{F}) \rightarrow (Y, \mathcal{D}^\sigma)$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .  $\square$

**Remark 4.9** The morphism  $\sigma : (Y, \mathcal{D}^\sigma) \rightarrow (X, \mathcal{D})$  is an equalizer of  $\phi_\sigma : (X, \mathcal{D}) \rightarrow (\{0, 1\}, \mathcal{D}_{coarse, \{0, 1\}})$  and a composition  $(X, \mathcal{D}) \xrightarrow{o_X} (\{1\}, \mathcal{D}_{coarse, \{1\}}) \xrightarrow{t} (\{0, 1\}, \mathcal{D}_{coarse, \{0, 1\}})$  by (4.5). In particular,  $\sigma : (Y, \mathcal{D}^\sigma) \rightarrow (X, \mathcal{D})$  is a strong monomorphism in  $\mathcal{P}_F(\mathcal{C}, J)$  by (4.3).

**Proposition 4.10**  $t : (\{1\}, \mathcal{D}_{coarse, \{1\}}) \rightarrow (\{0, 1\}, \mathcal{D}_{coarse, \{0, 1\}})$  is a strong subobject classifier in  $\mathcal{P}_F(\mathcal{C}, J)$ .

*Proof.* Let  $\sigma : (Y, \mathcal{F}) \rightarrow (X, \mathcal{D})$  be a strong monomorphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . We denote by  $i : \sigma(Y) \rightarrow X$  the inclusion map. It follows from (4.8) that there exists a morphism  $\phi_\sigma : (X, \mathcal{D}) \rightarrow (\{0, 1\}, \mathcal{D}_{coarse, \{0, 1\}})$  such that the following diagram is cartesian.

$$\begin{array}{ccc} (\sigma(Y), \mathcal{D}^i) & \xrightarrow{o_{\sigma(Y)}} & (\{1\}, \mathcal{D}_{coarse, \{1\}}) \\ \downarrow i & & \downarrow t \\ (X, \mathcal{D}) & \xrightarrow{\phi_\sigma} & (\{0, 1\}, \mathcal{D}_{coarse, \{0, 1\}}) \end{array}$$

Then, the following diagram is also cartesian by (4.7).

$$\begin{array}{ccc} (Y, \mathcal{F}) & \xrightarrow{o_Y} & (\{1\}, \mathcal{D}_{coarse, \{1\}}) \\ \downarrow \sigma & & \downarrow t \\ (X, \mathcal{D}) & \xrightarrow{\phi_\sigma} & (\{0, 1\}, \mathcal{D}_{coarse, \{0, 1\}}) \end{array}$$

Suppose that a map  $\psi : (X, \mathcal{D}) \rightarrow (\{0, 1\}, \mathcal{D}_{coarse, \{0, 1\}})$  also makes the following diagram cartesian.

$$\begin{array}{ccc} (Y, \mathcal{F}) & \xrightarrow{o_Y} & (\{1\}, \mathcal{D}_{coarse, \{1\}}) \\ \downarrow \sigma & & \downarrow t \\ (X, \mathcal{D}) & \xrightarrow{\psi} & (\{0, 1\}, \mathcal{D}_{coarse, \{0, 1\}}) \end{array}$$

Since the forgetful functor  $\Gamma : \mathcal{P}_F(\mathcal{C}, J) \rightarrow \mathit{Set}$  has a left adjoint,  $\Gamma$  preserves limits. Hence

$$\begin{array}{ccc} Y & \xrightarrow{o_Y} & \{1\} \\ \downarrow \sigma & & \downarrow t \\ X & \xrightarrow{\psi} & \{0, 1\} \end{array}$$

is a cartesian square in  $\mathit{Set}$ . Since  $\psi\sigma = to_Y$ , we have  $\psi(x) = 1$  if  $x \in \sigma(Y)$ . If  $\psi(x) = 1$  for  $x \in X$ , we define a map  $f : \{1\} \rightarrow X$  by  $f(1) = x$ . Then we have  $\psi f = tid_{\{1\}}$  which implies that there exists a map  $\bar{f} : \{1\} \rightarrow Y$  which satisfies  $\sigma\bar{f} = f$ . Thus  $x = f(1) = \sigma(\bar{f}(1)) \in \sigma(Y)$ . Therefore  $\psi = \phi_\sigma$  holds and this shows the uniqueness of  $\phi_\sigma$ .  $\square$

By (2.11), (2.14), (3.18) and (4.10), we have the following result.

**Theorem 4.11**  $\mathcal{P}_F(\mathcal{C}, J)$  is a quasi-topos.

**Proposition 4.12**  $\pi : (X, \mathcal{D}) \rightarrow (Y, \mathcal{E})$  is an epimorphism in  $\mathcal{P}_F(\mathcal{C}, J)$  if and only if  $\pi : X \rightarrow Y$  is surjective.

*Proof.* It is clear that  $\pi : (X, \mathcal{D}) \rightarrow (Y, \mathcal{E})$  is an epimorphism in  $\mathcal{P}_F(\mathcal{C}, J)$  if  $\pi : X \rightarrow Y$  is surjective. Assume that  $\pi : (X, \mathcal{D}) \rightarrow (Y, \mathcal{E})$  is an epimorphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . We denote by  $\sigma : \pi(X) \rightarrow Y$  the inclusion map. Since  $\sigma : (\pi(X), \mathcal{E}^\sigma) \rightarrow (Y, \mathcal{E})$  is a strong monomorphism by (4.9), there exists a morphism  $\phi_\sigma : (Y, \mathcal{E}) \rightarrow (\{0, 1\}, \mathcal{D}_{disc, \{0, 1\}})$  such that the following left diagram is cartesian.

$$\begin{array}{ccc} (\pi(X), \mathcal{E}^\sigma) & \xrightarrow{o_{\pi(X)}} & (\{1\}, \mathcal{D}_{coarse, \{1\}}) \\ \downarrow \sigma & & \downarrow t \\ (Y, \mathcal{E}) & \xrightarrow{\phi_\sigma} & (\{0, 1\}, \mathcal{D}_{coarse, \{0, 1\}}) \end{array} \quad \begin{array}{ccc} (\pi(X), \mathcal{E}^\sigma) & \xrightarrow{o_{\pi(X)}} & (\{1\}, \mathcal{D}_{coarse, \{1\}}) \\ \nearrow \bar{\pi} & \downarrow \sigma & \searrow o_Y \\ (X, \mathcal{D}) & \xrightarrow{\pi} & (Y, \mathcal{E}) & \xrightarrow{\phi_\sigma} & (\{0, 1\}, \mathcal{D}_{coarse, \{0, 1\}}) \\ & & \downarrow t & & \downarrow t \end{array}$$

Let  $\bar{\pi} : X \rightarrow \pi(X)$  be the surjection induced by  $\pi$ . Then  $\bar{\pi} : (X, \mathcal{D}) \rightarrow (\pi(X), \mathcal{E}^\sigma)$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . We consider a composition  $to_Y : (Y, \mathcal{E}) \rightarrow (\{0, 1\}, \mathcal{D}_{coarse, \{0, 1\}})$  which is a constant map whose image is  $\{1\}$ . Since  $\phi_\sigma\pi = \phi_\sigma\sigma\bar{\pi} = to_{\pi(X)}\bar{\pi}$ ,  $\phi_\sigma\pi$  is also a constant map to  $\{1\}$ . Thus we have  $\phi_\sigma\pi = to_Y\pi$ . Since  $\pi$  is an epimorphism, we have  $\phi_\sigma = to_Y$ , in other words,  $\phi_\sigma$  is a constant map to  $\{1\}$ . Therefore  $\pi(X) = \phi_\sigma^{-1}(\{1\}) = Y$  and  $\pi$  is surjective.  $\square$

## 5 Groupoids associated with epimorphisms

Let  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (B, \mathcal{B}))$  be an object  $\mathcal{P}_F(\mathcal{C}, J)_{(B, \mathcal{B})}^{(2)}$  such that  $\pi$  is an epimorphism. Then,  $\pi$  is surjective by (4.12), hence  $\pi^{-1}(x)$  is not an empty set for any  $x \in B$ . We denote by  $i_x : \pi^{-1}(x) \rightarrow E$  the inclusion map. We define a set  $G_1(\mathbf{E})(x, y)$  for  $x, y \in B$  by

$$G_1(\mathbf{E})(x, y) = \{\varphi \in \mathcal{P}_F(\mathcal{C}, J)((\pi^{-1}(x), \mathcal{E}^{i_x}), (\pi^{-1}(y), \mathcal{E}^{i_y})) \mid \varphi \text{ is an isomorphism.}\}$$

Put  $G_1(\mathbf{E}) = \coprod_{x, y \in B} G_1(\mathbf{E})(x, y)$  and define maps  $\sigma_{\mathbf{E}}, \tau_{\mathbf{E}} : G_1(\mathbf{E}) \rightarrow B$ ,  $\iota_{\mathbf{E}} : G_1(\mathbf{E}) \rightarrow G_1(\mathbf{E})$  and  $\varepsilon_{\mathbf{E}} : B \rightarrow G_1(\mathbf{E})$

by  $\sigma_{\mathbf{E}}(\varphi) = x$ ,  $\tau_{\mathbf{E}}(\varphi) = y$ ,  $\iota_{\mathbf{E}}(\varphi) = \varphi^{-1}$  if  $\varphi \in G_1(\mathbf{E})(x, y)$  and  $\varepsilon_{\mathbf{E}}(x) = id_{\pi^{-1}(x)}$ . Let

$$\begin{array}{ccc} G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) & \xrightarrow{\text{pr}_2} & G_1(\mathbf{E}) \\ \downarrow \text{pr}_1 & & \downarrow \sigma_{\mathbf{E}} \\ G_1(\mathbf{E}) & \xrightarrow{\tau_{\mathbf{E}}} & B \end{array}$$

be a cartesian square. In other words,  $G_1(\mathbf{E}) \times_B G_1(\mathbf{E})$  is given by

$$G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) = \{(\varphi, \psi) \in G_1(\mathbf{E}) \times G_1(\mathbf{E}) \mid \tau_{\mathbf{E}}(\varphi) = \sigma_{\mathbf{E}}(\psi)\}$$

as a set. We define a map  $\mu_{\mathbf{E}} : G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) \rightarrow G_1(\mathbf{E})$  by  $\mu_{\mathbf{E}}(\varphi, \psi) = \psi\varphi$ .

We consider the following cartesian squares.

$$\begin{array}{ccc} E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) & \xrightarrow{\text{pr}_{G_1(\mathbf{E})}^{\sigma}} & G_1(\mathbf{E}) \\ \downarrow \text{pr}_E^{\sigma} & & \downarrow \sigma_{\mathbf{E}} \\ E & \xrightarrow{\pi} & B \end{array} \quad \begin{array}{ccc} E \times_B^{\tau_{\mathbf{E}}} G_1(\mathbf{E}) & \xrightarrow{\text{pr}_{G_1(\mathbf{E})}^{\tau}} & G_1(\mathbf{E}) \\ \downarrow \text{pr}_E^{\tau} & & \downarrow \tau_{\mathbf{E}} \\ E & \xrightarrow{\pi} & B \end{array}$$

Hence  $E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E})$  and  $E \times_B^{\tau_{\mathbf{E}}} G_1(\mathbf{E})$  are given as follows as sets.

$$E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) = \{(e, \varphi) \in E \times G_1(\mathbf{E}) \mid \pi(e) = \sigma_{\mathbf{E}}(\varphi)\}, \quad E \times_B^{\tau_{\mathbf{E}}} G_1(\mathbf{E}) = \{(e, \varphi) \in E \times G_1(\mathbf{E}) \mid \pi(e) = \tau_{\mathbf{E}}(\varphi)\}$$

There exists unique map  $id_E \times_B \iota_{\mathbf{E}} : E \times_B^{\tau_{\mathbf{E}}} G_1(\mathbf{E}) \rightarrow E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E})$  that makes the following diagram commute.

$$\begin{array}{ccc} E \times_B^{\tau_{\mathbf{E}}} G_1(\mathbf{E}) & \xrightarrow{\text{pr}_{G_1(\mathbf{E})}^{\tau}} & G_1(\mathbf{E}) \\ \downarrow \text{pr}_E^{\tau} & \searrow id_E \times_B \iota_{\mathbf{E}} & \swarrow \iota_{\mathbf{E}} \\ E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) & \xrightarrow{\text{pr}_{G_1(\mathbf{E})}^{\sigma}} & G_1(\mathbf{E}) \\ \downarrow \text{pr}_E^{\sigma} & & \downarrow \sigma_{\mathbf{E}} \\ E & \xrightarrow{\pi} & B \end{array}$$

We define a map  $\hat{\xi}_{\mathbf{E}} : E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) \rightarrow E$  by  $\hat{\xi}_{\mathbf{E}}(e, \varphi) = i_{\tau_{\mathbf{E}}(\varphi)}\varphi(e)$ . Let  $\Sigma_{\mathbf{E}}$  the set of all the-ologies  $\mathcal{L}$  on  $G_1(\mathbf{E})$  which satisfy  $\mathcal{E}^{\text{pr}_E^{\sigma}} \cap \mathcal{L}^{\text{pr}_{G_1(\mathbf{E})}^{\sigma}} \subset \mathcal{E}^{\hat{\xi}_{\mathbf{E}}}$ ,  $\mathcal{E}^{\text{pr}_E^{\tau}} \cap \mathcal{L}^{\text{pr}_{G_1(\mathbf{E})}^{\tau}} \subset \mathcal{E}^{\hat{\xi}_{\mathbf{E}}(id_E \times_B \iota_{\mathbf{E}})}$  and  $\mathcal{L} \subset \mathcal{B}^{\sigma_{\mathbf{E}}} \cap \mathcal{B}^{\tau_{\mathbf{E}}}$ . We note that the  $\mathcal{L} \in \Sigma_{\mathbf{E}}$  if and only if following maps are morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ .

$$\begin{aligned} \hat{\xi}_{\mathbf{E}} &: (E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}), \mathcal{E}^{\text{pr}_E^{\sigma}} \cap \mathcal{L}^{\text{pr}_{G_1(\mathbf{E})}^{\sigma}}) \rightarrow (E, \mathcal{E}) \\ \hat{\xi}_{\mathbf{E}}(id_E \times_B \iota_{\mathbf{E}}) &: (E \times_B^{\tau_{\mathbf{E}}} G_1(\mathbf{E}), \mathcal{E}^{\text{pr}_E^{\tau}} \cap \mathcal{L}^{\text{pr}_{G_1(\mathbf{E})}^{\tau}}) \rightarrow (E, \mathcal{E}) \\ \sigma_{\mathbf{E}}, \tau_{\mathbf{E}} &: (G_1(\mathbf{E}), \mathcal{L}) \rightarrow (B, \mathcal{B}) \end{aligned}$$

**Proposition 5.1**  $\Sigma_{\mathbf{E}}$  is not empty.

*Proof.* It suffices to show that the discrete the-ology  $\mathcal{D}_{disc, G_1(\mathbf{E})}$  on  $G_1(\mathbf{E})$  belongs to  $\Sigma_{\mathbf{E}}$ . It follows from (1.17) that  $\mathcal{D}_{disc, G_1(\mathbf{E})} \subset \mathcal{B}^{\sigma_{\mathbf{E}}} \cap \mathcal{B}^{\tau_{\mathbf{E}}}$  holds. For  $U \in \text{Ob}\mathcal{C}$ , suppose that  $\psi \in \mathcal{E}^{\text{pr}_E^{\sigma}} \cap \mathcal{D}_{disc, G_1(\mathbf{E})}^{\text{pr}_{G_1(\mathbf{E})}^{\sigma}} \cap F_{E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E})}(U)$ . Then, we have  $\text{pr}_E^{\sigma}\psi \in \mathcal{E} \cap F_E(U)$  and  $\text{pr}_{G_1(\mathbf{E})}^{\sigma}\psi \in \mathcal{D}_{disc, G_1(\mathbf{E})} \cap F_{G_1(\mathbf{E})}(U)$ . Hence there exists a covering  $(U_j \xrightarrow{g_j} U)_{i \in J}$  such that  $F_{G_1(\mathbf{E})}(g_j)(\text{pr}_{G_1(\mathbf{E})}^{\sigma}\psi) : F(U_j) \rightarrow G_1(\mathbf{E})$  is a constant map for every  $i \in J$  by (1.17). Let us denote by  $\alpha_j \in G_1(\mathbf{E})$  the image of  $F_{G_1(\mathbf{E})}(g_j)(\text{pr}_{G_1(\mathbf{E})}^{\sigma}\psi)$  and put  $x_j = \sigma_{\mathbf{E}}(\alpha_j)$ ,  $y_j = \tau_{\mathbf{E}}(\alpha_j)$ . Then we have  $\alpha_j \in G_1(\mathbf{E})(x_j, y_j)$  and the image of  $F_E(g_j)(\text{pr}_E^{\sigma}\psi) = \text{pr}_E^{\sigma}\psi F(g_j) : F(U_j) \rightarrow E$  is contained in  $\pi^{-1}(x_j)$ . Hence we have a map  $\zeta_j : F(U_j) \rightarrow \pi^{-1}(x_j)$  satisfying  $i_{x_j}\zeta_j = F_E(g_j)(\text{pr}_E^{\sigma}\psi) \in \mathcal{E} \cap F_E(U_j)$ , which shows  $\zeta_j \in \mathcal{E}^{i_{x_j}} \cap F_{\pi^{-1}(x_j)}(U_j)$ . Since we have an equality

$$F_{E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E})}(g_j)(\psi) = (i_{x_j}\zeta_j, F_{G_1(\mathbf{E})}(g_j)(\text{pr}_{G_1(\mathbf{E})}^{\sigma}\psi)) : F(U_j) \rightarrow E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}),$$

it follows that the following equality holds.

$$F_E(g_j)(F_{\hat{\xi}_{\mathbf{E}}}(\psi)) = F_{\hat{\xi}_{\mathbf{E}}}(F_{E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E})}(g_j)(\psi)) = \hat{\xi}_{\mathbf{E}}(i_{x_j}\zeta_j, F_{G_1(\mathbf{E})}(g_j)(\text{pr}_{G_1(\mathbf{E})}^{\sigma}\psi)) = i_{y_j}\alpha_j\zeta_j = F_{i_{y_j}}(F_{\alpha_j}(\zeta_j))$$

Since  $\alpha_j : (\pi^{-1}(x_j), \mathcal{E}^{i_{x_j}}) \rightarrow (\pi^{-1}(y_j), \mathcal{E}^{i_{y_j}})$  and  $i_{y_j} : (\pi^{-1}(y_j), \mathcal{E}^{i_{y_j}}) \rightarrow (E, \mathcal{E})$  are morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ , we have  $F_{i_{y_j}}(F_{\alpha_j}(\zeta_j)) \in \mathcal{E} \cap F_E(U_j)$  for any  $i \in J$ . Therefore  $F_{\hat{\xi}_{\mathbf{E}}}(\psi) \in \mathcal{E} \cap F_E(U)$  holds and we see that  $\mathcal{E}^{\text{pr}_E^{\sigma}} \cap \mathcal{D}_{disc, G_1(\mathbf{E})}^{\text{pr}_{G_1(\mathbf{E})}^{\sigma}} \subset \mathcal{E}^{\hat{\xi}_{\mathbf{E}}}$  holds.

For  $U \in \text{Ob}\mathcal{C}$ , suppose that  $\psi \in \mathcal{E}^{\text{pr}_E^{\tau}} \cap \mathcal{D}_{disc, G_1(\mathbf{E})}^{\text{pr}_{G_1(\mathbf{E})}^{\tau}} \cap F_{E \times_B^{\tau_{\mathbf{E}}} G_1(\mathbf{E})}(U)$ . Then, we have  $\text{pr}_E^{\tau}\psi \in \mathcal{E} \cap F_E(U)$  and  $\text{pr}_{G_1(\mathbf{E})}^{\tau}\psi \in \mathcal{D}_{disc, G_1(\mathbf{E})} \cap F_{G_1(\mathbf{E})}(U)$ . Hence there exists a covering  $(U_j \xrightarrow{g_j} U)_{i \in J}$  such that  $F_{G_1(\mathbf{E})}(g_j)(\text{pr}_{G_1(\mathbf{E})}^{\tau}\psi) : F(U_j) \rightarrow G_1(\mathbf{E})$  is a constant map for every  $i \in J$  by (1.17). We denote by  $\alpha_j \in G_1(\mathbf{E})$  the image of

$F_{G_1(\mathbf{E})}(g_j)(\text{pr}_{G_1(\mathbf{E})}^\tau \psi)$  and put  $x_j = \sigma_{\mathbf{E}}(\alpha_j)$ ,  $y_j = \tau_{\mathbf{E}}(\alpha_j)$ . Then we have  $\alpha_j \in G_1(\mathbf{E})(x_j, y_j)$  and the image of  $F_E(g_j)(\text{pr}_E^\tau \psi) = \text{pr}_E^\tau \psi F(g_j) : F(U_j) \rightarrow E$  is contained in  $\pi^{-1}(y_j)$ . Hence we have a map  $\zeta_j : F(U_j) \rightarrow \pi^{-1}(y_j)$  satisfying  $i_{y_j} \zeta_j = F_E(g_j)(\text{pr}_E^\tau \psi) \in \mathcal{E} \cap F_E(U_j)$ , which shows  $\zeta_j \in \mathcal{E}^{i_{y_j}} \cap F_{\pi^{-1}(y_j)}(U_j)$ . Since we have an equality

$$F_{E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E})}(g_j)(\psi) = (i_{y_j} \zeta_j, F_{G_1(\mathbf{E})}(g_j)(\text{pr}_{G_1(\mathbf{E})}^\tau \psi)) : F(U_j) \rightarrow E \times_B^{\tau_{\mathbf{E}}} G_1(\mathbf{E}),$$

it follows that the following equality holds.

$$\begin{aligned} F_E(g_j)(F_{\hat{\xi}_{\mathbf{E}}(id_E \times_B \iota_{\mathbf{E}})}(\psi)) &= F_{\hat{\xi}_{\mathbf{E}}(id_E \times_B \iota_{\mathbf{E}})}(F_{E \times_B^{\tau_{\mathbf{E}}} G_1(\mathbf{E})}(g_j)(\psi)) = \hat{\xi}_{\mathbf{E}}(id_E \times_B \iota_{\mathbf{E}})(i_{y_j} \zeta_j, F_{G_1(\mathbf{E})}(g_j)(\text{pr}_{G_1(\mathbf{E})}^\tau \psi)) \\ &= \hat{\xi}_{\mathbf{E}}(i_{y_j} \zeta_j, \iota_{\mathbf{E}} F_{G_1(\mathbf{E})}(g_j)(\text{pr}_{G_1(\mathbf{E})}^\tau \psi)) = i_{x_j} \alpha_j^{-1} \zeta_j = F_{i_{x_j}}(F_{\alpha_j^{-1}}(\zeta_j)) \end{aligned}$$

Since  $\alpha_j^{-1} : (\pi^{-1}(y_j), \mathcal{E}^{i_{y_j}}) \rightarrow (\pi^{-1}(x_j), \mathcal{E}^{i_{x_j}})$  and  $i_{x_j} : (\pi^{-1}(x_j), \mathcal{E}^{i_{x_j}}) \rightarrow (E, \mathcal{E})$  are morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ , we have  $F_{i_{x_j}}(F_{\alpha_j^{-1}}(\zeta_j)) \in \mathcal{E} \cap F_E(U_j)$  for any  $i \in J$ . Therefore  $F_{\hat{\xi}_{\mathbf{E}}(id_E \times_B \iota_{\mathbf{E}})}(\psi) \in \mathcal{E} \cap F_E(U)$  holds and we see that  $\mathcal{E}^{\text{pr}_E^\tau} \cap \mathcal{D}_{disc, G_1(\mathbf{E})}^{\text{pr}_{G_1(\mathbf{E})}^\tau} \subset \mathcal{E}^{\hat{\xi}_{\mathbf{E}}(id_E \times_B \iota_{\mathbf{E}})}$  holds.  $\square$

For  $U \in \text{Ob } \mathcal{C}$ , we consider the following conditions (G1), (G2), (G3) on an element  $\gamma$  of  $F_{G_1(\mathbf{E})}(U)$ .

(G1) If  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$ ,  $g \in \mathcal{C}(W, V)$  and  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfy  $\pi \lambda F(g) = \sigma_{\mathbf{E}} \gamma F(f)$ , a composition

$$F(W) \xrightarrow{(\lambda F(g), \gamma F(f))} E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_{\mathbf{E}}} E \text{ belongs to } \mathcal{E} \cap F_E(W).$$

(G2) If  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$ ,  $g \in \mathcal{C}(W, V)$  and  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfy  $\pi \lambda F(g) = \tau_{\mathbf{E}} \gamma F(f)$ , a composition

$$F(W) \xrightarrow{(\lambda F(g), \iota_{\mathbf{E}} \gamma F(f))} E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_{\mathbf{E}}} E \text{ belongs to } \mathcal{E} \cap F_E(W).$$

(G3) Compositions  $F(U) \xrightarrow{\gamma} G_1(\mathbf{E}) \xrightarrow{\sigma_{\mathbf{E}}} B$  and  $F(U) \xrightarrow{\gamma} G_1(\mathbf{E}) \xrightarrow{\tau_{\mathbf{E}}} B$  belong to  $\mathcal{B} \cap F_B(U)$ .

Define a set  $\mathcal{G}_{\mathbf{E}}$  of  $F$ -parametrizations of a set  $G_1(\mathbf{E})$  so that  $\mathcal{G}_{\mathbf{E}} \cap F_{G_1(\mathbf{E})}(U)$  is a subset of  $F_{G_1(\mathbf{E})}(U)$  consisting of elements which satisfy the above conditions (G1), (G2) and (G3) for any  $U \in \text{Ob } \mathcal{C}$ .

**Remark 5.2** The conditions (G1), (G2) and (G3) on  $\gamma \in F_{G_1(\mathbf{E})}(U)$  above are equivalent to the following conditions (G1'), (G2') and (G3'), respectively.

(G1') If  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$ ,  $g \in \mathcal{C}(W, V)$  and  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfy  $\pi \lambda F(g) = \sigma_{\mathbf{E}} \gamma F(f)$ , then  $\gamma$  satisfies  $((\lambda F(g), \gamma F(f)) : F(W) \rightarrow E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E})) \in \mathcal{E}^{\hat{\xi}_{\mathbf{E}}} \cap F_{E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E})}(W)$ .

(G2') If  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$ ,  $g \in \mathcal{C}(W, V)$  and  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfy  $\pi \lambda F(g) = \tau_{\mathbf{E}} \gamma F(f)$ , then  $\gamma$  satisfies  $((\lambda F(g), \gamma F(f)) : F(W) \rightarrow E \times_B^{\tau_{\mathbf{E}}} G_1(\mathbf{E})) \in \mathcal{E}^{\hat{\xi}_{\mathbf{E}}(id_E \times_B \iota_{\mathbf{E}})} \cap F_{E \times_B^{\tau_{\mathbf{E}}} G_1(\mathbf{E})}(W)$ .

(G3')  $\gamma \in \mathcal{B}^{\sigma_{\mathbf{E}}} \cap \mathcal{B}^{\tau_{\mathbf{E}}} \cap F_{G_1(\mathbf{E})}(U)$

**Proposition 5.3**  $\mathcal{G}_{\mathbf{E}}$  is a the-ology on  $G_1(\mathbf{E})$ .

*Proof.* For  $\gamma \in F_{G_1(\mathbf{E})}(1_{\mathcal{C}})$ , put  $s = \sigma_{\mathbf{E}}(\gamma(*))$ ,  $t = \tau_{\mathbf{E}}(\gamma(*))$ . We take  $V, W \in \text{Ob } \mathcal{C}$ ,  $o_W \in \mathcal{C}(W, 1_{\mathcal{C}})$ ,  $g \in \mathcal{C}(W, V)$ . Assume that  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfies  $\pi \lambda F(g) = \sigma_{\mathbf{E}} \gamma F(o_W)$ . Then, the image of  $\lambda F(g) : F(W) \rightarrow E$  is contained in  $\pi^{-1}(s)$  hence there exists a map  $\zeta : F(W) \rightarrow \pi^{-1}(s)$  which satisfies  $\lambda F(g) = i_s \zeta$ . Since  $\lambda F(g) \in \mathcal{E} \cap F_E(W)$ , we have  $\zeta \in \mathcal{E}^{i_s} \cap F_{\pi^{-1}(s)}(W)$ . We note that  $\gamma(*) : (\pi^{-1}(s), \mathcal{E}^{i_s}) \rightarrow (\pi^{-1}(t), \mathcal{E}^{i_t})$  and  $i_t : (\pi^{-1}(t), \mathcal{E}^{i_t}) \rightarrow (E, \mathcal{E})$  are morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ . It follows that a composition  $F(W) \xrightarrow{(\lambda F(g), \gamma F(o_W))} E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_{\mathbf{E}}} E$  coincides with a composition  $F(W) \xrightarrow{\zeta} \pi^{-1}(s) \xrightarrow{\gamma(*)} \pi^{-1}(t) \xrightarrow{i_t} E$  which belongs to  $\mathcal{E} \cap F_E(W)$ . Therefore  $\gamma$  satisfies (G1). Assume that  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfies  $\pi \lambda F(g) = \tau_{\mathbf{E}} \gamma F(o_W)$ . Then, the image of  $\lambda F(g) : F(W) \rightarrow E$  is contained in  $\pi^{-1}(t)$  hence there exists a map  $\zeta : F(W) \rightarrow \pi^{-1}(t)$  which satisfies  $\lambda F(g) = i_t \zeta$ . Since  $\lambda F(g) \in \mathcal{E} \cap F_E(W)$ , we have  $\zeta \in \mathcal{E}^{i_t} \cap F_{\pi^{-1}(t)}(W)$ . Note that  $\iota_{\mathbf{E}}(\gamma(*)) : (\pi^{-1}(t), \mathcal{E}^{i_t}) \rightarrow (\pi^{-1}(s), \mathcal{E}^{i_s})$  and  $i_t : (\pi^{-1}(t), \mathcal{E}^{i_t}) \rightarrow (E, \mathcal{E})$  are morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ . It follows that a composition  $F(W) \xrightarrow{(\lambda F(g), \iota_{\mathbf{E}} \gamma F(o_W))} E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_{\mathbf{E}}} E$  coincides with a composition  $F(W) \xrightarrow{\zeta} \pi^{-1}(t) \xrightarrow{\iota_{\mathbf{E}}(\gamma(*))} \pi^{-1}(s) \xrightarrow{i_t} E$  which belongs to  $\mathcal{E} \cap F_E(W)$ . Therefore  $\gamma$  satisfies (G2). Since  $F_{\sigma_{\mathbf{E}}}(\gamma), F_{\tau_{\mathbf{E}}}(\gamma) \in F_B(1_{\mathcal{C}}) \subset \mathcal{B}$ , we have  $\gamma \in \mathcal{B}^{\sigma_{\mathbf{E}}} \cap \mathcal{B}^{\tau_{\mathbf{E}}}$ . Hence  $\gamma$  satisfies (G3). Thus we have  $\mathcal{G}_{\mathbf{E}} \supset F_{G_1(\mathbf{E})}(1_{\mathcal{C}})$ .

Let  $h : Z \rightarrow U$  be a morphism in  $\mathcal{C}$ . For  $\gamma \in \mathcal{G}_{\mathbf{E}} \cap F_{G_1(\mathbf{E})}(U)$ , we take  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, Z)$  and  $g \in \mathcal{C}(W, V)$ . Assume that  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfies  $\pi \lambda F(g) = \sigma_{\mathbf{E}} F_{G_1(\mathbf{E})}(h)(\gamma) F(f)$ . Since  $\pi \lambda F(g) = \sigma_{\mathbf{E}} \gamma F(hf)$  and  $\gamma$  satisfies (G1), a composition  $F(W) \xrightarrow{(\lambda F(g), \gamma F(hf))} E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_{\mathbf{E}}} E$  belongs to  $\mathcal{E} \cap F_E(W)$ . This shows that  $F_{G_1(\mathbf{E})}(h)(\gamma)$  satisfies (G1). Assume that  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfy  $\pi \lambda F(g) = \tau_{\mathbf{E}} F_{G_1(\mathbf{E})}(h)(\gamma) F(f)$ . Since  $\pi \lambda F(g) = \tau_{\mathbf{E}} \gamma F(hf)$  and  $\gamma$  satisfies (G2), a composition  $F(W) \xrightarrow{(\lambda F(g), \iota_{\mathbf{E}} \gamma F(hf))} E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_{\mathbf{E}}} E$  belongs



to  $\mathcal{E} \cap F_E(W)$ . This shows that  $F_{G_1(\mathbf{E})}(h)(\gamma)$  satisfies (G2). Since  $\gamma$  satisfies (G2), compositions  $F(Z) \xrightarrow{\gamma F(h)} G_1(\mathbf{E}) \xrightarrow{\sigma_E} B$  and  $F(U) \xrightarrow{\gamma F(h)} G_1(\mathbf{E}) \xrightarrow{\tau_E} B$  belong to  $\mathcal{B} \cap F_B(U)$ , which implies that  $F_{G_1(\mathbf{E})}(h)(\gamma) = \gamma F(h)$  satisfies (G3). Thus we have  $F_{G_1(\mathbf{E})}(h)(\gamma) = \gamma F(h) \in \mathcal{G}_E \cap F_{G_1(\mathbf{E})}(Z)$ .

For  $\gamma \in F_{G_1(\mathbf{E})}(U)$ , suppose that there exists  $R \in J(U)$  such that  $F_{G_1(\mathbf{E})}(j)(\gamma) \in \mathcal{G}_E \cap F_{G_1(\mathbf{E})}(\text{dom}(j))$  for any  $j \in R$ . We take  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$  and  $g \in \mathcal{C}(W, V)$ . If we put

$$h_f^{-1}(R) = \{k \in \text{Mor } \mathcal{C} \mid \text{codom}(k) = W, fk \in R\},$$

then we have  $h_f^{-1}(R) \in J(W)$  and  $F_{G_1(\mathbf{E})}(fk)(\gamma) \in \mathcal{G}_E \cap F_{G_1(\mathbf{E})}(\text{dom}(k))$  for any  $k \in h_f^{-1}(R)$ . Assume that  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfies  $\pi \lambda F(g) = \sigma_E \gamma F(f)$ . Hence the following composition belongs to  $\mathcal{E} \cap F_E(W)$  for any  $k \in h_f^{-1}(R)$ .

$$F(\text{dom}(k)) \xrightarrow{(\lambda F(gk), F_{G_1(\mathbf{E})}(fk)(\gamma))} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$$

Since the above composition coincides with the following composition

$$F(\text{dom}(k)) \xrightarrow{F(k)} F(W) \xrightarrow{(\lambda F(g), \gamma F(f))} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$$

for any  $k \in h_f^{-1}(R)$ , it follows that a composition  $F(W) \xrightarrow{(\lambda F(g), \gamma F(f))} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$  belongs to  $\mathcal{E} \cap F_E(W)$ , namely  $\gamma$  satisfies (G1). Assume that  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfies  $\pi \lambda F(g) = \tau_E \gamma F(f)$ . Hence the following composition belongs to  $\mathcal{E} \cap F_E(W)$  for any  $k \in h_f^{-1}(R)$ .

$$F(\text{dom}(k)) \xrightarrow{(\lambda F(gk), \iota_E F_{G_1(\mathbf{E})}(fk)(\gamma))} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$$

Since the above composition coincides with the following composition

$$F(\text{dom}(k)) \xrightarrow{F(k)} F(W) \xrightarrow{(\lambda F(g), \iota_E \gamma F(f))} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$$

for any  $k \in h_f^{-1}(R)$ , it follows that a composition  $F(W) \xrightarrow{(\lambda F(g), \iota_E \gamma F(f))} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$  belongs to  $\mathcal{E} \cap F_E(W)$ , namely  $\gamma$  satisfies (G2). Since  $F_{G_1(\mathbf{E})}(j)(\gamma) \in \mathcal{G}_E \cap F_{G_1(\mathbf{E})}(\text{dom}(j))$  for any  $j \in R$ , compositions  $F(\text{dom}(j)) \xrightarrow{F_{G_1(\mathbf{E})}(j)(\gamma)} G_1(\mathbf{E}) \xrightarrow{\sigma_E} B$  and  $F(\text{dom}(j)) \xrightarrow{F_{G_1(\mathbf{E})}(j)(\gamma)} G_1(\mathbf{E}) \xrightarrow{\tau_E} B$  belong to  $\mathcal{B} \cap F_B(\text{dom}(j))$ .

