A theory of plots

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Contents of this slide

§1. Recollections on Grothendieck site

We denote by $\mathcal{S}et$ the category of sets and maps.

For a category $\mathscr{C}$, we call a functor $\mathscr{C}^{op} \to \mathcal{S}et$ presheaf on $\mathscr{C}$.

For an object X of $\mathscr{C}$, let $h_X : \mathscr{C}^{op} \to \mathcal{S}et$ be a functor defined by $h_X(U) = \mathscr{C}(U, X)$ for an object U of $\mathscr{C}$ and

$$h_X(f : U \to V) = (f^* : \mathscr{C}(V, X) \to \mathscr{C}(U, X))$$

for a morphism $f : U \to V$ in $\mathscr{C}$.

Here, $\mathscr{C}(U, X)$ denotes the set of morphisms in $\mathscr{C}$ from U to X.

We call $h_X : \mathscr{C}^{op} \to \mathcal{S}et$ the presheaf on $\mathscr{C}$ represented by X.

For a morphism $\varphi : X \to Y$ in $\mathscr{C}$, let $h_\varphi : h_X \to h_Y$ be a natural transformation defined by $(h_\varphi)_U = \varphi_* : \mathscr{C}(U, X) \to \mathscr{C}(U, Y)$.

Definition 1.1

Let $\mathscr{C}$ be a category.

(1) A full subcategory $\mathscr{D}$ of $\mathscr{C}$ is called a sieve if it satisfies the following condition.

If $U \in \mathrm{Ob}\mathscr{C}$ and $\mathscr{C}(U, V) \neq \emptyset$ for some $V \in \mathrm{Ob}\mathscr{D}$, then $U \in \mathrm{Ob}\mathscr{D}$.

(2) For $X \in \mathrm{Ob}\mathscr{C}$, sieves of $\mathscr{C}/X$ is called a sieve on X.

For set valued functors $F, G : \mathscr{C} \to \mathscr{S}et$, if $F(U)$ is a subset of $G(U)$ for any object U of $\mathscr{C}$ and the inclusion map $i_U : F(U) \to G(U)$ defines a natural transformation $i : F \to G$, we call F a subfunctor of G. If F is a subfunctor of G, we denote this by $F \subset G$.

Remark 1.2
For a sieve R is on X, $\mathrm{Ob}\,R$ is a set of morphisms in $\mathscr{C}$ whose targets are X.
If we put $R(Y)=\{f:Y\to X \mid f\in \mathrm{Ob}\,R\}$ for $Y\in \mathrm{Ob}\mathscr{C}$, then R is a subfunctor of the presheaf $h_X:\mathscr{C}^{op}\to \mathscr{S}et$ represented by X.
Namely, $R\mapsto R(\text{-})$ gives a bijective correspondence between the set of sieves on X and the set of subfunctors of h_X.
Thus we identify a sieve on X with a subfunctor of h_X.

For a morphism f in a category $\mathscr{C}$, let us denote by $\mathrm{dom}(f)$ the source of f and $\mathrm{codom}(f)$ the target of f.

Definition 1.3

Let $\mathscr{C}$ be a category. For each $X \in \mathrm{Ob}\mathscr{C}$, a set $J(X)$ of sieves on X is given. If the following conditions are satisfied, a correspondence $J : X \mapsto J(X)$ is called a (Grothendieck) topology on $\mathscr{C}$. A category $\mathscr{C}$ with a topology J is called a site which we denote by $(\mathscr{C}, J)$.

(T1) For any $X \in \mathrm{Ob}\mathscr{C}$, $h_X \in J(X)$.

(T2) For any $X \in \mathrm{Ob}\mathscr{C}$, $R \in J(X)$ and morphism $f : Y \to X$ of $\mathscr{C}$, a subfunctor $h_f^{-1}(R)$ of h_Y defined below belongs to $J(Y)$.

$$h_f^{-1}(R)(Z) = \{ g : Z \to Y \mid fg \in R(Z) \}$$

(T3) A sieve S on X belongs to $J(X)$, if there exists $R \in J(X)$ such that $h_f^{-1}(S) \in J(\mathrm{dom}(f))$ for any $f \in \mathrm{Ob}\, R$.

Proposition 1.4

Consider the following conditions on J.

(T3') A sieve S on X belongs to $J(X)$, if there exists $R \in J(X)$ such that S is a subfunctor of R and $h_f^{-1}(S) \in J(\mathrm{dom}(f))$ for $f \in \mathrm{Ob}\, R$.

(T4) A sieve S on X belongs to $J(X)$ if it has a subfunctor which belongs to $J(X)$.

(T5) Suppose that $R \in J(X)$ and that $R_f \in J(\mathrm{dom}(f))$ is given for each $f \in \mathrm{Ob}\, R$. Then, $\{fg \mid f \in \mathrm{Ob}\, R, g \in \mathrm{Ob}\, R_f\} \in J(X)$.

(1) (T2) and (T3) imply (T4). (T1) and (T3) imply (T5).

(2) (T4) and (T5) imply (T3). (T3') and (T4) imply (T3).

For subfunctors G and H of a presheaf F on $\mathscr{C}$, let us denote by $G \cap H$ a subfunctor of F defined by $(G \cap H)(X) = G(X) \cap H(X)$.

Proposition 1.5

If $R, S \in J(X)$, then $R \cap S \in J(X)$.

Definition 1.6

Let J, J' be topologies on $\mathscr{C}$. If $J(X) \subset J'(X)$ for any $X \in \mathrm{Ob}\,\mathscr{C}$, J' is said to be finer than J, or J be coarser than J'.

Hence the set of all topologies on $\mathscr{C}$ is an ordered set.

Let $(J_i)_{i\in I}$ be a family of topologies on $\mathscr{C}$. We set $J(X)=\bigcap_{i\in I} J_i(X)$ for each $X\in \mathrm{Ob}\,\mathscr{C}$, then J is a topology on $\mathscr{C}$ and $J=\inf\{J_i \mid i\in I\}$. If T is the set of all topologies on $\mathscr{C}$ that are finer than every J_i, then $\sup\{J_i \mid i\in I\}=\inf T$.

A topology J on $\mathscr{C}$ given by $J(X)=$(the set of all sieves on X) is the finest topology on $\mathscr{C}$. On the other hand, a topology J given by $J(X)=\{h_X\}$ is the coarsest topology.

Proposition 1.7

For a set R of morphisms in $\mathscr{C}$ with target X, we put

$$\bar{R} = \bigcup_{f \in R} \mathrm{Im}(h_f : h_{\mathrm{dom}(f)} \to h_X).$$

In other words, $\bar{R}$ is the set of all morphisms of the form fg such that $f \in R$, $g \in \mathrm{Mor}\,\mathscr{C}$ and $\mathrm{codom}(g) = \mathrm{dom}(f)$.

Then, $\bar{R}$ is the smallest sieve containing R.

Definition 1.8

Let $(\mathscr{C}, J)$ be a site.

(1) For a set R of morphisms in $\mathscr{C}$ with target X, we call $\bar{R}$ the sieve generated by R.

(2) A family of morphisms $(f_i : X_i \to X)_{i \in I}$ is called a covering of X if the sieve generated by f_i's belongs to $J(X)$.

Let $\mathscr{C}$ be a category. Suppose that, for each object X, a set $P(X)$ of families of morphisms of $\mathscr{C}$ with target X is given. Then, there is the coarsest topology J_P on $\mathscr{C}$ such that for each object X, every element of $P(X)$ is a covering. In fact, J_P is the intersection of all topologies satisfying the above condition.

We call J_P the topology generated by P.

Definition 1.9

Let $\mathscr{C}$ be a category. For each $X \in \mathrm{Ob}\mathscr{C}$, a set $P(X)$ of families of morphisms of $\mathscr{C}$ with target X is given. If the following conditions (P1), (P2) and (P3) are satisfied, the correspondence $P: X \mapsto P(X)$ is called a basis for a (Grothendieck) topology on $\mathscr{C}$.

(P1) For any $X \in \mathrm{Ob}\mathscr{C}$, $\{id_X\} \in P(X)$.

(P2) If $(f_i: X_i \to X)_{i \in I} \in P(X)$, then for any morphism $f: Y \to X$ in $\mathscr{C}$, there exists $(g_j: Y_j \to Y)_{j \in I'} \in P(Y)$ such that for each $j \in I'$, fg_j factors through some f_i.

(P3) If $(f_i: X_i \to X)_{i \in I} \in P(X)$ and $(g_{ij}: X_{ij} \to X_i)_{j \in I_i} \in P(X_i)$ for each $i \in I$ are given, then $(f_i g_{ij}: X_{ij} \to X)_{(i,j) \in K} \in P(X)$, where $K = \{(i,j) \mid i \in I, j \in I_i\}$.

Proposition 1.10

Let $\mathscr{C}$ be a category and J a topology on $\mathscr{C}$. For each $X \in \operatorname{Ob}\mathscr{C}$, let $P(X)$ be the set of all coverings of X. Then P is a basis for a topology.

Proposition 1.11

(1) Let P be a basis for a topology on $\mathscr{C}$ and J_P the topology generated by P. Then, we have

$$J_P(X) = \{R \subset h_X \mid R \supset S \text{ for some } S \in P(X)\}.$$

(2) For a topology J on $\mathscr{C}$, let P be as in (1.10). Then the topology generated by P coincides with J.

We denote by $\hat{\mathscr{C}}$ the category of presheaves on $\mathscr{C}$ below.

Proposition 1.12

Let $S=(f_i:X_i\to X)_{i\in I}$ be a family of morphisms in $\mathscr{C}$.

For each $i\in I$, we regard f_i as an element of $\bar{S}(X_i)$.

For a presheaf F on $\mathscr{C}$, define a map $\Phi:\hat{\mathscr{C}}(\bar{S},F)\to\prod_{i\in I}F(X_i)$ by $\Phi(\varphi)=(\varphi_{X_i}(f_i))_{i\in I}$. Then, Φ is injective and its image consists of families $(x_i)_{i\in I}$ which satisfy the following condition for any $i,j\in I$ and any object Z of $\mathscr{C}$.

"If $f_iu=f_jv$ for $u:Z\to X_i$ and $v:Z\to X_j$, then $F(u)(x_i)=F(v)(x_j)$."

§2. Plots on a set

Definition 2.1
Let $\mathscr{C}$ be a category and $F:\mathscr{C}\to\mathscr{S}et$ a functor.
For a set X, we define a presheaf F_X on $\mathscr{C}$ to be a composition

$$\mathscr{C}^{op}\xrightarrow{F^{op}}\mathscr{S}et^{op}\xrightarrow{h_X}\mathscr{S}et.$$

Here we denote by $F^{op}:\mathscr{C}^{op}\to\mathscr{S}et^{op}$ a functor defined by
$F^{op}(U)=F(U)$ for $U\in\mathrm{Ob}\,\mathscr{C}$ and $F^{op}(f)=F(f)$ for $f\in\mathrm{Mor}\,\mathscr{C}$.
An element of $\coprod_{U\in\mathrm{Ob}\mathscr{C}}F_X(U)$ is called an F-parametrization of X.