Since the above compositions coincides with compositions  $F(\text{dom}(j)) \xrightarrow{F(j)} F(U) \xrightarrow{\gamma} G_1(\mathbf{E}) \xrightarrow{\sigma_E} B$  and  $F(\text{dom}(j)) \xrightarrow{F(j)} F(U) \xrightarrow{\gamma} G_1(\mathbf{E}) \xrightarrow{\tau_E} B$  respectively for any  $j \in R$ , it follows that compositions  $F(U) \xrightarrow{\gamma} G_1(\mathbf{E}) \xrightarrow{\sigma_E} B$  and  $F(U) \xrightarrow{\gamma} G_1(\mathbf{E}) \xrightarrow{\tau_E} B$  belong to  $\mathcal{B} \cap F_B(U)$ . Hence  $\gamma$  satisfies (G3) and we have  $\gamma \in \mathcal{G}_E \cap F_{G_1(\mathbf{E})}(U)$ .  $\square$

**Proposition 5.4**  $\mathcal{G}_E$  is maximum element of  $\Sigma_E$ .

*Proof.* For  $U \in \text{Ob } \mathcal{C}$  and  $\delta \in \mathcal{E}^{\text{pr}_E^\sigma} \cap \mathcal{G}_E^{\text{pr}_{G_1(\mathbf{E})}^\sigma} \cap F_{E \times_B^{\sigma_E} G_1(\mathbf{E})}(U)$ ,  $\pi \text{pr}_E^\sigma \delta = \sigma_E \text{pr}_{G_1(\mathbf{E})}^\sigma \delta$  holds and it follows from  $\text{pr}_E^\sigma \delta \in \mathcal{E} \cap F_E(U)$  and  $\text{pr}_{G_1(\mathbf{E})}^\sigma \delta \in \mathcal{G}_E \cap F_{G_1(\mathbf{E})}(U)$  that the following composition belongs to  $\mathcal{E} \cap F_E(U)$ .

$$F(U) \xrightarrow{\delta = (\text{pr}_E^\sigma \delta, \text{pr}_{G_1(\mathbf{E})}^\sigma \delta)} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$$

That is, we have  $\delta \in \mathcal{E}^{\hat{\xi}_E} \cap F_{E \times_B^{\sigma_E} G_1(\mathbf{E})}(U)$ . It follows that  $\mathcal{E}^{\text{pr}_E^\sigma} \cap \mathcal{G}_E^{\text{pr}_{G_1(\mathbf{E})}^\sigma} \subset \mathcal{E}^{\hat{\xi}_E}$  holds. For  $U \in \text{Ob } \mathcal{C}$  and  $\delta' \in \mathcal{E}^{\text{pr}_E^\tau} \cap \mathcal{G}_E^{\text{pr}_{G_1(\mathbf{E})}^\tau} \cap F_{E \times_B^{\tau_E} G_1(\mathbf{E})}(U)$ ,  $\pi \text{pr}_E^\tau \delta' = \tau_E \text{pr}_{G_1(\mathbf{E})}^\tau \delta'$  holds and it follows from  $\text{pr}_E^\tau \delta' \in \mathcal{E} \cap F_E(U)$  and  $\text{pr}_{G_1(\mathbf{E})}^\tau \delta' \in \mathcal{G}_E \cap F_{G_1(\mathbf{E})}(U)$  that the following composition belongs to  $\mathcal{E} \cap F_E(U)$ .

$$F(U) \xrightarrow{(\text{id}_E \times_B \iota_E) \delta' = (\text{pr}_E^\tau \delta', \iota_E \text{pr}_{G_1(\mathbf{E})}^\tau \delta')} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$$

That is, we have  $\delta' \in \mathcal{E}^{\hat{\xi}_E(\text{id}_E \times_B \iota_E)} \cap F_{E \times_B^{\sigma_E} G_1(\mathbf{E})}(U)$ . It follows that  $\mathcal{E}^{\text{pr}_E^\tau} \cap \mathcal{G}_E^{\text{pr}_{G_1(\mathbf{E})}^\tau} \subset \mathcal{E}^{\hat{\xi}_E(\text{id}_E \times_B \iota_E)}$  holds.  $\mathcal{G}_E \subset \mathcal{B}^{\sigma_E} \cap \mathcal{B}^{\tau_E}$  holds by (G3') of (5.2). Therefore  $\mathcal{G}_E$  belongs to  $\Sigma_E$ .

Let  $\mathcal{L}$  be an element of  $\Sigma_E$ . For  $U \in \text{Ob } \mathcal{C}$  and  $\gamma \in \mathcal{L} \cap F_{G_1(\mathbf{E})}(U)$ , we take  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$  and  $g \in \mathcal{C}(W, V)$ . Assume that  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfies  $\pi \lambda F(g) = \sigma_E \gamma F(f)$  and put  $\delta = (\lambda F(g), \gamma F(f))$ . Then we have  $\text{pr}_E^\sigma \delta = \lambda F(g) \in \mathcal{E} \cap F_E(W)$  and  $\text{pr}_{G_1(\mathbf{E})}^\sigma \delta = \gamma F(f) \in \mathcal{L} \cap F_{G_1(\mathbf{E})}(W)$ . It follows that we have  $\delta \in \mathcal{E}^{\text{pr}_E^\sigma} \cap \mathcal{L}^{\text{pr}_{G_1(\mathbf{E})}^\sigma} \cap F_{E \times_B^{\sigma_E} G_1(\mathbf{E})}(W) \subset \mathcal{E}^{\hat{\xi}_E} \cap F_{E \times_B^{\sigma_E} G_1(\mathbf{E})}(W)$ , which shows that  $\gamma$  satisfies (G1). Assume that  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfies  $\pi \lambda F(g) = \tau_E \gamma F(f)$  and put  $\delta' = (\lambda F(g), \gamma F(f))$ . Then we have  $\text{pr}_E^\tau \delta' = \lambda F(g) \in \mathcal{E} \cap F_E(W)$  and  $\text{pr}_{G_1(\mathbf{E})}^\tau \delta' = \gamma F(f) \in \mathcal{L} \cap F_{G_1(\mathbf{E})}(W)$ . It follows that  $\delta'$  belongs to  $\mathcal{E}^{\text{pr}_E^\tau} \cap \mathcal{L}^{\text{pr}_{G_1(\mathbf{E})}^\tau} \cap F_{E \times_B^{\tau_E} G_1(\mathbf{E})}(W)$  which is contained in  $\mathcal{E}^{\hat{\xi}_E(\text{id}_E \times_B \iota_E)} \cap F_{E \times_B^{\sigma_E} G_1(\mathbf{E})}(W)$ . This implies that  $\gamma$  satisfies (G2). Since  $\mathcal{L} \subset \mathcal{B}^{\sigma_E} \cap \mathcal{B}^{\tau_E}$ ,  $\gamma$  satisfies (G3). Thus we have  $\gamma \in \mathcal{G}_E$  which implies  $\mathcal{L} \subset \mathcal{G}_E$ .  $\square$

We consider the following cartesian square.

$$\begin{array}{ccc}
E \times_B^{\sigma_E} G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) & \xrightarrow{\text{pr}_{12}} & E \times_B^{\sigma_E} G_1(\mathbf{E}) \\
\downarrow \text{pr}_3 & & \downarrow \tau_E \text{pr}_{G_1(\mathbf{E})} \\
G_1(\mathbf{E}) & \xrightarrow{\sigma_E} & B
\end{array} \quad (i)$$

Then, we have  $E \times_B^{\sigma_E} G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) = \{(e, \varphi, \psi) \in E \times G_1(\mathbf{E}) \times G_1(\mathbf{E}) \mid \pi(e) = \sigma_E(\varphi), \tau_E(\varphi) = \sigma_E(\psi)\}$  as a set. It follows from the definition of  $\hat{\xi}_E$  that the following diagram is commutative.

$$\begin{array}{ccc}
E \times_B^{\sigma_E} G_1(\mathbf{E}) & \xrightarrow{\hat{\xi}_E} & E \\
\downarrow \text{pr}_{G_1(\mathbf{E})}^\sigma & & \downarrow \pi \\
G_1(\mathbf{E}) & \xrightarrow{\tau_E} & B
\end{array} \quad (ii)$$

There exists unique map  $\hat{\xi}_E \times_B \text{id}_{G_1(\mathbf{E})} : E \times_B^{\sigma_E} G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) \rightarrow E \times_B^{\sigma_E} G_1(\mathbf{E})$  that makes the following diagram commute by the commutativity of diagrams (i) and (ii) above.

$$\begin{array}{ccccc}
E \times_B^{\sigma_E} G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) & & & & \\
\downarrow \text{pr}_{12} & \searrow \hat{\xi}_E \times_B \text{id}_{G_1(\mathbf{E})} & & \searrow \text{pr}_3 & \\
E \times_B^{\sigma_E} G_1(\mathbf{E}) & \xrightarrow{\hat{\xi}_E} & E & \xrightarrow{\pi} & B \\
& & \downarrow \text{pr}_E^\sigma & & \downarrow \sigma_E \\
& & G_1(\mathbf{E}) & \xrightarrow{\sigma_E} & B
\end{array}$$

We define maps  $\text{pr}_{23} : E \times_B^{\sigma_E} G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) \rightarrow G_1(\mathbf{E}) \times_B G_1(\mathbf{E})$  and  $\text{pr}_E : E \times_B^{\sigma_E} G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) \rightarrow E$  by  $\text{pr}_{23}(e, \varphi, \psi) = (\varphi, \psi)$  and  $\text{pr}_E(e, \varphi, \psi) = e$ , respectively. Then, there exists unique map

$$\text{id}_E \times_B \mu_E : E \times_B^{\sigma_E} G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) \rightarrow E \times_B^{\sigma_E} G_1(\mathbf{E})$$

that makes the following diagram commute.

$$\begin{array}{ccccc}
E \times_B^{\sigma_E} G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) & \xrightarrow{\text{pr}_{23}} & G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) & & \\
\downarrow \text{pr}_E & \searrow \text{id}_E \times_B \mu_E & \downarrow \text{pr}_E^\sigma & \swarrow \mu_E & \downarrow \text{pr}_1 \\
E & \xrightarrow{\pi} & E & \xrightarrow{\pi} & B \\
& & \downarrow \text{pr}_E^\sigma & & \downarrow \sigma_E \\
& & G_1(\mathbf{E}) & \xrightarrow{\sigma_E} & B
\end{array}$$

Let  $\iota_E^{(2)} : G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) \rightarrow G_1(\mathbf{E}) \times_B G_1(\mathbf{E})$  be the unique map that makes the following diagram commute.

$$\begin{array}{ccccc}
G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) & \xrightarrow{\text{pr}_1} & G_1(\mathbf{E}) & & \\
\downarrow \text{pr}_2 & \searrow \iota_E^{(2)} & \downarrow \text{pr}_1 & \swarrow \iota_E & \downarrow \tau_E \\
G_1(\mathbf{E}) & \xrightarrow{\sigma_E} & G_1(\mathbf{E}) & \xrightarrow{\sigma_E} & B \\
& & \downarrow \text{pr}_1 & & \downarrow \tau_E \\
& & G_1(\mathbf{E}) & \xrightarrow{\tau_E} & B
\end{array}$$

We note that  $\iota_E^{(2)}$  maps  $(\varphi, \psi) \in G_1(\mathbf{E}) \times_B G_1(\mathbf{E})$  to  $(\iota_E(\psi), \iota_E(\varphi))$ . It is easy to verify the following fact.

**Lemma 5.5** *The following diagrams are commutative.*

$$\begin{array}{ccccc}
E \times_B^{\sigma_E} G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) & \xrightarrow{id_E \times_B \mu_E} & E \times_B^{\sigma_E} G_1(\mathbf{E}) & & E & \xrightarrow{id_E} & G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) & \xrightarrow{\mu_E} & G_1(\mathbf{E}) \\
\downarrow \hat{\xi}_E \times_B id_{G_1(\mathbf{E})} & & \downarrow \hat{\xi}_E & & \downarrow (id_E, \varepsilon_E \pi) & \searrow & \downarrow \iota_E^{(2)} & & \downarrow \iota_E \\
E \times_B^{\sigma_E} G_1(\mathbf{E}) & \xrightarrow{\hat{\xi}_E} & E & & E \times_B^{\sigma_E} G_1(\mathbf{E}) & \xrightarrow{\hat{\xi}_E} & E & & G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) & \xrightarrow{\mu_E} & G_1(\mathbf{E})
\end{array}$$

**Proposition 5.6** *The structure maps  $\sigma_E, \tau_E : (G_1(\mathbf{E}), \mathcal{G}_E) \rightarrow (B, \mathcal{B})$ ,  $\varepsilon_E : (B, \mathcal{B}) \rightarrow (G_1(\mathbf{E}), \mathcal{G}_E)$ ,  $\mu_E : (G_1(\mathbf{E}) \times_B G_1(\mathbf{E}), \mathcal{G}_E^{\text{pr}_1} \cap \mathcal{G}_E^{\text{pr}_2}) \rightarrow (G_1(\mathbf{E}), \mathcal{G}_E)$  and  $\iota_E : (G_1(\mathbf{E}), \mathcal{G}_E) \rightarrow (G_1(\mathbf{E}), \mathcal{G}_E)$  of the groupoid  $(B, G_1(\mathbf{E}))$  are morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ .*

*Proof.* It follows from (G3) that  $\sigma_E, \tau_E : (G_1(\mathbf{E}), \mathcal{G}_E) \rightarrow (B, \mathcal{B})$  are morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ . For  $U \in \text{Ob } \mathcal{C}$  and  $x \in \mathcal{B} \cap F_B(U)$ , we take  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$  and  $g \in \mathcal{C}(W, V)$ . Assume that  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfies  $\pi \lambda F(g) = \sigma_E(F_{\varepsilon_E})_U(x)F(f)$ . It follows from the definitions of  $\varepsilon_E$  and  $\hat{\xi}_E$  that the composition

$$F(W) \xrightarrow{(\lambda F(g), (F_{\varepsilon_E})_U(x)F(f))} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$$

coincides with  $\lambda F(g)$  which belongs to  $\mathcal{E} \cap F_E(W)$ . Hence  $(F_{\varepsilon_E})_U(x)$  satisfies (G1). Assume that  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfies  $\pi \lambda F(g) = \tau_E(F_{\varepsilon_E})_U(x)F(f)$ . It follows from the definitions of  $\varepsilon_E$  and  $\hat{\xi}_E$  that the composition

$$F(W) \xrightarrow{(\lambda F(g), (F_{\varepsilon_E})_U(x)F(f))} E \times_B^{\tau_E} G_1(\mathbf{E}) \xrightarrow{id_E \times_B \iota_E} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$$

coincides with  $\lambda F(g)$  which belongs to  $\mathcal{E} \cap F_E(W)$ . It follows that  $(F_{\varepsilon_E})_U(x)$  satisfies (G2). Since we have  $\sigma_E(F_{\varepsilon_E})_U(x) = \tau_E(F_{\varepsilon_E})_U(x) = x \in \mathcal{B} \cap F_B(U)$ ,  $(F_{\varepsilon_E})_U(x)$  satisfies (G3). Therefore  $(F_{\varepsilon_E})_U(x)$  belongs to  $\mathcal{G}_E \cap F_{G_1(\mathbf{E})}(U)$  and  $\varepsilon_E : (B, \mathcal{B}) \rightarrow (G_1(\mathbf{E}), \mathcal{G}_E)$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

For  $U \in \text{Ob } \mathcal{C}$  and  $\gamma \in \mathcal{G}_E \cap F_{G_1(\mathbf{E})}(U)$ , we take  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$  and  $g \in \mathcal{C}(W, V)$ . Assume that  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfies  $\pi \lambda F(g) = \sigma_E(F_{\iota_E})_U(\gamma)F(f)$ . Then,  $\pi \lambda F(g) = \tau_E \gamma F(f)$  holds and a composition

$$F(W) \xrightarrow{(\lambda F(g), (F_{\iota_E})_U(\gamma)F(f))} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$$

coincides with  $F(W) \xrightarrow{(\lambda F(g), \iota_E \gamma F(f))} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$  which belongs to  $\mathcal{E} \cap F_E(W)$  since  $\gamma$  satisfies (G2). Hence  $(F_{\iota_E})_U(\gamma)$  satisfies (G1). Assume that  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfies  $\pi \lambda F(g) = \tau_E(F_{\iota_E})_U(\gamma)F(f)$ . Then,  $\pi \lambda F(g) = \sigma_E \gamma F(f)$  holds and a composition  $F(W) \xrightarrow{(\lambda F(g), \iota_E (F_{\iota_E})_U(\gamma)F(f))} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$  coincides with

$$F(W) \xrightarrow{(\lambda F(g), \gamma F(f))} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$$

which belongs to  $\mathcal{E} \cap F_E(W)$  since  $\gamma$  satisfies (G1). Hence  $(F_{\iota_E})_U(\gamma)$  satisfies (G2). Since  $\gamma$  satisfies (G3), we have  $\sigma_E(F_{\iota_E})_U(\gamma) = \tau_E \in \mathcal{B} \cap F_B(U)$  and  $\tau_E(F_{\iota_E})_U(\gamma) = \sigma_E \in \mathcal{B} \cap F_B(U)$ . Thus  $(F_{\iota_E})_U(\gamma)$  also satisfies (G3) and  $(F_{\iota_E})_U(\gamma) \in \mathcal{G}_E \cap F_{G_1(\mathbf{E})}(U)$ . Therefore  $\iota_E : (G_1(\mathbf{E}), \mathcal{G}_E) \rightarrow (G_1(\mathbf{E}), \mathcal{G}_E)$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

For  $U \in \text{Ob } \mathcal{C}$  and  $(\alpha, \beta) \in \mathcal{G}_E^{\text{pr}_1} \cap \mathcal{G}_E^{\text{pr}_2} \cap F_{G_1(\mathbf{E}) \times_B G_1(\mathbf{E})}(U)$ , we take  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$  and  $g \in \mathcal{C}(W, V)$ . We note that  $\alpha, \beta \in \mathcal{G}_E \cap F_{G_1(\mathbf{E})}(U)$  and that  $\tau_E \alpha = \sigma_E \beta$  holds. Assume that  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfies  $\pi \lambda F(g) = \sigma_E(F_{\mu_E})_U((\alpha, \beta))F(f)$ . Since  $(F_{\mu_E})_U((\alpha, \beta))F(f) = \mu_E(\alpha, \beta)F(f)$  holds, a composition

$$F(W) \xrightarrow{(\lambda F(g), (F_{\mu_E})_U((\alpha, \beta))F(f))} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$$

coincides with the following composition.

$$F(W) \xrightarrow{(\lambda F(g), \alpha F(f), \beta F(f))} E \times_B^{\sigma_E} G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) \xrightarrow{id_E \times_B \mu_E} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$$

By the commutativity of the left diagram of (5.5), the above composition coincides with a composition

$$F(W) \xrightarrow{((F_{\hat{\xi}_E})_W(\lambda F(g), \alpha F(f), \beta F(f)))} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E.$$

Since  $\hat{\xi}_E : (E \times_B^{\sigma_E} G_1(\mathbf{E}), \mathcal{E}^{\text{pr}_E} \cap \mathcal{G}_E^{\text{pr}_{G_1(\mathbf{E})}}) \rightarrow (E, \mathcal{E})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$  and  $(\lambda F(g), \alpha F(f))$  belongs to  $\mathcal{E}^{\text{pr}_E} \cap \mathcal{G}_E^{\text{pr}_{G_1(\mathbf{E})}} \cap F_{E \times_B^{\sigma_E} G_1(\mathbf{E})}(W)$ , the above composition belongs to  $\mathcal{E} \cap F_E(W)$ . Hence  $(F_{\mu_E})_U((\alpha, \beta))$  satisfies (G1).

Assume that  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfies  $\pi \lambda F(g) = \tau_E(F_{\mu_E})_U((\alpha, \beta))F(f)$ . Since an equality

$$\iota_E(F_{\mu_E})_U((\alpha, \beta))F(f) = \iota_E \mu_E(\alpha, \beta)F(f) = \mu_E \iota_E^{(2)}(\alpha, \beta)F(f) = \mu_E(\iota_E \beta, \iota_E \alpha)F(f)$$

holds by the commutativity of the left diagram of (5.5), Then, a composition

$$F(W) \xrightarrow{(\lambda F(g), \iota_{\mathbf{E}}(F_{\mu_{\mathbf{E}}})_U((\alpha, \beta))F(f))} E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_{\mathbf{E}}} E \cdots (*)$$

coincides with the following composition.

$$F(W) \xrightarrow{(\lambda F(g), \iota_{\mathbf{E}}\beta F(f), \iota_{\mathbf{E}}\alpha F(f))} E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) \xrightarrow{id_E \times_B \mu_E} E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_{\mathbf{E}}} E$$

The following diagram is commutative by the commutativity of the left diagram of (5.5).

$$\begin{array}{ccc} F(W) & \xrightarrow{(\lambda F(g), \iota_{\mathbf{E}}\beta F(f), \iota_{\mathbf{E}}\alpha F(f))} E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) & \xrightarrow{id_E \times_B \mu_E} E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) \\ & \searrow^{((F_{\hat{\xi}_{\mathbf{E}}})_W(\lambda F(g), (F_{\iota_{\mathbf{E}}})_W(\beta F(f))), (F_{\iota_{\mathbf{E}}})_W(\alpha F(f)))} & \downarrow \hat{\xi}_{\mathbf{E} \times_B id_{G_1(\mathbf{E})}} \\ & & E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_{\mathbf{E}}} E \end{array}$$

Since  $\iota_{\mathbf{E}} : (G_1(\mathbf{E}), \mathcal{G}_{\mathbf{E}}) \rightarrow (G_1(\mathbf{E}), \mathcal{G}_{\mathbf{E}})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ ,  $(F_{\iota_{\mathbf{E}}})_W(\beta F(f))$  and  $(F_{\iota_{\mathbf{E}}})_W(\alpha F(f))$  belongs to  $\mathcal{G}_{\mathbf{E}} \cap F_{G_1(\mathbf{E})}(W)$ . Thus we have  $(\lambda F(g), (F_{\iota_{\mathbf{E}}})_W(\beta F(f))) \in \mathcal{E}^{\text{pr}_E} \cap \mathcal{G}_{\mathbf{E}}^{\text{pr}_{G_1(\mathbf{E})}} \cap F_{E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E})}(W)$ . Since  $\hat{\xi}_{\mathbf{E}} : (E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}), \mathcal{E}^{\text{pr}_E} \cap \mathcal{G}_{\mathbf{E}}^{\text{pr}_{G_1(\mathbf{E})}}) \rightarrow (E, \mathcal{E})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ ,  $(F_{\hat{\xi}_{\mathbf{E}}})_W(\lambda F(g), (F_{\iota_{\mathbf{E}}})_W(\beta F(f)))$  belongs to  $\mathcal{E} \cap F_E(W)$ . Then, it follows that  $((F_{\hat{\xi}_{\mathbf{E}}})_W(\lambda F(g), (F_{\iota_{\mathbf{E}}})_W(\beta F(f))), (F_{\iota_{\mathbf{E}}})_W(\alpha F(f)))$  also belongs to  $\mathcal{E}^{\text{pr}_E} \cap \mathcal{G}_{\mathbf{E}}^{\text{pr}_{G_1(\mathbf{E})}} \cap F_{E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E})}(W)$ . Finally, the image of  $((F_{\hat{\xi}_{\mathbf{E}}})_W(\lambda F(g), (F_{\iota_{\mathbf{E}}})_W(\beta F(f))), (F_{\iota_{\mathbf{E}}})_W(\alpha F(f)))$  by  $(F_{\hat{\xi}_{\mathbf{E}}})_W : F_{E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E})}(W) \rightarrow F_E(W)$  belongs to  $\mathcal{E} \cap F_E(W)$ . Therefore the composition  $(*)$  belongs to  $\mathcal{E} \cap F_E(W)$  and  $(F_{\mu_{\mathbf{E}}})_U((\alpha, \beta))$  satisfies (G2).

Since both  $\alpha$  and  $\beta$  satisfy (G3), it follows that both  $\sigma_{\mathbf{E}}(F_{\mu_{\mathbf{E}}})_U((\alpha, \beta)) = \sigma_{\mathbf{E}}\alpha$  and  $\tau_{\mathbf{E}}(F_{\mu_{\mathbf{E}}})_U((\alpha, \beta)) = \tau_{\mathbf{E}}\beta$  belongs to  $\mathcal{B} \cap F_B(U)$ , which shows that  $(F_{\mu_{\mathbf{E}}})_U((\alpha, \beta))$  satisfies (G3). Hence  $\mu_{\mathbf{E}}$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .  $\square$

**Definition 5.7** Let  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (B, \mathcal{B}))$  be an object of  $\mathcal{P}_F(\mathcal{C}, J)_{(B, \mathcal{B})}^{(2)}$  such that  $\pi$  is an epimorphism. We call the groupoid  $((B, \mathcal{B}), (G_1(\mathbf{E}), \mathcal{G}_{\mathbf{E}}); \sigma_{\mathbf{E}}, \tau_{\mathbf{E}}, \varepsilon_{\mathbf{E}}, \mu_{\mathbf{E}}, \iota_{\mathbf{E}})$  in  $\mathcal{P}_F(\mathcal{C}, J)$  the groupoid associated with  $\mathbf{E}$  and denote this groupoid by  $\mathbf{G}(\mathbf{E})$ .

Let us denote by  $\text{Epi}_{\mathcal{C}}(\mathcal{P}_F(\mathcal{C}, J))$  a subcategory of  $\mathcal{P}_F(\mathcal{C}, J)^{(2)}$  whose objects are epimorphisms in  $\mathcal{P}_F(\mathcal{C}, J)$  and morphisms are cartesian morphisms in the fibered category  $\varphi : \mathcal{P}_F(\mathcal{C}, J)^{(2)} \rightarrow \mathcal{P}_F(\mathcal{C}, J)$  of morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ .

**Example 5.8** For an object  $(X, \mathcal{X})$  of  $\mathcal{P}_F(\mathcal{C}, J)$ , we denote by  $o_X : (X, \mathcal{X}) \rightarrow (\{1\}, \mathcal{D}_{\text{coarse}, \{1\}})$  the unique morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . Since  $o_X$  is an epimorphism, we regard this as an object  $\mathbf{O}_X$  of  $\text{Epi}_{\mathcal{C}}(\mathcal{P}_F(\mathcal{C}, J))$ . The groupoid  $\mathbf{G}(\mathbf{O}_X) = ((\{1\}, \mathcal{D}_{\text{coarse}, \{1\}}), (G_1(\mathbf{O}_X), \mathcal{G}_{\mathbf{O}_X}); \sigma_{\mathbf{O}_X}, \tau_{\mathbf{O}_X}, \varepsilon_{\mathbf{O}_X}, \mu_{\mathbf{O}_X}, \iota_{\mathbf{O}_X})$  is given as follows.

We put  $\text{End}(X, \mathcal{X}) = \mathcal{P}_F(\mathcal{C}, J)((X, \mathcal{X}), (X, \mathcal{X}))$  and define a subset  $\text{Aut}(X, \mathcal{X})$  of  $\text{End}(X, \mathcal{X})$  by

$$\text{Aut}(X, \mathcal{X}) = \{\varphi \in \text{End}(X, \mathcal{X}) \mid \varphi \text{ is an isomorphism.}\}$$

Then,  $G_1(\mathbf{O}_X)$  is identified with  $\text{Aut}(X, \mathcal{X})$  as a set. The source  $\sigma_{\mathbf{O}_X}$  and the target  $\tau_{\mathbf{O}_X}$  are the unique map  $G_1(\mathbf{O}_X) \rightarrow \{1\}$ . The unit  $\varepsilon_{\mathbf{O}_X} : \{1\} \rightarrow G_1(\mathbf{O}_X)$  maps 1 to  $id_X$ . The composition  $\mu_{\mathbf{O}_X} : G_1(\mathbf{O}_X) \times G_1(\mathbf{O}_X) \rightarrow G_1(\mathbf{O}_X)$  maps  $(\varphi, \psi)$  to  $\psi\varphi$  and the inverse  $\iota_{\mathbf{O}_X} : G_1(\mathbf{O}_X) \rightarrow G_1(\mathbf{O}_X)$  maps  $\varphi$  to  $\varphi^{-1}$ .

We denote by  $\alpha_X : X \times G_1(\mathbf{O}_X) \rightarrow X$  the map defined by  $\alpha_X(x, \varphi) = \varphi(x)$ . Then, the the-ology  $\mathcal{G}_{\mathbf{O}_X}$  on  $G_1(\mathbf{O}_X) = \text{Aut}(X, \mathcal{X})$  is described as follows.

For  $U \in \text{Ob } \mathcal{C}$ ,  $\mathcal{G}_{\mathbf{O}_X} \cap F_{G_1(\mathbf{O}_X)}(U)$  is a subset of  $F_{G_1(\mathbf{O}_X)}(U)$  consisting of elements  $\gamma$  which satisfy the following condition (G).

(G) For  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, U)$ ,  $g \in \mathcal{C}(W, V)$  and  $\lambda \in \mathcal{X} \cap F_X(V)$ , the following compositions belong to  $\mathcal{X} \cap F_X(W)$ .

$$F(W) \xrightarrow{(\lambda F(g), \gamma F(f))} X \times G_1(\mathbf{O}_X) \xrightarrow{\alpha_X} X \quad F(W) \xrightarrow{(\lambda F(g), \iota_{\mathbf{O}_X} \gamma F(f))} X \times G_1(\mathbf{O}_X) \xrightarrow{\alpha_X} X$$

Let  $((G, \mathcal{G}); \varepsilon, \mu, \iota)$  be a group object in  $\mathcal{P}_F(\mathcal{C}, J)$  with structure morphisms  $\varepsilon : (\{1\}, \mathcal{D}_{\text{disc}, \{1\}}) \rightarrow (G, \mathcal{G})$ ,  $\mu : (G \times G, \mathcal{G}^{p_1} \cap \mathcal{G}^{p_2}) \rightarrow (G, \mathcal{G})$  and  $\iota : (G, \mathcal{G}) \rightarrow (G, \mathcal{G})$  in  $\mathcal{P}_F(\mathcal{C}, J)$  which make the following diagrams commute. Here,  $p_i : G \times G \rightarrow G$  denotes the projection onto the  $i$ -th component for  $i = 1, 2$ .

$$\begin{array}{ccccc} G \times G \times G & \xrightarrow{\mu \times id_G} & G \times G & & G \times G & \xrightarrow{\iota, id_G} & G & & G \\ \downarrow id_X \times \mu & & \downarrow \mu & & \downarrow \mu & & \downarrow \mu & & \downarrow o_G \\ G \times G & \xrightarrow{\mu} & G & & G & \xrightarrow{\varepsilon} & G & & G \\ \uparrow (id_G, o_G) & & \uparrow id_G & & \uparrow (o_G, id_G) & & \uparrow \varepsilon & & \uparrow \varepsilon \\ G & \xrightarrow{id_G} & G & & G & \xrightarrow{\varepsilon} & G & & G \end{array}$$

For an object  $(B, \mathcal{B})$  of  $\mathcal{P}_F(\mathcal{C}, J)$ , we define a groupoid  $\mathbf{G}_{G,B}$  in  $\mathcal{P}_F(\mathcal{C}, J)$  as follows. Put  $G_1 = B \times G \times B$  and let  $\sigma_{G,B}, \tau_{G,B} : G_1 \rightarrow B$  and  $\text{pr}_G : G_1 \rightarrow G$  be the projections given by  $\sigma_{G,B}(x, g, y) = x$ ,  $\tau_{G,B}(x, g, y) = y$  and  $\text{pr}_G(x, g, y) = g$ . Define maps  $\varepsilon_{G,B} : B \rightarrow G_1$  by  $\varepsilon_{G,B}(x) = (x, \varepsilon(1), x)$ . Consider a cartesian square

$$\begin{array}{ccc} G_1 \times_B G_1 & \xrightarrow{\text{pr}_2} & G_1 \\ \downarrow \text{pr}_1 & & \downarrow \sigma_{G,B} \\ G_1 & \xrightarrow{\tau_{G,B}} & B \end{array}$$

Then,  $G_1 \times_B G_1 = \{(x, g, y), (z, h, w) \in G_1 \times G_1 \mid y = z\}$  holds as a set. Define maps  $\mu_{G,B} : G_1 \times_B G_1 \rightarrow G_1$  and  $\iota_{G,B} : G_1 \rightarrow G_1$  by  $\mu_{G,B}((x, g, y), (z, h, w)) = (x, \mu(g, h), w)$  and  $\iota_{G,B}(x, g, y) = (y, \iota(g), x)$ , respectively. It is clear that  $\sigma_{G,B}, \tau_{G,B} : (G_1, \mathcal{B}^{\sigma_{G,B}} \cap \mathcal{G}^{\text{pr}_G} \cap \mathcal{B}^{\tau_{G,B}}) \rightarrow (B, \mathcal{B})$  and  $\text{pr}_G : (G_1, \mathcal{B}^{\sigma_{G,B}} \cap \mathcal{G}^{\text{pr}_G} \cap \mathcal{B}^{\tau_{G,B}}) \rightarrow (G, \mathcal{G})$  are morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ . Since  $\sigma_{G,B} \varepsilon_{G,B} = \tau_{G,B} \varepsilon_{G,B} = \text{id}_B$  and the following diagram is commutative, it follows that  $\varepsilon_{G,B} : (B, \mathcal{B}) \rightarrow (G_1, \mathcal{B}^{\sigma_{G,B}} \cap \mathcal{G}^{\text{pr}_G} \cap \mathcal{B}^{\tau_{G,B}})$  is also a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

$$\begin{array}{ccc} (B, \mathcal{B}) & \xrightarrow{\varepsilon_{G,B}} & (G_1, \mathcal{B}^{\sigma_{G,B}} \cap \mathcal{G}^{\text{pr}_G} \cap \mathcal{B}^{\tau_{G,B}}) \\ \downarrow \sigma_B & & \downarrow \text{pr}_G \\ (\{1\}, \mathcal{D}_{\text{disc}, \{1\}}) & \xrightarrow{\varepsilon} & (G, \mathcal{G}) \end{array}$$

We note that  $\sigma_{G,B} \mu_{G,B} = \sigma_{G,B} \text{pr}_1$  and  $\tau_{G,B} \mu_{G,B} = \tau_{G,B} \text{pr}_2$  hold and that the following diagram commutes.

$$\begin{array}{ccc} G_1 \times_B G_1 & \xrightarrow{(\text{pr}_G, \text{pr}_G)} & G \times G \\ \downarrow \mu_{G,B} & & \downarrow \mu \\ G_1 & \xrightarrow{\text{pr}_G} & G \end{array}$$

Since  $\sigma_{G,B}, \tau_{G,B}, (\text{pr}_G, \text{pr}_G)$  and  $\mu$  are morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ , it follows that

$$\mu_{G,B} : (G_1 \times_B G_1, (\mathcal{B}^{\sigma_{G,B}} \cap \mathcal{G}^{\text{pr}_G} \cap \mathcal{B}^{\tau_{G,B}})^{\text{pr}_1} \cap (\mathcal{B}^{\sigma_{G,B}} \cap \mathcal{G}^{\text{pr}_G} \cap \mathcal{B}^{\tau_{G,B}})^{\text{pr}_2}) \rightarrow (G_1, \mathcal{B}^{\sigma_{G,B}} \cap \mathcal{G}^{\text{pr}_G} \cap \mathcal{B}^{\tau_{G,B}})$$

is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . We also have  $\sigma_{G,B} \iota_{G,B} = \tau_{G,B}, \tau_{G,B} \iota_{G,B} = \sigma_{G,B}$  and  $\text{pr}_G \iota_{G,B} = \iota \text{pr}_G$  which imply that  $\iota_{G,B} : (G_1, \mathcal{B}^{\sigma_{G,B}} \cap \mathcal{G}^{\text{pr}_G} \cap \mathcal{B}^{\tau_{G,B}}) \rightarrow (G_1, \mathcal{B}^{\sigma_{G,B}} \cap \mathcal{G}^{\text{pr}_G} \cap \mathcal{B}^{\tau_{G,B}})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . It is easy to verify that  $((B, \mathcal{B}), (B \times G \times B, \mathcal{B}^{\sigma_{G,B}} \cap \mathcal{B}^{\tau_{G,B}} \cap \mathcal{G}^{\text{pr}_G}); \sigma_{G,B}, \tau_{G,B}, \varepsilon_{G,B}, \mu_{G,B}, \iota_{G,B})$  is a groupoid in  $\mathcal{P}_F(\mathcal{C}, J)$ .