We note that F_X is given by $F_X(U)=\mathscr{S}et(F(U),X)$ for $U\in\mathrm{Ob}\,\mathscr{C}$ and $F_X(f)(\alpha)=\alpha F(f)$ for $(f:U\to V)\in\mathrm{Mor}\,\mathscr{C}$ and $\alpha\in F_X(V)$.

Definition 2.2

Let $(\mathscr{C}, J)$ be a site, X a set and $F:\mathscr{C} \to \mathscr{S}et$ a functor. Assume that $\mathscr{C}$ has a terminal object $1_{\mathscr{C}}$ and that $F(1_{\mathscr{C}})$ consists of a single element. If a subset $\mathscr{D}$ of $\coprod_{U \in \mathrm{Ob}\mathscr{C}} F_X(U)$ satisfies the following conditions, we call $\mathscr{D}$ a the-ology on X.

(i) $\mathscr{D} \supset F_X(1_{\mathscr{C}})$

(ii) For a morphism $f:U \to V$ in $\mathscr{C}$, the map $F_X(f):F_X(V) \to F_X(U)$ induced by f maps $\mathscr{D} \cap F_X(V)$ into $\mathscr{D} \cap F_X(U)$.

(iii) For an object U of $\mathscr{C}$, an element x of $F_X(U)$ belongs to $\mathscr{D} \cap F_X(U)$ if there exists a covering $(f_i:U_i \to U)_{i \in I}$ such that $F_X(f_i):F_X(U) \to F_X(U_i)$ maps x into $\mathscr{D} \cap F_X(U_i)$ for any $i \in I$.

We call a pair $(X,\mathscr{D})$ a the-ological object and call an element of $\mathscr{D}$ an F-plot of $(X,\mathscr{D})$.

Proposition 2.3

Condition (iii) is of (2.2) is equivalent to the following condition if we assume condition (ii).

(iii') For an object U of $\mathscr{C}$, an element x of $F_X(U)$ belongs to $\mathscr{D}\cap F_X(U)$ if there exists $R\in J(U)$ such that $F_X(f):F_X(U)\to F_X(\mathrm{dom}(f))$ maps x into $\mathscr{D}\cap F_X(\mathrm{dom}(f))$ for any $f\in R$.

For a map $\varphi:X\to Y$ and a functor $F:\mathscr{C}\to\mathscr{S}et$, we define a morphism $F_\varphi:F_X\to F_Y$ of presheaves by

$$(F_\varphi)_U=\varphi_*:F_X(U)=\mathscr{S}et(F(U),X)\to\mathscr{S}et(F(U),Y)=F_Y(U).$$

Definition 2.4

Let $(\mathscr{C}, J)$ be a site, X a set and $F : \mathscr{C} \to \mathscr{S}et$ a functor.

(1) Let $(X, \mathscr{D})$ and $(Y, \mathscr{E})$ be the-ological objects.
If the map $(F_\varphi)_U : F_X(U) \to F_Y(U)$ induced by a map $\varphi : X \to Y$
maps $\mathscr{D} \cap F_X(U)$ into $\mathscr{E} \cap F_Y(U)$ for each $U \in \mathrm{Ob}\mathscr{C}$,
we call φ a morphism of F-$(\mathscr{C}, J)$-ological objects.
We denote this by $\varphi : (X, \mathscr{D}) \to (Y, \mathscr{E})$.

(2) We define a category $\mathscr{P}_F(\mathscr{C}, J)$ of the-ological objects as follows. Objects of $\mathscr{P}_F(\mathscr{C}, J)$ are the-ological objects and morphisms of $\mathscr{P}_F(\mathscr{C}, J)$ are morphism of the-ological objects.

For a the-ological object $(X, \mathscr{D})$ and $U \in \mathrm{Ob}\mathscr{C}$, we put $F_{\mathscr{D}}(U) = \mathscr{D} \cap F_X(U)$. Then $U \mapsto F_{\mathscr{D}}(U)$ defines a presheaf $F_{\mathscr{D}}$ on $\mathscr{E}$.

Remark 2.5

Let $\varphi : (X, \mathscr{D}) \to (Y, \mathscr{E})$ be a morphism of the-ological objects. It follows from the definition of a morphism of the-ological objects that $(F_\varphi)_U : F_X(U) \to F_Y(U)$ defines a map $(F_\varphi)_U : F_{\mathscr{D}}(U) \to F_{\mathscr{E}}(U)$ which is natural in $U \in \mathrm{Ob}\mathscr{C}$. Thus we have a morphism $F_\varphi : F_{\mathscr{D}} \to F_{\mathscr{E}}$ of presheaves.

Definition 2.6

For the-ologies $\mathscr{D}$ and $\mathscr{E}$ on X, we say that $\mathscr{D}$ is finer than $\mathscr{E}$ and that $\mathscr{E}$ is coarser than $\mathscr{D}$ if $\mathscr{D} \subset \mathscr{E}$.

Remark 2.7

We put $\mathscr{D}_{coarse,X} = \coprod_{U \in \mathrm{Ob}\mathscr{C}} F_X(U)$. It is clear that $\mathscr{D}_{coarse,X}$ is the coarsest the-ology on X. For a map $f: Y \to X$ and a the-ology $\mathscr{E}$ on Y, $f: (Y, \mathscr{E}) \to (X, \mathscr{D}_{coarse,X})$ is a morphism of the-ologies.

Proposition 2.8

Let $(\mathscr{D}_i)_{i \in I}$ be a family of the-ologies on a set X. Then, $\bigcap_{i \in I} \mathscr{D}_i$ is a the-ology on X that is the finest the-ology among the-ologies on X which are coarser than $\mathscr{D}_i$ for any $i \in I$.

For a set X, we denote by $\mathscr{P}_F(\mathscr{C}, J)_X$ a subcategory of $\mathscr{P}_F(\mathscr{C}, J)$ consisting of objects of the form $(X, \mathscr{D})$ and morphisms of the form $id_X : (X, \mathscr{D}) \to (X, \mathscr{E})$. Then, $\mathscr{P}_F(\mathscr{C}, J)_X$ is regarded as an ordered set of the-ologies on X.

We often denote by $\mathscr{D}$ an object $(X, \mathscr{D})$ of $\mathscr{P}_F(\mathscr{C}, J)_X$ for short.

It follows from (2.7) that $(X, \mathscr{D}_{coarse, X})$ is the maximum (terminal) object of $\mathscr{P}_F(\mathscr{C}, J)_X$.

Corollary 2.9

$\mathscr{P}_F(\mathscr{C}, J)_X$ is complete as an ordered set.

Proposition 2.10

Let $\mathscr{S}$ be a subset of $\coprod_{U \in \mathrm{Ob}\mathscr{C}} F_X(U)$ which contains $F_X(1_{\mathscr{C}})$.

For $f \in \mathrm{Mor}\mathscr{C}$, define a subset $\mathscr{S}_f$ of $F_X(\mathrm{dom}(f))$ by

$$\mathscr{S}_f = F_X(f)(\mathscr{S} \cap F_X(\mathrm{codom}(f))).$$

For $U \in \mathrm{Ob}\mathscr{C}$, we define a subset $\mathscr{S}(U)$ of $F_X(U)$ by

$$\mathscr{S}(U) = \left\{ x \in F_X(U) \,\middle|\, \begin{array}{l} \textit{There exists } R \in J(U) \textit{ such that} \\ \qquad F_X(g)(x) \in \bigcup_{f \in \mathrm{Mor}\mathscr{C}} \mathscr{S}_f \textit{ for all } g \in R. \end{array} \right\}.$$

If we put $\mathscr{G}(\mathscr{S}) = \coprod_{U \in \mathrm{Ob}\,\mathscr{C}} \mathscr{S}(U)$ and $\Sigma = \{\mathscr{D} \in \mathscr{P}_F(\mathscr{C}, J)_X \mid \mathscr{D} \supset \mathscr{S}\}$,

then we have $\mathscr{G}(\mathscr{S}) = \inf \Sigma \in \mathscr{P}_F(\mathscr{C}, J)_X$.

Remark 2.11

(1) For $U \in \mathrm{Ob}\mathscr{C}$, the subset $\mathscr{S}(U)$ of $F_X(U)$ defined in (2.10) coincides with the following set.

$$\left\{ x \in F_X(U) \,\middle|\, \begin{array}{r} \textit{There exists a covering } (U_i \xrightarrow{g_i} U)_{i\in I} \textit{ such that} \\ F_X(g_i)(x) \in \bigcup_{f \in \mathrm{Mor}\mathscr{C}} \mathscr{S}_f \textit{ for all } i \in I. \end{array} \right\}$$

(2) Let Σ be a non-empty subset of $\mathscr{P}_F(\mathscr{C}, J)_X$ and put $\mathscr{S}(\Sigma) = \bigcup_{\mathscr{D} \in \Sigma} \mathscr{D}$. Then $\mathscr{S}(\Sigma)(U)$ coincides with the following set.

$$\left\{ x \in F_X(U) \,\middle|\, \begin{array}{r} \textit{There exists a covering } (U_i \xrightarrow{g_i} U)_{i\in I} \textit{ such that} \\ F_X(g_i)(x) \in \bigcup_{\mathscr{D} \in \Sigma} \mathscr{D} \textit{ for all } i \in I. \end{array} \right\}$$

Hence $\sup \Sigma = \mathscr{G}(\mathscr{S}(\Sigma)) = \bigcup_{U \in \mathscr{C}} \mathscr{S}(\Sigma)(U)$ holds.

Definition 2.12
For a subset $\mathscr{S}$ of $\coprod_{U \in \mathrm{Ob}\mathscr{C}} F_X(U)$ containing $F_X(1_{\mathscr{C}})$, we call $\mathscr{G}(\mathscr{S})$ defined in (2.10) the the-ology generated by $\mathscr{S}$.

Definition 2.13
Let $(\mathscr{C}, J)$ be a site and X a set. We put $\mathscr{D}_{disc,X} = \bigcap_{\mathscr{D} \in \mathrm{Ob}\mathscr{P}_F(\mathscr{C},J)_X} \mathscr{D}$ and call this the discrete the-ology on X. $\mathscr{D}_{disc,X}$ is the finest the-ology on X.

Remark 2.14
For any map $f: X \to Y$ and a the-ology $\mathscr{E}$ on Y, $f: (X, \mathscr{D}_{disc,X}) \to (Y, \mathscr{E})$ is a morphism of the-ologies.

Remark 2.15

(1) Since $\mathscr{D}_{disc,X} \supset F_X(1_{\mathscr{C}})$, $\mathscr{D}_{disc,X}$ contains the image of the map $F_X(o_U): F_X(1_{\mathscr{C}}) \to F_X(U)$ induced by the unique map $o_U : U \to 1_{\mathscr{C}}$ for any $U \in \mathrm{Ob}\mathscr{C}$. Hence every constant map in $F_X(U)$ belongs to $\mathscr{D}_{disc,X}$.

(2) Let $\mathscr{S}_{const}$ be the set of all constant maps in $\coprod_{U \in \mathrm{Ob}\mathscr{C}} F_X(U)$. Then $\mathscr{S}_{const} = \bigcup_{f \in \mathrm{Mor}\mathscr{C}} (\mathscr{S}_{const})_f$. Thus $\mathscr{D}_{disc,X} \cap F_X(U) = \mathscr{D}(\mathscr{S}_{const}) \cap F_X(U)$ coincides with the following set.

$$\{x \in F_X(U) \mid \textit{There exists a covering } (U_i \xrightarrow{g_i} U)_{i \in I} \textit{ such that } F_X(g_i)(x) \textit{ is a contant map for all } i \in I.\}$$

§3. Category of F-plots

For a map $f: X \to Y$ and $(Y, \mathscr{E}) \in \mathrm{Ob}\, \mathscr{P}_F(\mathscr{C}, J)$, we define an the-ology $\mathscr{E}^f$ on X to be the coarsest the-ology such that $f: (X, \mathscr{E}^f) \to (Y, \mathscr{E})$ is a morphism of the-ologies.

Proposition 3.1

For a map $f: X \to Y$ and $(Y, \mathscr{E}) \in \mathrm{Ob}\, \mathscr{P}_F(\mathscr{C}, J)$, $\mathscr{E}^f$ is as follows.

$$\mathscr{E}^f = \coprod_{U \in \mathrm{Ob}\mathscr{C}} (F_f)^{-1}(\mathscr{E} \cap F_Y(U)) = \coprod_{U \in \mathrm{Ob}\mathscr{C}} \left\{ \varphi \in F_X(U) \mid f\varphi \in \mathscr{E} \cap F_Y(U) \right\}$$

Proposition 3.2

Let $(\mathscr{E}_i)_{i \in I}$ a family of the-ologies on a set Y, For a map $f: X \to Y$, $\left(\bigcap_{i \in I} \mathscr{E}_i \right)^f = \bigcap_{i \in I} \mathscr{E}_i^f$ holds.

We define a forgetful functor $\Gamma : \mathscr{P}_F(\mathscr{C}, J) \to \mathscr{S}et$ by $\Gamma(X, \mathscr{D}) = X$ for $(X, \mathscr{D}) \in \mathrm{Ob}\mathscr{P}_F(\mathscr{C}, J)$ and $\Gamma(\varphi : (X, \mathscr{D}) \to (Y, \mathscr{E})) = (\varphi : X \to Y)$ for a morphism $\varphi : (X, \mathscr{D}) \to (Y, \mathscr{E})$ in $\mathscr{P}_F(\mathscr{C}, J)$.

It is clear that Γ is faithful. In other words, if we put

$$\mathscr{P}_F(\mathscr{C}, J)_f((X, \mathscr{D}), (Y, \mathscr{E})) = \Gamma^{-1}(f) \cap \mathscr{P}_F(\mathscr{C}, J)((X, \mathscr{D}), (Y, \mathscr{E}))$$

for a map $f : X \to Y$ and $(X, \mathscr{D}), (Y, \mathscr{E}) \in \mathrm{Ob}\,\mathscr{P}_F(\mathscr{C}, J)$, $\mathscr{P}_F(\mathscr{C}, J)_f((X, \mathscr{D}), (Y, \mathscr{E}))$ has at most one element.

$\mathscr{P}_F(\mathscr{C}, J)_f((X, \mathscr{D}), (Y, \mathscr{E}))$ is not empty if and only if $\mathscr{D} \subset \mathscr{E}^f$ which is equivalent that $\mathscr{P}_F(\mathscr{C}, J)_X((X, \mathscr{D}), (X, \mathscr{E}^f))$ is not empty.

Proposition 3.3
For maps $f: X \to Y$, $g: W \to X$ and an object $(Y, \mathscr{E})$ of $\mathscr{P}_F(\mathscr{C}, J)_Y$, $\mathscr{E}^{fg} = (\mathscr{E}^f)^g$ holds and $\Gamma: \mathscr{P}_F(\mathscr{C}, J) \to \mathscr{S}et$ is a fibered category.

In fact, $f: (X, \mathscr{E}^f) \to (Y, \mathscr{E})$ is unique cartesian morphism over a map $f: X \to Y$ whose target is $(Y, \mathscr{E})$. Hence the inverse image functor

$$f^*: \mathscr{P}_F(\mathscr{C}, J)_Y \to \mathscr{P}_F(\mathscr{C}, J)_X$$

associated with f is given by $f^*(Y, \mathscr{E}) = (X, \mathscr{E}^f)$ and

$$f^*(id_Y: (Y, \mathscr{E}) \to (Y, \mathscr{G})) = (id_X: (X, \mathscr{E}^f) \to (X, \mathscr{G}^f)).$$

It is clear that $\mathscr{E}^{fg} = (\mathscr{E}^f)^g$ holds, which implies $(fg)^* = g^* f^*$.

For a map $f\colon X \to Y$ and $(X, \mathscr{D}) \in \operatorname{Ob} \mathscr{P}_F(\mathscr{C}, J)$, we define a the-ology $\mathscr{D}_f$ on Y to be the finest the-ology such that $f\colon (X, \mathscr{D}) \to (Y, \mathscr{D}_f)$ is a morphism of the-ologies, that is, $\mathscr{D}_f = \bigcap_{\mathscr{E} \in \Sigma} \mathscr{E}$, where

$$\Sigma = \left\{ \mathscr{E} \in \operatorname{Ob} \mathscr{P}_F(\mathscr{C}, J)_Y \mid \mathscr{E} \supset \coprod_{U \in \operatorname{Ob} \mathscr{C}} (F_f)_U(\mathscr{D} \cap F_X(U)) \right\}.$$

Remark 3.4

For $U \in \mathrm{Ob}\mathscr{C}$, the subset $\mathcal{S}(U)$ of $F_X(U)$ defined in (2.9) is the set of elements x of $F_X(U)$ which satisfy the following condition (*) if $f: X \to Y$ is surjective.

(*) There exists $R \in J(U)$ such that, for each $h \in R$, there exists $y \in \mathscr{D} \cap F_X(\mathrm{dom}(h))$ which satisfies $F_Y(h)(x) = (F_f)_{\mathrm{dom}(h)}(y)$.

If we put $\mathscr{G}(\mathcal{S}) = \coprod_{U \in \mathrm{Ob}\mathscr{C}} \mathcal{S}(U)$, we have $\mathscr{D}_f = \mathscr{G}(\mathcal{S})$.

Proposition 3.5

$\Gamma : \mathscr{P}_F(\mathscr{C}, J) \to \mathscr{S}et$ is a bifibered category.

For a map $f : X \to Y$, define a functor $f_* : \mathscr{P}_F(\mathscr{C}, J)_X \to \mathscr{P}_F(\mathscr{C}, J)_Y$ as follows. For $(X, \mathscr{D}) \in \mathrm{Ob}\mathscr{P}_F(\mathscr{C}, J)_X$, we put $f_*(X, \mathscr{D}) = (Y, \mathscr{D}_f)$.

If $(X, \mathscr{D}), (X, \mathscr{D}') \in \mathrm{Ob}\mathscr{P}_F(\mathscr{C}, J)_X$ satisfies $\mathscr{D} \subset \mathscr{D}'$, then $\mathscr{D}_f \subset \mathscr{D}'_f$ holds. Hence, for a morphism $id_X : (X, \mathscr{D}) \to (X, \mathscr{D}')$ in $\mathscr{P}_F(\mathscr{C}, J)_X$, we put $f_*(id_X : (X, \mathscr{D}) \to (X, \mathscr{D}')) = (id_Y : (Y, \mathscr{D}_f) \to (Y, \mathscr{D}'_f))$.

It can be verified that $\mathscr{P}_F(\mathscr{C}, J)_Y(f_*(X, \mathscr{D}), (Y, \mathscr{E}))$ is not empty if and only if $\mathscr{P}_F(\mathscr{C}, J)_Y((X, \mathscr{D}), f^*(Y, \mathscr{E}))$ is not empty.

This shows that f_* is a left adjoint of f^*.

Proposition 3.6

Let $p : \mathscr{F} \to \mathscr{C}$ be a prefibered category. If $\mathscr{F}_X$ has an initial object for any object X of $\mathscr{C}$, then p has a left adjoint.

Corollary 3.7

Let $p : \mathscr{F} \to \mathscr{C}$ be a bifibered category. If $\mathscr{F}_X$ has a terminal object for any object X of $\mathscr{C}$, then p has a right adjoint.

Corollary 3.8

$\Gamma : \mathscr{P}_F(\mathscr{C}, J) \to \mathscr{S}et$ has left and right adjoints.

Let $\{(X_i, \mathscr{D}_i)\}_{i\in I}$ be a family of objects of $\mathscr{P}_F(\mathscr{C}, J)$.

We denote by $\mathrm{pr}_i : \prod_{j\in I} X_j \to X_i$ the projection to the i-th component and $\iota_i : X_i \to \coprod_{j\in I} X_j$ the inclusion to the i-th summand.

Put $\mathscr{D}^I = \bigcap_{j\in I} \mathscr{D}_i^{\mathrm{pr}_i}$. Then, $\mathscr{D}^I$ is the finest the-ology such that $\mathrm{pr}_i : \left(\prod_{j\in I} X_j, \mathscr{D}^I\right) \to (X_i, \mathscr{D}_i)$ is a morphism in $\mathscr{P}_F(\mathscr{C}, J)$ for any $i \in I$.

Let $\mathscr{D}_I$ be the coarsest the-ology on $\coprod_{j\in I} X_j$ such that $\iota_i : (X_i, \mathscr{D}_i) \to \left(\coprod_{j\in I} X_j, \mathscr{D}_I\right)$ is a morphism in $\mathscr{P}_F(\mathscr{C}, J)$ for any $i \in I$.

If we put $\mathscr{S}_I = \left\{ \mathscr{E} \in \mathrm{Ob}\, \mathscr{P}_F(\mathscr{C}, J)_{\coprod_{j\in I} X_j} \,\middle|\, \mathscr{E} \supset \bigcup_{j\in I} (\mathscr{D}_j)_{\iota_j} \right\}$, then $\mathscr{D}_I = \bigcap_{\mathscr{E}\in\mathscr{S}_I} \mathscr{E}$.

Proposition 3.9

(1) $\left(\left(\prod_{j\in I} X_j, \mathscr{D}^I\right) \xrightarrow{\mathrm{pr}_i} (X_i, \mathscr{D}_i)\right)_{i\in I}$ is a product of $\{(X_i, \mathscr{D}_i)\}_{i\in I}$.

(2) $\left((X_i, \mathscr{D}_i) \xrightarrow{\iota_i} \left(\coprod_{j\in I} X_j, \mathscr{D}_I\right)\right)_{i\in I}$ is a coproduct of $\{(X_i, \mathscr{D}_i)\}_{i\in I}$.

Proposition 3.10

Let $f, g : (X, \mathscr{D}) \to (Y, \mathscr{E})$ be morphisms in $\mathscr{P}_F(\mathscr{C}, J)$. Then, equalizers and coequalizers of f and g exist.

In fact, if $Z \xrightarrow{i} X$ is an equalizer of f and g in the category of sets, then $(Z, \mathscr{D}^i) \xrightarrow{i} (X, \mathscr{D})$ is an equalizer of f and g in $\mathscr{P}_F(\mathscr{C}, J)$.