**Definition 5.9** *The groupoid  $((B, \mathcal{B}), (B \times G \times B, \mathcal{B}^{\sigma_{G,B}} \cap \mathcal{G}^{\text{pr}_G} \cap \mathcal{B}^{\tau_{G,B}}); \sigma_{G,B}, \tau_{G,B}, \varepsilon_{G,B}, \mu_{G,B}, \iota_{G,B})$  in  $\mathcal{P}_F(\mathcal{C}, J)$  constructed above is called the trivial groupoid associated with  $((G, \mathcal{G}); \varepsilon, \mu, \iota)$  and  $(B, \mathcal{B})$ .*

Let  $(X, \mathcal{X})$  and  $(B, \mathcal{B})$  be objects of  $\mathcal{P}_F(\mathcal{C}, J)$ . Let us denote by  $\text{pr}_X : X \times B \rightarrow X$  and  $\text{pr}_B : X \times B \rightarrow B$  the projections. Then we have an object  $\mathbf{X} = ((X \times B, \mathcal{X}^{\text{pr}_X} \cap \mathcal{B}^{\text{pr}_B}) \xrightarrow{\text{pr}_B} (B, \mathcal{B}))$  of  $\text{Epi}_c(\mathcal{P}_F(\mathcal{C}, J))$ . We also have a group object  $G_1(\mathbf{O}_X) = \text{Aut}(X, \mathcal{X})$  in  $\mathcal{P}_F(\mathcal{C}, J)$  with unit  $\varepsilon_{\mathbf{O}_X} : \{1\} \rightarrow G_1(\mathbf{O}_X)$ , product  $\mu_{\mathbf{O}_X} : G_1(\mathbf{O}_X) \times G_1(\mathbf{O}_X) \rightarrow G_1(\mathbf{O}_X)$  and inverse  $\iota_{\mathbf{O}_X} : G_1(\mathbf{O}_X) \rightarrow G_1(\mathbf{O}_X)$  as we considered in (5.8).

**Proposition 5.10** *The groupoid  $\mathbf{G}(\mathbf{X}) = ((B, \mathcal{B}), (G_1(\mathbf{X}), \mathcal{G}_X); \sigma_X, \tau_X, \varepsilon_X, \mu_X, \iota_X)$  in  $\mathcal{P}_F(\mathcal{C}, J)$  associated with  $\mathbf{X}$  is isomorphic to the trivial groupoid associated with  $((G_1(\mathbf{O}_X), \mathcal{G}_{\mathbf{O}_X}); \varepsilon_{\mathbf{O}_X}, \mu_{\mathbf{O}_X}, \iota_{\mathbf{O}_X})$  and  $(B, \mathcal{B})$ .*

*Proof.* We denote by  $i_x : \text{pr}_B^{-1}(x) \rightarrow X \times B$  the inclusion map for  $x \in B$ . Then,  $\text{pr}_X i_x : \text{pr}_B^{-1}(x) \rightarrow X$  is a bijection and  $\text{pr}_B i_x : \text{pr}_B^{-1}(x) \rightarrow B$  is a constant map to  $\{x\}$ . Hence we have  $\mathcal{B}^{\text{pr}_B i_x} = \mathcal{D}_{\text{disc}, \text{pr}_B^{-1}(x)}$  and the following equality.

$$(\mathcal{X}^{\text{pr}_X} \cap \mathcal{B}^{\text{pr}_B})^{i_x} = \mathcal{X}^{\text{pr}_X i_x} \cap \mathcal{B}^{\text{pr}_B i_x} = \mathcal{X}^{\text{pr}_X i_x} \cap \mathcal{D}_{\text{disc}, \text{pr}_B^{-1}(x)} = \mathcal{X}^{\text{pr}_X i_x}.$$

Therefore  $\text{pr}_X i_x : (\text{pr}_B^{-1}(x), (\mathcal{X}^{\text{pr}_X} \cap \mathcal{B}^{\text{pr}_B})^{i_x}) \rightarrow (X, \mathcal{X})$  is an isomorphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

We put  $G = G_1(\mathbf{O}_X) = \text{Aut}(X, \mathcal{X})$  and  $G_1 = B \times G \times B$  for short and define a map  $\zeta_1 : G_1 \rightarrow G_1(\mathbf{X})$  by  $\zeta_1(x, y, \psi) = (\text{pr}_X i_y)^{-1} \psi (\text{pr}_X i_x)$ . Then,  $\zeta_1$  is bijective. In fact, the inverse  $\zeta_1^{-1} : G_1(\mathbf{X}) \rightarrow G_1$  of  $\zeta_1$  is given by  $\zeta_1^{-1}(\varphi) = (\sigma_X(\varphi), \tau_X(\varphi), (\text{pr}_X i_{\tau_X(\varphi)} \varphi (\text{pr}_X i_{\sigma_X(\varphi)})^{-1})$ . The following diagrams are commutative, hence  $(\text{id}_B, \zeta_1) : (B, G_1) \rightarrow (B, G_1(\mathbf{X}))$  is a morphism of groupoids. Here  $\zeta_1 \times_B \zeta_1 : G_1 \times_B G_1 \rightarrow G_1(\mathbf{X}) \times_B G_1(\mathbf{X})$  maps  $(\varphi, \psi)$  to  $(\zeta_1(\varphi), \zeta_1(\psi))$ .

$$\begin{array}{ccccccc} B \xleftarrow{\sigma_{G,B}} G_1 & \xrightarrow{\tau_{G,B}} & B & B \xrightarrow{\varepsilon_{G,B}} & G_1 & G_1 \times_B G_1 \xrightarrow{\mu_{G,B}} & G_1 & G_1 \xrightarrow{\iota_{G,B}} & G_1 \\ \downarrow \text{id}_B & \downarrow \zeta_1 & \downarrow \text{id}_B & \downarrow \text{id}_B & \downarrow \zeta_1 & \downarrow \zeta_1 \times_B \zeta_1 & \downarrow \zeta_1 & \downarrow \zeta_1 & \downarrow \zeta_1 \\ B \xleftarrow{\sigma_X} G_1(\mathbf{X}) & \xrightarrow{\tau_X} & B & B \xrightarrow{\varepsilon_X} & G_1(\mathbf{X}) & G_1(\mathbf{X}) \times_B G_1(\mathbf{X}) \xrightarrow{\mu_X} & G_1(\mathbf{X}) & G_1(\mathbf{X}) \xrightarrow{\iota_X} & G_1(\mathbf{X}) \end{array}$$

It remains to show that  $\zeta_1 : (G_1, \mathcal{B}^{\sigma_{G,B}} \cap \mathcal{B}^{\tau_{G,B}} \cap \mathcal{G}_{O_X}^{\text{pr}_G}) \rightarrow (G_1(\mathbf{X}), \mathcal{G}_{\mathbf{X}})$  and its inverse are morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ . We consider the following cartesian squares.

$$\begin{array}{ccc} (X \times B) \times_B G_1 & \xrightarrow{\text{pr}_{G_1}} & G_1 \\ \downarrow \text{pr}_{X \times B} & & \downarrow \sigma_{G,B} \\ X \times B & \xrightarrow{\text{pr}_B} & B \end{array} \quad \begin{array}{ccc} (X \times B) \times_B^{\sigma_{\mathbf{X}}} G_1(\mathbf{X}) & \xrightarrow{\text{pr}_{G_1(\mathbf{X})}^{\sigma_{\mathbf{X}}}} & G_1(\mathbf{X}) \\ \downarrow \text{pr}_{X \times B}^{\sigma_{\mathbf{X}}} & & \downarrow \sigma_{\mathbf{X}} \\ X \times B & \xrightarrow{\text{pr}_B} & B \end{array}$$

Then  $(X \times B) \times_B G_1$  is given by  $(X \times B) \times_B G_1 = \{(u, z), (x, y, \psi) \in (X \times B) \times G_1 \mid z = x\}$  as a set. Define maps  $\hat{\alpha}_X : (X \times B) \times_B G_1 \rightarrow X \times B$  and  $id_{X \times B} \times_B \zeta_1 : (X \times B) \times_B G_1 \rightarrow (X \times B) \times_B^{\sigma_{\mathbf{X}}} G_1(\mathbf{X})$  by  $\hat{\alpha}_X((u, x), (x, y, \psi)) = (\psi(u), y)$  and  $(id_{X \times B} \times_B \zeta_1)((u, x), (x, y, \psi)) = ((u, x), \gamma_1(x, y, \psi))$ , respectively. Since projections  $\text{pr}_{X \times B}, \text{pr}_{G_1}, \text{pr}_X, \text{pr}_G, \tau_{G,B}$  and the right  $G$ -action  $\alpha_X$  on  $X$  are morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ , it follows that  $\hat{\alpha}_X = (\alpha_X(\text{pr}_X \text{pr}_{X \times B}, \text{pr}_G \text{pr}_{G_1}), \tau_{G,B} \text{pr}_{G_1})$  is also a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . Let  $U$  be an object of  $\mathcal{C}$  and  $\gamma \in \mathcal{B}^{\sigma_{G,B}} \cap \mathcal{B}^{\tau_{G,B}} \cap \mathcal{G}_{O_X}^{\text{pr}_G} \cap F_{G_1}(U)$ . We take  $V, W \in \text{Ob } \mathcal{C}$  and  $f \in \mathcal{C}(W, U), g \in \mathcal{C}(W, V)$ .

Assume that  $\lambda \in \mathcal{X}^{\text{pr}_X} \cap \mathcal{B}^{\text{pr}_B} \cap F_{X \times B}(V)$  satisfies  $\text{pr}_B \lambda F(g) = \sigma_{\mathbf{X}}(F_{\zeta_1})_U(\gamma) F(f)$ . Then, we have  $\text{pr}_B \lambda F(g) = \sigma_{\mathbf{X}} \zeta_1 \gamma F(f) = \sigma_{G,B} \gamma F(f)$ , hence there exists a map  $(\lambda F(g), \gamma F(f)) : F(W) \rightarrow (X \times B) \times_B G_1$  such that the following diagram is commutative. Here  $id_{X \times B} \times_B \zeta_1 : (X \times B) \times_B G_1 \rightarrow (X \times B) \times_B^{\sigma_{\mathbf{X}}} G_1(\mathbf{X})$  is given by  $(id_{X \times B} \times_B \zeta_1)((u, x), \alpha) = ((u, x), \zeta_1(\alpha))$ .

$$\begin{array}{ccccc} F(W) & \xrightarrow{(\lambda F(g), \gamma F(f))} & (X \times B) \times_B G_1 & \xrightarrow{\hat{\alpha}_X} & X \times B \\ & \searrow (\lambda F(g), \zeta_1 \gamma F(f)) & \downarrow id_{X \times B} \times_B \zeta_1 & & \downarrow id_{X \times B} \\ & & (X \times B) \times_B^{\sigma_{\mathbf{X}}} G_1(\mathbf{X}) & \xrightarrow{\hat{\xi}_X} & X \times B \end{array}$$

Since  $\hat{\alpha}_X$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ ,  $F(W) \xrightarrow{(\lambda F(g), \zeta_1 \gamma F(f))} (X \times B) \times_B^{\sigma_{\mathbf{X}}} G_1(\mathbf{X}) \xrightarrow{\hat{\xi}_X} X \times B$  belongs to  $\mathcal{X}^{\text{pr}_X} \cap \mathcal{B}^{\text{pr}_B} \cap F_{X \times B}(W)$  by the commutativity of the above diagram. This shows that  $\gamma$  satisfies (G1).

Assume that  $\lambda \in \mathcal{X}^{\text{pr}_X} \cap \mathcal{B}^{\text{pr}_B} \cap F_{X \times B}(V)$  satisfies  $\text{pr}_B \lambda F(g) = \tau_{\mathbf{X}}(F_{\zeta_1})_U(\gamma) F(f)$ . Then, we have  $\text{pr}_B \lambda F(g) = \tau_{\mathbf{X}} \zeta_1 \gamma F(f) = \sigma_{G,B} \iota_{G,B} \gamma F(f)$  and there exists a map  $(\lambda F(g), \iota_{G,B} \gamma F(f)) : F(W) \rightarrow (X \times B) \times_B G_1$  such that the following diagram is commutative.

$$\begin{array}{ccccc} F(W) & \xrightarrow{(\lambda F(g), \iota_{G,B} \gamma F(f))} & (X \times B) \times_B G_1 & \xrightarrow{\hat{\alpha}_X} & X \times B \\ & \searrow (\lambda F(g), \iota_{\mathbf{X}} \zeta_1 \gamma F(f)) & \downarrow id_{X \times B} \times_B \zeta_1 & & \downarrow id_{X \times B} \\ & & (X \times B) \times_B^{\sigma_{\mathbf{X}}} G_1(\mathbf{X}) & \xrightarrow{\hat{\xi}_X} & X \times B \end{array}$$

Since  $\hat{\alpha}_X$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ ,  $F(W) \xrightarrow{(\lambda F(g), \iota_{\mathbf{X}} \zeta_1 \gamma F(f))} (X \times B) \times_B^{\sigma_{\mathbf{X}}} G_1(\mathbf{X}) \xrightarrow{\hat{\xi}_X} X \times B$  belongs to  $\mathcal{X}^{\text{pr}_X} \cap \mathcal{B}^{\text{pr}_B} \cap F_{X \times B}(W)$  by the commutativity of the above diagram. This shows that  $\gamma$  satisfies (G2).

Since  $\gamma \in \mathcal{B}^{\sigma_{G,B}} \cap \mathcal{B}^{\tau_{G,B}} \cap \mathcal{G}_{O_X}^{\text{pr}_G} \cap F_{G_1}(U)$ , both  $\sigma_X \zeta_1 \gamma = \sigma_{G,B} \gamma$  and  $\tau_X \zeta_1 \gamma = \tau_{G,B} \gamma$  belong to  $\mathcal{B}$ . Thus  $\gamma$  satisfies (G3) and  $\zeta_1$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

For  $\gamma \in \mathcal{G}_{\mathbf{X}} \cap F_{G_1(\mathbf{X})}(U)$ , both  $\sigma_{G,B}((F_{\zeta_1^{-1}})_U(\gamma)) = \sigma_{\mathbf{X}} \gamma$  and  $\tau_{G,B}((F_{\zeta_1^{-1}})_U(\gamma)) = \tau_{\mathbf{X}} \gamma$  belong to  $\mathcal{B} \cap F_B(U)$  since  $\gamma$  satisfies (G3). We put  $\gamma' = \text{pr}_G((F_{\zeta_1^{-1}})_U(\gamma))$  and take  $U, W \in \text{Ob } \mathcal{C}, f \in \mathcal{C}(W, U), g \in \mathcal{C}(W, V)$  and  $\lambda \in \mathcal{X} \cap F_X(V)$ . Define  $\lambda' \in \mathcal{X}^{\text{pr}_X} \cap \mathcal{B}^{\text{pr}_B} \cap F_{X \times B}(W)$  by  $\lambda' = (\lambda F(g), \sigma_{\mathbf{X}} \gamma F(f))$ . Then we have  $\text{pr}_B \lambda' F(id_W) = \sigma_{\mathbf{X}} \gamma F(f)$  and the following diagram is commutative.

$$\begin{array}{ccccc} & & (X \times B) \times_B^{\sigma_{\mathbf{X}}} G_1(\mathbf{X}) & \xrightarrow{\hat{\xi}_X} & X \times B \\ & \nearrow (\lambda' F(id_W), \gamma F(f)) & \downarrow id_{X \times B} \times_B \zeta_1^{-1} & & \downarrow id_{X \times B} \\ F(W) & \xrightarrow{(\lambda' F(id_W), \zeta_1^{-1} \gamma F(f))} & (X \times B) \times_B G_1 & \xrightarrow{\hat{\alpha}_X} & X \times B \\ & \searrow (\lambda F(g), \gamma' F(f)) & \downarrow (\text{pr}_X \text{pr}_{X \times B}, \text{pr}_G \text{pr}_{G_1}) & & \downarrow \text{pr}_X \\ & & X \times G & \xrightarrow{\alpha_X} & X \end{array}$$

Since  $\gamma$  satisfies (G1) for  $\mathbf{E} = \mathbf{X}$ , it follows from the commutativity of the above diagram that a composition  $F(W) \xrightarrow{(\lambda F(g), \gamma' F(f))} X \times G \xrightarrow{\alpha_X} X$  belong to  $\mathcal{X} \cap F_X(W)$ .

Define  $\lambda'' \in \mathcal{X}^{\text{pr}_X} \cap \mathcal{B}^{\text{pr}_B} \cap F_{X \times B}(W)$  by  $\lambda'' = (\lambda F(g), \tau_{\mathbf{X}} \gamma F(f))$ . Then we have  $\text{pr}_B \lambda'' F(id_W) = \tau_{\mathbf{X}} \gamma F(f)$  and the following diagram is commutative.

$$\begin{array}{ccccc}
& & (X \times B) \times_B^{\sigma_X} G_1(\mathbf{X}) & \xrightarrow{\hat{\xi}_X} & X \times B \\
& \nearrow^{(\lambda'' F(id_W), \iota_X \gamma F(f))} & \downarrow id_{X \times B \times B} \zeta_1^{-1} & & \downarrow id_{X \times B} \\
F(W) & \xrightarrow{(\lambda'' F(id_W), \iota_{G, B} \zeta_1^{-1} \gamma F(f))} & (X \times B) \times_B G_1 & \xrightarrow{\hat{\alpha}_X} & X \times B \\
& \searrow_{(\lambda F(g), \iota_{\mathcal{O}_X} \gamma' F(f))} & \downarrow (\text{pr}_X \text{pr}_{X \times B}, \text{pr}_G \text{pr}_{G_1}) & & \downarrow \text{pr}_X \\
& & X \times G & \xrightarrow{\alpha_X} & X
\end{array}$$

Since  $\gamma$  satisfies (G2) for  $\mathbf{E} = \mathbf{X}$ , it follows from the commutativity of the above diagram that a composition  $F(W) \xrightarrow{(\lambda F(g), \iota_{\mathcal{O}_X} \gamma' F(f))} X \times G \xrightarrow{\alpha_X} X$  belong to  $\mathcal{X} \cap F_X(W)$ . Therefore  $\gamma'$  satisfies condition (G) in (5.8) which implies that  $\gamma' = \text{pr}_G((F_{\zeta_1^{-1}})_U(\gamma))$  belongs to  $\mathcal{G}_{\mathcal{O}_X} \cap F_{G_1}(\mathcal{O}_X)(U)$ . We conclude that  $(F_{\zeta_1^{-1}})_U(\gamma) = \zeta_1^{-1} \gamma$  belongs to  $\mathcal{B}^{\sigma_X} \cap \mathcal{B}^{\tau_X} \cap \mathcal{G}_{\mathcal{O}_X}^{\text{pr}_G} \cap F_{G_1}(U)$ . Thus  $\zeta_1^{-1}$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .  $\square$

Let  $\mathbf{D} = ((D, \mathcal{D}) \xrightarrow{\rho} (A, \mathcal{A}))$  and  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (B, \mathcal{B}))$  be objects of  $\text{Epi}_c(\mathcal{P}_F(\mathcal{C}, J))$  and  $\xi = \langle \xi, f \rangle : \mathbf{D} \rightarrow \mathbf{E}$  a morphism in  $\text{Epi}_c(\mathcal{P}_F(\mathcal{C}, J))$ . For  $x \in A$  and  $y \in B$ , we denote by  $j_x : \rho^{-1}(x) \rightarrow D$  and  $i_y : \pi^{-1}(y) \rightarrow E$  the inclusion maps, respectively. Let  $\xi_x : \rho^{-1}(x) \rightarrow \pi^{-1}(f(x))$  be the map obtained from  $\xi : D \rightarrow E$  by restricting the source and the target, namely  $\xi_x$  is the unique map that makes the following diagram commute.

$$\begin{array}{ccc}
\rho^{-1}(x) & \xrightarrow{\xi_x} & \pi^{-1}(f(x)) \\
\downarrow j_x & & \downarrow i_{f(x)} \\
D & \xrightarrow{\xi} & E
\end{array}$$

**Lemma 5.11**  $\xi_x : (\rho^{-1}(x), \mathcal{D}^{j_x}) \rightarrow (\pi^{-1}(f(x)), \mathcal{E}^{i_{f(x)}})$  is an isomorphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

*Proof.* We consider the inverse image  $f^*(\mathbf{E}) = ((A \times_B E, \mathcal{A}^{\pi_f} \cap \mathcal{E}^{f_\pi}) \xrightarrow{\pi_f} (A, \mathcal{A}))$  of  $\mathbf{E}$  by  $f$  which is also an object of  $\text{Epi}_c(\mathcal{P}_F(\mathcal{C}, J))$ . We have a natural cartesian morphism  $\alpha_f(\mathbf{E}) = \langle f_\pi, f \rangle : f^*(\mathbf{E}) \rightarrow \mathbf{E}$ .

$$\begin{array}{ccc}
A \times_B E & \xrightarrow{f_\pi} & E \\
\downarrow \pi_f & & \downarrow \pi \\
A & \xrightarrow{f} & B
\end{array}$$

For  $x \in A$ , we denote by  $i_x^f : \pi_f^{-1}(x) \rightarrow A \times_B E$  the inclusion map. Since we have  $\pi_f^{-1}(x) = \{x\} \times \pi^{-1}(f(x))$  in  $A \times_B E$ , there is a bijection  $f_x : \pi_f^{-1}(x) \rightarrow \pi^{-1}(f(x))$  which makes the following diagram commute.

$$\begin{array}{ccc}
\pi_f^{-1}(x) & \xrightarrow{f_x} & \pi^{-1}(f(x)) \\
\downarrow i_x^f & & \downarrow i_{f(x)} \\
A \times_B E & \xrightarrow{f_\pi} & E
\end{array}$$

Since  $\pi_f i_x^f : \pi_f^{-1}(x) \rightarrow A$  is a constant map to  $\{x\}$ ,  $\mathcal{A}^{\pi_f i_x^f}$  coincides with  $\mathcal{D}_{\text{coarse}, \pi_f^{-1}(x)}$ . Therefore we have  $(\mathcal{A}^{\pi_f} \cap \mathcal{E}^{f_\pi})^{i_x^f} = \mathcal{A}^{\pi_f i_x^f} \cap \mathcal{E}^{f_\pi i_x^f} = \mathcal{E}^{i_{f(x)} f_x}$  and it follows that  $f_x : (\pi_f^{-1}(x), (\mathcal{A}^{\pi_f} \cap \mathcal{E}^{f_\pi})^{i_x^f}) \rightarrow (\pi^{-1}(f(x)), \mathcal{E}^{i_{f(x)}})$  is an isomorphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

Since  $\xi$  is cartesian,  $(\rho, \xi) : (D, \mathcal{D}) \rightarrow (A \times_B E, \mathcal{A}^{\pi_f} \cap \mathcal{E}^{f_\pi})$  is an isomorphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . Put  $\xi_f = (\rho, \xi)$  and we have an isomorphism  $\xi_f = \langle \xi_f, id_A \rangle : \mathbf{D} \rightarrow f^*(\mathbf{E})$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{X})}^{(2)}$  that satisfies  $\alpha_f(\mathbf{E}) \xi_f = \xi$ . Then  $\pi_f \xi_f = \rho$  holds and we have an isomorphism  $\xi_{f,x} : (\rho^{-1}(x), \mathcal{D}^{j_x}) \rightarrow (\pi_f^{-1}(x), (\mathcal{A}^{\pi_f} \cap \mathcal{E}^{f_\pi})^{i_x^f})$  for each  $x \in A$  by restricting the source and the target of  $\xi_f$ . Since  $\xi = f_\pi \xi_f$ , we have  $\xi_x = f_x \xi_{f,x}$  which implies that  $\xi_x : (\rho^{-1}(x), \mathcal{D}^{j_x}) \rightarrow (\pi^{-1}(f(x)), \mathcal{E}^{i_{f(x)}})$  is an isomorphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .  $\square$

**Remark 5.12** Since  $\xi_f : (D, \mathcal{D}) \rightarrow (A \times_B E, \mathcal{A}^{\pi_f} \cap \mathcal{E}^{f_\pi})$  is an isomorphism in  $\mathcal{P}_F(\mathcal{C}, J)$  which satisfies  $\pi_f \xi_f = \rho$  and  $f_\pi \xi_f = \xi$ ,  $\mathcal{D} = (\mathcal{A}^{\pi_f} \cap \mathcal{E}^{f_\pi})^{\xi_f} = \mathcal{A}^{\pi_f \xi_f} \cap \mathcal{E}^{f_\pi \xi_f} = \mathcal{A}^\rho \cap \mathcal{E}^\xi$  holds.

By (5.11), we can define a bijection  $\xi_{x,y} : G_1(\mathbf{D})(x, y) \rightarrow G_1(\mathbf{E})(f(x), f(y))$  by  $\xi_{x,y}(\varphi) = \xi_y \varphi \xi_x^{-1}$  for  $x, y \in A$ . We also define a map  $\xi_1 : G_1(\mathbf{D}) \rightarrow G_1(\mathbf{E})$  by  $\xi_1(\varphi) = \xi_{x,y}(\varphi)$  where  $x = \sigma_{\mathbf{D}}(\varphi)$  and  $y = \tau_{\mathbf{D}}(\varphi)$ . Note that a pair  $(f, \xi_1)$  of maps is a morphism  $\mathbf{G}(\mathbf{D}) \rightarrow \mathbf{G}(\mathbf{E})$  of groupoids, that is, the following diagrams are commutative. Here,  $\xi_1 \times_f \xi_1 : G_1(\mathbf{D}) \times_A G_1(\mathbf{D}) \rightarrow G_1(\mathbf{E}) \times_B G_1(\mathbf{E})$  maps  $(\varphi, \psi)$  to  $(\xi_1(\varphi), \xi_1(\psi))$ .

$$\begin{array}{ccccccc}
A \xleftarrow{\sigma_D} G_1(\mathbf{D}) \xrightarrow{\tau_D} A & A \xrightarrow{\varepsilon_D} G_1(\mathbf{D}) & G_1(\mathbf{D}) \times_A G_1(\mathbf{D}) \xrightarrow{\mu_D} G_1(\mathbf{D}) & G_1(\mathbf{D}) \xrightarrow{\iota_D} G_1(\mathbf{D}) \\
\downarrow f & \downarrow \xi_1 & \downarrow \xi_1 \times_f \xi_1 & \downarrow \xi_1 & \downarrow \xi_1 & \downarrow \xi_1 \\
B \xleftarrow{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\tau_E} B & B \xrightarrow{\varepsilon_E} G_1(\mathbf{E}) & G_1(\mathbf{E}) \times_B G_1(\mathbf{E}) \xrightarrow{\mu_E} G_1(\mathbf{E}) & G_1(\mathbf{E}) \xrightarrow{\iota_E} G_1(\mathbf{E}) \\
\downarrow f & \downarrow \xi_1 & \downarrow \xi_1 & \downarrow \xi_1 & \downarrow \xi_1 & \downarrow \xi_1
\end{array}$$

Define a map  $\xi \times_f \xi_1 : D \times_A^{\sigma_D} G_1(\mathbf{D}) \rightarrow E \times_B^{\sigma_E} G_1(\mathbf{E})$  by  $(\xi \times_f \xi_1)(e, \varphi) = (\xi(e), \xi_1(\varphi))$ . Then, the following diagram is commutative.

$$\begin{array}{ccc}
D \times_A^{\sigma_D} G_1(\mathbf{D}) & \xrightarrow{\hat{\xi}_D} & D \\
\downarrow \xi \times_f \xi_1 & & \downarrow \xi \\
E \times_B^{\sigma_E} G_1(\mathbf{E}) & \xrightarrow{\hat{\xi}_E} & E
\end{array}$$

**Lemma 5.13**  $\xi_1 : (G_1(\mathbf{D}), \mathcal{G}_D) \rightarrow (G_1(\mathbf{E}), \mathcal{G}_E)$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . It follows that a pair of morphisms  $(f, \xi_1) : \mathbf{G}(\mathbf{D}) \rightarrow \mathbf{G}(\mathbf{E})$  is a morphism of groupoids in  $\mathcal{P}_F(\mathcal{C}, J)$ .

*Proof.* For  $U \in \text{Ob } \mathcal{C}$  and  $\gamma \in \mathcal{G}_D \cap F_{G_1(\mathbf{D})}(U)$ , we verify that  $(F_{\xi_1})_U(\gamma) = \xi_1 \gamma$  satisfies the conditions (G1), (G2) and (G3). We take objects  $V, W$  of  $\mathcal{C}$  and morphisms  $f : W \rightarrow U$  and  $g : W \rightarrow V$  in  $\mathcal{C}$ . Assume that  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfies  $\pi \lambda F(g) = \sigma_E \xi_1 \gamma F(f)$ . Since the outer rectangle of the following diagram is commutative and the lower right rectangle is cartesian in  $\mathcal{P}_F(\mathcal{C}, J)$ , there exists unique  $F$ -plot  $\lambda_1 \in \mathcal{D} \cap F_D(W)$  that satisfies  $\rho \lambda_1 = \sigma_D \gamma F(f)$  and  $\xi \lambda_1 = \lambda F(g)$ .

$$\begin{array}{ccccc}
F(W) & \xrightarrow{F(g)} & F(V) & & \\
\downarrow F(f) & \searrow \lambda_1 & \downarrow \lambda & & \\
D \times_A^{\sigma_D} G_1(\mathbf{D}) & \xrightarrow{\text{pr}_D^{\sigma_D}} & D & \xrightarrow{\xi} & E \\
\downarrow \text{pr}_{G_1(\mathbf{D})}^{\sigma_D} & & \downarrow \rho & & \downarrow \pi \\
F(U) & \xrightarrow{\gamma} & G_1(\mathbf{D}) & \xrightarrow{\sigma_D} & A & \xrightarrow{f} & B \\
\downarrow \xi_1 & & \downarrow \xi_1 & & \downarrow \xi_1 & & \downarrow \xi_1 \\
G_1(\mathbf{E}) & & G_1(\mathbf{E}) & & G_1(\mathbf{E}) & & G_1(\mathbf{E})
\end{array}$$

Since  $\gamma$  satisfies (G1) for  $\mathbf{D}$ , the following composition belongs to  $\mathcal{D} \cap F_D(W)$ .

$$F(W) \xrightarrow{\lambda_2 = (\lambda_1 F(id_W), \gamma F(f))} D \times_A^{\sigma_D} G_1(\mathbf{D}) \xrightarrow{\hat{\xi}_D} D$$

Since  $\xi : (D, \mathcal{D}) \rightarrow (E, \mathcal{E})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$  and the following diagram is commutative, a composition  $F(W) \xrightarrow{(\lambda F(g), \xi_1 \gamma F(f))} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$  belongs to  $\mathcal{E} \cap F_E(W)$ . Hence  $\xi_1 \gamma$  satisfies (G1).

$$\begin{array}{ccc}
F(W) & \xrightarrow{(\lambda_1 F(id_W), \gamma F(f))} & D \times_A^{\sigma_D} G_1(\mathbf{D}) \xrightarrow{\hat{\xi}_D} D \\
\searrow (\lambda F(g), \xi_1 \gamma F(f)) & & \downarrow \xi \times_f \xi_1 \quad \downarrow \xi \\
& & E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E
\end{array}$$

Assume that  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfies  $\pi \lambda F(g) = \tau_E \xi_1 \gamma F(f)$ . Since the outer rectangle of the following diagram is commutative and the lower right rectangle is cartesian in  $\mathcal{P}_F(\mathcal{C}, J)$ , there exists unique  $F$ -plot  $\lambda_3 \in \mathcal{D} \cap F_D(W)$  that satisfies  $\rho \lambda_3 = \sigma_D \iota_D \gamma F(f)$  and  $\xi \lambda_3 = \lambda F(g)$ .



$$\begin{array}{ccccc}
F(W) & \xrightarrow{F(g)} & & & F(V) \\
\downarrow F(f) & \searrow \lambda_3 & & & \downarrow \lambda \\
F(U) & \xrightarrow{\lambda_4} & D \times_A^{\sigma_D} G_1(D) & \xrightarrow{\text{pr}_D^\sigma} & D \xrightarrow{\xi} E \\
\downarrow \gamma & & \downarrow \text{pr}_{G_1(D)}^\sigma & & \downarrow \rho \\
G_1(D) & \xrightarrow{\iota_D} & G_1(D) & \xrightarrow{\sigma_D} & A \xrightarrow{f} B \\
& & \downarrow \xi_1 & & \downarrow \pi \\
& & G_1(E) & \xrightarrow{\sigma_E} & B \\
& \searrow \xi_1 & \uparrow \iota_E & \nearrow \tau_E & \\
& & G_1(E) & & 
\end{array}$$

Since  $\gamma$  satisfies (G2) for  $D$ , the following composition belongs to  $\mathcal{D} \cap F_D(W)$ .

$$F(W) \xrightarrow{\lambda_4 = (\lambda_3 F(id_W), \iota_D \gamma F(f))} D \times_A^{\sigma_D} G_1(D) \xrightarrow{\hat{\xi}_D} D$$

Since  $\xi : (D, \mathcal{D}) \rightarrow (E, \mathcal{E})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$  and the following diagram is commutative, a composition  $F(W) \xrightarrow{(\lambda F(g), \iota_E \xi_1 \gamma F(f))} E \times_B^{\sigma_E} G_1(E) \xrightarrow{\hat{\xi}_E} E$  belongs to  $\mathcal{E} \cap F_E(W)$ . Hence  $\xi_1 \gamma$  satisfies (G2).

$$\begin{array}{ccccc}
F(W) & \xrightarrow{(\lambda_3 F(id_W), \iota_D \gamma F(f))} & D \times_A^{\sigma_D} G_1(D) & \xrightarrow{\hat{\xi}_D} & D \\
& \searrow (\lambda F(g), \iota_E \xi_1 \gamma F(f)) & \downarrow \xi \times_f \xi_1 & & \downarrow \xi \\
& & E \times_B^{\sigma_E} G_1(E) & \xrightarrow{\hat{\xi}_E} & E
\end{array}$$

Since  $\gamma$  satisfies (G3) for  $D$ ,  $\sigma_D \gamma, \tau_D \gamma \in F_A(U)$  belong to  $\mathcal{A} \cap F_A(U)$ . Since  $f : (A, \mathcal{A}) \rightarrow (B, \mathcal{B})$  is a morphism in  $\mathcal{P}_E(\mathcal{C}, J)$ ,  $(F_f)_U(\sigma_D \gamma)$  and  $(F_f)_U(\tau_D \gamma)$  belong to  $\mathcal{B} \cap F_B(U)$ . On the other hand, since  $(F_f)_U(\sigma_D \gamma) = f \sigma_D \gamma = \sigma_E \xi_1 \gamma$  and  $(F_f)_U(\tau_D \gamma) = f \tau_D \gamma = \tau_E \xi_1 \gamma$  hold,  $\xi_1 \gamma$  satisfies (G3).  $\square$

We denote by  $\text{Grp}(\mathcal{P}_F(\mathcal{C}, J))$  the category of groupoids in  $\mathcal{P}_F(\mathcal{C}, J)$ . That is, objects of  $\text{Grp}(\mathcal{P}_F(\mathcal{C}, J))$  are groupoids in  $\mathcal{P}_F(\mathcal{C}, J)$  and morphisms of  $\text{Grp}(\mathcal{P}_F(\mathcal{C}, J))$  are morphisms of groupoids. Define a functor

$$\mathbf{Gr} : \text{Epi}_c(\mathcal{P}_F(\mathcal{C}, J)) \rightarrow \text{Grp}(\mathcal{P}_F(\mathcal{C}, J))$$

as follows. For an object  $E = ((E, \mathcal{E}) \xrightarrow{\pi} (B, \mathcal{B}))$  of  $\text{Epi}_c(\mathcal{P}_F(\mathcal{C}, J))$ , let  $\mathbf{Gr}(E)$  be the groupoid  $G(E)$  associated with  $E$  as we defined in (5.7). For a morphism  $\xi = \langle \xi, f \rangle : D \rightarrow E$  in  $\text{Epi}_c(\mathcal{P}_F(\mathcal{C}, J))$ , we put  $\mathbf{Gr}(\xi) = (f, \xi_1) : G(D) \rightarrow G(E)$ . Then  $\mathbf{Gr}(\xi)$  is a morphism in  $\text{Grp}(\mathcal{P}_F(\mathcal{C}, J))$  by (5.13).