If $Y \xrightarrow{q} W$ is a coequalizer of f and g in the category of sets, then $(Y, \mathscr{E}) \xrightarrow{q} (W, \mathscr{E}_q)$ is a coequalizer of f and g in $\mathscr{P}_F(\mathscr{C}, J)$.

§4. Fibered category of morphisms

For a category $\mathscr{C}$, let $\mathscr{C}^{(2)}$ be the category of morphisms in $\mathscr{C}$ defined as follows.

Put $\mathrm{Ob}\mathscr{C}^{(2)} = \mathrm{Mor}\,\mathscr{C}$ and a morphism from $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ to $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ is a pair $\langle \xi : E \to F, f : X \to Y \rangle$ of morphisms in $\mathscr{C}$ which satisfies $\rho\xi = f\pi$.

The composition of morphisms $\langle \xi, f \rangle : \boldsymbol{E} \to \boldsymbol{F}$ and $\langle \zeta, g \rangle : \boldsymbol{F} \to \boldsymbol{G}$ is defined to be $\langle \zeta\xi, gf \rangle : \boldsymbol{E} \to \boldsymbol{G}$.

$$
\begin{array}{ccc}
E & \xrightarrow{\ \xi\ } & F \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
\qquad\qquad
\begin{array}{ccccc}
E & \xrightarrow{\ \xi\ } & F & \xrightarrow{\ \zeta\ } & G \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \chi} \\
X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & Z
\end{array}
$$

Define a functor $\wp : \mathscr{C}^{(2)} \to \mathscr{C}$ by $\wp(E \overset{\pi}{\to} X) = X$ and $\wp(\langle \xi, f \rangle) = f$.
For an object X of $\mathscr{C}$, we denote by $\mathscr{C}_X^{(2)}$ a subcategory of $\mathscr{C}^{(2)}$ given as follows.

$$\mathrm{Ob}\mathscr{C}_X^{(2)} = \{ \boldsymbol{E} \in \mathrm{Ob}\, \mathscr{C}^{(2)} \mid \wp(\boldsymbol{E}) = X \}$$

$$\mathrm{Mor}\, \mathscr{C}_X^{(2)} = \{ \boldsymbol{\xi} \in \mathrm{Mor}\, \mathscr{C}^{(2)} \mid \wp(\boldsymbol{\xi}) = id_X \}$$

We mention that $\mathscr{C}_X^{(2)}$ is often denoted by $\mathscr{C}/X$ in literatures.

For a morphism $f : X \to Y$ in $\mathscr{C}$, an object $\boldsymbol{E}$ of $\mathscr{C}_X^{(2)}$ and an object $\boldsymbol{F}$ of $\mathscr{C}_Y^{(2)}$, we denote by $\mathscr{C}_f^{(2)}(\boldsymbol{E}, \boldsymbol{F})$ the set of all morphisms $\boldsymbol{\xi} : \boldsymbol{E} \to \boldsymbol{F}$ in $\mathscr{C}^{(2)}$ such that $\wp(\boldsymbol{\xi}) = f$.

If $\mathscr{C}$ has finite limits, $\wp:\mathscr{C}^{(2)}\to\mathscr{C}$ is a fibered category as we explain below.

For a morphism $f:X\to Y$ in $\mathscr{C}$ and an object $\boldsymbol{F}=(F\xrightarrow{\rho}Y)$ of $\mathscr{C}^{(2)}_Y$, consider the following cartesian square in $\mathscr{C}$.

$$\begin{array}{ccc} F\times_Y X & \xrightarrow{f_\rho} & F \\ \downarrow{\scriptstyle \rho_f} & & \downarrow{\scriptstyle \rho} \\ X & \xrightarrow{f} & Y \end{array}$$

We put $f^*(\boldsymbol{F})=(F\times_Y X\xrightarrow{\rho_f}X)$ and $\boldsymbol{\alpha}_f(\boldsymbol{F})=\langle f_\rho,f\rangle:f^*(\boldsymbol{F})\to\boldsymbol{F}$.

Proposition 4.1

$\boldsymbol{\alpha}_f(\boldsymbol{F})$ is a cartesian morphism, that is, for any object $\boldsymbol{G}$ of $\mathscr{C}^{(2)}_X$ the map $\boldsymbol{\alpha}_f(\boldsymbol{F})_*:\mathscr{C}^{(2)}_X(\boldsymbol{G},f^*(\boldsymbol{F}))\to\mathscr{C}^{(2)}_f(\boldsymbol{G},\boldsymbol{F})$ defined by $\boldsymbol{\alpha}_f(\boldsymbol{F})_*(\xi)=\boldsymbol{\alpha}_f(\boldsymbol{F})\xi$ is bijective.

For objects $\boldsymbol{E}, \boldsymbol{F}$ of $\mathscr{C}_Y^{(2)}$ and a morphism $\boldsymbol{\varphi}: \boldsymbol{E} \to \boldsymbol{F}$ in $\mathscr{C}_Y^{(2)}$, let $f^*(\boldsymbol{\varphi}): f^*(\boldsymbol{E}) \to f^*(\boldsymbol{F})$ be the unique morphism in $\mathscr{C}_X^{(2)}$ that is mapped to a composition $f^*(\boldsymbol{E}) \xrightarrow{\boldsymbol{\alpha}_f(\boldsymbol{E})} \boldsymbol{E} \xrightarrow{\boldsymbol{\varphi}} \boldsymbol{F}$ by the bijection

$$\boldsymbol{\alpha}_f(\boldsymbol{F})_* : \mathscr{C}_X^{(2)}(f^*(\boldsymbol{E}), f^*(\boldsymbol{F})) \to \mathscr{C}_f^{(2)}(f^*(\boldsymbol{E}), \boldsymbol{F})$$

given in (4.1). Thus we have the inverse image functor

$$f^* : \mathscr{C}_Y^{(2)} \to \mathscr{C}_X^{(2)}$$

associated with a morphism $f: X \to Y$ in $\mathscr{C}$. It follows from the definition of f^* that the bijection in (4.1) is natural in $\boldsymbol{F}$.

For morphisms $f: X \to Y$, $g: Z \to X$ in $\mathscr{C}$ and an object E of $\mathscr{C}_Y^{(2)}$, let $c_{f,g}(E): g^*(f^*(E)) \to (fg)^*(E)$ be the unique morphism in $\mathscr{C}_Z^{(2)}$ that is mapped to a composition $g^*(f^*(E)) \xrightarrow{\alpha_g(f^*(E))} f^*(E) \xrightarrow{\alpha_f(E)} E$ by the following bijection given in (4.1).

$$\alpha_{fg}(E)_*: \mathscr{C}_Z^{(2)}(g^*(f^*(E)), (fg)^*(E)) \to \mathscr{C}_{fg}^{(2)}(g^*(f^*(E)), E)$$

Proposition 4.2

$c_{f,g}(E)$ is an isomorphism in $\mathscr{C}_Z^{(2)}$. Hence $\wp: \mathscr{C}^{(2)} \to \mathscr{C}$ is a fibered category.

For a morphism $f\colon X \to Y$ in $\mathscr{C}$, define a functor $f_*\colon \mathscr{C}^{(2)}_X \to \mathscr{C}^{(2)}_Y$ by $f_*(\boldsymbol{E}) = (E \xrightarrow{f\rho} Y)$ and $f_*(\langle \xi, id_X \rangle) = \langle \xi, id_Y \rangle : f_*(\boldsymbol{E}) \to f_*(\boldsymbol{F})$ for an object $\boldsymbol{E} = (E \xrightarrow{\rho} X)$ of $\mathscr{C}^{(2)}_X$ and a morphism $\langle \xi, id_X \rangle : \boldsymbol{E} \to \boldsymbol{F}$ in $\mathscr{C}^{(2)}_X$.

Proposition 4.3

$f_*\colon \mathscr{C}^{(2)}_X \to \mathscr{C}^{(2)}_Y$ is a left adjoint of $f^*\colon \mathscr{C}^{(2)}_Y \to \mathscr{C}^{(2)}_X$.
Hence $\wp\colon \mathscr{C}^{(2)} \to \mathscr{C}$ is a bifibered category.

For an object $\boldsymbol{E}$ of $\mathscr{C}^{(2)}_X$ and an object $\boldsymbol{F}$ of $\mathscr{C}^{(2)}_Y$, we define a map $\Phi_{\boldsymbol{E},\boldsymbol{F}}\colon \mathscr{C}^{(2)}_f(\boldsymbol{E},\boldsymbol{F}) \to \mathscr{C}^{(2)}_Y(f_*(\boldsymbol{E}),\boldsymbol{F})$ by $\Phi_{\boldsymbol{E},\boldsymbol{F}}(\langle \xi, f \rangle) = \langle \xi, id_Y \rangle$, which is a natural bijection. It follows from (4.1) that we have a natural bijection $\Phi_{\boldsymbol{E},\boldsymbol{F}}\,\alpha_f(\boldsymbol{F})_*\colon \mathscr{C}^{(2)}_X(\boldsymbol{E}, f^*(\boldsymbol{F})) \to \mathscr{C}^{(2)}_Y(f_*(\boldsymbol{E}),\boldsymbol{F})$.

§5. Locally cartesian closedness

$\mathscr{P}_F(\mathscr{C}, J)$ is complete and cocomplete by (3.9) and (3.10), in particular $\mathscr{P}_F(\mathscr{C}, J)$ has finite limits.

Hence we can consider the fibered category

$$\wp : \mathscr{P}_F(\mathscr{C}, J)^{(2)} \to \mathscr{P}_F(\mathscr{C}, J)$$

of morphisms in $\mathscr{P}_F(\mathscr{C}, J)$ by (4.2).

It follows from (4.3) that the inverse image functors of this fibered category have left adjoints.

We show that the inverse image functors also have right adjoints below.

Let $\varphi:(X,\mathscr{D})\to(Y,\mathscr{F})$ be a morphism in $\mathscr{P}_F(\mathscr{C},J)$ and $\boldsymbol{E}=((E,\mathscr{E})\xrightarrow{\pi}(X,\mathscr{D}))$ an object of $\mathscr{P}_F(\mathscr{C},J)^{(2)}$.

For $y\in Y$, we denote by $\iota_y:\varphi^{-1}(y)\to X$ the inclusion map and consider a the-ology $\mathscr{D}^{\iota_y}$ on $\varphi^{-1}(y)$.

We define a subset $E(\varphi;y)$ of $\mathscr{P}_F(\mathscr{C},J)((\varphi^{-1}(y),\mathscr{D}^{\iota_y}),(E,\mathscr{E}))$ by

$$E(\varphi;y)=\{\alpha\in\mathscr{P}_F(\mathscr{C},J)((\varphi^{-1}(y),\mathscr{D}^{\iota_y}),(E,\mathscr{E}))\mid\pi\alpha=\iota_y\}$$

if $\varphi^{-1}(y)\neq\varnothing$ and $E(\varphi;y)=\varnothing$ if $\varphi^{-1}(y)=\varnothing$.

Put $E(\varphi)=\coprod_{y\in Y}E(\varphi;y)$ and define map $\varphi_{!\boldsymbol{E}}:E(\varphi)\to Y$ by $\varphi_{!\boldsymbol{E}}(\alpha)=y$ if $\alpha\in E(\varphi;y)$. Note that the image of $\varphi_{!\boldsymbol{E}}$ coincides with the image of φ.

We consider the following cartesian square (*) in $\mathscr{Set}$.

$$(*)\qquad \begin{array}{ccc} E(\varphi)\times_Y X & \xrightarrow{\tilde{\varphi}_E} & E(\varphi) \\ \Big\downarrow{\scriptstyle \widetilde{\varphi_{!E}}} & & \Big\downarrow{\scriptstyle \varphi_{!E}} \\ X & \xrightarrow{\quad\varphi\quad} & Y \end{array}$$

Define a map $\varepsilon_E^{\varphi}: E(\varphi)\times_Y X \to E$ by $\varepsilon_E^{\varphi}(\alpha, x) = \alpha(x)$ if $\alpha \in E(\varphi; y)$ and $x \in \varphi^{-1}(y)$ for $y \in Y$.

Then, ε_E^{φ} makes the following diagram commute.

$$\begin{array}{ccc} E(\varphi)\times_Y X & \xrightarrow{\varepsilon_E^{\varphi}} & E \\ & \searrow{\scriptstyle \widetilde{\varphi_{!E}}} & \Big\downarrow{\scriptstyle \pi} \\ & & X \end{array}$$

Let $\Sigma_{E,\varphi}$ the set of all the-ologies $\mathscr{L}$ on $E(\varphi)$ such that $\mathscr{L} \subset \mathscr{F}^{\varphi_{!E}}$ and $\mathscr{D}^{\widetilde{\varphi_{!E}}} \cap \mathscr{L}^{\tilde{\varphi}_E} \subset \mathscr{E}^{\varepsilon_E^{\varphi}}$ hold.

Note that $\mathscr{L} \in \Sigma_{E,\varphi}$ if and only if $\varphi_{!E} : (E(\varphi), \mathscr{L}) \to (Y, \mathscr{F})$ and $\varepsilon_E^{\varphi} : (E(\varphi) \times_Y X, \mathscr{D}^{\widetilde{\varphi_{!E}}} \cap \mathscr{L}^{\tilde{\varphi}_E}) \to (E, \mathscr{E})$ are morphisms in $\mathscr{P}_F(\mathscr{C}, J)$.

Proposition 5.1

$\Sigma_{E,\varphi}$ is not empty.

In fact, the discrete the-ology $\mathscr{D}_{disc,\, E(\varphi)}$ on $E(\varphi)$ belongs to $\Sigma_{E,\varphi}$.

For $U \in \mathrm{Ob}\,\mathscr{C}$, we consider the following condition (LE) on an element γ of $F_{E(\varphi)}(U)$.

(LE) If $V, W \in \mathrm{Ob}\,\mathscr{C}$, $f \in \mathscr{C}(W, U)$, $g \in \mathscr{C}(W, V)$ and $\psi \in \mathscr{D} \cap F_X(V)$ satisfy $\varphi\psi F(g) = \varphi_{!E}\gamma F(f)$, a composition

$$F(W) \xrightarrow{(\gamma F(f), \psi F(g))} E(\varphi) \times_Y X \xrightarrow{\varepsilon_E^{\varphi}} E$$

belongs to $\mathscr{E} \cap F_E(W)$ and a composition $F(U) \xrightarrow{\gamma} E(\varphi) \xrightarrow{\varphi_{!E}} Y$ belongs to $\mathscr{F} \cap F_Y(U)$.

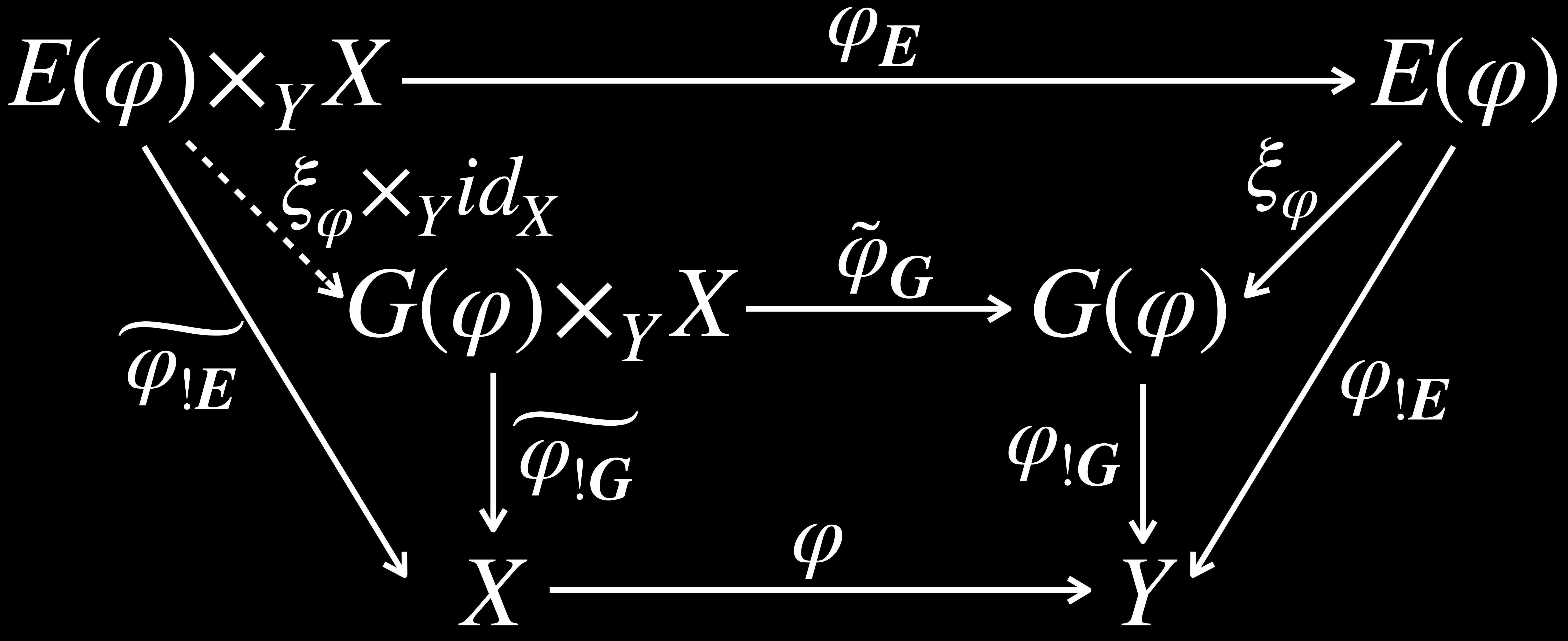
Define a set $\mathscr{D}_{E,\varphi}$ of F-parametrizations of a set $E(\varphi)$ so that $\mathscr{D}_{E,\varphi} \cap F_{E(\varphi)}(U)$ is a subset of $F_{E(\varphi)}(U)$ consisting of elements which satisfy the above condition (LE) for any $U \in \mathrm{Ob}\,\mathscr{C}$.

Proposition 5.2
$\mathscr{D}_{\boldsymbol{E},\varphi}$ is a the-ology on $E(\varphi)$.

Proposition 5.3
$\mathscr{D}_{\boldsymbol{E},\varphi}$ is maximum element of $\Sigma_{\boldsymbol{E},\varphi}$.

Let $\boldsymbol{E}=((E,\mathscr{E})\xrightarrow{\pi}(X,\mathscr{D}))$, $\boldsymbol{G}=((G,\mathscr{G})\xrightarrow{\rho}(X,\mathscr{D}))$ be objects of $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})}$ and $\varphi:(X,\mathscr{D})\to(Y,\mathscr{F})$ a morphism in $\mathscr{P}_F(\mathscr{C},J)$.
Let $\langle\xi,id_X\rangle:\boldsymbol{E}\to\boldsymbol{G}$ be a morphism in $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})}$.
If $\alpha\in E(\varphi;y)$ for $y\in Y$, we have $\rho\xi\alpha=\pi\alpha=\iota_y$, hence $\xi\alpha\in G(\varphi;y)$.
Thus we can define a map $\xi_\varphi:E(\varphi)\to G(\varphi)$ by $\varphi(\xi)(\alpha)=\xi\alpha$.

We consider the following diagram whose outer trapezoid and lower rectangle are cartesian.



Since the right triangle of the above diagram is commutative, there exists unique map $\xi_\varphi \times_Y id_X : E(\varphi) \times_Y X \to G(\varphi) \times_Y X$ that makes the above diagram commutative.

Proposition 5.4

$\xi_\varphi : (E(\varphi), \mathscr{D}_{\boldsymbol{E},\varphi}) \to (G(\varphi), \mathscr{D}_{\boldsymbol{G},\varphi})$ is a morphism in $\mathscr{P}_F(\mathscr{C}, J)$ and the following diagram is commutative.

$$\begin{array}{ccc} E(\varphi) \times_Y X & \xrightarrow{\varepsilon^\varphi_{\boldsymbol{E}}} & E \\ \big\downarrow{\scriptstyle \xi_\varphi \times_Y id_X} & & \big\downarrow{\scriptstyle \xi} \\ G(\varphi) \times_Y X & \xrightarrow{\varepsilon^\varphi_{\boldsymbol{G}}} & G \end{array}$$

Remark 5.5

Let $\boldsymbol{E}=((E,\mathscr{E})\stackrel{\pi}{\to}(X,\mathscr{D}))$, $\boldsymbol{G}=((G,\mathscr{G})\stackrel{\rho}{\to}(X,\mathscr{D}))$, $\boldsymbol{H}=((X,\mathscr{H})\stackrel{\chi}{\to}(X,\mathscr{D}))$ be objects of $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})}$ and $\langle\xi,id_X\rangle:\boldsymbol{E}\to\boldsymbol{G}$, $\langle\zeta,id_X\rangle:\boldsymbol{G}\to\boldsymbol{H}$ be morphisms in $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})}$.

For a morphism $\varphi:(X,\mathscr{D})\to(Y,\mathscr{F})$, it follows from the definition of ξ_φ that $(\zeta\xi)_\varphi:E(\varphi)\to H(\varphi)$ coincides with a composition

$E(\varphi)\stackrel{\xi_\varphi}{\to}G(\varphi)\stackrel{\zeta_\varphi}{\to}H(\varphi)$.

We also note that $(id_E)_\varphi$ coincides with the identity map of $E(\varphi)$.

We define a functor $\varphi_! : \mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})} \to \mathscr{P}_F(\mathscr{C},J)^{(2)}_{(Y,\mathscr{E})}$ by putting

$$\varphi_!(\boldsymbol{E}) = ((E(\varphi), \mathscr{D}_{\boldsymbol{E},\varphi}) \xrightarrow{\varphi_{!\boldsymbol{E}}} (Y, \mathscr{F}))$$

for an object $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (X, \mathscr{D}))$ of $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})}$ and

$$\varphi_!(\langle \xi, id_X \rangle) = \langle \xi_\varphi, id_Y \rangle : \varphi_!(\boldsymbol{E}) \to \varphi_!(\boldsymbol{G})$$

for a morphism $\langle \xi, id_X \rangle : \boldsymbol{E} \to \boldsymbol{G}$ in $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})}$.