Let  $C = ((C, \mathcal{C}) \xrightarrow{\chi} (H, \mathcal{H}))$  and  $D = ((D, \mathcal{D}) \xrightarrow{\rho} (A, \mathcal{A}))$  be objects of  $\text{Epi}_c(\mathcal{P}_F(\mathcal{C}, J))$  and  $\zeta = \langle \zeta, g \rangle : C \rightarrow D$  a morphism in  $\text{Epi}_c(\mathcal{P}_F(\mathcal{C}, J))$ . We denote by  $k_x : \chi^{-1}(x) \rightarrow C$ ,  $j_y : \rho^{-1}(y) \rightarrow D$  the inclusion maps for  $x \in H$  and  $y \in A$ . We have an isomorphism  $\zeta_x : (\chi^{-1}(x), \mathcal{C}^{k_x}) \rightarrow (\rho^{-1}(g(x)), \mathcal{D}^{j_{g(x)}})$  in  $\mathcal{P}_F(\mathcal{C}, J)$  such that the following diagram is commutative.

$$\begin{array}{ccccc}
\chi^{-1}(x) & \xrightarrow{\zeta_x} & \rho^{-1}(g(x)) & \xrightarrow{\xi_{g(x)}} & \pi^{-1}(f(g(x))) \\
\downarrow k_x & & \downarrow j_{g(x)} & & \downarrow i_{f(g(x))} \\
C & \xrightarrow{\zeta} & D & \xrightarrow{\xi} & E
\end{array}$$

We put  $\mathbf{Gr}(\zeta) = (g, \zeta_1)$  and  $\mathbf{Gr}(\xi \zeta) = (fg, (\xi \zeta)_1)$ . Then,  $(\xi \zeta)_1 : G_1(C) \rightarrow G_1(E)$  maps  $\varphi \in G_1(C)(x, y)$  to  $(\xi_{g(y)} \zeta_y) \varphi (\xi_{g(x)} \zeta_x)^{-1} = \xi_{g(y)} (\zeta_y \varphi \zeta_x^{-1}) \xi_{g(x)}^{-1} = \xi_1(\zeta_1(\varphi))$  by the commutativity of the above diagram. It follows that  $\mathbf{Gr}(\xi \zeta) = \mathbf{Gr}(\xi) \mathbf{Gr}(\zeta)$  holds. If  $\mathbf{id}_E$  is the identity morphism of  $E$ , it is clear that  $\mathbf{Gr}(\mathbf{id}_E)$  is the identity morphism of  $G(E)$ . Thus we verified that  $\mathbf{Gr}$  is a functor from  $\text{Epi}_c(\mathcal{P}_F(\mathcal{C}, J))$  to  $\text{Grp}(\mathcal{P}_F(\mathcal{C}, J))$ .

## 6 Fibrations

**Definition 6.1** Let  $G = ((G_0, \mathcal{G}_0), (G_1, \mathcal{G}_1); \sigma, \tau, \varepsilon, \mu, \iota)$  be a groupoid in  $\mathcal{P}_F(\mathcal{C}, J)$ . We denote by  $\text{pr}_\sigma, \text{pr}_\tau : G_0 \times G_0 \rightarrow G_0$  the projections given by  $\text{pr}_\sigma(x, y) = x$  and  $\text{pr}_\tau(x, y) = y$ . If a map  $(\sigma, \tau) : G_1 \rightarrow G_0 \times G_0$  given by

$(\sigma, \tau)(\varphi) = (\sigma(\varphi), \tau(\varphi))$  is an epimorphism and the the-ology  $(\mathcal{G}_1)_{(\sigma, \tau)}$  on  $G_0 \times G_0$  coincides with  $\mathcal{G}_0^{\text{pr}_\sigma} \cap \mathcal{G}_0^{\text{pr}_\tau}$ , we say that  $\mathbf{G}$  is fibrating ([6], 8.4). Let  $\mathbf{E}$  be an object of  $\text{Epi}_c(\mathcal{P}_F(\mathcal{C}, J))$ . If the groupoid  $\mathbf{G}(\mathbf{E})$  associated with  $\mathbf{E}$  (5.7) is fibrating, we call  $\mathbf{E}$  a fibration ([6], 8.8).

**Remark 6.2** If  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (B, \mathcal{B}))$  is a fibration, then, since  $(\sigma_{\mathbf{E}}, \tau_{\mathbf{E}}) : G_1(\mathbf{E}) \rightarrow B \times B$  is surjective,  $G_1(\mathbf{E})(x, y)$  is not empty for any  $x, y \in B$ . Hence fibers  $(\pi^{-1}(x), \mathcal{E}^{ix})$  of  $\pi$  are all isomorphic.

**Lemma 6.3** Let  $(X, \mathcal{X})$  and  $(B, \mathcal{B})$  be objects of  $\mathcal{P}_F(\mathcal{C}, J)$ . We denote the projections by  $\text{pr}_X : X \times B \rightarrow X$  and  $\text{pr}_B : X \times B \rightarrow B$ . Then  $\mathcal{B}$  coincides with  $(\mathcal{X}^{\text{pr}_X} \cap \mathcal{B}^{\text{pr}_B})_{\text{pr}_B}$ .

*Proof.* Since  $\text{pr}_B : (X \times B, \mathcal{X}^{\text{pr}_X} \cap \mathcal{B}^{\text{pr}_B}) \rightarrow (B, \mathcal{B})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ , we have  $(\mathcal{X}^{\text{pr}_X} \cap \mathcal{B}^{\text{pr}_B})_{\text{pr}_B} \subset \mathcal{B}$ . We choose  $a \in X$ . For  $U \in \text{Ob } \mathcal{C}$  and  $\gamma \in \mathcal{B} \cap F_B(U)$ , define  $\bar{\gamma} : F(U) \rightarrow X \times B$  by  $\bar{\gamma}(x) = (a, \gamma(x))$ . Since  $\text{pr}_X \bar{\gamma}$  is a constant map and  $\text{pr}_Y \bar{\gamma} = \gamma$ , we have  $\bar{\gamma} \in \mathcal{X}^{\text{pr}_X} \cap \mathcal{B}^{\text{pr}_B} \cap F_{X \times B}(U)$ . Hence, for any  $h \in h_U$ ,  $\bar{\gamma}F(h) \in \mathcal{X}^{\text{pr}_X} \cap \mathcal{B}^{\text{pr}_B} \cap F_{X \times B}(\text{dom}(h))$  satisfies  $F_B(h)(\gamma) = (F_{\text{pr}_B})_{\text{dom}(h)}(\bar{\gamma}F(h))$ . This implies that  $\gamma$  belongs to  $(\mathcal{X}^{\text{pr}_X} \cap \mathcal{B}^{\text{pr}_B})_{\text{pr}_B}$  by (2.4). Thus we conclude that  $(\mathcal{X}^{\text{pr}_X} \cap \mathcal{B}^{\text{pr}_B})_{\text{pr}_B} = \mathcal{B}$  holds.  $\square$

**Proposition 6.4** Let  $\xi : \mathbf{D} \rightarrow \mathbf{E}$  be a morphism in  $\text{Epi}_c(\mathcal{P}_F(\mathcal{C}, J))$ . If  $\mathbf{E}$  is a fibration, so is  $\mathbf{D}$ .

*Proof.* We put  $\mathbf{D} = ((D, \mathcal{D}) \xrightarrow{\rho} (A, \mathcal{A}))$ ,  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (B, \mathcal{B}))$  and  $\xi = \langle \xi, f \rangle : \mathbf{D} \rightarrow \mathbf{E}$ . It follows from (5.13) that  $\xi$  induces a morphism  $\mathbf{Gr}(\xi) = (f, \xi_1) : \mathbf{G}(\mathbf{D}) \rightarrow \mathbf{G}(\mathbf{E})$  of groupoids. Then, the following diagram is commutative.

$$\begin{array}{ccc} G_1(\mathbf{D}) & \xrightarrow{\xi_1} & G_1(\mathbf{E}) \\ \downarrow (\sigma_{\mathbf{D}}, \tau_{\mathbf{D}}) & & \downarrow (\sigma_{\mathbf{E}}, \tau_{\mathbf{E}}) \\ A \times A & \xrightarrow{f \times f} & B \times B \end{array}$$

For  $x, y \in A$ , since  $(\sigma_{\mathbf{E}}, \tau_{\mathbf{E}}) : G_1(\mathbf{E}) \rightarrow B \times B$  is surjective, there exists  $\varphi \in G_1(\mathbf{E})$  which satisfies  $\sigma_{\mathbf{E}}(\varphi) = f(x)$  and  $\tau_{\mathbf{E}}(\varphi) = f(y)$ . Since there is a bijection  $\xi_{x,y} : G_1(\mathbf{D})(x, y) \rightarrow G_1(\mathbf{E})(f(x), f(y))$  by (5.11), there exists  $\psi \in G_1(\mathbf{D})(x, y)$  which satisfies  $\sigma_{\mathbf{D}}(\psi) = x$  and  $\tau_{\mathbf{D}}(\psi) = y$ . Hence  $(\sigma_{\mathbf{D}}, \tau_{\mathbf{D}}) : G_1(\mathbf{E}) \rightarrow A \times A$  is surjective.

We denote by  $\text{pr}_{A_i} : A \times A \rightarrow A$  and  $\text{pr}_{B_i} : B \times B \rightarrow B$  the projections onto the  $i$ -th component. Since  $\sigma_{\mathbf{D}}, \tau_{\mathbf{D}} : (G_1(\mathbf{D}), \mathcal{G}_{\mathbf{D}}) \rightarrow (A, \mathcal{A})$  are morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ ,  $(\sigma_{\mathbf{D}}, \tau_{\mathbf{D}}) : (G_1(\mathbf{D}), \mathcal{G}_{\mathbf{D}}) \rightarrow (A \times A, \mathcal{A}^{\text{pr}_{A1}} \cap \mathcal{A}^{\text{pr}_{A2}})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . On the other hand, since  $(\mathcal{G}_{\mathbf{D}})_{(\sigma_{\mathbf{D}}, \tau_{\mathbf{D}})}$  is the finest the-ology on  $A \times A$  such that  $(\sigma_{\mathbf{D}}, \tau_{\mathbf{D}}) : (G_1(\mathbf{D}), \mathcal{G}_{\mathbf{D}}) \rightarrow (A \times A, (\mathcal{G}_{\mathbf{D}})_{(\sigma_{\mathbf{D}}, \tau_{\mathbf{D}})})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ ,  $(\mathcal{G}_{\mathbf{D}})_{(\sigma_{\mathbf{D}}, \tau_{\mathbf{D}})} \subset \mathcal{A}^{\text{pr}_{A1}} \cap \mathcal{A}^{\text{pr}_{A2}}$  holds. For  $U \in \text{Ob } \mathcal{C}$  and  $\gamma \in \mathcal{A}^{\text{pr}_{A1}} \cap \mathcal{A}^{\text{pr}_{A2}} \cap F_{A \times A}(U)$ , since

$$f \times f : (A \times A, \mathcal{A}^{\text{pr}_{A1}} \cap \mathcal{A}^{\text{pr}_{A2}}) \rightarrow (B \times B, \mathcal{B}^{\text{pr}_{B1}} \cap \mathcal{B}^{\text{pr}_{B2}})$$

is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ ,  $(F_{f \times f})_U(\gamma) \in \mathcal{B}^{\text{pr}_{B1}} \cap \mathcal{B}^{\text{pr}_{B2}} \cap F_{B \times B}(U)$ . Since  $\mathcal{B}^{\text{pr}_{B1}} \cap \mathcal{B}^{\text{pr}_{B2}} = (\mathcal{G}_{\mathbf{E}})_{(\sigma_{\mathbf{E}}, \tau_{\mathbf{E}})}$  by the assumption, we have  $(F_{f \times f})_U(\gamma) \in (\mathcal{G}_{\mathbf{E}})_{(\sigma_{\mathbf{E}}, \tau_{\mathbf{E}})} \cap F_{B \times B}(U)$ . It follows from (2.4) that there exists  $R \in J(U)$  such that, for any  $h \in R$ , there exists  $\varphi_h \in \mathcal{G}_{\mathbf{E}} \cap F_{G_1(\mathbf{E})}(\text{dom}(h))$  which makes the following diagram commute.

$$\begin{array}{ccccc} & & F(\text{dom}(h)) & & \\ & \swarrow F(h) & & \searrow \varphi_h & \\ F(U) & & G_1(\mathbf{D}) & \xrightarrow{\xi_1} & G_1(\mathbf{E}) \\ & \searrow \gamma & \downarrow (\sigma_{\mathbf{D}}, \tau_{\mathbf{D}}) & & \downarrow (\sigma_{\mathbf{E}}, \tau_{\mathbf{E}}) \\ & & A \times A & \xrightarrow{f \times f} & B \times B \end{array}$$

We define a map  $\psi_h : F(\text{dom}(h)) \rightarrow G_1(\mathbf{D})$  as follows. For  $u \in F(\text{dom}(h))$ , put  $F_{A \times A}(h)(\gamma) = (x, y)$ . It follows from the commutativity of the above diagram that  $\varphi_h(u)$  belongs to  $G_1(\mathbf{E})(f(x), f(y))$ . It follows from (5.11) that we can define  $\psi_h(u) \in G_1(\mathbf{D})(x, y)$  by  $\psi_h(u) = \xi_y^{-1} \varphi_h(u) \xi_x$ . In order to show that  $\psi_h$  belongs to  $\mathcal{G}_{\mathbf{D}} \cap F_{G_1(\mathbf{D})}(\text{dom}(h))$ , we take  $V, W \in \text{Ob } \mathcal{C}$ ,  $f \in \mathcal{C}(W, \text{dom}(h))$  and  $g \in \mathcal{C}(W, V)$ . Assume that  $\lambda \in \mathcal{D} \cap F_D(V)$  satisfies  $\rho \lambda F(g) = \sigma_{\mathbf{D}} \psi_h F(f)$ . Since  $(\sigma_{\mathbf{D}}, \tau_{\mathbf{D}}) \psi_h = \gamma F(h)$  and  $\xi_1 \psi_h = \varphi_h$ , the following diagrams are commutative.

$$\begin{array}{ccc} & & G_1(\mathbf{E}) \\ & \nearrow \varphi_h & \uparrow \xi_1 \\ F(\text{dom}(h)) & \xrightarrow{\psi_h} & G_1(\mathbf{D}) \\ \downarrow F(h) & \searrow (\sigma_{\mathbf{D}}, \tau_{\mathbf{D}}) & \downarrow \tau_{\mathbf{D}} \\ F(U) & \xrightarrow{\gamma} & A \times A \xrightarrow{\text{pr}_{A2}} A \end{array} \quad \begin{array}{ccccc} & & G_1(\mathbf{D}) & \xrightarrow{\tau_{\mathbf{D}}} & A \\ & \nearrow \psi_h F(f) & \uparrow \text{pr}_{G_1(\mathbf{D})}^\sigma & & \uparrow \rho \\ F(W) & \xrightarrow{(\lambda F(g), \psi_h F(f))} & D \times_A^{\sigma_D} G_1(\mathbf{D}) & \xrightarrow{\hat{\xi}_{\mathbf{D}}} & D \\ & \searrow (\lambda F(g), \varphi_h F(f)) & \downarrow \xi \times_f \xi_1 & & \downarrow \xi \\ & & E \times_B^{\sigma_E} G_1(\mathbf{E}) & \xrightarrow{\hat{\xi}_{\mathbf{E}}} & E \end{array}$$

Since  $(F_{\tau_D})_{\text{dom}(u)}(\psi_h) = (F_{\text{pr}_{A_2}})_{\text{dom}(u)}(F_{A \times A}(h)(\gamma))$  and  $F_{A \times A}(h)(\gamma) \in \mathcal{A}^{\text{pr}_{A_1}} \cap \mathcal{A}^{\text{pr}_{A_2}} \cap F_{A \times A}(\text{dom}(h))$ , it follows from the commutativity of the above diagram that  $(F_{\hat{\xi}_D})_W((\lambda F(g), \psi_h F(f)))$  belongs to  $\mathcal{A}^\rho \cap F_D(W)$ . On the other hand, since  $\lambda \in \mathcal{D} \cap F_D(V)$ ,  $\varphi_h \in \mathcal{G}_E \cap F_{G_1(E)}(\text{dom}(h))$ ,  $(\lambda F(g), \varphi_h F(f)) : F(W) \rightarrow E \times_B^{\sigma_E} G_1(E)$  belongs to  $\mathcal{E}^{\text{pr}_E} \cap \mathcal{G}_E^{\text{pr}_{G_1(E)}} \cap F_{E \times_B^{\sigma_E} G_1(E)}(W)$ . Since  $\hat{\xi}_E : (E \times_B^{\sigma_E} G_1(E), \mathcal{E}^{\text{pr}_E} \cap \mathcal{G}_E^{\text{pr}_{G_1(E)}}) \rightarrow (E, \mathcal{E})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ ,  $(F_{\hat{\xi}_D})_W((\lambda F(g), \psi_h F(f)))$  belongs to  $\mathcal{E}^\xi \cap F_D(W)$  by the commutativity of the above diagram. Thus we have  $(F_{\hat{\xi}_D})_W((\lambda F(g), \psi_h F(f))) \in \mathcal{A}^\rho \cap \mathcal{E}^\xi \cap F_D(W) = \mathcal{D} \cap F_D(W)$  by (5.12) and  $\psi_h$  satisfies (G1).

Assume that  $\lambda \in \mathcal{D} \cap F_D(V)$  satisfies  $\rho \lambda F(g) = \tau_D \psi_h F(f)$ . Since  $(\sigma_D, \tau_D) \psi_h = \gamma F(h)$  and  $\xi_1 \psi_h = \varphi_h$ , the following diagrams are commutative.

$$\begin{array}{ccc}
& G_1(\mathbf{E}) & \xrightarrow{\iota_E} G_1(\mathbf{E}) \\
\varphi_h \nearrow & \uparrow \xi_1 & \uparrow \xi_1 \\
F(\text{dom}(h)) & \xrightarrow{\psi_h} G_1(\mathbf{D}) & \xrightarrow{\iota_D} G_1(\mathbf{D}) \\
\downarrow F(h) & \downarrow (\sigma_D, \tau_D) & \downarrow \tau_D \\
F(U) & \xrightarrow{\gamma} A \times A & \xrightarrow{\text{pr}_{A_1}} A
\end{array}
\qquad
\begin{array}{ccc}
& & G_1(\mathbf{D}) \xrightarrow{\tau_D} A \\
\iota_D \psi_h F(f) \nearrow & \uparrow \text{pr}_{G_1(\mathbf{D})} & \uparrow \rho \\
F(W) & \xrightarrow{(\lambda F(g), \iota_D \psi_h F(f))} D \times_A^{\sigma_D} G_1(\mathbf{D}) & \xrightarrow{\hat{\xi}_D} D \\
\downarrow (\lambda F(g), \iota_E \varphi_h F(f)) & \downarrow \xi \times_f \xi_1 & \downarrow \xi \\
& & E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E
\end{array}$$

Since  $(F_{\tau_D})_{\text{dom}(u)}(\iota_D \psi_h) = (F_{\text{pr}_{A_1}})_{\text{dom}(u)}(F_{A \times A}(h)(\gamma))$  and  $F_{A \times A}(h)(\gamma) \in \mathcal{A}^{\text{pr}_{A_1}} \cap \mathcal{A}^{\text{pr}_{A_2}} \cap F_{A \times A}(\text{dom}(h))$ , it follows from the commutativity of the above diagram that  $(F_{\hat{\xi}_D})_W((\lambda F(g), \psi_h F(f)))$  belongs to  $\mathcal{A}^\rho \cap F_D(W)$ . Since  $\lambda \in \mathcal{D} \cap F_D(V)$ ,  $\iota_E \varphi_h \in \mathcal{G}_E \cap F_{G_1(E)}(\text{dom}(h))$ ,  $(\lambda F(g), \iota_E \varphi_h F(f)) : F(W) \rightarrow E \times_B^{\sigma_E} G_1(E)$  belongs to  $\mathcal{E}^{\text{pr}_E} \cap \mathcal{G}_E^{\text{pr}_{G_1(E)}} \cap F_{E \times_B^{\sigma_E} G_1(E)}(W)$ . Since  $\hat{\xi}_E : (E \times_B^{\sigma_E} G_1(E), \mathcal{E}^{\text{pr}_E} \cap \mathcal{G}_E^{\text{pr}_{G_1(E)}}) \rightarrow (E, \mathcal{E})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ ,  $(F_{\hat{\xi}_D})_W((\lambda F(g), \iota_D \psi_h F(f)))$  belongs to  $\mathcal{E}^\xi \cap F_D(W)$  by the commutativity of the above diagram. Thus we have  $(F_{\hat{\xi}_D})_W((\lambda F(g), \iota_D \psi_h F(f))) \in \mathcal{A}^\rho \cap \mathcal{E}^\xi \cap F_D(W) = \mathcal{D} \cap F_D(W)$  by (5.12) and  $\psi_h$  satisfies (G2).

By  $(\sigma_D, \tau_D) \psi_h = \gamma F(h)$ ,  $\sigma_D \psi_h = (F_{\text{pr}_{A_1}})_{\text{dom}(h)}(F_{A \times A}(h)(\gamma))$  and  $\tau_D \psi_h = (F_{\text{pr}_{A_2}})_{\text{dom}(h)}(F_{A \times A}(h)(\gamma))$  hold. Since  $F_{A \times A}(h)(\gamma) \in \mathcal{A}^{\text{pr}_{A_1}} \cap \mathcal{A}^{\text{pr}_{A_2}} \cap F_{A \times A}(\text{dom}(h))$ , we have  $(F_{\text{pr}_{A_i}})_{\text{dom}(h)}(F_{A \times A}(h)(\gamma)) \in \mathcal{A} \cap F_A(\text{dom}(h))$  for  $i = 1, 2$ . Hence both  $\sigma_D \psi_h$  and  $\tau_D \psi_h$  belong to  $\mathcal{A} \cap F_A(\text{dom}(h))$ , which shows that  $\psi_h$  satisfies (G3). Therefore we have  $\phi_h \in \mathcal{G}_E \cap F_{G_1(\mathbf{D})}(\text{dom}(h))$  and it follows from (2.4) and  $F_{A \times A}(h)(\gamma) = (F_{(\sigma_D, \tau_D)})_{\text{dom}(h)}(\psi_h)$  that  $\gamma$  belongs to  $(\mathcal{G}_E)_{(\sigma_E, \tau_E)} \cap F_{A \times A}(U)$ . Thus we conclude that  $(\mathcal{G}_D)_{(\sigma_D, \tau_D)} = \mathcal{A}^{\text{pr}_{A_1}} \cap \mathcal{A}^{\text{pr}_{A_2}}$  holds.  $\square$

**Example 6.5** Let  $((G, \mathcal{G}); \varepsilon, \mu, \iota)$  be a group in  $\mathcal{P}_F(\mathcal{C}, J)$  and  $(B, \mathcal{B})$  an object of  $\mathcal{P}_F(\mathcal{C}, J)$ . Consider the trivial groupoid  $\mathbf{G}_{G,B} = ((B, \mathcal{B}), (B \times G \times B, \mathcal{B}^{\sigma_{G,B}} \cap \mathcal{B}^{\tau_{G,B}} \cap \mathcal{G}^{\text{pr}_G}); \sigma_{G,B}, \tau_{G,B}, \varepsilon_{G,B}, \mu_{G,B}, \iota_{G,B})$  in  $\mathcal{P}_F(\mathcal{C}, J)$  associated with  $((G, \mathcal{G}); \varepsilon, \mu, \iota)$  and  $(B, \mathcal{B})$ . Since  $(\sigma_{G,B}, \tau_{G,B}) : B \times G \times B \rightarrow B \times B$  is a projection, it follows from (6.3) that  $\mathbf{G}_{G,B}$  is fibrating. Hence  $\mathbf{X} = ((X \times B, \mathcal{X}^{\text{pr}_X} \cap \mathcal{B}^{\text{pr}_B}) \xrightarrow{\text{pr}_B} (B, \mathcal{B}))$  is a fibration by (5.10). We call  $\mathbf{X}$  a product fibration.

**Definition 6.6** Let  $\mathcal{C}$  be a category with a terminal object  $1_{\mathcal{C}}$ . For an object  $U$  of  $\mathcal{C}$ , we say that a functor  $F : \mathcal{C} \rightarrow \text{Set}$  is  $U$ -pointed if  $F : \mathcal{C}(1_{\mathcal{C}}, U) \rightarrow \text{Set}(F(1_{\mathcal{C}}), F(U))$  is surjective. If  $F$  is  $U$ -pointed for any object  $U$  of  $\mathcal{C}$ , we say that  $F$  is pointed.

**Proposition 6.7** If a category  $\mathcal{C}$  has a terminal object  $1_{\mathcal{C}}$ , then the functor  $h^{1_{\mathcal{C}}} : \mathcal{C} \rightarrow \text{Set}$  defined by  $h^{1_{\mathcal{C}}}(U) = \mathcal{C}(1_{\mathcal{C}}, U)$  and  $h^{1_{\mathcal{C}}}(f : U \rightarrow V) = (f_* : \mathcal{C}(1_{\mathcal{C}}, U) \rightarrow \mathcal{C}(1_{\mathcal{C}}, V))$  is pointed.

*Proof.* For an object  $U$  of  $\mathcal{C}$  and  $\alpha \in \text{Set}(h^{1_{\mathcal{C}}}(1_{\mathcal{C}}), h^{1_{\mathcal{C}}}(U))$ , put  $f = \alpha(id_{1_{\mathcal{C}}}) \in h^{1_{\mathcal{C}}}(U) = \mathcal{C}(1_{\mathcal{C}}, U)$ . Then, we have  $h^{1_{\mathcal{C}}}(f)(id_{1_{\mathcal{C}}}) = id_{1_{\mathcal{C}}} f = f = \alpha(id_{1_{\mathcal{C}}})$  which shows  $h^{1_{\mathcal{C}}}(f) = \alpha$ . Hence  $h^{1_{\mathcal{C}}}$  is pointed.  $\square$

**Definition 6.8** Let  $(\mathcal{C}, J)$  be a site. For an object  $U$  of  $\mathcal{C}$ , we say that a functor  $F : \mathcal{C} \rightarrow \text{Set}$  is  $U$ -local if  $F$  satisfies the following condition (L). If  $F$  is  $U$ -local for any object  $U$  of  $\mathcal{C}$ , we say that  $F$  is local.

- (L) For an object  $V$  of  $\mathcal{C}$  and a map  $\alpha : F(V) \rightarrow F(U)$ , if there exists a covering  $(V_i \xrightarrow{f_i} V)_{i \in I}$  of  $V$  such that  $F(f_i)_* : \text{Set}(F(V), F(U)) \rightarrow \text{Set}(F(V_i), F(U))$  maps  $\alpha$  into the image of  $F : \mathcal{C}(V_i, U) \rightarrow \text{Set}(F(V_i), F(U))$  for any  $i \in I$ , then  $\alpha$  belongs to the image of  $F : \mathcal{C}(V, U) \rightarrow \text{Set}(F(V), F(U))$ .

**Remark 6.9** Let  $\mathcal{C}$  be a category and  $F : \mathcal{C} \rightarrow \text{Set}$  a functor. For an object  $U$  of  $\mathcal{C}$ , we define a subset  $\mathcal{F}_U$  of  $\prod_{V \in \text{Ob } \mathcal{C}} F_{F(U)}(V)$  by  $\mathcal{F}_U = \prod_{V \in \text{Ob } \mathcal{C}} \text{Im}(F : \mathcal{C}(V, U) \rightarrow \text{Set}(F(V), F(U))) = F_{F(U)}(V)$ . Then, it is easy to verify that  $\mathcal{F}_U$  satisfies condition (ii) of (1.2).

(1) Assume that  $\mathcal{C}$  has a terminal object  $1_{\mathcal{C}}$ . Since  $\mathcal{F}_U \cap F_{F(U)}(1_{\mathcal{C}}) = \text{Im}(F : \mathcal{C}(1_{\mathcal{C}}, U) \rightarrow F_{F(U)}(1_{\mathcal{C}}))$ ,  $F$  is  $U$ -pointed if and only if  $\mathcal{F}_U$  satisfies condition (i) of (1.2).

(2) For a site  $(\mathcal{C}, J)$ ,  $F$  is  $U$ -local if and only if  $\mathcal{F}_U$  satisfies condition (iii) of (1.2).

Thus  $\mathcal{F}_U$  is a the-ology on  $F(U)$  if and only if  $F$  is  $U$ -pointed and  $U$ -local. Assume that  $F$  is pointed and local. For an object  $V$  of  $\mathcal{C}$ , a morphism  $f : U \rightarrow W$  in  $\mathcal{C}$  and  $\varphi \in \mathcal{F}_U \cap F_{F(U)}(V)$ , since there exists  $g \in \mathcal{C}(V, U)$  such that  $F(g) = \varphi$ , we have  $(F_{F(f)})_V(\varphi) = F(f)\varphi = F(f)F(g) = F(fg) \in \mathcal{F}_W \cap F_{F(W)}(V)$ . It follows that  $(F_{F(f)})_V : F_{F(U)}(V) \rightarrow F_{F(W)}(V)$  maps  $\mathcal{F}_U \cap F_{F(U)}(V)$  into  $\mathcal{F}_W \cap F_{F(W)}(V)$ . We define a functor  $\tilde{F} : \mathcal{C} \rightarrow \mathcal{P}_F(\mathcal{C}, J)$  by  $\tilde{F}(U) = (F(U), \mathcal{F}_U)$  for  $U \in \text{Ob } \mathcal{C}$  and  $\tilde{F}(f : U \rightarrow W) = (F(f) : (F(U), \mathcal{F}_U) \rightarrow (F(W), \mathcal{F}_W))$  for a morphism  $f : U \rightarrow W$  in  $\mathcal{C}$ . Then  $\Gamma\tilde{F} = F$  holds.

**Example 6.10** Define a category  $\mathcal{C}^\infty$  as follows. Objects of  $\mathcal{C}^\infty$  are open sets of  $n$  dimensional Euclidean space  $\mathbf{R}^n$  for some  $n \geq 0$ . Morphisms of  $\mathcal{C}^\infty$  are  $C^\infty$ -maps. For  $U \in \text{Ob } \mathcal{C}^\infty$ , let  $P_\infty(U)$  be the set of families  $(U_i \xrightarrow{f_i} U)_{i \in I}$  of open embeddings such that  $U = \bigcup_{i \in I} f_i(U_i)$ . It is easy to verify that  $P_\infty$  is a pretopology on  $\mathcal{C}^\infty$ . We give a Grothendieck topology  $J_\infty$  on  $\mathcal{C}^\infty$  generated by  $P_\infty$ . Then, the forgetful functor  $F : \mathcal{C}^\infty \rightarrow \text{Set}$  is pointed and local. For a set  $X$ , an  $F$ - $(\mathcal{C}^\infty, J_\infty)$ -ology on  $X$  is usually called a diffeology on  $X$  and an  $F$ - $(\mathcal{C}^\infty, J_\infty)$ -ological object is called a diffeological space.

**Example 6.11** Let  $k$  be an algebraically closed field. We denote by  $\text{Aff}_k$  the category of affine varieties over  $k$ . For  $V \in \text{Ob } \text{Aff}_k$ , let  $P_{\text{Aff}_k}(V)$  be the set of families  $(V_i \xrightarrow{f_i} V)_{i \in I}$  of Zariski open embeddings such that  $V = \bigcup_{i \in I} f_i(V_i)$ . It is easy to verify that  $P_{\text{Aff}_k}$  is a pretopology on  $\text{Aff}_k$ . We give a Grothendieck topology  $J_{\text{Aff}_k}$  on  $\text{Aff}_k$  generated by  $P_{\text{Aff}_k}$ . Then, the forgetful functor  $F : \text{Aff}_k \rightarrow \text{Set}$  is pointed and local.

**Proposition 6.12** Let  $(X, \mathcal{X})$  be an object of  $\mathcal{P}_F(\mathcal{C}, J)$ . Suppose that  $F : \mathcal{C} \rightarrow \text{Set}$  is  $U$ -pointed and  $U$ -local for an object  $U$  of  $\mathcal{C}$ . Then, a map  $\varphi : F(U) \rightarrow X$  is an  $F$ -plot if and only if  $\varphi : (F(U), \mathcal{F}_U) \rightarrow (X, \mathcal{X})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

*Proof.* Assume that  $\varphi : F(U) \rightarrow X$  is an  $F$ -plot, namely,  $\varphi \in \mathcal{D} \cap F_X(U)$ . For  $V \in \text{Ob } \mathcal{C}$  and  $\psi \in \mathcal{F}_U \cap F_{F(U)}(V)$ , there exists  $f \in \mathcal{C}(V, U)$  such that  $F(f) = \psi$ . Then, we have  $(F_\varphi)_V(\psi) = \varphi F(f) = F_X(f)(\varphi) \in \mathcal{D} \cap F_X(V)$ , which shows that  $\varphi : (F(U), \mathcal{F}_U) \rightarrow (X, \mathcal{X})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

Conversely, assume that  $\varphi : (F(U), \mathcal{F}_U) \rightarrow (X, \mathcal{X})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . Since  $\text{id}_{F(U)} = F(\text{id}_U)$  belongs to  $\mathcal{F}_U \cap F_{F(U)}(U)$ , we have  $\varphi = \varphi \text{id}_{F(U)} = (F_\varphi)_U(\text{id}_{F(U)}) \in \mathcal{D} \cap F_X(U)$ . Hence  $\varphi$  is an  $F$ -plot.  $\square$

**Lemma 6.13** For an object  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (B, \mathcal{B}))$  of  $\mathcal{P}_F(\mathcal{C}, J)^{(2)}$ , the following diagram in  $\mathcal{P}_F(\mathcal{C}, J)$  is cartesian.

$$\begin{array}{ccc} (E \times_B^{\sigma_E} G_1(\mathbf{E}), \mathcal{E}^{\text{pr}_E^\sigma} \cap \mathcal{G}_E^{\text{pr}_{G_1(\mathbf{E})}^\sigma}) & \xrightarrow{\hat{\xi}_E} & (E, \mathcal{E}) \\ \downarrow \text{pr}_{G_1(\mathbf{E})}^\sigma & & \downarrow \pi \\ (G_1(\mathbf{E}), \mathcal{G}_E) & \xrightarrow{\tau_E} & (B, \mathcal{B}) \end{array}$$

*Proof.* Since  $\pi \hat{\xi}_E = \tau_E \text{pr}_{G_1(\mathbf{E})}^\sigma$  holds, we have  $\pi \hat{\xi}_E(\text{id}_{E \times B} \iota_E) = \tau_E \text{pr}_{G_1(\mathbf{E})}^\sigma(\text{id}_{E \times B} \iota_E) = \tau_E \iota_E \text{pr}_{G_1(\mathbf{E})}^\tau = \sigma_E \text{pr}_{G_1(\mathbf{E})}^\tau$ . Hence there exist morphisms

$$\begin{aligned} \kappa &: (E \times_B^{\sigma_E} G_1(\mathbf{E}), \mathcal{E}^{\text{pr}_E^\sigma} \cap \mathcal{G}_E^{\text{pr}_{G_1(\mathbf{E})}^\sigma}) \rightarrow (E \times_B^{\tau_E} G_1(\mathbf{E}), \mathcal{E}^{\text{pr}_E^\tau} \cap \mathcal{G}_E^{\text{pr}_{G_1(\mathbf{E})}^\tau}) \\ \lambda &: (E \times_B^{\tau_E} G_1(\mathbf{E}), \mathcal{E}^{\text{pr}_E^\tau} \cap \mathcal{G}_E^{\text{pr}_{G_1(\mathbf{E})}^\tau}) \rightarrow (E \times_B^{\sigma_E} G_1(\mathbf{E}), \mathcal{E}^{\text{pr}_E^\sigma} \cap \mathcal{G}_E^{\text{pr}_{G_1(\mathbf{E})}^\sigma}) \end{aligned}$$

in  $\mathcal{P}_F(\mathcal{C}, J)$  that make the following diagrams commute.

$$\begin{array}{ccccc} (E \times_B^{\sigma_E} G_1(\mathbf{E}), \mathcal{E}^{\text{pr}_E^\sigma} \cap \mathcal{G}_E^{\text{pr}_{G_1(\mathbf{E})}^\sigma}) & & & & \\ & \searrow \kappa & & \searrow \hat{\xi}_E & \\ & & (E \times_B^{\tau_E} G_1(\mathbf{E}), \mathcal{E}^{\text{pr}_E^\tau} \cap \mathcal{G}_E^{\text{pr}_{G_1(\mathbf{E})}^\tau}) & \xrightarrow{\text{pr}_E^\tau} & (E, \mathcal{E}) \\ & \searrow \text{pr}_{G_1(\mathbf{E})}^\sigma & \downarrow \text{pr}_{G_1(\mathbf{E})}^\tau & & \downarrow \pi \\ & & (G_1(\mathbf{E}), \mathcal{G}_E) & \xrightarrow{\tau_E} & (B, \mathcal{B}) \end{array}$$

$$\begin{array}{ccc}
(E \times_B^{\tau_E} G_1(\mathbf{E}), \mathcal{E}^{\text{pr}_E^{\tau}} \cap \mathcal{G}_{\mathbf{E}}^{\text{pr}_{G_1(\mathbf{E})}^{\tau}}) & \xrightarrow{\hat{\xi}_{\mathbf{E}}(id_E \times B \iota_E)} & (E, \mathcal{E}) \\
\downarrow \text{pr}_{G_1(\mathbf{E})}^{\tau} & \searrow \lambda & \downarrow \text{pr}_E^{\sigma} \\
(E \times_B^{\sigma_E} G_1(\mathbf{E}), \mathcal{E}^{\text{pr}_E^{\sigma}} \cap \mathcal{G}_{\mathbf{E}}^{\text{pr}_{G_1(\mathbf{E})}^{\sigma}}) & \xrightarrow{\text{pr}_E^{\sigma}} & (E, \mathcal{E}) \\
\downarrow \text{pr}_{G_1(\mathbf{E})}^{\sigma} & & \downarrow \pi \\
(G_1(\mathbf{E}), \mathcal{G}_{\mathbf{E}}) & \xrightarrow{\sigma_E} & (B, \mathcal{B})
\end{array}$$

Since  $\kappa$  maps  $(x, \varphi) \in E \times_B^{\sigma_E} G_1(\mathbf{E})$  to  $(\varphi(x), \varphi) \in E \times_B^{\tau_E} G_1(\mathbf{E})$  and  $\lambda$  maps  $(y, \psi) \in E \times_B^{\tau_E} G_1(\mathbf{E})$  to  $(\psi^{-1}(y), \psi) \in E \times_B^{\sigma_E} G_1(\mathbf{E})$ ,  $\lambda$  is the inverse of  $\kappa$ . It follows that  $\kappa$  is an isomorphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . Since the lower rectangle of the upper diagram is cartesian, the assertion follows.  $\square$

Let  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (B, \mathcal{B}))$  be a fibration. For  $b \in B$ , define a map  $\iota_b : B \rightarrow B \times B$  by  $\iota_b(x) = (b, x)$ . We denote by  $\text{pr}_{B_i} : B \times B \rightarrow B$  the projection onto the  $i$ -th component for  $i = 1, 2$ . Since  $\text{pr}_{B_1} \iota_b$  is a constant map and  $\text{pr}_{B_2} \iota_b$  is the identity map of  $B$ ,  $\iota_b : (B, \mathcal{B}) \rightarrow (B \times B, \mathcal{B}^{\text{pr}_{B_1}} \cap \mathcal{B}^{\text{pr}_{B_2}})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . For  $U \in \text{Ob } \mathcal{C}$  and  $\gamma \in \mathcal{B} \cap F_B(U)$ , since  $(F_{\iota_b})_U(\gamma) \in \mathcal{B}^{\text{pr}_{B_1}} \cap \mathcal{B}^{\text{pr}_{B_2}} = (\mathcal{G}_{\mathbf{E}})_{(\sigma_E, \tau_E)}$ , it follows from (2.4) that there exists  $R \in J(U)$  such that, for each  $h \in R$ , there exists  $\gamma_h \in \mathcal{G}_{\mathbf{E}} \cap F_{G_1(\mathbf{E})}(\text{dom}(h))$  which satisfies  $F_{B \times B}(h)((F_{\iota_b})_U(\gamma)) = (F_{(\sigma_E, \tau_E)})_{\text{dom}(h)}(\gamma_h)$ . For  $u \in F(\text{dom}(h))$ , since  $\gamma_h(u)$  belongs to  $G_1(\mathbf{E})(b, \gamma(F(h)(u)))$  by the commutativity of the following diagram,  $\pi((\gamma_h(u))(e)) = \gamma(F(h)(u))$  holds for  $e \in \pi^{-1}(b)$ .

$$\begin{array}{ccc}
F(\text{dom}(h)) & \xrightarrow{\gamma_h} & G_1(\mathbf{E}) \\
\downarrow F(h) & & \downarrow (\sigma_E, \tau_E) \\
F(U) & \xrightarrow{\gamma} & B \xrightarrow{\iota_b} B \times B
\end{array}$$

We denote by  $\text{pr}_{\pi^{-1}(b)} : \pi^{-1}(b) \times F(\text{dom}(h)) \rightarrow \pi^{-1}(b)$  and  $\text{pr}_{F(\text{dom}(h))} : \pi^{-1}(b) \times F(\text{dom}(h)) \rightarrow F(\text{dom}(h))$  the projections onto the first and second components, respectively. We also denote by  $\iota_b : \pi^{-1}(b) \rightarrow E$  the inclusion map. For  $(e, u) \in \pi^{-1}(b) \times F(\text{dom}(h))$ , since  $\pi(e) = b = \sigma_E \gamma_h(u)$  by the commutativity of the above diagram, we have a map  $(\iota_b \text{pr}_{\pi^{-1}(b)}, \gamma_h \text{pr}_{F(\text{dom}(h))}) : \pi^{-1}(b) \times F(\text{dom}(h)) \rightarrow E \times_B^{\sigma_E} G_1(\mathbf{E})$ . Let us denote by  $\bar{\gamma}_h : \pi^{-1}(b) \times F(\text{dom}(h)) \rightarrow E$  a composition  $\pi^{-1}(b) \times F(\text{dom}(h)) \xrightarrow{(\iota_b \text{pr}_{\pi^{-1}(b)}, \gamma_h \text{pr}_{F(\text{dom}(h))})} E \times_B^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_{\mathbf{E}}} E$ . Then  $\bar{\gamma}_h(e, u) = (\gamma_h(u))(e)$  holds for  $(e, u) \in \pi^{-1}(b) \times F(\text{dom}(h))$ .

**Lemma 6.14** *The following diagram is cartesian in the category of sets.*

$$\begin{array}{ccc}
\pi^{-1}(b) \times F(\text{dom}(h)) & \xrightarrow{\bar{\gamma}_h} & E \\
\downarrow \text{pr}_{F(\text{dom}(h))} & & \downarrow \pi \\
F(\text{dom}(h)) & \xrightarrow{\gamma^{F(h)}} & B
\end{array}$$

*Proof.* We note that  $\pi \bar{\gamma}_h = \gamma^{F(h)} \text{pr}_{F(\text{dom}(h))}$  holds by the definition of  $\bar{\gamma}_h$ . Assume that  $(e, u) \in E \times F(\text{dom}(h))$  satisfies  $\gamma^{F(h)}(u) = \pi(e)$ , namely  $e \in \pi^{-1}(\gamma^{F(h)}(u))$ . Since  $\gamma_h(u) : \pi^{-1}(b) \rightarrow \pi^{-1}(\gamma^{F(h)}(u))$  is surjective, there exists  $e' \in \pi^{-1}(b)$  which maps to  $e$  by  $\gamma_h(u)$ . Hence we have  $\bar{\gamma}_h(e', u) = (\gamma_h(u))(e') = e$ . Suppose that  $(e'', u') \in \pi^{-1}(b) \times F(\text{dom}(h))$  satisfies  $\text{pr}_{F(\text{dom}(h))}(e'', u') = u$  and  $\bar{\gamma}_h(e'', u') = e$ . It is clear that  $u' = u$ , hence we have  $(\gamma_h(u))(e'') = \bar{\gamma}_h(e'', u') = e = (\gamma_h(u))(e')$ . Since  $\gamma_h(u) : \pi^{-1}(b) \rightarrow \pi^{-1}(\gamma^{F(h)}(u))$  is injective, it follows that  $e'' = e'$ . Thus the assertion follows.  $\square$

**Lemma 6.15** *If  $F : \mathcal{C} \rightarrow \text{Set}$  is pointed and local, the following diagram is cartesian in  $\mathcal{P}_F(\mathcal{C}, J)$ .*

$$\begin{array}{ccc}
(\pi^{-1}(b) \times F(\text{dom}(h)), (\mathcal{E}^{\iota_b})^{\text{pr}_{\pi^{-1}(b)}} \cap \mathcal{F}_{\text{dom}(h)}^{\text{pr}_{F(\text{dom}(h))}}) & \xrightarrow{\bar{\gamma}_h} & (E, \mathcal{E}) \\
\downarrow \text{pr}_{F(\text{dom}(h))} & & \downarrow \pi \\
(F(\text{dom}(h)), \mathcal{F}_{\text{dom}(h)}) & \xrightarrow{\gamma^{F(h)}} & (B, \mathcal{B})
\end{array}$$

*Proof.* Since  $\gamma$  is an  $F$ -plot, so is  $\gamma^{F(h)}$ , hence  $\gamma^{F(h)} : (F(\text{dom}(h)), \mathcal{F}_{\text{dom}(h)}) \rightarrow (B, \mathcal{B})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$  by (6.12). Since  $\gamma_h$  is an  $F$ -plot,  $\gamma_h : (F(\text{dom}(h)), \mathcal{F}_{\text{dom}(h)}) \rightarrow (G_1(\mathbf{E}), \mathcal{G}_{\mathbf{E}})$  is a morphism

in  $\mathcal{P}_F(\mathcal{C}, J)$  hence so is  $\gamma_h \text{pr}_{F(\text{dom}(h))} : (\pi^{-1}(b) \times F(\text{dom}(h)), (\mathcal{E}^{i_b})^{\text{pr}_{\pi^{-1}(b)}} \cap \mathcal{F}_{\text{dom}(h)}^{\text{pr}_{F(\text{dom}(h))}}) \rightarrow (G_1(\mathbf{E}), \mathcal{G}_{\mathbf{E}})$ .  $i_b \text{pr}_{\pi^{-1}(b)} : (\pi^{-1}(b) \times F(\text{dom}(h)), (\mathcal{E}^{i_b})^{\text{pr}_{\pi^{-1}(b)}} \cap \mathcal{F}_{\text{dom}(h)}^{\text{pr}_{F(\text{dom}(h))}}) \rightarrow (E, \mathcal{E})$  is also a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . Thus  $(i_b \text{pr}_{\pi^{-1}(b)}, \gamma_h \text{pr}_{F(\text{dom}(h))}) : (\pi^{-1}(b) \times F(\text{dom}(h)), (\mathcal{E}^{i_b})^{\text{pr}_{\pi^{-1}(b)}} \cap \mathcal{F}_{\text{dom}(h)}^{\text{pr}_{F(\text{dom}(h))}}) \rightarrow (E \times_B^{\sigma_E} G_1(\mathbf{E}), \mathcal{E}^{\text{pr}_E} \cap \mathcal{G}_{\mathbf{E}}^{\sigma_E})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . Since  $\hat{\xi}_{\mathbf{E}} : (E \times_B^{\sigma_E} G_1(\mathbf{E}), \mathcal{E}^{\text{pr}_E} \cap \mathcal{G}_{\mathbf{E}}^{\sigma_E}) \rightarrow (E, \mathcal{E})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ , we see that  $\bar{\gamma}_h = \hat{\xi}_{\mathbf{E}}(i_b \text{pr}_{\pi^{-1}(b)}, \gamma_h \text{pr}_{F(\text{dom}(h))}) : (\pi^{-1}(b) \times F(\text{dom}(h)), (\mathcal{E}^{i_b})^{\text{pr}_{\pi^{-1}(b)}} \cap \mathcal{F}_{\text{dom}(h)}^{\text{pr}_{F(\text{dom}(h))}}) \rightarrow (E, \mathcal{E})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . It is clear that the following projection is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

$$\text{pr}_{F(\text{dom}(h))} : (\pi^{-1}(b) \times F(\text{dom}(h)), (\mathcal{E}^{i_b})^{\text{pr}_{\pi^{-1}(b)}} \cap \mathcal{F}_{\text{dom}(h)}^{\text{pr}_{F(\text{dom}(h))}}) \rightarrow (F(\text{dom}(h)), \mathcal{F}_{\text{dom}(h)})$$

Hence  $(\mathcal{E}^{i_b})^{\text{pr}_{\pi^{-1}(b)}} \cap \mathcal{F}_{\text{dom}(h)}^{\text{pr}_{F(\text{dom}(h))}}$  is contained in  $\mathcal{E}^{\bar{\gamma}_h} \cap \mathcal{F}_{\text{dom}(h)}^{\text{pr}_{F(\text{dom}(h))}}$ .

For  $U \in \text{Ob } \mathcal{C}$  and  $\alpha \in \mathcal{E}^{\bar{\gamma}_h} \cap \mathcal{F}_{\text{dom}(h)}^{\text{pr}_{F(\text{dom}(h))}} \cap F_{\pi^{-1}(b) \times F(\text{dom}(h))}(U)$ , put  $\alpha_1 = \text{pr}_{\pi^{-1}(b)}\alpha$  and  $\alpha_2 = \text{pr}_{F(\text{dom}(h))}\alpha$ . Since  $\hat{\xi}_{\mathbf{E}}(i_b \alpha_1, \gamma_h \alpha_2) = \bar{\gamma}_h \alpha \in \mathcal{E} \cap F_E(U)$ , we have  $(i_b \alpha_1, \gamma_h \alpha_2) \in \mathcal{E}^{\hat{\xi}_{\mathbf{E}}} \cap F_{E \times_B^{\sigma_E} G_1(\mathbf{E})}(U)$ . On the other hand, since  $\gamma_h \alpha_2 = (F_{\gamma_h \text{pr}_{F(\text{dom}(h))}})_U(\alpha) \in \mathcal{G}_{\mathbf{E}}$ , we also have  $(i_b \alpha_1, \gamma_h \alpha_2) \in \mathcal{G}_{\mathbf{E}}^{\text{pr}_{G_1(\mathbf{E})}} \cap F_{E \times_B^{\sigma_E} G_1(\mathbf{E})}(U)$ . Therefore  $(i_b \alpha_1, \gamma_h \alpha_2)$  belongs to  $\mathcal{E}^{\hat{\xi}_{\mathbf{E}}} \cap \mathcal{G}_{\mathbf{E}}^{\text{pr}_{G_1(\mathbf{E})}} \cap F_{E \times_B^{\sigma_E} G_1(\mathbf{E})}(U) = \mathcal{E}^{\text{pr}_E} \cap \mathcal{G}_{\mathbf{E}}^{\text{pr}_{G_1(\mathbf{E})}} \cap F_{E \times_B^{\sigma_E} G_1(\mathbf{E})}(U)$  by (6.13). Thus we have  $i_b \alpha_1 = \text{pr}_E^{\sigma}(i_b \alpha_1, \gamma_h \alpha_2) \in \mathcal{E} \cap F_E(U)$  which implies  $\alpha_1 \in \mathcal{E}^{i_b} \cap F_{\pi^{-1}(b)}(U)$ . It follows that  $\alpha$  belongs to  $(\mathcal{E}^{i_b})^{\text{pr}_{\pi^{-1}(b)}} \cap \mathcal{F}_{\text{dom}(h)}^{\text{pr}_{F(\text{dom}(h))}} \cap F_{\pi^{-1}(b) \times F(\text{dom}(h))}(U)$  and that  $\mathcal{E}^{\bar{\gamma}_h} \cap \mathcal{F}_{\text{dom}(h)}^{\text{pr}_{F(\text{dom}(h))}} \subset (\mathcal{E}^{i_b})^{\text{pr}_{\pi^{-1}(b)}} \cap \mathcal{F}_{\text{dom}(h)}^{\text{pr}_{F(\text{dom}(h))}}$  holds. We conclude that  $\mathcal{E}^{\bar{\gamma}_h} \cap \mathcal{F}_{\text{dom}(h)}^{\text{pr}_{F(\text{dom}(h))}}$  coincides with  $(\mathcal{E}^{i_b})^{\text{pr}_{\pi^{-1}(b)}} \cap \mathcal{F}_{\text{dom}(h)}^{\text{pr}_{F(\text{dom}(h))}}$ . Since a diagram

$$\begin{array}{ccc} (\pi^{-1}(b) \times F(\text{dom}(h)), \mathcal{E}^{\bar{\gamma}_h} \cap \mathcal{F}_{\text{dom}(h)}^{\text{pr}_{F(\text{dom}(h))}}) & \xrightarrow{\bar{\gamma}_h} & (E, \mathcal{E}) \\ \downarrow \text{pr}_{F(\text{dom}(h))} & & \downarrow \pi \\ (F(\text{dom}(h)), \mathcal{F}_{\text{dom}(h)}) & \xrightarrow{\gamma^{F(h)}} & (B, \mathcal{B}) \end{array}$$

is cartesian by (6.14), the assertion follows.  $\square$

Assume that the lower right rectangle of the following diagram is cartesian. Then, there exists unique map  $\hat{\gamma}_h : \pi^{-1}(b) \times F(\text{dom}(h)) \rightarrow F(U) \times_B E$  that makes the following diagram commute.

$$\begin{array}{ccccc} \pi^{-1}(b) \times F(\text{dom}(h)) & & & & \\ \downarrow \text{pr}_{F(\text{dom}(h))} & \searrow \hat{\gamma}_h & \xrightarrow{\bar{\gamma}_h} & & \\ & F(U) \times_B E & \xrightarrow{\gamma_{\pi}} & E & \\ & \downarrow \pi_{\gamma} & & \downarrow \pi & \\ F(\text{dom}(h)) & \xrightarrow{F(h)} & F(U) & \xrightarrow{\gamma} & B \end{array}$$

**Proposition 6.16** *We assume that  $F : \mathcal{C} \rightarrow \text{Set}$  is pointed and local. Consider objects*

$$\begin{aligned} \gamma^*(\mathbf{E}) &= ((F(U) \times_B E, \mathcal{F}_U^{\pi_{\gamma}} \cap \mathcal{E}^{\gamma_{\pi}}) \xrightarrow{\pi_{\gamma}} (F(U), \mathcal{F}_U)) \\ \mathbf{G} &= ((\pi^{-1}(b) \times F(\text{dom}(h)), (\mathcal{E}^{i_b})^{\text{pr}_{\pi^{-1}(b)}} \cap \mathcal{F}_{\text{dom}(h)}^{\text{pr}_{F(\text{dom}(h))}}) \xrightarrow{\text{pr}_{F(\text{dom}(h))}} (F(\text{dom}(h)), \mathcal{F}_{\text{dom}(h)})) \end{aligned}$$

of  $\mathcal{P}_F(\mathcal{C}, J)$ . Then,  $\gamma_h = \langle \hat{\gamma}_h, F(h) \rangle : \mathbf{G} \rightarrow \gamma^*(\mathbf{E})$  is cartesian morphism in  $\mathcal{P}_F(\mathcal{C}, J)^{(2)}$ .

*Proof.* Since  $\bar{\gamma}_h = \gamma_{\pi} \hat{\gamma}_h$ , the outer rectangle of the following diagram is cartesian by (6.15). Since the right rectangle of the following diagram is also cartesian, it follows that the left rectangle of the following diagram is cartesian.

$$\begin{array}{ccccc} (\pi^{-1}(b) \times F(\text{dom}(h)), (\mathcal{E}^{i_b})^{\text{pr}_{\pi^{-1}(b)}} \cap \mathcal{F}_{\text{dom}(h)}^{\text{pr}_{F(\text{dom}(h))}}) & \xrightarrow{\hat{\gamma}_h} & (F(U) \times_B E, \mathcal{F}_U^{\pi_{\gamma}} \cap \mathcal{E}^{\gamma_{\pi}}) & \xrightarrow{\gamma_{\pi}} & (E, \mathcal{E}) \\ \downarrow \text{pr}_{F(\text{dom}(h))} & & \downarrow \pi_{\gamma} & & \downarrow \pi \\ (F(\text{dom}(h)), \mathcal{F}_{\text{dom}(h)}) & \xrightarrow{F(h)} & (F(U), \mathcal{F}_U) & \xrightarrow{\gamma} & (B, \mathcal{B}) \end{array}$$

□

Let  $\zeta_1, \zeta_2 : \mathbf{D} \rightarrow \mathbf{E}$  be morphisms in  $\text{Epi}_c(\mathcal{P}_F(\mathcal{C}, J))$ . Put  $\mathbf{D} = ((D, \mathcal{D}) \xrightarrow{\rho} (A, \mathcal{A}))$ ,  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (B, \mathcal{B}))$  and  $\zeta_k = \langle \zeta_k, f_k \rangle$  for  $k = 1, 2$ . For  $a \in A$  and  $b \in B$ , we denote by  $j_a : \rho^{-1}(a) \rightarrow D$ ,  $i_b : \pi^{-1}(b) \rightarrow E$  the inclusion maps. It follows from (5.11) that the morphisms  $\zeta_{k,x} : (\rho^{-1}(x), \mathcal{D}^{j_x}) \rightarrow (\pi^{-1}(f_k(x)), \mathcal{E}^{i_{f_k(x)}})$  ( $k = 1, 2$ ) obtained by restricting  $\zeta_k : (D, \mathcal{D}) \rightarrow (E, \mathcal{E})$  are isomorphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ . Thus we have an isomorphism  $\zeta_{2,x} \zeta_{1,x}^{-1} : (\pi^{-1}(f_1(x)), \mathcal{E}^{i_{f_1(x)}}) \rightarrow (\pi^{-1}(f_2(x)), \mathcal{E}^{i_{f_2(x)}})$  in  $\mathcal{P}_F(\mathcal{C}, J)$ . We define a map  $\tilde{\zeta} : A \rightarrow G_1(\mathbf{E})$  by  $\tilde{\zeta}(x) = \zeta_{2,x} \zeta_{1,x}^{-1}$ . Then,  $\sigma_{\mathbf{E}} \tilde{\zeta}(x) = f_1(x)$  and  $\tau_{\mathbf{E}} \tilde{\zeta}(x) = f_2(x)$  hold and the following diagram is commutative.

$$\begin{array}{ccc} & & G_1(\mathbf{E}) \\ & \nearrow \tilde{\zeta} & \downarrow (\sigma_{\mathbf{E}}, \tau_{\mathbf{E}}) \\ A & \xrightarrow{(f_1, f_2)} & B \times B \end{array}$$

**Lemma 6.17**  $\tilde{\zeta} : (A, \mathcal{A}) \rightarrow (G_1(\mathbf{E}), \mathcal{G}_{\mathbf{E}})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ .

*Proof.* We denote by  $f_j^*(\mathbf{E}) = ((A \times_B^{f_j} E, \mathcal{A}^{\pi_{f_j}} \cap \mathcal{E}^{(f_j)\pi}) \xrightarrow{\pi_{f_j}} (A, \mathcal{A}))$  the inverse image of  $\mathbf{E}$  by  $f_j$ . Then, the following left diagram is cartesian and the right one is also cartesian by the assumption.

$$\begin{array}{ccc} (A \times_B^{f_j} E, \mathcal{A}^{\pi_{f_j}} \cap \mathcal{E}^{(f_j)\pi}) & \xrightarrow{(f_j)\pi} & (E, \mathcal{E}) \\ \downarrow \pi_{f_j} & & \downarrow \pi \\ (A, \mathcal{A}) & \xrightarrow{f_j} & (B, \mathcal{B}) \end{array} \quad \begin{array}{ccc} (D, \mathcal{D}) & \xrightarrow{\zeta_j} & (E, \mathcal{E}) \\ \downarrow \rho & & \downarrow \pi \\ (A, \mathcal{A}) & \xrightarrow{f_j} & (B, \mathcal{B}) \end{array}$$

Hence there exists unique isomorphism  $(\rho, \zeta_j) : (D, \mathcal{D}) \rightarrow (A \times_B^{f_j} E, \mathcal{A}^{\pi_{f_j}} \cap \mathcal{E}^{(f_j)\pi})$  in  $\mathcal{P}_F(\mathcal{C}, J)$  that makes the following diagram commute.

$$\begin{array}{ccc} (D, \mathcal{D}) & \xrightarrow{\zeta_j} & (E, \mathcal{E}) \\ \searrow (\rho, \zeta_j) & & \downarrow \pi \\ (A \times_B^{f_j} E, \mathcal{A}^{\pi_{f_j}} \cap \mathcal{E}^{(f_j)\pi}) & \xrightarrow{(f_j)\pi} & (E, \mathcal{E}) \\ \downarrow \rho & & \downarrow \pi \\ (A, \mathcal{A}) & \xrightarrow{f_j} & (B, \mathcal{B}) \end{array}$$

We put  $\psi_j = (\rho, \zeta_j)$ , then  $\psi_j(x) = (\rho(x), \zeta_{j,\rho(x)}(x))$  holds for  $x \in D$  and the inverse

$$\psi_j^{-1} : (A \times_B^{f_j} E, \mathcal{A}^{\pi_{f_j}} \cap \mathcal{E}^{(f_j)\pi}) \rightarrow (D, \mathcal{D})$$

of  $\psi_j$  is given by  $\psi_j^{-1}(a, e) = \zeta_{j,a}^{-1}(e)$ . Hence  $\psi_k \psi_j^{-1} : (A \times_B^{f_j} E, \mathcal{A}^{\pi_{f_j}} \cap \mathcal{E}^{(f_j)\pi}) \rightarrow (A \times_B^{f_k} E, \mathcal{A}^{\pi_{f_k}} \cap \mathcal{E}^{(f_k)\pi})$  for  $(j, k) = (1, 2), (2, 1)$  are given by  $\psi_k \psi_j^{-1}(a, e) = \psi_k(\zeta_{j,a}^{-1}(e)) = (\rho(\zeta_{j,a}^{-1}(e)), \zeta_{k,\rho(\zeta_{j,a}^{-1}(e))}(\zeta_{j,a}^{-1}(e))) = (a, \zeta_{k,a} \zeta_{j,a}^{-1}(e))$ . Thus we have  $\psi_2 \psi_1^{-1}(a, e) = (a, \tilde{\zeta}(a)(e)) = (a, \hat{\xi}_{\mathbf{E}}(e, \tilde{\zeta}(a)))$  and  $\psi_1 \psi_2^{-1}(a, e) = (a, \tilde{\zeta}(a)^{-1}(e)) = (a, \hat{\xi}_{\mathbf{E}}(e, (\iota_{\mathbf{E}} \tilde{\zeta})(a)))$ . We note that  $\pi(f_1)\pi = f_1 \pi_{f_1} = \sigma_{\mathbf{E}} \tilde{\zeta} \pi_{f_1}$  and  $\pi(f_2)\pi = f_2 \pi_{f_2} = \tau_{\mathbf{E}} \tilde{\zeta} \pi_{f_2} = \sigma_{\mathbf{E}} \iota_{\mathbf{E}} \tilde{\zeta} \pi_{f_2}$  holds and that the following diagrams are commutative.

$$\begin{array}{ccc} A \times_B^{f_1} E & \xrightarrow{\psi_2 \psi_1^{-1}} & A \times_B^{f_2} E \\ \downarrow ((f_1)\pi, \tilde{\zeta} \pi_{f_1}) & & \downarrow (f_2)\pi \\ E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) & \xrightarrow{\hat{\xi}_{\mathbf{E}}} & E \end{array} \quad \begin{array}{ccc} A \times_B^{f_2} E & \xrightarrow{\psi_1 \psi_2^{-1}} & A \times_B^{f_1} E \\ \downarrow ((f_2)\pi, \iota_{\mathbf{E}} \tilde{\zeta} \pi_{f_2}) & & \downarrow (f_1)\pi \\ E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) & \xrightarrow{\hat{\xi}_{\mathbf{E}}} & E \end{array}$$

Since compositions

$$\begin{array}{ccc} (A \times_B^{f_1} E, \mathcal{A}^{\pi_{f_1}} \cap \mathcal{E}^{(f_1)\pi}) & \xrightarrow{\psi_2 \psi_1^{-1}} & (A \times_B^{f_2} E, \mathcal{A}^{\pi_{f_2}} \cap \mathcal{E}^{(f_2)\pi}) \xrightarrow{(f_2)\pi} (E, \mathcal{E}), \\ (A \times_B^{f_2} E, \mathcal{A}^{\pi_{f_2}} \cap \mathcal{E}^{(f_2)\pi}) & \xrightarrow{\psi_1 \psi_2^{-1}} & (A \times_B^{f_1} E, \mathcal{A}^{\pi_{f_1}} \cap \mathcal{E}^{(f_1)\pi}) \xrightarrow{(f_1)\pi} (E, \mathcal{E}) \end{array}$$

are morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ , so are the following.

$$\hat{\xi}_{\mathbf{E}}((f_1)_\pi, \tilde{\zeta}\pi_{f_1}) : (A \times_B^{f_1} E, \mathcal{A}^{\pi_{f_1}} \cap \mathcal{E}^{(f_1)_\pi}) \rightarrow (E, \mathcal{E}), \quad \hat{\xi}_{\mathbf{E}}((f_2)_\pi, \iota_{\mathbf{E}}\tilde{\zeta}\pi_{f_2}) : (A \times_B^{f_2} E, \mathcal{A}^{\pi_{f_2}} \cap \mathcal{E}^{(f_2)_\pi}) \rightarrow (E, \mathcal{E})$$

For  $U \in \text{Ob } \mathcal{C}$  and  $\gamma \in \mathcal{A} \cap F_A(U)$ , we verify that  $(F_{\tilde{\zeta}})_U(\gamma) = \tilde{\zeta}\gamma$  satisfies the conditions (G1), (G2) and (G3). We take  $V, W \in \text{Ob } \mathcal{C}$ ,  $h \in \mathcal{C}(W, U)$ ,  $k \in \mathcal{C}(W, V)$ . Assume that  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfies  $\pi\lambda F(k) = \sigma_{\mathbf{E}}\tilde{\zeta}\gamma F(h)$ . Then,  $f_1\gamma F(h) = \sigma_{\mathbf{E}}\tilde{\zeta}\gamma F(h) = \pi\lambda F(k)$  holds and the following diagram is commutative.

$$\begin{array}{ccccc} & & & A \times_B^{f_1} E & \xrightarrow{\psi_2\psi_1^{-1}} & A \times_B^{f_2} E \\ & \nearrow^{(\gamma F(h), \lambda F(k))} & & \downarrow^{((f_1)_\pi, \tilde{\zeta}\pi_{f_1})} & & \downarrow^{(f_2)_\pi} \\ F(W) & \xrightarrow{(\lambda F(k), \tilde{\zeta}\gamma F(h))} & E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) & \xrightarrow{\hat{\xi}_{\mathbf{E}}} & E \end{array}$$

Since  $(\gamma F(h), \lambda F(k)) : F(W) \rightarrow A \times_B^{f_1} E$  belongs to  $\mathcal{A}^{\pi_{f_1}} \cap \mathcal{E}^{(f_1)_\pi} \cap F_{A \times_B^{f_1} E}(W)$  and  $\hat{\xi}_{\mathbf{E}}((f_1)_\pi, \tilde{\zeta}\pi_{f_1})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ , a composition  $F(W) \xrightarrow{(\lambda F(k), \tilde{\zeta}\gamma F(h))} E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_{\mathbf{E}}} E$  belongs to  $\mathcal{E} \cap F_E(W)$  by the commutativity of the above diagram. Thus  $\tilde{\zeta}\gamma$  satisfies the condition (G1).