It follows from (5.3) and (5.4) that we have a natural transformation $\boldsymbol{\varepsilon}^{\varphi} : \varphi^*\varphi_! \to id_{\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})}}$ defined by

$$\boldsymbol{\varepsilon}^{\varphi}_{\boldsymbol{E}} = \langle \varepsilon^{\varphi}_{\boldsymbol{E}}, id_X \rangle : \left(\left(E(\varphi) \times_Y X, \mathscr{D}^{\tilde{\varphi}_E}_{\boldsymbol{E},\varphi} \cap \mathscr{D}^{\widetilde{\varphi_{!E}}} \right) \xrightarrow{\widetilde{\varphi_{!E}}} (X, \mathscr{D}) \right)$$
$$\longrightarrow ((E, \mathscr{E}) \xrightarrow{\pi} (X, \mathscr{D})).$$

For an object $\boldsymbol{G} = ((G, \mathscr{G}) \xrightarrow{\rho} (Y, \mathscr{F}))$ of $\mathscr{P}_F(\mathscr{C}, J)^{(2)}_{(Y,\mathscr{F})}$, we consider the following cartesian square in $\mathscr{P}_F(\mathscr{C}, J)$.

$$\begin{array}{ccc} (G \times_Y X, \mathscr{G}^{\varphi_\rho} \cap \mathscr{D}^{\rho_\varphi}) & \xrightarrow{\varphi_\rho} & (G, \mathscr{G}) \\ \downarrow \rho_\varphi & & \downarrow \rho \\ (X, \mathscr{D}) & \xrightarrow{\varphi} & (Y, \mathscr{F}) \end{array}$$

Then, we have $\varphi^*(\boldsymbol{G}) = (G \times_Y X, \mathscr{G}^{\varphi_\rho} \cap \mathscr{D}^{\rho_\varphi}) \xrightarrow{\rho_\varphi} (X, \mathscr{D}))$.

We note that, for $y \in Y$, $(X \times_Y G)(\varphi; y)$ is a subset of

$$\mathscr{P}_F(\mathscr{C}, J)((\varphi^{-1}(y), \mathscr{D}^{\iota_y}), (G \times_Y X, \mathscr{G}^{\varphi_\rho} \cap \mathscr{D}^{\rho_\varphi}))$$

consisting of elements of the form (λ, ι_y) such that $\lambda : \varphi^{-1}(y) \to G$ satisfies $\lambda(\varphi^{-1}(y)) \subset \rho^{-1}(y)$.

For $v \in G$, let us denote by $c_v : \varphi^{-1}(\rho(v)) \to G$ the constant map whose image is $\{v\}$. Then we have $c_v(\varphi^{-1}(\rho(v))) = \{v\} \subset \rho^{-1}(\rho(v))$ which implies $(c_v, \iota_{\rho(v)}) \in (G \times_Y X)(\varphi)$.

Define a map $\eta_G^\varphi : G \to (G \times_Y X)(\varphi)$ by $\eta_G^\varphi(v) = (c_v, \iota_{\rho(v)})$.

Then, η_G^φ makes the following diagram commute.

$$
\begin{array}{ccc}
G & \xrightarrow{\eta_G^\varphi} & (G \times_Y X)(\varphi) \\
& \underset{\rho}{\searrow} & \downarrow \varphi_{!\varphi^*(G)} \\
& & Y
\end{array}
$$

Proposition 5.6

$\eta_G^\varphi : (G, \mathcal{G}) \to ((G \times_Y X)(\varphi), \mathcal{D}_{\varphi^*(G),\varphi})$ is a morphism in $\mathcal{P}_F(\mathcal{C}, J)$.

For objects $\boldsymbol{E}=((E,\mathscr{E})\xrightarrow{\pi}(Y,\mathscr{F}))$, $\boldsymbol{G}=((G,\mathscr{G})\xrightarrow{\rho}(Y,\mathscr{F}))$ of $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(Y,\mathscr{F})}$ and a morphism $\varphi:(X,\mathscr{D})\to(Y,\mathscr{F})$ in $\mathscr{P}_F(\mathscr{C},J)$, we consider the following cartesian squares in $\mathscr{P}_F(\mathscr{C},J)$.

$$\begin{array}{ccc}
(E\times_Y X,\mathscr{E}^{\varphi_\pi}\cap\mathscr{D}^{\pi_\varphi}) & \xrightarrow{\varphi_\pi} & (E,\mathscr{E}) \\
\Big\downarrow{\scriptstyle \pi_\varphi} & & \Big\downarrow{\scriptstyle \pi} \\
(X,\mathscr{D}) & \xrightarrow{\varphi} & (Y,\mathscr{F})
\end{array}$$

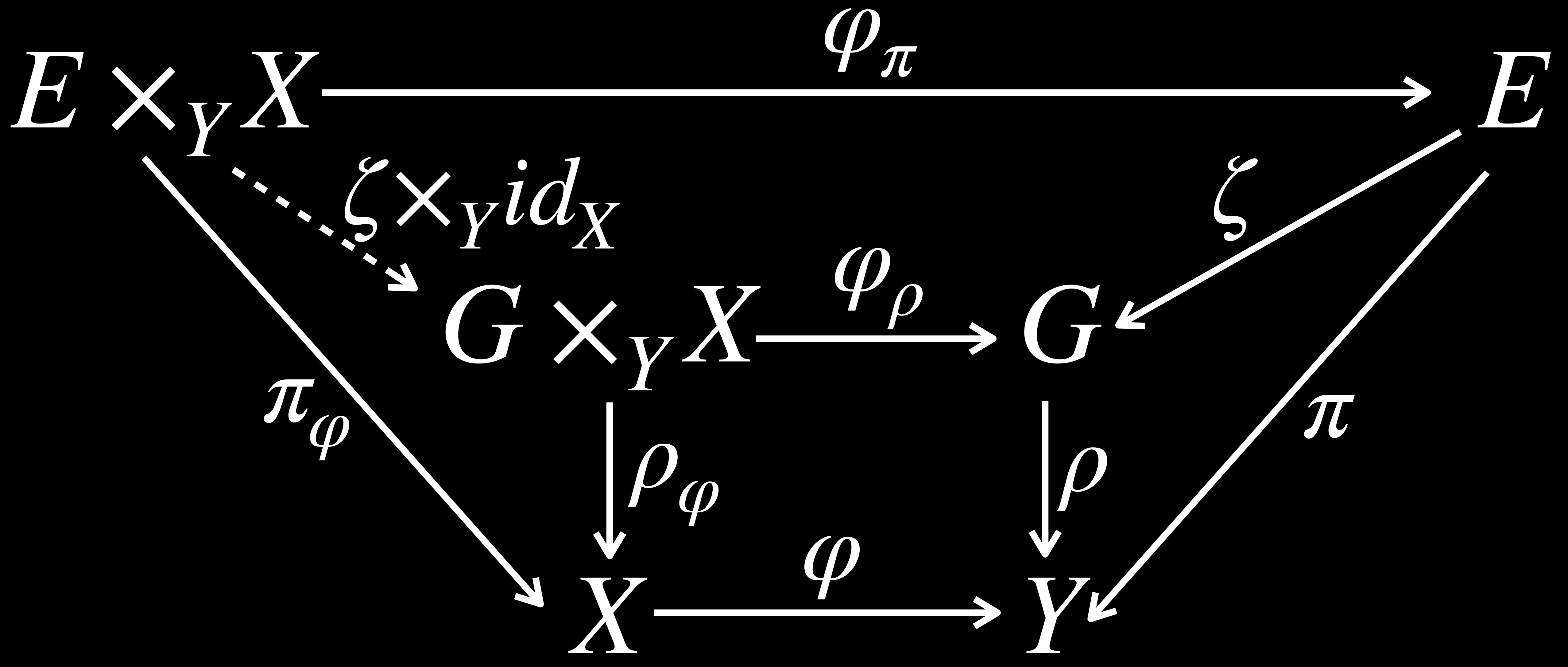
$$\begin{array}{ccc}
(G\times_Y X,\mathscr{G}^{\varphi_\rho}\cap\mathscr{D}^{\rho_\varphi}) & \xrightarrow{\varphi_\rho} & (G,\mathscr{G}) \\
\Big\downarrow{\scriptstyle \rho_\varphi} & & \Big\downarrow{\scriptstyle \rho} \\
(X,\mathscr{D}) & \xrightarrow{\varphi} & (Y,\mathscr{F})
\end{array}$$

Let $\langle \zeta, id_Y \rangle : \boldsymbol{E} \to \boldsymbol{G}$ be a morphism in $\mathscr{P}_F(\mathscr{C}, J)^{(2)}_{(Y,\mathscr{F})}$.

Since $\rho\zeta = \pi$ holds, there exists unique morphism

$$\zeta \times_Y id_X : (E \times_Y X, \mathscr{E}^{\varphi_\pi} \cap \mathscr{D}^{\pi_\varphi}) \to (G \times_Y X, \mathscr{G}^{\varphi_\rho} \cap \mathscr{D}^{\rho_\varphi})$$

in $\mathscr{P}_F(\mathscr{C}, J)$ that makes the following diagram commutative.



The following result is easily verified from the definitions of $\eta^\varphi_{\boldsymbol{E}}$, $\eta^\varphi_{\boldsymbol{G}}$ and $(\zeta \times_Y id_X)_\varphi$.

Proposition 5.7
For a morphism $\langle\zeta, id_Y\rangle : ((E,\mathscr{E}) \xrightarrow{\pi} (Y,\mathscr{F})) \to ((G,\mathscr{G}) \xrightarrow{\rho} (Y,\mathscr{F}))$ in $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(Y,\mathscr{F})}$, the following diagram is commutative.

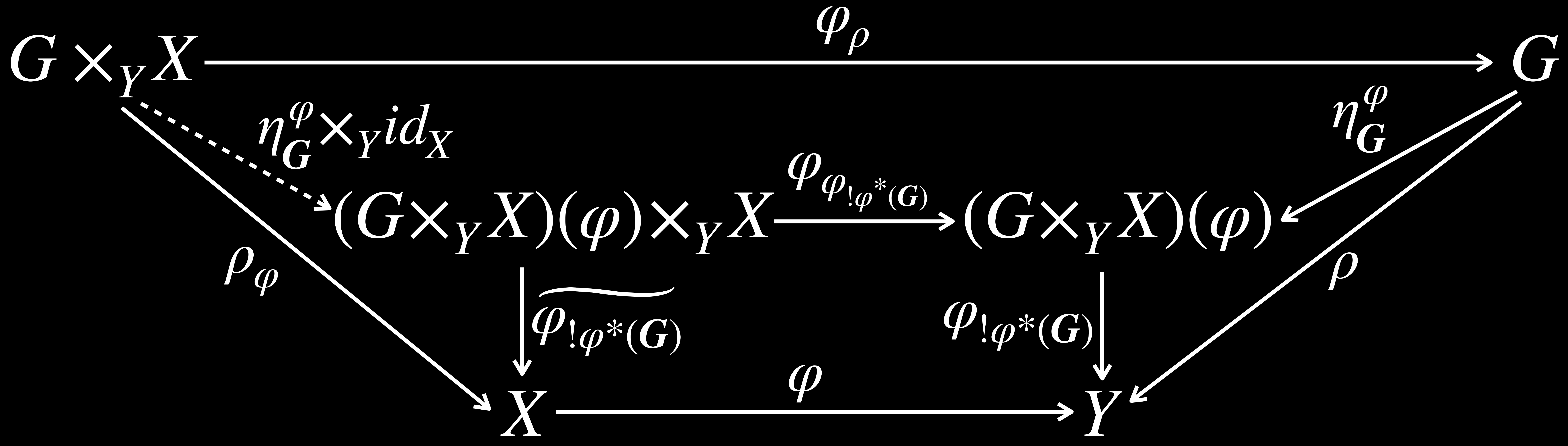
$$\begin{array}{ccc} E & \xrightarrow{\eta^{\varphi}_E} & (E\times_Y X)(\varphi) \\ \downarrow \zeta & & \downarrow (\zeta\times_Y id_X)_{\varphi} \\ G & \xrightarrow{\eta^{\varphi}_G} & (G\times_Y X)(\varphi) \end{array}$$

It follows from (5.6) and (5.7) that we have a natural transformation $\boldsymbol{\eta}^{\varphi} : id_{\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(Y,\mathscr{F})}} \to \varphi_!\varphi^*$ defined by

$$\boldsymbol{\eta}^{\varphi}_{\boldsymbol{G}} = \langle \eta^{\varphi}_G, id_Y\rangle : ((G,\mathscr{G}) \xrightarrow{\rho} (Y,\mathscr{F})) \to ((G\times_Y X)(\varphi) \xrightarrow{\varphi_{!\varphi^*(G)}} (Y,\mathscr{F}))$$

for an object $\boldsymbol{G} = ((G,\mathscr{G}) \xrightarrow{\rho} (Y,\mathscr{F}))$ of $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(Y,\mathscr{F})}$.

Consider the following diagram, where the outer trapezoid and the lower rectangle are cartesian.



Since the right triangle of the above diagram is commutative, there exists unique map

$$\eta_G^\varphi \times_Y id_X : G \times_Y X \to (G \times_Y X)(\varphi) \times_Y X$$

that makes the above diagram commute.

Lemma 5.8

For objects $\boldsymbol{E}=((E,\mathscr{E})\xrightarrow{\pi}(Y,\mathscr{F}))$, $\boldsymbol{G}=((G,\mathscr{G})\xrightarrow{\rho}(Y,\mathscr{F}))$ of $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(Y,\mathscr{F})}$ and a morphism $\varphi:(X,\mathscr{D})\to(Y,\mathscr{F})$ in $\mathscr{P}_F(\mathscr{C},J)$, the following compositions are both identity maps.

$$E(\varphi)\xrightarrow{\eta^{\varphi}_{\varphi_!(\boldsymbol{E})}}(E(\varphi)\times_Y X)(\varphi)\xrightarrow{(\varepsilon^{\varphi}_{\boldsymbol{E}})_{\varphi}}E(\varphi)$$

$$G\times_Y X\xrightarrow{\eta^{\varphi}_{\boldsymbol{G}}\times_Y id_X}(G\times_Y X)(\varphi)\times_Y X\xrightarrow{\varepsilon^{\varphi}_{\varphi^*(\boldsymbol{G})}}G\times_Y X$$

For an object $\boldsymbol{G}=((G,\mathscr{G})\overset{\rho}{\to}(Y,\mathscr{F}))$ of $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(Y,\mathscr{F})}$ and an object $\boldsymbol{E}=((E,\mathscr{E})\overset{\pi}{\to}(X,\mathscr{D}))$ of $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})}$, since compositions

$$\varphi_!(\boldsymbol{E})\xrightarrow{\eta^{\varphi}_{\varphi_!(\boldsymbol{E})}}\varphi_!\varphi^*\varphi_!(\boldsymbol{E})\xrightarrow{\varphi_!(\varepsilon^{\varphi}_{\boldsymbol{E}})}\varphi_!(\boldsymbol{E}),$$

$$\varphi^*(\boldsymbol{G})\xrightarrow{\varphi^*(\eta^{\varphi}_{\boldsymbol{G}})}\varphi^*\varphi_!\varphi^*(\boldsymbol{G})\xrightarrow{\varepsilon^{\varphi}_{\varphi^*(\boldsymbol{G})}}\varphi^*(\boldsymbol{G})$$

are both identity morphisms by (5.8), we have the following result.

Proposition 5.9

$\varphi_!:\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})}\to\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(Y,\mathscr{E})}$ is a right adjoint of the inverse image functor $\varphi^*:\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(Y,\mathscr{D})}\to\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{E})}$.

Hence $\mathscr{P}_F(\mathscr{C},J)$ is locally cartesian closed.

Remark 5.10 ([10], Proposition A.16.22)

Let $\boldsymbol{E}=((Y,\mathscr{E})\xrightarrow{\pi}(X,\mathscr{D}))$, $\boldsymbol{F}=((Z,\mathscr{F})\xrightarrow{\rho}(X,\mathscr{D}))$ and $\boldsymbol{G}=((W,\mathscr{G})\xrightarrow{\chi}(X,\mathscr{D}))$ be objects of $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})}$.

It follows from (4.3) and (5.7) that there exist natural bijections

$$\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})}(\rho_*\rho^*(\boldsymbol{E}),\boldsymbol{G})\to\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(Z,\mathscr{F})}(\rho^*(\boldsymbol{E}),\rho^*(\boldsymbol{G})),$$

$$\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(Z,\mathscr{F})}(\rho^*(\boldsymbol{E}),\rho^*(\boldsymbol{G}))\to\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})}(\boldsymbol{E},\rho_!\rho^*(\boldsymbol{G})).$$

We note that the product $\boldsymbol{E}\times\boldsymbol{F}$ of $\boldsymbol{E}$ and $\boldsymbol{F}$ in $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})}$ is given by $\boldsymbol{E}\times\boldsymbol{F}=\rho_*\rho^*(\boldsymbol{E})$.

Hence if we put $\boldsymbol{G}^{\boldsymbol{F}}=\rho_!\rho^*(\boldsymbol{G})$, we have a natural bijection

$$\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})}(\boldsymbol{E}\times\boldsymbol{F},\boldsymbol{G})\to\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})}(\boldsymbol{E},\boldsymbol{G}^{\boldsymbol{F}}).$$

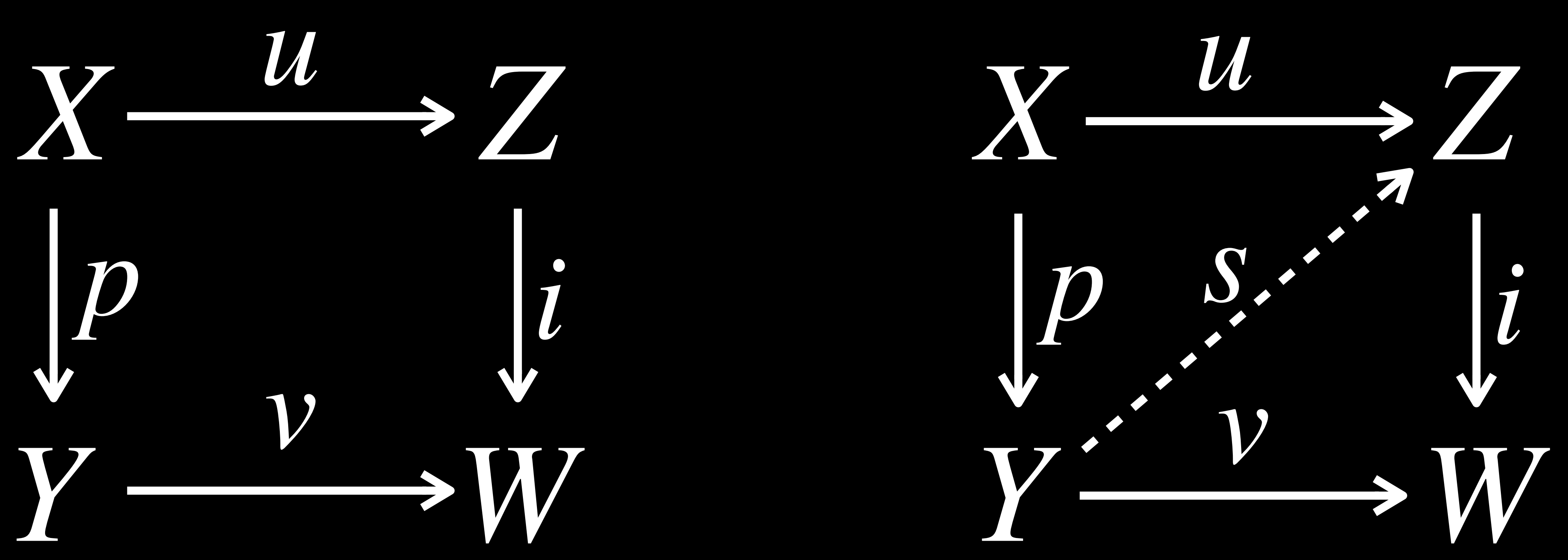
This shows that $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})}$ is cartesian closed.

§6. Strong subobject classifier

Definition 6.1

Let $\mathscr{C}$ be a category.

(1) Two morphisms $p: X \to Y$ and $i: Z \to W$ in $\mathscr{C}$ are said to be orthogonal if the following left diagram is commutative, there exits unique morphism $s: Y \to Z$ that makes the following right diagram commute.



If p and i are orthogonal, we denote this by $p \perp i$.

(2) For a class C of morphisms in $\mathscr{C}$, we put

$$C^{\perp} = \{ i \in \mathrm{Mor}\,\mathscr{C} \mid p \perp i \text{ } \textit{if} \text{ } p \in C \},$$

$$^{\perp}C = \{ p \in \mathrm{Mor}\,\mathscr{C} \mid p \perp i \text{ } \textit{if} \text{ } i \in C \}.$$

(3) Let E be the class of all epimorphisms in $\mathscr{C}$. A monomorphism $i: Z \to W$ in $\mathscr{C}$ is called a strong monomorphism if i belongs to $E^{\perp}$.

(4) Let M be the class of all monomorphisms in $\mathscr{C}$. An epimorphism $p: X \to Y$ in $\mathscr{C}$ is called a strong epimorphism if p belongs to $^{\perp}M$.

Proposition 6.2

Let C be a class of morphisms in $\mathscr{C}$.

(1) If D is a class of morphisms in $\mathscr{C}$ which contains C, then $C^{\perp} \supset D^{\perp}$ and ${}^{\perp}C \supset {}^{\perp}D$.

(2) $C \subset {}^{\perp}(C^{\perp})$ and $C \subset ({}^{\perp}C)^{\perp}$ hold.

(3) $({}^{\perp}(C^{\perp}))^{\perp} = C^{\perp}$ and ${}^{\perp}(({}^{\perp}C)^{\perp}) = {}^{\perp}C$ hold.

Proposition 6.3

(1) If $i : Z \to W$ is an equalizer of $f, g : W \to V$, then i is a strong monomorphism.

(2) If $p : X \to Y$ is a coequalizer of $f, g : U \to X$, then p is a strong epimorphism.