Assume that  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfies  $\pi\lambda F(k) = \tau_{\mathbf{E}}\tilde{\zeta}\gamma F(h)$ . Then,  $f_2\gamma F(h) = \tau_{\mathbf{E}}\tilde{\zeta}\gamma F(h) = \pi\lambda F(k)$  holds and the following diagram is commutative.

$$\begin{array}{ccccc} & & & A \times_B^{f_2} E & \xrightarrow{\psi_1\psi_2^{-1}} & A \times_B^{f_1} E \\ & \nearrow^{(\gamma F(h), \lambda F(k))} & & \downarrow^{((f_2)_\pi, \iota_{\mathbf{E}}\tilde{\zeta}\pi_{f_2})} & & \downarrow^{(f_1)_\pi} \\ F(W) & \xrightarrow{(\lambda F(k), \iota_{\mathbf{E}}\tilde{\zeta}\gamma F(h))} & E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) & \xrightarrow{\hat{\xi}_{\mathbf{E}}} & E \end{array}$$

Since  $(\gamma F(h), \lambda F(k)) : F(W) \rightarrow A \times_B^{f_2} E$  belongs to  $\mathcal{A}^{\pi_{f_2}} \cap \mathcal{E}^{(f_2)_\pi} \cap F_{A \times_B^{f_2} E}(W)$  and  $\hat{\xi}_{\mathbf{E}}((f_2)_\pi, \iota_{\mathbf{E}}\tilde{\zeta}\pi_{f_2})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ , a composition  $F(W) \xrightarrow{(\lambda F(k), \iota_{\mathbf{E}}\tilde{\zeta}\gamma F(h))} E \times_B^{\sigma_{\mathbf{E}}} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_{\mathbf{E}}} E$  belongs to  $\mathcal{E} \cap F_E(W)$  by the commutativity of the above diagram. Thus  $\tilde{\zeta}\gamma$  satisfies the condition (G2).

Since we have  $\sigma_{\mathbf{E}}\tilde{\zeta} = f_1$  and  $\tau_{\mathbf{E}}\tilde{\zeta} = f_2$  and  $f_1, f_2 : (A, \mathcal{A}) \rightarrow (B, \mathcal{B})$  are morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ , compositions  $F(U) \xrightarrow{\tilde{\zeta}\gamma} G_1(\mathbf{E}) \xrightarrow{\sigma_{\mathbf{E}}} B$  and  $F(U) \xrightarrow{\tilde{\zeta}\gamma} G_1(\mathbf{E}) \xrightarrow{\tau_{\mathbf{E}}} B$  belong to  $\mathcal{B} \cap F_B(U)$ . Hence  $\tilde{\zeta}\gamma$  satisfies the condition (G3).  $\square$

**Proposition 6.18** ([6], 8.9) *We assume that  $F : \mathcal{C} \rightarrow \text{Set}$  is pointed and local. An object  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (B, \mathcal{B}))$  of  $\text{Epi}_{\mathcal{C}}(\mathcal{P}_F(\mathcal{C}, J))$  is a fibration if and only if the following condition (P) is satisfied.*

(P) *There exists an object  $(T, \mathcal{T})$  of  $\mathcal{P}_F(\mathcal{C}, J)$  such that, for any  $U \in \text{Ob } \mathcal{C}$  and  $\gamma \in \mathcal{B} \cap F_B(U)$ , there exists a covering  $(U_i \xrightarrow{f_i} U)_{i \in I}$  of  $U$  such that the inverse image  $(\gamma F(f_i))^*(\mathbf{E})$  of  $\mathbf{E}$  by  $\gamma F(f_i) : F(U_i) \rightarrow B$  is isomorphic to a product fibration  $\text{pr}_{F(U_i)} : (T \times F(U_i), \mathcal{T}^{\text{pr}_T} \cap \mathcal{F}_{U_i}^{\text{pr}_{F(U_i)}}) \rightarrow (F(U_i), \mathcal{F}_{U_i})$  for any  $i \in I$ . Here  $\text{pr}_T : T \times F(U_i) \rightarrow T$  and  $\text{pr}_{F(U_i)} : T \times F(U_i) \rightarrow F(U_i)$  denote the projections.*

*Proof.* If  $\mathbf{E}$  is a fibration, the condition (P) follows from (6.2) and (6.16).

Suppose that  $\mathbf{E}$  satisfies the condition (P). Since  $(\sigma_{\mathbf{E}}, \tau_{\mathbf{E}}) : (G_1(\mathbf{E}), \mathcal{G}_{\mathbf{E}}) \rightarrow (B \times B, \mathcal{B}^{\text{pr}_{B1}} \cap \mathcal{B}^{\text{pr}_{B2}})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$  and  $(\mathcal{G}_{\mathbf{E}})_{(\sigma_{\mathbf{E}}, \tau_{\mathbf{E}})}$  is the finest the-ology on  $B \times B$ ,  $(\mathcal{G}_{\mathbf{E}})_{(\sigma_{\mathbf{E}}, \tau_{\mathbf{E}})}$  is contained in  $\mathcal{B}^{\text{pr}_{B1}} \cap \mathcal{B}^{\text{pr}_{B2}}$ . For  $U \in \text{Ob } \mathcal{C}$ , assume that  $\gamma \in \mathcal{B}^{\text{pr}_{B1}} \cap \mathcal{B}^{\text{pr}_{B2}} \cap F_{B \times B}(U)$ . We put  $\gamma_j = \text{pr}_{B_j}\gamma \in \mathcal{B} \cap F_B(U)$  for  $j = 1, 2$ . There exist coverings  $(U_{ji} \xrightarrow{f_{ji}} U)_{i \in I_j}$  of  $U$  for  $j = 1, 2$  such that, for any  $i \in I_j$ , the inverse image  $(\gamma_j F(f_{ji}))^*(\mathbf{E})$  of  $\mathbf{E}$  by  $\gamma_j F(f_{ji}) : F(U_{ji}) \rightarrow B$  is isomorphic to the following product fibration by (P).

$$\text{pr}_{F(U_{ji})} : (T \times F(U_{ji}), \mathcal{T}^{\text{pr}_T} \cap \mathcal{F}_{U_{ji}}^{\text{pr}_{F(U_{ji})}}) \rightarrow (F(U_{ji}), \mathcal{F}_{U_{ji}})$$

Let  $R_j \in J(U)$  be the sieve generated by  $(U_{ji} \xrightarrow{f_{ji}} U)_{i \in I_j}$  and put  $R = R_1 \cap R_2$ . Then  $R \in J(U)$  and, for any  $h \in R$  and  $j = 1, 2$ , there exists  $i \in I_j$  and  $g_{ji} \in \mathcal{C}(\text{dom}(h), U_{ji})$  which satisfies  $h = f_{ji}g_{ji}$ . Since the inverse image of a product fibration is also a product fibration, the inverse image  $(\gamma_j F(h))^*(\mathbf{E})$  of  $\mathbf{E}$  by  $\gamma_j F(h) : F(\text{dom}(h)) \rightarrow B$  is isomorphic to the following product fibration for any  $h \in R$  and  $j = 1, 2$ .

$$\mathbf{P}_h = ((T \times F(\text{dom}(h)), \mathcal{T}^{\text{pr}_T} \cap \mathcal{F}_{\text{dom}(h)}^{\text{pr}_{F(\text{dom}(h))}}) \xrightarrow{\text{pr}_{F(\text{dom}(h))}} (F(\text{dom}(h)), \mathcal{F}_{\text{dom}(h)}))$$

Hence there exists a cartesian morphism  $\gamma_{h,j} = \langle \gamma_{h,j}, \gamma_j F(h) \rangle : \mathbf{P}_h \rightarrow \mathbf{E}$ . We apply (6.17) to these cartesian morphisms  $\gamma_{h,1}$  and  $\gamma_{h,2}$ . Then, we have a map  $\tilde{\gamma}_h : F(\text{dom}(h)) \rightarrow G_1(\mathbf{E})$  which makes the following diagram commute.



$$\begin{array}{ccc}
F(\text{dom}(h)) & \xrightarrow{\tilde{\gamma}_h} & G_1(\mathbf{E}) \\
\downarrow F(h) & & \downarrow (\sigma_{\mathbf{E}}, \tau_{\mathbf{E}}) \\
F(U) & \xrightarrow{\gamma} & B \times B
\end{array}$$

In particular, if  $\gamma : F(U) \rightarrow B \times B$  is a constant map to  $(b_1, b_2)$ , then  $\gamma$  is an  $F$ -plot of  $B \times B$  and we have  $(\sigma_{\mathbf{E}}, \tau_{\mathbf{E}})\gamma_h(x) = \gamma F(h) = (b_1, b_2)$ , hence  $(\sigma_{\mathbf{E}}, \tau_{\mathbf{E}}) : G_1(\mathbf{E}) \rightarrow B \times B$  is surjective. It follows from (6.17) that  $\tilde{\gamma}_h : (F(\text{dom}(h)), \mathcal{F}_{\text{dom}(h)}) \rightarrow (G_1(\mathbf{E}), \mathcal{G}_{\mathbf{E}})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ , hence it belongs to  $\mathcal{G}_{\mathbf{E}} \cap F_{G_1(\mathbf{E})}(\text{dom}(h))$  by (6.12). This implies that  $\gamma$  belongs to  $(\mathcal{G}_{\mathbf{E}})_{(\sigma_{\mathbf{E}}, \tau_{\mathbf{E}})}$  by (2.4). Therefore we conclude that  $(\mathcal{G}_{\mathbf{E}})_{(\sigma_{\mathbf{E}}, \tau_{\mathbf{E}})}$  coincides with  $\mathcal{B}^{\text{pr}B_1} \cap \mathcal{B}^{\text{pr}B_2}$  and that  $\mathbf{E}$  is a fibration.  $\square$

## 7 $F$ -topology

Let  $\mathcal{Top}$  be the category of topological spaces and continuous maps. We denote by  $\mathcal{U} : \mathcal{Top} \rightarrow \mathcal{Set}$  the forgetful functor. For a functor  $F : \mathcal{C} \rightarrow \mathcal{Set}$ , we assume in this section that there exists a functor  $F_{\mathcal{T}} : \mathcal{C} \rightarrow \mathcal{Top}$  which satisfies  $F = \mathcal{U}F_{\mathcal{T}}$ .

**Definition 7.1** For an object  $(X, \mathcal{D})$  of  $\mathcal{P}_F(\mathcal{C}, J)$ , we define a set  $\mathcal{O}_{(X, \mathcal{D})}$  of subsets of  $X$  by

$$\mathcal{O}_{(X, \mathcal{D})} = \{O \subset X \mid \alpha^{-1}(O) \text{ is an open set of } F_{\mathcal{T}}(U) \text{ for any } U \in \text{Ob}\mathcal{C} \text{ and } \alpha \in \mathcal{D} \cap F_X(U)\}.$$

It is easy to verify that  $\mathcal{O}_{(X, \mathcal{D})}$  is a topology on  $X$ . In fact,  $\mathcal{O}_{(X, \mathcal{D})}$  is the coarsest topology on  $X$  such that  $\alpha : F_{\mathcal{T}}(U) \rightarrow X$  is continuous for any  $U \in \text{Ob}\mathcal{C}$  and  $\alpha \in \mathcal{D} \cap F_X(U)$ . We call  $\mathcal{O}_{(X, \mathcal{D})}$  the  $F$ -topology on  $X$  associated with  $\mathcal{D}$ .

Let  $\varphi : (X, \mathcal{D}) \rightarrow (Y, \mathcal{E})$  be a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . For  $O \in \mathcal{O}_{(Y, \mathcal{E})}$  and  $U \in \text{Ob}\mathcal{C}$ ,  $\alpha \in \mathcal{D} \cap F_X(U)$ , since  $\varphi\alpha = (F_{\varphi})_U(\alpha) \in \mathcal{E} \cap F_Y(U)$  holds, we have  $\alpha^{-1}(\varphi^{-1}(O)) = (\varphi\alpha)^{-1}(O)$  which is an open set of  $F_{\mathcal{T}}(U)$ . Hence we have  $\varphi^{-1}(O) \in \mathcal{O}_{(X, \mathcal{D})}$  and  $\varphi : (X, \mathcal{O}_{(X, \mathcal{D})}) \rightarrow (Y, \mathcal{O}_{(Y, \mathcal{E})})$  is a continuous map. Define a functor  $\mathcal{T} : \mathcal{P}_F(\mathcal{C}, J) \rightarrow \mathcal{Top}$  by  $\mathcal{T}((X, \mathcal{D})) = (X, \mathcal{O}_{(X, \mathcal{D})})$  and  $\mathcal{T}(\varphi : (X, \mathcal{D}) \rightarrow (Y, \mathcal{E})) = (\varphi : (X, \mathcal{O}_{(X, \mathcal{D})}) \rightarrow (Y, \mathcal{O}_{(Y, \mathcal{E})}))$ .

**Definition 7.2** For a topological space  $(X, \mathcal{O})$ , we define a set  $\mathcal{D}_{(X, \mathcal{O})}$  of  $F$ -parametrizations as follows.

$$\mathcal{D}_{(X, \mathcal{O})} = \coprod_{U \in \text{Ob}\mathcal{C}} \{\alpha \in F_X(U) \mid \alpha : F_{\mathcal{T}}(U) \rightarrow X \text{ is continuous.}\}$$

If  $\mathcal{D}_{(X, \mathcal{O})}$  is a the-ology on  $X$ , we call an element of  $\mathcal{D}_{(X, \mathcal{O})}$  an  $F$ - $(X, \mathcal{O})$ -plot.

The following proposition gives a sufficient condition for  $\mathcal{D}_{(X, \mathcal{O})}$  being a the-ology on  $X$ .

**Proposition 7.3** Let  $(X, \mathcal{O})$  be a topological space. If the following condition (C) is satisfied for  $(X, \mathcal{O})$ , then  $\mathcal{D}_{(X, \mathcal{O})}$  is a the-ology on  $X$ .

(C) For any  $U \in \text{Ob}\mathcal{C}$ , a map  $\alpha : F_{\mathcal{T}}(U) \rightarrow X$  is continuous if there exists a covering  $(U_i \xrightarrow{f_i} U)_{i \in I}$  of  $U$  such that compositions  $F_{\mathcal{T}}(U_i) \xrightarrow{F_{\mathcal{T}}(f_i)} F_{\mathcal{T}}(U) \xrightarrow{\alpha} X$  are continuous for any  $i \in I$ .

*Proof.* Since  $F(1_{\mathcal{C}})$  has only one element, every map from  $F_{\mathcal{T}}(1_{\mathcal{C}})$  to  $X$  is continuous. Hence  $\mathcal{D}_{(X, \mathcal{O})} \supset F_X(1_{\mathcal{C}})$  holds. For a morphism  $f : U \rightarrow V$  in  $\mathcal{C}$  and  $\alpha \in \mathcal{D}_{(X, \mathcal{O})} \cap F_X(V)$ , since  $F_{\mathcal{T}}(f) : F_{\mathcal{T}}(V) \rightarrow F_{\mathcal{T}}(U)$  is continuous, so is  $F_X(f)(\alpha) = \alpha F_{\mathcal{T}}(f) : F_{\mathcal{T}}(U) \rightarrow X$ . It follows that  $F_X(f)(\alpha) \in \mathcal{D}_{(X, \mathcal{O})} \cap F_X(U)$ . For an object  $U$  of  $\mathcal{C}$ , suppose that there exists a covering  $(U_i \xrightarrow{f_i} U)_{i \in I}$  such that  $F_X(f_i) : F_X(U) \rightarrow F_X(U_i)$  maps  $\alpha \in F_X(U)$  into  $\mathcal{D}_{(X, \mathcal{O})} \cap F_X(U_i)$  for any  $i \in I$ . Then,  $\alpha F_{\mathcal{T}}(f_i) = F_X(f_i)(\alpha) : F_{\mathcal{T}}(U_i) \rightarrow X$  is continuous for any  $i \in I$ . Hence  $\alpha : F_{\mathcal{T}}(U) \rightarrow X$  is continuous and belongs to  $\mathcal{D}_{(X, \mathcal{O})} \cap F_X(U)$ .  $\square$

**Remark 7.4** We consider the following condition (Q) on  $F_{\mathcal{T}} : \mathcal{C} \rightarrow \mathcal{Top}$ .

(Q) For any  $U \in \text{Ob}\mathcal{C}$ , there exists a covering  $(U_i \xrightarrow{f_i} U)_{i \in I}$  of  $U$  such that the map  $\coprod_{i \in I} F_{\mathcal{T}}(U_i) \rightarrow F_{\mathcal{T}}(U)$  induced by the family  $(F_{\mathcal{T}}(U_i) \xrightarrow{F_{\mathcal{T}}(f_i)} F_{\mathcal{T}}(U))_{i \in I}$  of maps is a quotient map.

If the condition (Q) is satisfied, the condition (C) of (7.3) is satisfied for any topological space  $(X, \mathcal{O})$ .

**Lemma 7.5** Let  $(X, \mathcal{O}_X)$ ,  $(Y, \mathcal{O}_Y)$  and  $(Z, \mathcal{O}_Z)$  be topological spaces. For continuous maps  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ , if  $gf : X \rightarrow Z$  is a quotient map, so is  $g$ .

*Proof.* For an open set  $O$  of  $Z$ , assume that  $g^{-1}(O)$  is an open set of  $Y$ . Then,  $f^{-1}(g^{-1}(O)) = (gf)^{-1}(O)$  is an open set by the continuity of  $f$ . It follows from the assumption that  $O$  is an open set of  $Z$ .  $\square$

**Proposition 7.6** *For an object  $U$  of  $\mathcal{C}$ , suppose that there exists a covering  $R$  of  $U$  such that the map  $\rho : \coprod_{f \in R} F_{\mathcal{T}}(\text{dom}(f)) \rightarrow F_{\mathcal{T}}(U)$  induced by the family  $(F_{\mathcal{T}}(\text{dom}(f)) \xrightarrow{F_{\mathcal{T}}(f)} F_{\mathcal{T}}(U))_{f \in R}$  of maps is a quotient map.*

*Let  $\bar{R}$  be the sieve on  $U$  generated by  $R$ . Then, the map  $\bar{\rho} : \coprod_{u \in \bar{R}} F_{\mathcal{T}}(\text{dom}(u)) \rightarrow F_{\mathcal{T}}(U)$  induced by the family  $(F_{\mathcal{T}}(\text{dom}(u)) \xrightarrow{F_{\mathcal{T}}(u)} F_{\mathcal{T}}(U))_{u \in \bar{R}}$  of maps is a quotient map.*

*Proof.* For  $u \in \bar{R}$ , there exist  $f_u \in R$  and  $g_u \in \text{Mor } \mathcal{C}$  such that  $\text{codom}(g_u) = \text{dom}(f_u)$  and  $u = f_u g_u$ . We put  $X = \coprod_{f \in R} F_{\mathcal{T}}(\text{dom}(f))$  and  $Y = \coprod_{u \in \bar{R}-R} F_{\mathcal{T}}(\text{dom}(u))$ , then we have  $X \amalg Y = \coprod_{u \in \bar{R}} F_{\mathcal{T}}(\text{dom}(u))$ . Let

$\rho' : \coprod_{u \in \bar{R}-R} F_{\mathcal{T}}(\text{dom}(u)) \rightarrow F_{\mathcal{T}}(U)$  be the map induced by the family  $(F_{\mathcal{T}}(\text{dom}(u)) \xrightarrow{F_{\mathcal{T}}(u)} F_{\mathcal{T}}(U))_{u \in \bar{R}-R}$  of maps. We denote by  $\iota_X : X \rightarrow X \amalg Y$  and  $\iota_Y : Y \rightarrow X \amalg Y$  the inclusion maps. Then  $\bar{\rho} : X \amalg Y \rightarrow F_{\mathcal{T}}(U)$  is the unique map that satisfy  $\bar{\rho} \iota_X = \rho$  and  $\bar{\rho} \iota_Y = \rho'$ . Since  $\rho$  is a quotient map, so is  $\bar{\rho}$  by (7.5).  $\square$

Thus we have the following result.

**Proposition 7.7** *The condition (Q) in (7.4) is equivalent to the following condition.*

(Q') *For any  $U \in \text{Ob } \mathcal{C}$ , there exists  $R \in J(U)$  such that the map  $\coprod_{f \in R} F_{\mathcal{T}}(\text{dom}(f)) \rightarrow F_{\mathcal{T}}(U)$  induced by the family  $(F_{\mathcal{T}}(\text{dom}(f)) \xrightarrow{F_{\mathcal{T}}(f)} F_{\mathcal{T}}(U))_{f \in R}$  of maps is a quotient map.*

**Proposition 7.8** (1) *For an object  $(X, \mathcal{D})$  of  $\mathcal{P}_F(\mathcal{C}, J)$ , we have  $\mathcal{D} \subset \mathcal{D}_{(X, \mathcal{O}_{(X, \mathcal{D})})}$ .*

(2) *For a topological space  $(X, \mathcal{O})$ ,  $\mathcal{O} \subset \mathcal{O}_{(X, \mathcal{D}_{(X, \mathcal{O})})}$  holds.*

*Proof.* (1) For  $U \in \text{Ob } \mathcal{C}$  and  $\alpha \in \mathcal{D} \cap F_X(U)$ , since  $\alpha : F_{\mathcal{T}}(U) \rightarrow X$  is continuous map with respect to the topology  $\mathcal{O}_{(X, \mathcal{D})}$  on  $X$ , it follows  $\alpha \in \mathcal{D}_{(X, \mathcal{O}_{(X, \mathcal{D})})} \cap F_X(U)$ . Therefore  $\mathcal{D} \subset \mathcal{D}_{(X, \mathcal{O}_{(X, \mathcal{D})})}$  holds.

(2) For  $U \in \text{Ob } \mathcal{C}$  and  $\alpha \in \mathcal{D}_{(X, \mathcal{O})} \cap F_X(U)$ , since  $\alpha : F_{\mathcal{T}}(U) \rightarrow X$  is continuous,  $\alpha^{-1}(O)$  is an open set of  $F_{\mathcal{T}}(U)$  for any  $O \in \mathcal{O}$ . By the definition of  $\mathcal{O}_{(X, \mathcal{D}_{(X, \mathcal{O})})}$ , we have  $\mathcal{O} \subset \mathcal{O}_{(X, \mathcal{D}_{(X, \mathcal{O})})}$ .  $\square$

Assume that  $(X, \mathcal{D}_{(X, \mathcal{O})})$  is an object of  $\mathcal{P}_F(\mathcal{C}, J)$  for any topological space  $(X, \mathcal{O})$ . Let  $(X, \mathcal{O}_X)$  and  $(Y, \mathcal{O}_Y)$  be topological spaces and  $f : X \rightarrow Y$  a continuous map. Then  $f : (X, \mathcal{D}_{(X, \mathcal{O}_X)}) \rightarrow (Y, \mathcal{D}_{(Y, \mathcal{O}_Y)})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$ . In fact, for  $U \in \text{Ob } \mathcal{C}$  and  $\alpha \in \mathcal{D}_{(X, \mathcal{O}_X)} \cap F_X(U)$ , since  $(F_f)_U(\alpha) = f\alpha : F_{\mathcal{T}}(U) \rightarrow Y$  is continuous,  $(F_f)_U(\alpha) \in \mathcal{D}_{(Y, \mathcal{O}_Y)} \cap F_Y(U)$  holds. We define a functor  $\mathcal{P} : \text{Top} \rightarrow \mathcal{P}_F(\mathcal{C}, J)$  by  $\mathcal{P}((X, \mathcal{O})) = (X, \mathcal{D}_{(X, \mathcal{O})})$  for an object  $(X, \mathcal{O})$  of  $\text{Top}$  and  $\mathcal{P}(f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)) = (f : (X, \mathcal{D}_{(X, \mathcal{O}_X)}) \rightarrow (Y, \mathcal{D}_{(Y, \mathcal{O}_Y)}))$  for a continuous map  $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ . We remark that  $\Gamma \mathcal{P} = \mathcal{U}$  and  $\mathcal{U} \mathcal{T} = \Gamma$  hold and that both  $\mathcal{P}$  and  $\mathcal{T}$  are faithful.

**Proposition 7.9** *Suppose that  $(X, \mathcal{D}_{(X, \mathcal{O})})$  is an object of  $\mathcal{P}_F(\mathcal{C}, J)$  for any topological space  $(X, \mathcal{O})$ . Then,  $\mathcal{P} : \text{Top} \rightarrow \mathcal{P}_F(\mathcal{C}, J)$  is a right adjoint of  $\mathcal{T} : \mathcal{P}_F(\mathcal{C}, J) \rightarrow \text{Top}$ .*

*Proof.* It follows from (1) of (7.8) that we have a morphism  $\eta_{(X, \mathcal{D})} : (X, \mathcal{D}) \rightarrow (X, \mathcal{D}_{(X, \mathcal{O}_{(X, \mathcal{D})})}) = \mathcal{P}\mathcal{T}((X, \mathcal{D}))$  in  $\mathcal{P}_F(\mathcal{C}, J)$  which is natural in  $(X, \mathcal{D}) \in \text{Ob } \mathcal{P}_F(\mathcal{C}, J)$ . It follows from (2) of (7.8) that we have a continuous bijection  $\varepsilon_{(X, \mathcal{O})} : \mathcal{T}\mathcal{P}((X, \mathcal{O})) = (X, \mathcal{O}_{(X, \mathcal{D}_{(X, \mathcal{O})})}) \rightarrow (X, \mathcal{O})$  which is natural in  $(X, \mathcal{O}) \in \text{Ob } \text{Top}$ . Then,  $\eta : \text{id}_{\mathcal{P}_F(\mathcal{C}, J)} \rightarrow \mathcal{P}\mathcal{T}$  and  $\varepsilon : \mathcal{T}\mathcal{P} \rightarrow \text{id}_{\text{Top}}$  are the unit and the counit of the adjunction  $\mathcal{T} \dashv \mathcal{P}$ , respectively.  $\square$

For a topological space  $(Y, \mathcal{O}_Y)$  and a map  $f : X \rightarrow Y$ , we put  $\mathcal{O}^f = \{O \subset X \mid O = f^{-1}(V) \text{ for some } V \in \mathcal{O}_Y\}$ . Then  $\mathcal{O}^f$  is the coarsest topology on  $X$  such that  $f : X \rightarrow Y$  is a continuous map.

**Proposition 7.10** *For a map  $f : X \rightarrow Y$  and an object  $(Y, \mathcal{E})$  of  $\mathcal{P}_F(\mathcal{C}, J)$ , consider the the-ology  $\mathcal{E}^f$  on  $X$ . Then, the  $F$ -topology  $\mathcal{O}_{(X, \mathcal{E}^f)}$  on  $X$  associated with  $\mathcal{E}^f$  is finer than  $\mathcal{O}_{(Y, \mathcal{E})}^f$ .*

*Proof.* For  $V \in \mathcal{O}_{(Y, \mathcal{E})}$ ,  $U \in \text{Ob } \mathcal{C}$  and  $\alpha \in \mathcal{E}^f \cap F_X(U)$ , since  $\alpha^{-1}(f^{-1}(V)) = (f\alpha)^{-1}(V)$  and  $f\alpha \in \mathcal{E} \cap F_Y(U)$ ,  $\alpha^{-1}(f^{-1}(V))$  is an open set of  $F_{\mathcal{T}}(U)$ . Hence we have  $f^{-1}(V) \in \mathcal{O}_{(X, \mathcal{E}^f)}$  which implies  $\mathcal{O}_{(Y, \mathcal{E})}^f \subset \mathcal{O}_{(X, \mathcal{E}^f)}$ .  $\square$

For a topological space  $(X, \mathcal{O}_X)$  and a map  $f : X \rightarrow Y$ , we put  $\mathcal{O}_f = \{O \subset Y \mid f^{-1}(O) \in \mathcal{O}_X\}$ . Then  $\mathcal{O}_f$  is the finest topology on  $Y$  such that  $f : X \rightarrow Y$  is a continuous map.

**Proposition 7.11** For a map  $f : X \rightarrow Y$  and an object  $(X, \mathcal{D})$  of  $\mathcal{P}_F(\mathcal{C}, J)$ , consider the the-ology  $\mathcal{D}_f$  on  $Y$ . Then, the  $F$ -topology  $\mathcal{O}_{(Y, \mathcal{D}_f)}$  on  $Y$  associated with  $\mathcal{D}_f$  is coarser than  $(\mathcal{O}_{(X, \mathcal{D})})_f$ . If  $F_{\mathcal{T}} : \mathcal{C} \rightarrow \text{Top}$  satisfies the following condition  $(Q'')$ ,  $\mathcal{O}_{(Y, \mathcal{D}_f)}$  coincides with  $(\mathcal{O}_{(X, \mathcal{D})})_f$ .

$(Q'')$  For any  $U \in \text{Ob } \mathcal{C}$  and  $R \in J(U)$ , the map  $\coprod_{f \in R} F_{\mathcal{T}}(\text{dom}(f)) \rightarrow F_{\mathcal{T}}(U)$  induced by the family

$$(F_{\mathcal{T}}(\text{dom}(h)) \xrightarrow{F_{\mathcal{T}}(h)} F_{\mathcal{T}}(U))_{h \in R} \text{ of maps is a quotient map.}$$

*Proof.* For  $O \in \mathcal{O}_{(Y, \mathcal{D}_f)}$ ,  $U \in \text{Ob } \mathcal{C}$  and  $\alpha \in \mathcal{D} \cap F_X(U)$ , since  $\alpha^{-1}(f^{-1}(O)) = (f\alpha)^{-1}(O)$  and  $f\alpha = (F_f)_U(\alpha)$  belongs to  $\mathcal{D}_f \cap F_Y(U)$ ,  $\alpha^{-1}(f^{-1}(O))$  is an open set of  $F_{\mathcal{T}}(U)$ . Hence we have  $f^{-1}(O) \in \mathcal{O}_{(X, \mathcal{D})}$  which shows  $O \in (\mathcal{O}_{(X, \mathcal{D})})_f$ . Therefore  $\mathcal{O}_{(Y, \mathcal{D}_f)} \subset (\mathcal{O}_{(X, \mathcal{D})})_f$  holds.

Assume that  $F_{\mathcal{T}}$  satisfies  $(Q'')$ . We take  $O \in (\mathcal{O}_{(X, \mathcal{D})})_f$ ,  $U \in \text{Ob } \mathcal{C}$  and  $\alpha \in \mathcal{D} \cap F_Y(U)$ . There exists  $R \in J(U)$  such that  $F_Y(h)(\alpha) \in \bigcup_{g \in \text{Mor } \mathcal{C}} \mathcal{S}_g$  for all  $h \in R$ . Then,  $F_Y(h)(\alpha) \in \mathcal{S}_{g_h}$  for some  $g_h \in \text{Mor } \mathcal{C}$  such that  $\text{dom}(g_h) = \text{dom}(h)$ . Assume that  $\text{codom}(g_h) \neq 1_{\mathcal{C}}$ . Since  $\mathcal{S}_{g_h} = (F_f)_{\text{dom}(g_h)}(F_X(g_h)(\mathcal{D} \cap F_X(\text{codom}(g_h))))$  by (2.4), there exists  $j_h \in \mathcal{D} \cap F_X(\text{codom}(g_h))$  such that  $F_Y(h)(\alpha) = (F_f)_{\text{dom}(g_h)}(F_X(g_h)(j_h))$ . Thus we have the following commutative diagram.

$$\begin{array}{ccccc} F(\text{dom}(g_h)) & \xlongequal{\quad} & F(\text{dom}(h)) & \xrightarrow{F(h)} & F(U) \\ \downarrow F(g_h) & & & & \downarrow \alpha \\ F(\text{codom}(g_h)) & \xrightarrow{j_h} & X & \xrightarrow{f} & Y \end{array}$$

Since  $j_h \in \mathcal{D}$  and  $f^{-1}(O) \in \mathcal{O}_{(X, \mathcal{D})}$ ,  $j_h^{-1}(f^{-1}(O))$  is an open set of  $F_{\mathcal{T}}(\text{codom}(g_h))$ . Then the continuity of  $F(g_h)$  implies that  $F(h)^{-1}(\alpha^{-1}(O)) = F(g_h)^{-1}(j_h^{-1}(f^{-1}(O)))$  is an open set of  $F(\text{dom}(h))$ . Consider the case  $\text{codom}(g_h) = 1_{\mathcal{C}}$ . Then,  $\mathcal{S}_{g_h} = F_Y(g_h)(F_Y(1_{\mathcal{C}}))$  by (2.4) and there exists a constant map  $j_h \in F_Y(1_{\mathcal{C}})$  such that  $\alpha F(h) = F_Y(h)(\alpha) = F_Y(g_h)(j_h) = j_h F(g_h)$  which is a constant map. It follows that  $F(h)^{-1}(\alpha^{-1}(O))$  coincides with  $F(\text{dom}(h))$  if  $O$  contains the image of  $j_h$  and that  $F(h)^{-1}(\alpha^{-1}(O))$  is empty otherwise. Therefore  $F(h)^{-1}(\alpha^{-1}(O))$  is an open set of  $F_{\mathcal{T}}(\text{dom}(h))$  for any  $h \in R$ . It follows from  $(Q'')$  that  $\alpha^{-1}(O)$  is an open set of  $F_{\mathcal{T}}(U)$  for any  $\alpha \in \mathcal{D}_f \cap F_Y(U)$ . Hence  $O \in \mathcal{O}_{(Y, \mathcal{D}_f)}$  holds and we have  $(\mathcal{O}_{(X, \mathcal{D})})_f \subset \mathcal{O}_{(Y, \mathcal{D}_f)}$ .  $\square$

## 8 Representations of groupoids in the category of plots

Let  $f : (X, \mathcal{X}) \rightarrow (Y, \mathcal{Y})$ ,  $g : (X, \mathcal{X}) \rightarrow (Z, \mathcal{Z})$ ,  $k : (W, \mathcal{W}) \rightarrow (X, \mathcal{X})$  be morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$  and  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (Y, \mathcal{Y}))$ ,  $\mathbf{D} = ((D, \mathcal{D}) \xrightarrow{\rho} (Z, \mathcal{Z}))$  objects of  $\mathcal{P}_F(\mathcal{C}, J)^{(2)}$ . It follows from (3.3) that there are isomorphisms  $\mathbf{c}_{f,k}(\mathbf{E})^{-1} : (fk)^*(\mathbf{E}) \rightarrow k^*(f^*(\mathbf{E}))$  and  $\mathbf{c}_{g,k}(\mathbf{D}) : k^*(g^*(\mathbf{D})) \rightarrow (gk)^*(\mathbf{D})$  in  $\mathcal{P}_F(\mathcal{C}, J)^{(2)}$ . Consider the following diagrams whose rectangles are all cartesian.

$$\begin{array}{ccc} (E \times_Y X) \times_X W & \xrightarrow{k\pi_f} & E \times_Y X \xrightarrow{f\pi} E \\ \downarrow (\pi_f)_k & & \downarrow \pi_f \quad \downarrow \pi \\ W & \xrightarrow{k} & X \xrightarrow{f} Y \end{array} \quad \begin{array}{ccc} E \times_Y W & \xrightarrow{(fk)\pi} & E \\ \downarrow \pi_{fk} & & \downarrow \pi \\ W & \xrightarrow{fk} & Y \end{array}$$

$$\begin{array}{ccc} (D \times_Z X) \times_X W & \xrightarrow{k\rho_g} & D \times_Z X \xrightarrow{g\rho} D \\ \downarrow (\rho_g)_k & & \downarrow \rho_g \quad \downarrow \rho \\ W & \xrightarrow{k} & X \xrightarrow{g} Z \end{array} \quad \begin{array}{ccc} D \times_Z W & \xrightarrow{(gk)\rho} & D \\ \downarrow \rho_{gk} & & \downarrow \rho \\ W & \xrightarrow{gk} & Z \end{array}$$

It follows from (3.3) and (3.4) that we have unique isomorphisms in  $\mathcal{P}_F(\mathcal{C}, J)$

$$\begin{aligned} \mathbf{c}_{f,k}(\mathbf{E})^{-1} &: (E \times_Y W, \mathcal{E}^{(fk)\pi} \cap \mathcal{W}^{\pi_{fk}}) \rightarrow ((E \times_Y X) \times_X W, (\mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi_f})^{k\pi_f} \cap \mathcal{W}^{(\pi_f)_k}) \\ \mathbf{c}_{g,k}(\mathbf{D}) &: ((D \times_Z X) \times_X W, (\mathcal{D}^{g\rho} \cap \mathcal{X}^{\rho_g})^{k\rho_g} \cap \mathcal{W}^{(\rho_g)_k}) \rightarrow (D \times_Z W, \mathcal{D}^{(gk)\rho} \cap \mathcal{W}^{\rho_{gk}}) \end{aligned}$$

that make following diagram commute.

$$\begin{array}{ccc}
E \times_Y W & \xrightarrow{(fk)_\pi} & E \\
\downarrow \pi_{fk} & \dashrightarrow^{c_{f,k}(\mathbf{E})^{-1}} & \downarrow \pi \\
(E \times_Y X) \times_X W & \xrightarrow{k_{\pi f}} & E \times_Y X \xrightarrow{f_\pi} E \\
\downarrow (\pi_f)_k & & \downarrow \pi_f \\
W & \xrightarrow{k} & X \xrightarrow{f} Y \\
& & \downarrow \pi
\end{array}
\quad
\begin{array}{ccc}
(D \times_Y X) \times_X W & \xrightarrow{k_{\rho g}} & D \times_Y X \\
\downarrow (\rho_g)_k & \dashrightarrow^{c_{g,k}(\mathbf{E})} & \downarrow g_\rho \\
D \times_Y W & \xrightarrow{(gk)_\rho} & D \\
\downarrow \rho_{gk} & & \downarrow \rho \\
W & \xrightarrow{fk} & Y
\end{array}$$

We note that  $\mathbf{c}_{f,k}(\mathbf{E})^{-1} = \langle c_{f,k}(\mathbf{E})^{-1}, id_W \rangle$  and  $\mathbf{c}_{g,k}(\mathbf{D}) = \langle c_{g,k}(\mathbf{D}), id_W \rangle$  hold. The following fact follows from the above diagrams.

**Proposition 8.1**  $c_{f,k}(\mathbf{E})^{-1}$  and  $c_{g,k}(\mathbf{D})$  are given by  $c_{f,k}(\mathbf{E})^{-1}(u, w) = (u, k(w), w)$  for  $(u, w) \in E \times_Y W$  and  $c_{g,k}(\mathbf{D})(v, x, w) = (v, w)$  for  $(v, x, w) \in (D \times_Z X) \times_X W$ , respectively.

For a morphism  $\xi : f^*(\mathbf{E}) \rightarrow g^*(\mathbf{D})$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(X, \mathcal{X})}^{(2)}$ , we define a morphism  $\xi_k : (fk)^*(\mathbf{E}) \rightarrow (gk)^*(\mathbf{D})$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(W, \mathcal{W})}^{(2)}$  to be a composition  $(fk)^*(\mathbf{E}) \xrightarrow{c_{f,k}(\mathbf{E})^{-1}} k^*(f^*(\mathbf{E})) \xrightarrow{k^*(\xi)} k^*(g^*(\mathbf{D})) \xrightarrow{c_{g,k}(\mathbf{D})} (gk)^*(\mathbf{D})$ . We put  $\xi = \langle \xi, id_X \rangle$ , where  $\xi : (E \times_Y X, \mathcal{E}^{f_\pi} \cap \mathcal{X}^{\pi_f}) \rightarrow (D \times_Z X, \mathcal{D}^{g_\rho} \cap \mathcal{X}^{\rho_g})$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$  which satisfies  $\rho_g \xi = \pi_f$ . Then, there exists unique morphism

$$\xi \times_X id_W : ((E \times_Y X) \times_X W, (\mathcal{E}^{f_\pi} \cap \mathcal{X}^{\pi_f})^{k_{\pi f}} \cap \mathcal{W}^{(\pi_f)_k}) \rightarrow ((D \times_Z X) \times_X W, (\mathcal{D}^{g_\rho} \cap \mathcal{X}^{\rho_g})^{k_{\rho g}} \cap \mathcal{W}^{(\rho_g)_k})$$

that makes the following diagram commute.

$$\begin{array}{ccccc}
W & \xleftarrow{(\pi_f)_k} & (E \times_Y X) \times_X W & \xrightarrow{k_{\pi f}} & E \times_Y X \\
\downarrow id_W & & \downarrow \xi \times_X id_W & & \downarrow \xi \\
W & \xleftarrow{(\rho_g)_k} & (D \times_Z X) \times_X W & \xrightarrow{k_{\rho g}} & D \times_Z X
\end{array}$$

Then, we have  $k^*(\xi) = \langle \xi \times_X id_W, id_W \rangle$ . We denote by  $\xi_k : (E \times_Y W, \mathcal{E}^{(fk)_\pi} \cap \mathcal{W}^{\pi_{fk}}) \rightarrow (D \times_Z X, \mathcal{D}^{(gk)_\rho} \cap \mathcal{W}^{\rho_{gk}})$  the following composition.

$$\begin{aligned}
(E \times_Y W, \mathcal{E}^{(fk)_\pi} \cap \mathcal{W}^{\pi_{fk}}) &\xrightarrow{c_{f,k}(\mathbf{E})^{-1}} ((E \times_Y X) \times_X W, (\mathcal{E}^{f_\pi} \cap \mathcal{X}^{\pi_f})^{k_{\pi f}} \cap \mathcal{W}^{(\pi_f)_k}) \xrightarrow{\xi \times_X id_W} \\
&((D \times_Z X) \times_X W, (\mathcal{D}^{g_\rho} \cap \mathcal{X}^{\rho_g})^{k_{\rho g}} \cap \mathcal{W}^{(\rho_g)_k}) \xrightarrow{c_{g,k}(\mathbf{D})} (D \times_Z W, \mathcal{D}^{(gk)_\rho} \cap \mathcal{W}^{\rho_{gk}})
\end{aligned}$$

It follows from the definition of  $\xi_k : (fk)^*(\mathbf{E}) \rightarrow (gk)^*(\mathbf{D})$  that  $\xi_k = \langle \xi_k, id_W \rangle$ . Since  $\rho_g \xi = \pi_f$ , we have  $\xi(u, x) = (g_\rho \xi(u, x), x)$  for  $(u, x) \in E \times_Y X$ . Thus we have the following result.

**Proposition 8.2**  $\xi_k$  maps  $(u, w) \in E \times_Y W$  to  $(g_\rho \xi(u, k(w)), w) \in D \times_Y W$ .

Let  $\mathbf{G} = ((G_0, \mathcal{G}_0), (G_1, \mathcal{G}_1); \sigma, \tau, \varepsilon, \mu, \iota)$  be a groupoid in  $\mathcal{P}_F(\mathcal{C}, J)$  and  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (G_0, \mathcal{G}_0))$  be an object of  $\mathcal{P}_F(\mathcal{C}, J)_{(G_0, \mathcal{G}_0)}^{(2)}$ . Recall that we consider the following cartesian square.

$$\begin{array}{ccc}
G_1 \times_{G_0} G_1 & \xrightarrow{\text{pr}_2} & G_1 \\
\downarrow \text{pr}_1 & & \downarrow \sigma \\
G_1 & \xrightarrow{\tau} & G_0
\end{array}$$

**Definition 8.3** We call a pair  $(\mathbf{E}, \xi)$  of object  $\mathbf{E}$  of  $\mathcal{P}_F(\mathcal{C}, J)_{(G_0, \mathcal{G}_0)}^{(2)}$  and a morphism  $\xi : \sigma^*(\mathbf{E}) \rightarrow \tau^*(\mathbf{E})$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(G_1, \mathcal{G}_1)}^{(2)}$  a representation of  $\mathbf{G}$  on  $\mathbf{E}$  if  $\xi$  satisfies the following conditions.

(A) The following diagram is commutative.

$$\begin{array}{ccc}
(\sigma \text{pr}_1)^*(\mathbf{E}) & \xrightarrow{\xi_{\text{pr}_1}} & (\tau \text{pr}_1)^*(\mathbf{E}) = (\sigma \text{pr}_2)^*(\mathbf{E}) \xrightarrow{\xi_{\text{pr}_2}} & (\tau \text{pr}_2)^*(\mathbf{E}) \\
\parallel & & & \parallel \\
(\sigma \mu)^*(\mathbf{E}) & \xrightarrow{\xi_\mu} & & (\tau \mu)^*(\mathbf{E})
\end{array}$$

(U)  $\xi_\varepsilon : id_{G_0}^*(\mathbf{E}) = (\sigma\varepsilon)^*(\mathbf{E}) \rightarrow (\tau\varepsilon)^*(\mathbf{E}) = id_{G_0}^*(\mathbf{E})$  coincides with the identity morphism of  $id_{G_0}^*(\mathbf{E}) = \mathbf{E}$ .

**Definition 8.4** Let  $(\mathbf{E}, \xi)$  and  $(\mathbf{D}, \zeta)$  be representations of  $\mathbf{G}$  on  $\mathbf{E}$  and  $\mathbf{D}$ , respectively. If a morphism  $\varphi : \mathbf{E} \rightarrow \mathbf{D}$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(G_0, \mathcal{G}_0)}^{(2)}$  makes the following diagram commute, we call  $\varphi$  a morphism of representations.

$$\begin{array}{ccc} \sigma^*(\mathbf{E}) & \xrightarrow{\xi} & \tau^*(\mathbf{E}) \\ \downarrow \sigma^*(\varphi) & & \downarrow \tau^*(\varphi) \\ \sigma^*(\mathbf{D}) & \xrightarrow{\zeta} & \tau^*(\mathbf{D}) \end{array}$$

We denote by  $\text{Rep}(\mathbf{G})$  the category whose objects are representations of  $\mathbf{G}$  and morphisms are morphisms of representations. We call  $\text{Rep}(\mathbf{G})$  the category of representations of  $\mathbf{G}$ .

Let  $\mathbf{G} = ((G_0, \mathcal{G}_0), (G_1, \mathcal{G}_1); \sigma, \tau, \varepsilon, \mu, \iota)$ ,  $\mathbf{H} = ((H_0, \mathcal{H}_0), (H_1, \mathcal{H}_1); \sigma', \tau', \varepsilon', \mu', \iota')$  be groupoids in  $\mathcal{P}_F(\mathcal{C}, J)$  and  $\mathbf{f} = (f_0, f_1) : \mathbf{H} \rightarrow \mathbf{G}$  a morphism of groupoids. For a representation  $(\mathbf{E}, \xi)$  of  $\mathbf{G}$  on  $\mathbf{E}$ , we define a morphism  $\xi_f : \sigma'^*(f_0^*(\mathbf{E})) \rightarrow \tau'^*(f_0^*(\mathbf{E}))$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(H_1, \mathcal{H}_1)}^{(2)}$  to be the following composition.

$$\sigma'^*(f_0^*(\mathbf{E})) \xrightarrow{c_{f_0, \sigma'}(\mathbf{E})} (f_0 \sigma')^*(\mathbf{E}) = (\sigma f_1)^*(\mathbf{E}) \xrightarrow{\xi_{f_1}} (\tau f_1)^*(\mathbf{E}) = (f_0 \tau')^*(\mathbf{E}) \xrightarrow{c_{f_0, \tau'}(\mathbf{E})^{-1}} \tau'^*(f_0^*(\mathbf{E}))$$

**Proposition 8.5** ([10],[11])  $(f_0^*(\mathbf{E}), \xi_f)$  is a representation of  $\mathbf{H}$  on  $f_0^*(\mathbf{E})$ .

**Proposition 8.6** ([10], [11]) Let  $(\mathbf{E}, \xi)$  and  $(\mathbf{D}, \zeta)$  be objects of  $\text{Rep}(\mathbf{G})$  and  $\varphi : (\mathbf{E}, \xi) \rightarrow (\mathbf{D}, \zeta)$  a morphism in  $\text{Rep}(\mathbf{G})$ . For a morphism  $\mathbf{f} = (f_0, f_1) : \mathbf{H} \rightarrow \mathbf{G}$  of groupoids in  $\mathcal{P}_F(\mathcal{C}, J)$ ,  $f_0^*(\varphi) : f_0^*(\mathbf{E}) \rightarrow f_0^*(\mathbf{D})$  defines a morphism  $f_0^*(\varphi) : (f_0^*(\mathbf{E}), \xi_f) \rightarrow (f_0^*(\mathbf{D}), \zeta_f)$  in  $\text{Rep}(\mathbf{H})$ .

(8.4) and (8.5) enable us to define the notion of restriction functor.

**Definition 8.7** Let  $\mathbf{G}$  and  $\mathbf{H}$  be groupoids in  $\mathcal{P}_F(\mathcal{C}, J)$ . For a morphism  $\mathbf{f} = (f_0, f_1) : \mathbf{H} \rightarrow \mathbf{G}$  of groupoids in  $\mathcal{P}_F(\mathcal{C}, J)$ , define a functor  $\mathbf{f}^* : \text{Rep}(\mathbf{G}) \rightarrow \text{Rep}(\mathbf{H})$  by  $\mathbf{f}^*(\mathbf{E}, \xi) = (f_0^*(\mathbf{E}), \xi_f)$  for an object  $(\mathbf{E}, \xi)$  of  $\text{Rep}(\mathbf{G})$  and  $\mathbf{f}^*(\varphi) = f_0^*(\varphi)$  for a morphism  $\varphi : (\mathbf{E}, \xi) \rightarrow (\mathbf{D}, \zeta)$  in  $\text{Rep}(\mathbf{G})$ . We call  $(f_0^*(\mathbf{E}), \xi_f)$  the restriction of  $(\mathbf{E}, \xi)$  along  $\mathbf{f}$  and  $\mathbf{f}^*$  the restriction functor associated with  $\mathbf{f}$ .

We consider the following diagrams whose rectangles are cartesian.

$$\begin{array}{ccc} (E \times_{G_0} H_0) \times_{H_0}^{\sigma'} H_1 & \xrightarrow{\sigma'_{\pi f_0}} & E \times_{G_0} H_0 & \xrightarrow{(f_0)\pi} & E \\ \downarrow (\pi_{f_0})_{\sigma'} & & \downarrow \pi_{f_0} & & \downarrow \pi \\ H_1 & \xrightarrow{\sigma'} & H_0 & \xrightarrow{f_0} & G_0 \end{array} \quad \begin{array}{ccc} (E \times_{G_0} H_0) \times_{H_0}^{\tau'} H_1 & \xrightarrow{\tau'_{\pi f_0}} & E \times_{G_0} H_0 & \xrightarrow{(f_0)\pi} & E \\ \downarrow (\pi_{f_0})_{\tau'} & & \downarrow \pi_{f_0} & & \downarrow \pi \\ H_1 & \xrightarrow{\tau'} & H_0 & \xrightarrow{f_0} & G_0 \end{array}$$

$$\begin{array}{ccc} (E \times_{G_0}^{\sigma} G_1) \times_{G_1} H_1 & \xrightarrow{(f_1)\pi_{\sigma}} & E \times_{G_0}^{\sigma} G_1 & \xrightarrow{\sigma_{\pi}} & E \\ \downarrow (\pi_{\sigma})_{f_1} & & \downarrow \pi_{\sigma} & & \downarrow \pi \\ H_1 & \xrightarrow{f_1} & G_1 & \xrightarrow{\sigma} & G_0 \end{array} \quad \begin{array}{ccc} (E \times_{G_0}^{\tau} G_1) \times_{G_1} H_1 & \xrightarrow{(f_1)\pi_{\tau}} & E \times_{G_0}^{\tau} G_1 & \xrightarrow{\tau_{\pi}} & E \\ \downarrow (\pi_{\tau})_{f_1} & & \downarrow \pi_{\tau} & & \downarrow \pi \\ H_1 & \xrightarrow{f_1} & G_1 & \xrightarrow{\tau} & G_0 \end{array}$$

The following result can be verified from the definition of  $\xi_f$ .

**Proposition 8.8** We put  $\xi_f = \langle \xi_f, id_{H_0} \rangle$  for a morphism

$\xi_f : ((E \times_{G_0} H_0) \times_{H_0}^{\sigma'} H_1, (\mathcal{E}^{(f_0)\pi} \cap \mathcal{H}_0^{\pi f_0})^{\sigma'_{\pi f_0}} \cap \mathcal{H}_1^{(\pi f_0)\sigma'}) \rightarrow ((E \times_{G_0} H_0) \times_{H_0}^{\tau'} H_1, (\mathcal{E}^{(f_0)\pi} \cap \mathcal{H}_0^{\pi f_0})^{\tau'_{\pi f_0}} \cap \mathcal{H}_1^{(\pi f_0)\tau'})$  in  $\mathcal{P}_F(\mathcal{C}, J)$ . Then,  $\xi_f$  maps  $((u, x), y) \in (E \times_{G_0} H_0) \times_{H_0}^{\sigma'} H_1$  to  $((\tau_{\pi}\xi(u, f_1(y)), \tau'(y)), y) \in (E \times_{G_0} H_0) \times_{H_0}^{\tau'} H_1$ .

Let  $\mathbf{f} = (f_0, f_1)$ ,  $\mathbf{g} = (g_0, g_1) : \mathbf{H} \rightarrow \mathbf{G}$  be morphisms of groupoids in  $\mathcal{P}_F(\mathcal{C}, J)$ . Suppose that a morphism  $\chi : (H_0, \mathcal{H}_0) \rightarrow (G_1, \mathcal{G}_1)$  in  $\mathcal{P}_F(\mathcal{C}, J)$  makes the following diagrams commute.

$$\begin{array}{ccc} G_0 & \xleftarrow{f_0} & H_0 & \xrightarrow{g_0} & G_0 \\ & \searrow \sigma & \downarrow \chi & \nearrow \tau & \\ & & G_1 & & \end{array} \quad \begin{array}{ccc} H_1 & \xrightarrow{(f_1, \chi\tau')} & G_1 \times_{G_0} G_1 \\ \downarrow (\chi\sigma', g_1) & & \downarrow \mu \\ G_1 \times_{G_0} G_1 & \xrightarrow{\mu} & G_1 \end{array}$$

For a representation  $(\mathbf{E}, \xi)$  of  $\mathbf{G}$ , we define a morphism  $\chi_{(\mathbf{E}, \xi)}^{\bullet} : f_0^*(\mathbf{E}) \rightarrow g_0^*(\mathbf{E})$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(H_0, \mathcal{H}_0)}^{(2)}$  to be  $\xi_{\chi} : f_0^*(\mathbf{E}) = (\sigma\chi)^*(\mathbf{E}) \rightarrow (\tau\chi)^*(\mathbf{E}) = g_0^*(\mathbf{E})$ .

**Proposition 8.9** ([10], [11])  $\chi_{(\mathbf{E}, \xi)}^{\bullet}$  defines a morphism of representations  $\chi_{(\mathbf{E}, \xi)}^{\bullet} : (f_0^*(\mathbf{E}), \xi_f) \rightarrow (g_0^*(\mathbf{E}), \xi_g)$  and the following diagram in  $\text{Rep}(\mathbf{H})$  commutes for a morphism  $\varphi : (\mathbf{E}, \xi) \rightarrow (\mathbf{D}, \zeta)$  of representations of  $\mathbf{G}$ .

$$\begin{array}{ccc} (f_0^*(\mathbf{E}), \xi_f) & \xrightarrow{f^*(\varphi)} & (f_0^*(\mathbf{D}), \zeta_f) \\ \downarrow \xi_{\chi} & & \downarrow \zeta_{\chi} \\ (g_0^*(\mathbf{E}), \xi_g) & \xrightarrow{g^*(\varphi)} & (g_0^*(\mathbf{D}), \zeta_g) \end{array}$$

Thus we have a natural transformation  $\chi^{\bullet} : f^{\bullet} \rightarrow g^{\bullet}$ .

Let  $f : (X, \mathcal{X}) \rightarrow (Y, \mathcal{Y})$ ,  $g : (X, \mathcal{X}) \rightarrow (Z, \mathcal{Z})$  and  $k : (V, \mathcal{V}) \rightarrow (X, \mathcal{X})$  be morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$  and  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (Y, \mathcal{Y}))$  an object of  $\mathcal{P}_F(\mathcal{C}, J)_{(Y, \mathcal{Y})}^{(2)}$ . We consider the following commutative diagram in  $\mathcal{P}_F(\mathcal{C}, J)$  whose outer trapezoid and lower rectangle are cartesian.

$$\begin{array}{ccccc} (E \times_Y V, \mathcal{E}^{(fk)\pi} \cap \mathcal{V}^{\pi_{fk}}) & & & & \\ \downarrow \pi_{fk} & \dashrightarrow^{id_E \times_Y k} & & \searrow^{(fk)\pi} & \\ & & (E \times_Y X, \mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi_f}) & \xrightarrow{f\pi} & (E, \mathcal{E}) \\ & & \downarrow \pi_f & & \downarrow \pi \\ (V, \mathcal{V}) & \xrightarrow{k} & (X, \mathcal{X}) & \xrightarrow{f} & (Y, \mathcal{Y}) \end{array}$$

There exists unique morphism  $id_E \times_Y k : (E \times_Y V, \mathcal{E}^{(fk)\pi} \cap \mathcal{V}^{\pi_{fk}}) \rightarrow (E \times_Y X, \mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi_f})$  that makes the above diagram commute. Since objects  $(gk)_*(fk)^*(\mathbf{E})$  and  $g_*f^*(\mathbf{E})$  of  $\mathcal{P}_F(\mathcal{C}, J)_{(Z, \mathcal{Z})}^{(2)}$  are given by

$$\begin{aligned} (gk)_*(fk)^*(\mathbf{E}) &= ((E \times_Y V, \mathcal{E}^{(fk)\pi} \cap \mathcal{V}^{\pi_{fk}}) \xrightarrow{gk\pi_{fk}} (Z, \mathcal{Z})) \\ g_*f^*(\mathbf{E}) &= ((E \times_Y X, \mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi_f}) \xrightarrow{g\pi_f} (Z, \mathcal{Z})), \end{aligned}$$

we define a morphism  $\mathbf{E}_k : (gk)_*(fk)^*(\mathbf{E}) \rightarrow g_*f^*(\mathbf{E})$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(Z, \mathcal{Z})}^{(2)}$  by  $\mathbf{E}_k = \langle id_E \times_Y k, id_Z \rangle$ . It is easy to verify the following fact.

**Proposition 8.10** For a morphism  $j : (U, \mathcal{U}) \rightarrow (V, \mathcal{V})$  in  $\mathcal{P}_F(\mathcal{C}, J)$ , a composition

$$(gkj)_*(fkj)^*(\mathbf{E}) \xrightarrow{\mathbf{E}_j} (gk)_*(fk)^*(\mathbf{E}) \xrightarrow{\mathbf{E}_k} g_*f^*(\mathbf{E})$$

coincides with  $\mathbf{E}_{kj} : (gkj)_*(fkj)^*(\mathbf{E}) \rightarrow g_*f^*(\mathbf{E})$ . Moreover,  $\mathbf{E}_k$  is natural in  $\mathbf{E}$ , that is, for a morphism  $\varphi : \mathbf{E} \rightarrow \mathbf{D}$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(Y, \mathcal{Y})}^{(2)}$ , the following diagram is commutative.

$$\begin{array}{ccc} (gk)_*(fk)^*(\mathbf{E}) & \xrightarrow{\mathbf{E}_k} & g_*f^*(\mathbf{E}) \\ \downarrow (gk)_*(fk)^*(\varphi) & & \downarrow g_*f^*(\varphi) \\ (gk)_*(fk)^*(\mathbf{D}) & \xrightarrow{\mathbf{D}_k} & g_*f^*(\mathbf{D}) \end{array}$$

Let  $f : (X, \mathcal{X}) \rightarrow (Y, \mathcal{Y})$ ,  $g : (X, \mathcal{X}) \rightarrow (Z, \mathcal{Z})$ ,  $h : (V, \mathcal{V}) \rightarrow (Z, \mathcal{Z})$  and  $i : (V, \mathcal{V}) \rightarrow (W, \mathcal{W})$  be morphisms in  $\mathcal{P}_F(\mathcal{C}, J)$ . We consider the following cartesian square in  $\mathcal{P}_F(\mathcal{C}, J)$ .

$$\begin{array}{ccc} (X \times_Z V, \mathcal{X}^{hg} \cap \mathcal{V}^{gh}) & \xrightarrow{gh} & (V, \mathcal{V}) \\ \downarrow h_g & & \downarrow h \\ (X, \mathcal{X}) & \xrightarrow{g} & (Z, \mathcal{Z}) \end{array}$$

For an object  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (Y, \mathcal{Y}))$  of  $\mathcal{P}_F(\mathcal{C}, J)_{(Y, \mathcal{Y})}^{(2)}$ , we consider the following commutative diagrams in  $\mathcal{P}_F(\mathcal{C}, J)$  whose rectangles are all cartesian.

$$\begin{array}{ccc}
(E \times_Y (X \times_Z V), \mathcal{E}^{(fh_g)\pi} \cap (\mathcal{X}^{h_g} \cap \mathcal{V}^{g_h})^{\pi_{fh_g}}) & \xrightarrow{(fh_g)\pi} & (E, \mathcal{E}) \\
\downarrow \pi_{fh_g} & & \downarrow \pi \\
(X \times_Z V, \mathcal{X}^{h_g} \cap \mathcal{V}^{g_h}) & \xrightarrow{fh_g} & (Y, \mathcal{Y}) \\
\\
((E \times_Y X) \times_Z V, (\mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi_f})^{h_{g\pi_f}} \cap \mathcal{V}^{(g\pi_f)_h}) & \xrightarrow{h_{g\pi_f}} & (E \times_Y X, \mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi_f}) \xrightarrow{f\pi} (E, \mathcal{E}) \\
\downarrow (g\pi_f)_h & & \downarrow \pi_f \\
(V, \mathcal{V}) & \xrightarrow{h} & (Z, \mathcal{Z}) \\
& & \downarrow g \\
& & (X, \mathcal{X}) \xrightarrow{f} (Y, \mathcal{Y}) \\
& & \downarrow \pi
\end{array}$$

Thus we have the following equalities.

$$\begin{aligned}
(ig_h)_*(fh_g)^*(\mathbf{E}) &= ((E \times_Y (X \times_Z V), \mathcal{E}^{(fh_g)\pi} \cap (\mathcal{X}^{h_g} \cap \mathcal{V}^{g_h})^{\pi_{fh_g}}) \xrightarrow{ig_h \pi_{fh_g}} (W, \mathcal{W})) \\
i_* h^* g_* f^*(\mathbf{E}) &= (((E \times_Y X) \times_Z V, (\mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi_f})^{h_{g\pi_f}} \cap \mathcal{V}^{(g\pi_f)_h}) \xrightarrow{i(g\pi_f)_h} (W, \mathcal{W}))
\end{aligned}$$

There exists unique morphism  $id_E \times_Y h_g : (E \times_Y (X \times_Z V), \mathcal{E}^{(fh_g)\pi} \cap (\mathcal{X}^{h_g} \cap \mathcal{V}^{g_h})^{\pi_{fh_g}}) \rightarrow (E \times_Y X, \mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi_f})$  that makes the following diagram commute.

$$\begin{array}{ccc}
(E \times_Y (X \times_Z V), \mathcal{E}^{(fh_g)\pi} \cap (\mathcal{X}^{h_g} \cap \mathcal{V}^{g_h})^{\pi_{fh_g}}) & \xrightarrow{id_E \times_Y h_g} & (E \times_Y X, \mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi_f}) \xrightarrow{f\pi} (E, \mathcal{E}) \\
\downarrow \pi_{fh_g} & \dashrightarrow & \downarrow \pi_f \\
(X \times_Z V, \mathcal{X}^{h_g} \cap \mathcal{V}^{g_h}) & \xrightarrow{h_g} & (X, \mathcal{X}) \xrightarrow{f} (Y, \mathcal{Y}) \\
& & \downarrow \pi
\end{array}$$

There exists unique morphism

$(id_E \times_Y h_g, g_h \pi_{fh_g}) : (E \times_Y (X \times_Z V), \mathcal{E}^{(fh_g)\pi} \cap (\mathcal{X}^{h_g} \cap \mathcal{V}^{g_h})^{\pi_{fh_g}}) \rightarrow ((E \times_Y X) \times_Z V, (\mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi_f})^{h_{g\pi_f}} \cap \mathcal{V}^{(g\pi_f)_h})$  that makes the following diagram commute.

$$\begin{array}{ccc}
(E \times_Y (X \times_Z V), \mathcal{E}^{(fh_g)\pi} \cap (\mathcal{X}^{h_g} \cap \mathcal{V}^{g_h})^{\pi_{fh_g}}) & \xrightarrow{id_E \times_Y h_g} & (E \times_Y X, \mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi_f}) \\
\downarrow \pi_{fh_g} & \dashrightarrow & \uparrow h_{g\pi_f} \\
(X \times_Z V, \mathcal{X}^{h_g} \cap \mathcal{V}^{g_h}) & \xrightarrow{g_h} & (V, \mathcal{V}) \\
& & \downarrow (g\pi_f)_h \\
& & ((E \times_Y X) \times_Z V, (\mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi_f})^{h_{g\pi_f}} \cap \mathcal{V}^{(g\pi_f)_h}) \xrightarrow{g\pi_f} (Z, \mathcal{Z}) \\
& & \uparrow h \\
& & (Y, \mathcal{Y})
\end{array}$$

Thus we have a morphism  $\langle (id_E \times_Y h_g, g_h \pi_{fh_g}), id_W \rangle : (ig_h)_*(fh_g)^*(\mathbf{E}) \rightarrow i_* h^* g_* f^*(\mathbf{E})$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(W, \mathcal{W})}^{(2)}$  which we denote by  $\theta_{f,g,h,i}(\mathbf{E})$  below.

**Proposition 8.11** ([11] Proposition 2.4.15)  $\theta_{f,g,h,i}(\mathbf{E}) : (ig_h)_*(fh_g)^*(\mathbf{E}) \rightarrow i_* h^* g_* f^*(\mathbf{E})$  is an isomorphism which is natural in  $\mathbf{E}$ .

*Proof.* There exists unique morphism

$$\pi_f \times_Z id_V : ((E \times_Y X) \times_Z V, (\mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi_f})^{h_{g\pi_f}} \cap \mathcal{V}^{(g\pi_f)_h}) \rightarrow (E \times_Y X, \mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi_f})$$

in  $\mathcal{P}_F(\mathcal{C}, J)$  that makes the following diagram commute.

$$\begin{array}{ccc}
((E \times_Y X) \times_Z V, (\mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi f})^{h_g \pi f} \cap \mathcal{V}^{(g\pi f)_h}) & \xrightarrow{h_g \pi f} & (E \times_Y X, \mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi f}) \\
& \searrow^{\pi_f \times_Z id_V} & \downarrow \pi_f \\
& & (X \times_Z V, \mathcal{X}^{h_g} \cap \mathcal{V}^{g_h}) \xrightarrow{h_g} (X, \mathcal{X}) \\
& \searrow_{(g\pi f)_h} & \downarrow g \\
& & (V, \mathcal{V}) \xrightarrow{h} (Z, \mathcal{Z})
\end{array}$$

Hence here exists unique morphism

$(f_\pi h_g \pi_f, \pi_f \times_Z id_V) : ((E \times_Y X) \times_Z V, (\mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi f})^{h_g \pi f} \cap \mathcal{V}^{(g\pi f)_h}) \rightarrow (E \times_Y (X \times_Z V), \mathcal{E}^{(fh_g)\pi} \cap (\mathcal{X}^{h_g} \cap \mathcal{V}^{g_h})^{\pi_f h_g})$  in  $\mathcal{P}_F(\mathcal{C}, J)$  that makes the following diagram commute.

$$\begin{array}{ccccc}
((E \times_Y X) \times_Z V, (\mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi f})^{h_g \pi f} \cap \mathcal{V}^{(g\pi f)_h}) & \xrightarrow{h_g \pi f} & (E \times_Y X, \mathcal{E}^{f\pi} \cap \mathcal{X}^{\pi f}) & \xrightarrow{f_\pi} & (E, \mathcal{E}) \\
& \searrow^{(f_\pi h_g \pi_f, \pi_f \times_Z id_V)} & \uparrow id_{E \times_Y h_g} & \searrow^{(fh_g)_\pi} & \downarrow \pi \\
& \searrow^{\pi_f \times_Z id_V} & (E \times_Y (X \times_Z V), \mathcal{E}^{(fh_g)\pi} \cap (\mathcal{X}^{h_g} \cap \mathcal{V}^{g_h})^{\pi_f h_g}) & \xrightarrow{\pi_f} & (Y, \mathcal{Y}) \\
& & \downarrow \pi_f h_g & \searrow & \uparrow f \\
& & (X \times_Z V, \mathcal{X}^{h_g} \cap \mathcal{V}^{g_h}) & \xrightarrow{h_g} & (X, \mathcal{X})
\end{array}$$

Thus we have a morphism  $\langle (f_\pi h_g \pi_f, \pi_f \times_Z id_V), id_W \rangle : i_* h^* g_* f^*(\mathbf{E}) \rightarrow (ig_h)_*(fh_g)^*(\mathbf{E})$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(W, \mathcal{W})}^{(2)}$  which is the inverse of  $\theta_{f,g,h,i}(\mathbf{E})$ . The naturality of  $\theta_{f,g,h,i}(\mathbf{E})$  in  $\mathbf{E}$  is clear from the definition of  $\theta_{f,g,h,i}(\mathbf{E})$ .  $\square$

**Remark 8.12**  $(id_{E \times_Y h_g}, g_h \pi_f h_g) : E \times_Y (X \times_Z V) \rightarrow (E \times_Y X) \times_Z V$  maps  $(u, (x, v)) \in E \times_Y (X \times_Z V)$  to  $((u, x), v) \in (E \times_Y X) \times_Z V$ .

For an object  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (G_0, \mathcal{G}_0))$  of  $\mathcal{P}_F(\mathcal{C}, J)_{(G_0, \mathcal{G}_0)}^{(2)}$  and a morphism  $\xi : \sigma^*(\mathbf{E}) \rightarrow \tau^*(\mathbf{E})$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(G_1, \mathcal{G}_1)}^{(2)}$ , we denote by  $\hat{\xi} : \tau_* \sigma^*(\mathbf{E}) \rightarrow \mathbf{E}$  the adjoint of  $\xi$  with respect to the adjunction  $\tau_* \dashv \tau^*$ .

**Proposition 8.13** ([11] Proposition 3.4.2)  $\xi$  satisfies condition (A) of (8.3) if and only if  $\hat{\xi}$  makes the following diagram commute.

$$\begin{array}{ccccc}
(\tau \text{pr}_2)_*(\sigma \text{pr}_1)^*(\mathbf{E}) & \xrightarrow{\theta_{\sigma, \tau, \sigma, \tau}(\mathbf{E})} & \tau_* \sigma^* \tau_* \sigma^*(\mathbf{E}) & \xrightarrow{\tau_* \sigma^*(\hat{\xi})} & \tau_* \sigma^*(\mathbf{E}) \\
\parallel & & & & \downarrow \hat{\xi} \\
(\tau \mu)_*(\sigma \mu)^*(\mathbf{E}) & \xrightarrow{\mathbf{E}_\mu} & \tau_* \sigma^*(\mathbf{E}) & \xrightarrow{\hat{\xi}} & \mathbf{E}
\end{array}$$

$\xi$  satisfies condition (U) of (8.3) if and only if a composition  $\mathbf{E} = (\tau \varepsilon)_*(\sigma \varepsilon)^*(\mathbf{E}) \xrightarrow{\mathbf{E}_\varepsilon} \tau_* \sigma^*(\mathbf{E}) \xrightarrow{\hat{\xi}} \mathbf{E}$  coincides with the identity morphism of  $\mathbf{E}$ .

**Remark 8.14** We consider the following diagrams whose rectangles are all cartesian.

$$\begin{array}{ccc}
E \times_{G_0}^{\sigma \mu} (G_1 \times_{G_0} G_1) & \xrightarrow{(\sigma \mu)_\pi} & E & E \times_{G_0}^{\tau} G_1 & \xrightarrow{\tau_\pi} & E \\
\downarrow \pi_{\sigma \mu} = \pi_{\sigma \text{pr}_1} & & \downarrow \pi & \downarrow \pi_\tau & & \downarrow \pi \\
G_1 \times_{G_0} G_1 & \xrightarrow{\sigma \mu = \sigma \text{pr}_1} & G_0 & G_1 & \xrightarrow{\tau} & G_0 \\
& & & & & \\
(E \times_{G_0}^{\sigma} G_1) \times_{G_0}^{\sigma} G_1 & \xrightarrow{\sigma_{\tau \pi \sigma}} & E \times_{G_0}^{\sigma} G_1 & \xrightarrow{\sigma_\pi} & E \\
\downarrow (\tau \pi_\sigma)_\sigma & & \downarrow \pi_\sigma & & \downarrow \pi \\
& & G_1 & \xrightarrow{\sigma} & G_0 \\
& & \downarrow \tau & & \\
G_1 & \xrightarrow{\sigma} & G_0 & & 
\end{array}$$

Then, we have the following equalities.

$$\begin{aligned}
\tau_* \sigma^*(\mathbf{E}) &= ((E \times_{G_0}^{\sigma} G_1, \mathcal{E}^{\sigma \pi} \cap \mathcal{G}_1^{\pi \sigma}) \xrightarrow{\tau \pi_\sigma} (G_0, \mathcal{G}_0)) \\
(\tau \text{pr}_2)_*(\sigma \text{pr}_1)^*(\mathbf{E}) &= (\tau \mu)_*(\sigma \mu)^*(\mathbf{E}) = ((E \times_{G_0}^{\sigma \mu} (G_1 \times_{G_0} G_1), \mathcal{E}^{(\sigma \mu)_\pi} \cap (\mathcal{G}_1^{\text{pr}_1} \cap \mathcal{G}_1^{\text{pr}_2})^{\pi_{\sigma \mu}}) \xrightarrow{\tau \mu \pi_{\sigma \mu}} (G_0, \mathcal{G}_0)) \\
\tau_* \sigma^* \tau_* \sigma^*(\mathbf{E}) &= ((E \times_{G_0}^{\sigma} G_1) \times_{G_0}^{\sigma} G_1, (\mathcal{E}^{\sigma \pi} \cap \mathcal{G}_1^{\pi \sigma})^{\sigma_{\tau \pi \sigma}} \cap \mathcal{G}_1^{(\tau \pi_\sigma)_\sigma}) \xrightarrow{\tau (\tau \pi_\sigma)_\sigma} (G_0, \mathcal{G}_0)
\end{aligned}$$



If we put  $\xi = \langle \xi, id_{G_1} \rangle$  and  $\hat{\xi} = \langle \hat{\xi}, id_{G_0} \rangle$  for morphisms  $\xi : (E \times_{G_0}^\sigma G_1, \mathcal{E}^{\sigma\pi} \cap \mathcal{G}_1^{\pi\sigma}) \rightarrow (E \times_{G_0}^\tau G_1, \mathcal{E}^{\tau\pi} \cap \mathcal{G}_1^{\pi\tau})$  and  $\hat{\xi} : (E \times_{G_0}^\sigma G_1, \mathcal{E}^{\sigma\pi} \cap \mathcal{G}_1^{\pi\sigma}) \rightarrow (E, \mathcal{E})$  in  $\mathcal{P}_F(\mathcal{C}, J)$ , then  $\hat{\xi}$  is a composition  $E \times_{G_0}^\sigma G_1 \xrightarrow{\hat{\xi}} E \times_{G_0}^\tau G_1 \xrightarrow{\tau\pi} E$  and  $\xi = \langle \hat{\xi}, \pi_\sigma \rangle$  holds. The diagram of (8.13) is commutative if and only if the following diagram is commutative.

$$\begin{array}{ccccc} E \times_{G_0}^{\sigma\text{pr}_1} (G_1 \times_{G_0} G_1) & \xrightarrow{(id_E \times_Y \text{pr}_1, \text{pr}_2 \pi \sigma \text{pr}_1)} & (E \times_{G_0}^\sigma G_1) \times_{G_0}^\sigma G_1 & \xrightarrow{\hat{\xi} \times_{G_0} id_{G_1}} & E \times_{G_0}^\sigma G_1 \\ \parallel & & & & \downarrow \hat{\xi} \\ E \times_{G_0}^{\sigma\mu} (G_1 \times_{G_0} G_1) & \xrightarrow{id_E \times_{G_0} \mu} & E \times_{G_0}^\sigma G_1 & \xrightarrow{\hat{\xi}} & E \end{array}$$

A composition  $\mathbf{E} = (\tau\varepsilon)_*(\sigma\varepsilon)^*(\mathbf{E}) \xrightarrow{\mathbf{E}_\varepsilon} \tau_*\sigma^*(\mathbf{E}) \xrightarrow{\hat{\xi}} \mathbf{E}$  coincides with the identity morphism of  $\mathbf{E}$  if and only if a composition  $E \xrightarrow{(id_E, \varepsilon\pi)} E \times_{G_0}^\sigma G_1 \xrightarrow{\hat{\xi}} E$  coincides with the identity morphism of  $E$ .

The next result follows from the naturality of the adjointness.

**Proposition 8.15** Let  $(\mathbf{E}, \xi)$  and  $(\mathbf{F}, \zeta)$  be representations of  $\mathbf{G}$ . A morphism  $\varphi : \mathbf{E} \rightarrow \mathbf{F}$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(G_0, \mathcal{G}_0)}^{(2)}$  makes the following left diagram commute if and only if it makes the following right diagram commute.

$$\begin{array}{ccc} \sigma^*(\mathbf{E}) & \xrightarrow{\xi} & \tau^*(\mathbf{E}) \\ \downarrow \sigma^*(\varphi) & & \downarrow \tau^*(\varphi) \\ \sigma^*(\mathbf{F}) & \xrightarrow{\zeta} & \tau^*(\mathbf{F}) \end{array} \quad \begin{array}{ccc} \tau_*\sigma^*(\mathbf{E}) & \xrightarrow{\hat{\xi}} & \mathbf{E} \\ \downarrow \tau_*\sigma^*(\varphi) & & \downarrow \varphi \\ \tau_*\sigma^*(\mathbf{F}) & \xrightarrow{\hat{\zeta}} & \mathbf{F} \end{array}$$

If a morphism  $\hat{\xi} : \tau_*\sigma^*(\mathbf{E}) \rightarrow \mathbf{E}$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(G_0, \mathcal{G}_0)}^{(2)}$  satisfies both conditions of (8.14), we also call a pair  $(\mathbf{E}, \hat{\xi} : \tau_*\sigma^*(\mathbf{E}) \rightarrow \mathbf{E})$  a representation of  $\mathbf{G}$  on  $\mathbf{E}$ .

**Example 8.16** For an object  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (B, \mathcal{B}))$  of  $\text{Epi}_c(\mathcal{P}_F(\mathcal{C}, J))$ , we consider the groupoid  $\mathbf{G}(\mathbf{E})$  associated with  $\mathbf{E}$ . We define a morphism  $\hat{\xi}_{\mathbf{E}} : \tau_{\mathbf{E}*}\sigma_{\mathbf{E}}^*(\mathbf{E}) \rightarrow \mathbf{E}$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(B, \mathcal{B})}^{(2)}$  by  $\hat{\xi}_{\mathbf{E}} = \langle \hat{\xi}_{\mathbf{E}}, id_B \rangle$ . It follows from (5.5) and (8.14) that  $(\mathbf{E}, \hat{\xi}_{\mathbf{E}})$  is a representation of  $\mathbf{G}(\mathbf{E})$  on  $\mathbf{E}$ . We call  $(\mathbf{E}, \hat{\xi}_{\mathbf{E}})$  the canonical representation of  $\mathbf{E}$ .

Let  $\mathbf{G} = ((G_0, \mathcal{G}_0), (G_1, \mathcal{G}_1); \sigma, \tau, \varepsilon, \mu, \nu)$  and  $\mathbf{H} = ((H_0, \mathcal{H}_0), (H_1, \mathcal{H}_1); \sigma', \tau', \varepsilon', \mu', \nu')$  be a groupoids in  $\mathcal{P}_F(\mathcal{C}, J)$  and  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (G_0, \mathcal{G}_0))$  an object of  $\mathcal{P}_F(\mathcal{C}, J)_{(G_0, \mathcal{G}_0)}^{(2)}$ . For a morphism  $f = (f_0, f_1) : \mathbf{H} \rightarrow \mathbf{G}$  of groupoids in  $\mathcal{P}_F(\mathcal{C}, J)$ , we consider the following diagram in  $\mathcal{P}_F(\mathcal{C}, J)$  whose rectangles are cartesian.

$$\begin{array}{ccccc} ((E \times_{G_0} H_0) \times_{H_0}^{\sigma'} H_1, (\mathcal{E}^{(f_0)\pi} \cap \mathcal{H}_0^{\pi f_0})^{\sigma'_{\pi f_0}} \cap \mathcal{H}_1^{(\pi f_0)\sigma'}) & \xrightarrow{\sigma'_{\pi f_0}} & (E \times_{G_0} H_0, \mathcal{E}^{(f_0)\pi} \cap \mathcal{H}_0^{\pi f_0}) & \xrightarrow{(f_0)\pi} & (E, \mathcal{E}) \\ \downarrow (\pi f_0)_{\sigma'} & & \downarrow \pi f_0 & & \downarrow \pi \\ (H_1, \mathcal{H}_1) & \xrightarrow{\sigma'} & (H_0, \mathcal{H}_0) & \xrightarrow{f_0} & (G_0, \mathcal{G}_0) \end{array}$$

There exists unique morphism

$$(f_0)\pi \times_{f_0} f_1 : ((E \times_{G_0} H_0) \times_{H_0}^{\sigma'} H_1, (\mathcal{E}^{(f_0)\pi} \cap \mathcal{H}_0^{\pi f_0})^{\sigma'_{\pi f_0}} \cap \mathcal{H}_1^{(\pi f_0)\sigma'}) \rightarrow (E \times_{G_0}^\sigma G_1, \mathcal{E}^{\text{pr}_E^\sigma} \cap \mathcal{G}_1^{\text{pr}_{G_1}^\sigma})$$

in  $\mathcal{P}_F(\mathcal{C}, J)$  that makes the following diagram commute.

$$\begin{array}{ccccc} (E \times_{G_0} H_0) \times_{H_0}^{\sigma'} H_1 & \xrightarrow{\sigma'_{\pi f_0}} & E \times_{G_0} H_0 & & \\ \downarrow (\pi f_0)_{\sigma'} & \searrow (f_0)\pi \times_{f_0} f_1 & \downarrow \pi f_0 & & \\ & E \times_{G_0}^\sigma G_1 & \xrightarrow{\text{pr}_E^\sigma} & E & \downarrow \pi \\ & \downarrow \text{pr}_{G_1}^\sigma & \downarrow \sigma' & \downarrow \pi & \downarrow f_0 \\ H_1 & \xrightarrow{f_1} & G_1 & \xrightarrow{\sigma} & G_0 \end{array}$$

Consider a representation  $(\mathbf{E}, \hat{\xi})$  of  $\mathbf{G}$  on  $\mathbf{E}$  and put  $\hat{\xi} = \langle \hat{\xi}, id_{G_0} \rangle$ . There exists unique morphism

$$\hat{\zeta} : ((E \times_{G_0} H_0) \times_{H_0}^{\sigma'} H_1, (\mathcal{E}^{(f_0)\pi} \cap \mathcal{H}_0^{\pi f_0})^{\sigma'_{f_0}} \cap \mathcal{H}_1^{(\pi f_0)\sigma'}) \rightarrow (E \times_{G_0} H_0, \mathcal{E}^{(f_0)\pi} \cap \mathcal{H}_0^{\pi f_0})$$

in  $\mathcal{P}_F(\mathcal{C}, J)$  that makes the following diagram commute.

$$\begin{array}{ccccc} (E \times_{G_0} H_0) \times_{H_0}^{\sigma'} H_1 & \xrightarrow{(f_0)\pi \times f_0 f_1} & E \times_{G_0}^{\sigma} G_1 & & \\ \downarrow (\pi f_0)\sigma' & \searrow \hat{\zeta} & \downarrow \text{pr}_{G_1}^{\sigma} & \xrightarrow{\hat{\xi}} & E \\ H_1 & \xrightarrow{f_1} & G_1 & \xrightarrow{\tau} & E \times_{G_0} H_0 \xrightarrow{(f_0)\pi} E \\ & \searrow \tau' & \downarrow \pi_{f_0} & \searrow & \downarrow \pi \\ & & H_0 & \xrightarrow{f_0} & G_0 \end{array}$$

Define a morphism  $\hat{\zeta} : \tau'_* \sigma'^*(f_0^*(\mathbf{E})) \rightarrow f_0^*(\mathbf{E})$  by  $\hat{\zeta} = \langle \hat{\zeta}, id_{H_0} \rangle$ .

**Proposition 8.17**  $(f_0^*(\mathbf{E}), \hat{\zeta})$  coincides with the restriction of the representation  $(\mathbf{E}, \hat{\xi})$  of  $\mathbf{G}$  on  $\mathbf{E}$  along  $f$ .

*Proof.* Let  $(f_0^*(\mathbf{E}), \xi_f)$  be the restriction of  $(\mathbf{E}, \xi)$  along  $f : \mathbf{H} \rightarrow \mathbf{G}$  and put  $\xi_f = \langle \xi_f, id_{H_0} \rangle$ . We denote by  $\hat{\xi}_f = \langle \hat{\xi}_f, id_{H_0} \rangle : \tau'_* \sigma'^*(f_0^*(\mathbf{E})) \rightarrow \mathbf{E}$  the adjoint of  $\xi_f$  with respect to the adjunction  $\tau'_* \dashv \tau'^*$ . It follows from (8.8) that  $\hat{\xi}_f$  maps  $((u, x), y) \in (E \times_{G_0} H_0) \times_{H_0}^{\sigma'} H_1$  to  $(\hat{\xi}(u, f_1(y)), \tau'(y)) \in E \times_{G_0} H_0$ . On the other hand,  $\hat{\zeta}$  also maps  $((u, x), y) \in (E \times_{G_0} H_0) \times_{H_0}^{\sigma'} H_1$  to  $(\hat{\xi}(u, f_1(y)), \tau'(y)) \in E \times_{G_0} H_0$  by the definition of  $\hat{\zeta}$ . Thus we have  $\hat{\xi}_f = \hat{\zeta}$ .  $\square$

**Proposition 8.18** Let  $\mathbf{E} = ((E, \mathcal{E}) \xrightarrow{\pi} (G_0, \mathcal{G}_0))$  be an object  $\text{Epi}_{\mathcal{C}}(\mathcal{P}_F(\mathcal{C}, J))$  and  $(\mathbf{E}, \hat{\xi} : \tau_* \sigma^*(\mathbf{E}) \rightarrow \mathbf{E})$  a representation of  $\mathbf{G} = ((G_0, \mathcal{G}_0), (G_1, \mathcal{G}_1); \sigma, \tau, \varepsilon, \mu, \iota)$  on  $\mathbf{E}$ . There exists a morphism  $\mathbf{f} : \mathbf{G} \rightarrow \mathbf{G}(\mathbf{E})$  of groupoids in  $\mathcal{P}_F(\mathcal{C}, J)$  such that  $(\mathbf{E}, \hat{\xi})$  coincides with the restriction of the canonical representation  $(\mathbf{E}, \hat{\xi}_{\mathbf{E}})$  along  $\mathbf{f}$ . Moreover, if  $\mathbf{g} = (id_{G_0}, g_1) : \mathbf{G} \rightarrow \mathbf{G}(\mathbf{E})$  is a morphism of groupoids in  $\mathcal{P}_F(\mathcal{C}, J)$  such that  $(\mathbf{E}, \hat{\xi})$  coincides with the restriction of the canonical representation  $(\mathbf{E}, \hat{\xi}_{\mathbf{E}})$  along  $\mathbf{g}$ , then  $\mathbf{g} = \mathbf{f}$  holds.

*Proof.* We put  $\hat{\xi} = \langle \hat{\xi}, id_{G_0} \rangle$ . Here,  $\hat{\xi}$  is a morphism in  $\mathcal{P}_F(\mathcal{C}, J)$  from  $(E \times_{G_0}^{\sigma} G_1, \mathcal{E}^{\sigma\pi} \cap \mathcal{G}_1^{\pi\sigma})$  to  $(E, \mathcal{E})$ . By the commutativity of the following diagram,  $\hat{\xi}(e, g) \in \pi^{-1}(\tau(g))$  holds for  $g \in G_1$  and  $e \in \pi^{-1}(\sigma(g))$ .

$$\begin{array}{ccc} E \times_{G_0}^{\sigma} G_1 & \xrightarrow{\hat{\xi}} & E \\ \downarrow \pi_{\sigma} & & \downarrow \pi \\ G_1 & \xrightarrow{\tau} & G_0 \end{array}$$

For  $g \in G_1$ ,  $U \in \text{Ob } \mathcal{C}$ ,  $\lambda \in F_{\pi^{-1}(\sigma(g))}(U) \cap \mathcal{E}^{i_{\pi^{-1}(\sigma(g))}}$ , we denote by  $c_g : F(U) \rightarrow G_1$  the constant map to  $g$  and define a map  $\lambda_g : F(U) \rightarrow E \times_{G_0}^{\sigma} G_1$  by  $\lambda_g = (i_{\pi^{-1}(\sigma(g))} \lambda, c_g)$ . Since  $\sigma_{\pi} \lambda_g = i_{\pi^{-1}(\sigma(g))} \lambda = (F_{i_{\pi^{-1}(\sigma(g))}})_U(\lambda) \in \mathcal{E}$  and  $\pi_{\sigma} \lambda = c_g \in \mathcal{G}_1$ ,  $\lambda_g$  belongs to  $\mathcal{E}^{\sigma\pi} \cap \mathcal{G}_1^{\pi\sigma}$ . We define a map  $\varphi_g : \pi^{-1}(\sigma(g)) \rightarrow \pi^{-1}(\tau(g))$  by  $\varphi_g(e) = \hat{\xi}(e, g)$ . If  $\lambda \in F_{\pi^{-1}(\sigma(g))}(U) \cap \mathcal{E}^{i_{\pi^{-1}(\sigma(g))}}$ , then we have  $(F_{i_{\pi^{-1}(\sigma(g))}})_{\varphi_g}(\lambda) = \hat{\xi} \lambda_g = (F_{\hat{\xi}})_U(\lambda_g) \in \mathcal{E}$ , which shows that  $\varphi_g$  defines a morphism  $\varphi_g : (\pi^{-1}(\sigma(g)), \mathcal{E}^{i_{\pi^{-1}(\sigma(g))}}) \rightarrow (\pi^{-1}(\tau(g)), \mathcal{E}^{i_{\pi^{-1}(\tau(g))}})$ . For  $(g, h) \in G_1 \times_{G_0}^{\sigma} G_1$ , it follows from the commutativity of the diagram of (8.14) that we have  $\varphi_h \varphi_g(e) = \hat{\xi}(\hat{\xi}(e, g), h) = \hat{\xi}(e, \mu(g, h)) = \varphi_{\mu(g, h)}(e)$ . This implies that  $\varphi_{\iota(g)} : \pi^{-1}(\tau(g)) \rightarrow \pi^{-1}(\sigma(g))$  is the inverse of  $\varphi_g$ , hence  $\varphi_g \in G_1(\mathbf{E})(\sigma(g), \tau(g)) \subset G_1(\mathbf{E})$ .

We define a map  $f_1 : G_1 \rightarrow G_1(\mathbf{E})$  by  $f_1(g) = \varphi_g$ . Then,  $f_1$  makes the following diagrams commute.

$$\begin{array}{ccc} E \times_{G_0}^{\sigma} G_1 \xrightarrow{\hat{\xi}} E & & G_0 \xleftarrow{\sigma} G_1 \xrightarrow{\tau} G_0 \\ \downarrow id_E \times_{G_0}^{\sigma} f_1 & \nearrow \hat{\xi}_{\mathbf{E}} & \swarrow \sigma_{\mathbf{E}} \quad \downarrow f_1 \quad \searrow \tau_{\mathbf{E}} \\ E \times_{G_0}^{\sigma} G_1(\mathbf{E}) & & G_1(\mathbf{E}) \end{array} \quad \begin{array}{ccc} G_1 \times_{G_0} G_1 \xrightarrow{\mu} G_1 \xleftarrow{\varepsilon} G_0 & & \\ \downarrow f_1 \times_{G_0} f_1 & \downarrow f_1 & \swarrow \varepsilon_{\mathbf{E}} \\ G_1(\mathbf{E}) \times_{G_0} G_1(\mathbf{E}) \xrightarrow{\mu_{\mathbf{E}}} G_1(\mathbf{E}) & & \end{array}$$

For  $U \in \text{Ob } \mathcal{C}$  and  $\gamma \in F_{G_1}(U) \cap \mathcal{G}_1$ , we verify  $(F_{f_1})(\gamma) = f_1 \gamma \in F_{G_1(\mathbf{E})}(U) \cap \mathcal{G}_{\mathbf{E}}$  below. It follows from the commutativity of the above middle diagram that the following compositions belong to  $\mathcal{G}_0 \cap F_{G_0}(U)$ .

$$F(U) \xrightarrow{f_1 \gamma} G_1(\mathbf{E}) \xrightarrow{\sigma_{\mathbf{E}}} G_0, \quad F(U) \xrightarrow{f_1 \gamma} G_1(\mathbf{E}) \xrightarrow{\tau_{\mathbf{E}}} G_0$$

Assume that  $V, W \in \text{Ob } \mathcal{C}$ ,  $j \in \mathcal{C}(W, U)$ ,  $k \in \mathcal{C}(W, V)$  and  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfy  $\pi\lambda F(k) = \sigma_E f_1 \gamma F(j)$ . Then,  $\pi\lambda F(k) = \sigma\gamma F(j)$  holds by the commutativity of the above middle diagram, there exists a morphism  $(\lambda F(k), \gamma F(j)) : F(W) \rightarrow E \times_{G_0}^\sigma G_1$  which makes the following diagram commute. It follows that a composition  $F(W) \xrightarrow{(\lambda F(k), f_1 \gamma F(j))} E \times_{G_0}^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$  belongs to  $\mathcal{E} \cap F_E(W)$ .

$$\begin{array}{ccc} & & E \times_{G_0}^\sigma G_1 \xrightarrow{\hat{\xi}} E \\ & \nearrow^{(\lambda F(k), \gamma F(j))} & \downarrow id_E \times_{G_0}^\sigma f_1 \\ F(W) & \xrightarrow{(\lambda F(k), f_1 \gamma F(j))} & E \times_{G_0}^\sigma G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E \end{array}$$

Assume that  $V, W \in \text{Ob } \mathcal{C}$ ,  $j \in \mathcal{C}(W, U)$ ,  $k \in \mathcal{C}(W, V)$  and  $\lambda \in \mathcal{E} \cap F_E(V)$  satisfy  $\pi\lambda F(k) = \tau_E f_1 \gamma F(j)$ . Then,  $\pi\lambda F(k) = \sigma\iota\gamma F(j)$  holds by the commutativity of the above middle diagram, there exists a morphism  $(\lambda F(k), \iota\gamma F(j)) : F(W) \rightarrow E \times_{G_0}^\sigma G_1$  which makes the following diagram commute. We note that  $f_1\iota = \iota_E f_1$  holds. It follows that a composition  $F(W) \xrightarrow{(\lambda F(k), \iota_E f_1 \gamma F(j))} E \times_{G_0}^{\sigma_E} G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E$  belongs to  $\mathcal{E} \cap F_E(W)$ .

$$\begin{array}{ccc} & & E \times_{G_0}^\sigma G_1 \xrightarrow{\hat{\xi}} E \\ & \nearrow^{(\lambda F(k), \iota\gamma F(j))} & \downarrow id_E \times_{G_0}^\sigma f_1 \\ F(W) & \xrightarrow{(\lambda F(k), f_1 \iota \gamma F(j))} & E \times_{G_0}^\sigma G_1(\mathbf{E}) \xrightarrow{\hat{\xi}_E} E \end{array}$$

Thus we conclude that  $f_1\gamma$  belongs to  $F_{G_1(\mathbf{E})}(U) \cap \mathcal{G}_E$  by the definition of  $\mathcal{G}_E$  and that we have a morphism  $\mathbf{f} = (id_{G_0}, f_1) : \mathbf{G} \rightarrow \mathbf{G}(\mathbf{E})$  of groupoids in  $\mathcal{P}_F(\mathcal{C}, J)$ .

We define  $\xi : E \times_{G_0}^\sigma G_1 \rightarrow E \times_{G_0}^\tau G_1$  and  $\xi_E : E \times_{G_0}^{\sigma_E} G_1(\mathbf{E}) \rightarrow E \times_{G_0}^{\tau_E} G_1(\mathbf{E})$  by  $\xi = (\hat{\xi}, \pi_\sigma)$  and  $\xi_E = (\hat{\xi}_E, \pi_{\sigma_E})$ , respectively. Consider a morphism  $\xi_E : \sigma_E^*(\mathbf{E}) \rightarrow \tau_E^*(\mathbf{E})$  in  $\mathcal{P}_F(\mathcal{C}, J)_{(G_1(\mathbf{E}), \mathcal{G}_E)}^{(2)}$  given by  $\xi_E = \langle \xi_E, id_{G_1(\mathbf{E})} \rangle$ . Note that  $(\xi_E)_\mathbf{f} = (\xi_E)_{f_1} : \sigma^*(\mathbf{E}) = (\sigma_E f_1)^*(\mathbf{E}) \rightarrow (\tau_E f_1)^*(\mathbf{E}) = \tau^*(\mathbf{E})$  and put  $(\xi_E)_\mathbf{f} = \langle (\xi_E)_\mathbf{f}, id_{G_1} \rangle$ . We consider the following diagrams whose rectangles are all cartesian.

$$\begin{array}{ccccc} E \times_{G_0}^\sigma G_1 & \xrightarrow{\sigma_\pi} & E \times_{G_0}^\tau G_1 & \xrightarrow{\pi} & E \\ \downarrow (id_E \times_{G_0} f_1, \pi_\sigma) & \searrow id_E \times_{G_0} f_1 & \downarrow id_E \times_{G_0} f_1 & \searrow (\sigma_E)_\pi & \downarrow \pi \\ (E \times_{G_0}^{\sigma_E} G_1(\mathbf{E})) \times_{G_1(\mathbf{E})} G_1 & \xrightarrow{(f_1)_{\pi_{\sigma_E}}} & E \times_{G_0}^{\sigma_E} G_1(\mathbf{E}) & \xrightarrow{(\sigma_E)_\pi} & E \\ \downarrow (\pi_{\sigma_E})_{f_1} & \searrow f_1 & \downarrow \pi_{\sigma_E} & \searrow \sigma_E & \downarrow \pi \\ G_1 & \xrightarrow{f_1} & G_1(\mathbf{E}) & \xrightarrow{\sigma_E} & G_0 \\ \\ (E \times_{G_0}^{\tau_E} G_1(\mathbf{E})) \times_{G_1(\mathbf{E})} G_1 & \xrightarrow{(f_1)_{\pi_{\tau_E}}} & E \times_{G_0}^{\tau_E} G_1(\mathbf{E}) & \xrightarrow{(\tau_E)_\pi} & E \\ \downarrow (\pi_{\tau_E})_{f_1} & \searrow f_1 & \downarrow \pi_{\tau_E} & \searrow \tau_E & \downarrow \pi \\ G_1 & \xrightarrow{f_1} & G_1(\mathbf{E}) & \xrightarrow{\tau_E} & G_0 \end{array}$$

Then,  $(\xi_E)_\mathbf{f}$  is the following composition.

$$\begin{aligned} E \times_{G_0}^\sigma G_1 & \xrightarrow{(id_E \times_{G_0} f_1, \pi_\sigma)} (E \times_{G_0}^{\sigma_E} G_1(\mathbf{E})) \times_{G_1(\mathbf{E})} G_1 \xrightarrow{\xi_E \times_{G_1(\mathbf{E})} id_{G_1}} (E \times_{G_0}^{\tau_E} G_1(\mathbf{E})) \times_{G_1(\mathbf{E})} G_1 \\ & \xrightarrow{((\tau_E)_\pi (f_1)_{\pi_{\tau_E}}, (\pi_{\tau_E})_{f_1})} E \times_{G_0}^\tau G_1 \end{aligned}$$

Since  $\hat{\xi}_E(id_E \times_{G_0} f_1) = \hat{\xi}$ , we have the following equalities by the commutativity of the above diagrams.

$$\begin{aligned} \tau_\pi(\xi_E)_\mathbf{f} &= \tau_\pi((\tau_E)_\pi (f_1)_{\pi_{\tau_E}}, (\pi_{\tau_E})_{f_1})(\xi_E \times_{G_1(\mathbf{E})} id_{G_1})(id_E \times_{G_0} f_1, \pi_\sigma) = (\tau_E)_\pi (f_1)_{\pi_{\tau_E}}(\xi_E(id_E \times_{G_0} f_1), \pi_\sigma) \\ &= (\tau_E)_\pi (f_1)_{\pi_{\tau_E}}((\hat{\xi}_E, \pi_{\sigma_E})(id_E \times_{G_0} f_1), \pi_\sigma) = (\tau_E)_\pi (f_1)_{\pi_{\tau_E}}((\hat{\xi}_E(id_E \times_{G_0} f_1), \pi_{\sigma_E}(id_E \times_{G_0} f_1)), \pi_\sigma) \\ &= (\tau_E)_\pi (f_1)_{\pi_{\tau_E}}((\hat{\xi}, f_1 \pi_\sigma), \pi_\sigma) = (\tau_E)_\pi(\hat{\xi}, f_1 \pi_\sigma) = \hat{\xi} = \tau_\pi \xi \\ \pi_\tau(\xi_E)_\mathbf{f} &= \pi_\tau((\tau_E)_\pi (f_1)_{\pi_{\tau_E}}, (\pi_{\tau_E})_{f_1})(\xi_E \times_{G_1(\mathbf{E})} id_{G_1})(id_E \times_{G_0} f_1, \pi_\sigma) = (\pi_{\tau_E})_{f_1}(\xi_E(id_E \times_{G_0} f_1), \pi_\sigma) \\ &= \pi_\sigma = \pi_\tau \xi \end{aligned}$$

Hence we have  $(\xi_E)_f = \xi$ , equivalently  $(\xi_E)_f = \langle \xi, id_{G_1} \rangle$ , which shows that  $(E, \hat{\xi})$  coincides with the restriction of the canonical representation  $(E, \hat{\xi}_E)$  along  $f$ .

For a morphism  $g = (id_{G_0}, g_1) : G \rightarrow G(E)$  of groupoids in  $\mathcal{P}_F(\mathcal{C}, J)$ , we consider the restriction  $(E, (\xi_E)_g)$  of the canonical representation  $(E, \xi_E)$  along  $g$ . We denote by  $(\hat{\xi}_E)_g = \langle (\hat{\xi}_E)_g, id_{G_0} \rangle : \tau_* \sigma^*(E) \rightarrow E$  the adjoint of  $(\xi_E)_G = \langle (\xi_E)_G, id_{G_1} \rangle : \sigma^*(E) \rightarrow \tau^*(E)$  with respect to the adjunction  $\tau_* \dashv \tau^*$ . It follows from (8.8) that  $(\hat{\xi}_E)_g$  maps  $(e, u) \in E \times_{G_0}^\sigma G_1$  to  $\hat{\xi}_E(e, g_1(u)) = g_1(u)(e) \in E$ . Assume that  $(E, (\xi_E)_g)$  coincides with  $(E, \hat{\xi})$ . Since  $(E, \hat{\xi})$  coincides with the restriction  $(\hat{\xi}_E)_f = \langle (\hat{\xi}_E)_f, id_{G_0} \rangle$  of the canonical representation of  $E$  along  $f$  and  $(\hat{\xi}_E)_f$  maps  $(e, u) \in E \times_{G_0}^\sigma G_1$  to  $\hat{\xi}_E(e, f_1(u)) = f_1(u)(e) \in E$ , it follows that  $g_1(u)(e) = f_1(u)(e)$  holds for any  $e \in \pi^{-1}(\sigma(u))$  and  $u \in G_1$ . Thus  $g_1(u) = f_1(u)$  holds for any  $u \in G_1$ , which shows  $g_1 = f_1$ , equivalently  $g = f$ .  $\square$

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