Definition 6.4

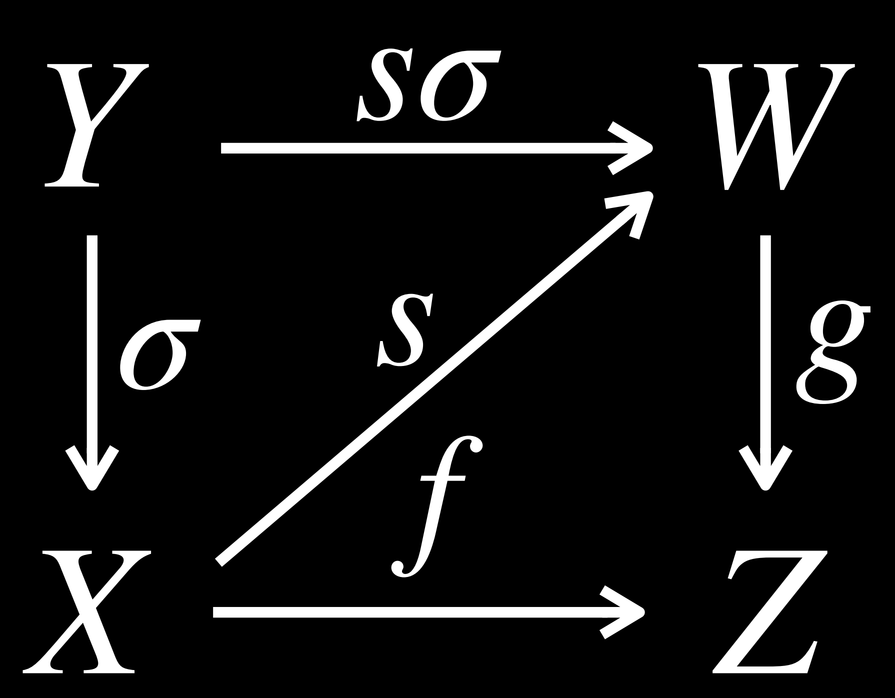
Let $\mathscr{C}$ be a category with a terminal object $1_{\mathscr{C}}$.
If a morphism $t : 1_{\mathscr{C}} \to \Omega$ satisfies the following condition, we call t a strong subobject classifier of $\mathscr{C}$.

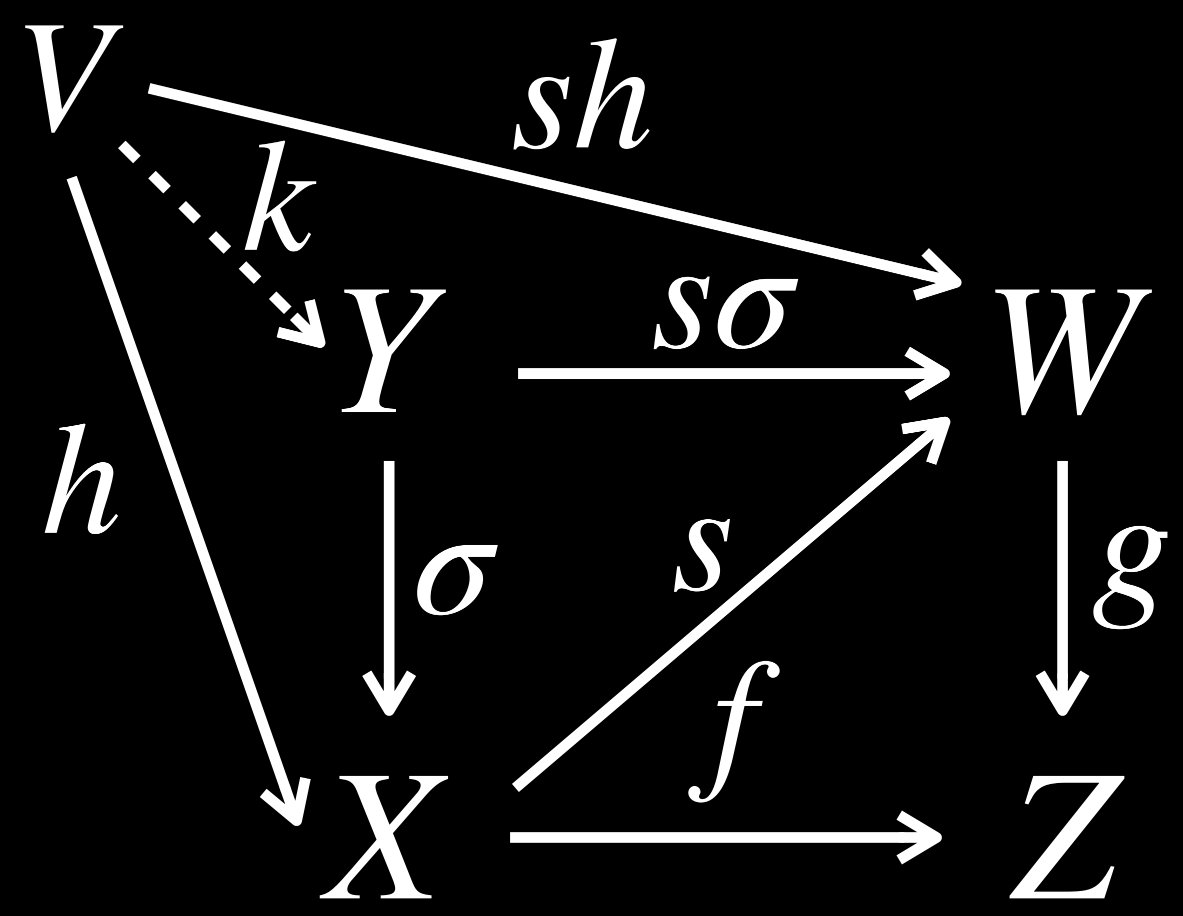
(*) For each strong monomorphism $\sigma : Y \rightarrowtail X$ in $\mathscr{C}$, there exists unique morphism $\phi_\sigma : X \to \Omega$ that makes the following square cartesian.

$$\begin{array}{ccc} Y & \xrightarrow{o_Y} & 1_{\mathscr{C}} \\ \Big\downarrow{\scriptstyle \sigma} & & \Big\downarrow{\scriptstyle t} \\ X & \xrightarrow{\phi_\sigma} & \Omega \end{array}$$

Remark 6.5

Assume that the outer rectangle of the following left diagram is cartesian. If $h:V\to X$ satisfies $fh=gsh$, then there exists unique morphism $k:V\to Y$ that satisfies $\sigma k=h$ by the assumption.





Hence if $\sigma:Y\to X$ is a monomorphism, σ is an equalizer of $f, gs:X\to Z$.

It follows that if $\mathscr{C}$ has a strong subobject classifier, each strong monomorphism in $\mathscr{C}$ is an equalizer of a certain pair of morphisms.

Proposition 6.6

A morphism $i:(Y,\mathscr{E})\to(X,\mathscr{D})$ in $\mathscr{P}_F(\mathscr{C},J)$ is a monomorphism if and only if $i:Y\to X$ is injective.

Proposition 6.7

Let $\sigma:(Y,\mathscr{F})\to(X,\mathscr{D})$ be a strong monomorphism in $\mathscr{P}_F(\mathscr{C},J)$ and denote by $i:\sigma(Y)\to X$ the inclusion map.
Then there is a surjection $\tilde{\sigma}:Y\to\sigma(Y)$ which satisfies $i\tilde{\sigma}=\sigma$.
This map gives an isomorphism $\tilde{\sigma}:(Y,\mathscr{F})\to(\sigma(Y),\mathscr{D}^i)$ in $\mathscr{P}_F(\mathscr{C},J)$.

Let $t:\{1\}\to\{0,1\}$ be an inclusion map. Then,

$$t:(\{1\},\mathscr{D}_{coarse,\{1\}})\to(\{0,1\},\mathscr{D}_{coarse,\{0,1\}})$$

is a morphism in $\mathscr{P}_F(\mathscr{C},J)$.

Proposition 6.8

Let $(X,\mathscr{D})$ be an object of $\mathscr{P}_F(\mathscr{C},J)$ and Y a subset of X.
We denote by $\sigma:Y\to X$ the inclusion map and define a map
$\phi_\sigma:X\to\{0,1\}$ by $\phi_\sigma(x)=\begin{cases}1 & x\in Y\\ 0 & x\notin Y\end{cases}$.
Then, the following diagram is a cartesian square in $\mathscr{P}_F(\mathscr{C},J)$.

$$\begin{array}{ccc}
(Y,\mathscr{D}^\sigma) & \xrightarrow{o_Y} & (\{1\},\mathscr{D}_{coarse,\{1\}}) \\
\downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle t} \\
(X,\mathscr{D}) & \xrightarrow{\phi_\sigma} & (\{0,1\},\mathscr{D}_{coarse,\{0,1\}})
\end{array}$$

Remark 6.9

The morphism $\sigma:(Y,\mathscr{D}^{\sigma})\to(X,\mathscr{D})$ is an equalizer of $\phi_{\sigma}:(X,\mathscr{D})\to(\{0,1\},\mathscr{D}_{coarse,\{0,1\}})$ and a composition $(X,\mathscr{D})\overset{o_X}{\to}(\{1\},\mathscr{D}_{coarse,\{1\}})\overset{t}{\to}(\{0,1\},\mathscr{D}_{coarse,\{0,1\}})$ by (6.5).

In particular, $\sigma:(Y,\mathscr{D}^{\sigma})\to(X,\mathscr{D})$ is a strong monomorphism in $\mathscr{P}_F(\mathscr{C},J)$ by (6.3).

Proposition 6.10

$t:(\{1\},\mathscr{D}_{coarse,\{1\}})\to(\{0,1\},\mathscr{D}_{coarse,\{0,1\}})$ is a strong subobject classifier in $\mathscr{P}_F(\mathscr{C},J)$.

By (3.9), (3.10), (5.9) and (6.10), we have the following result.

Theorem 6.11

$\mathscr{P}_F(\mathscr{C}, J)$ is a quasi-topos.

Proposition 6.12

$\pi : (X, \mathscr{D}) \to (Y, \mathscr{E})$ is an epimorphism in $\mathscr{P}_F(\mathscr{C}, J)$ if and only if $\pi : X \to Y$ is surjective.

§7. Groupoids associated with epimorphisms

Let $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (B, \mathscr{B}))$ be an object $\mathscr{P}_F(\mathscr{C}, J)^{(2)}_{(B,\mathscr{B})}$ such that π is an epimorphism. Then, π is surjective by (6.7), hence $\pi^{-1}(x)$ is not an empty set for any $x \in B$.

We denote by $i_x : \pi^{-1}(x) \to E$ the inclusion map.

Let $G_1(\boldsymbol{E})(x, y)$ be a subset of $\mathscr{P}_F(\mathscr{C}, J)((\pi^{-1}(x), \mathscr{E}^{i_x}), (\pi^{-1}(y), \mathscr{E}^{i_y}))$ consisting of elements which are isomorphisms for $x, y \in B$.

Put $G_1(\boldsymbol{E}) = \coprod_{x,y \in B} G_1(\boldsymbol{E})(x, y)$ and define maps $\sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}} : G_1(\boldsymbol{E}) \to B$, $\iota_{\boldsymbol{E}} : G_1(\boldsymbol{E}) \to G_1(\boldsymbol{E})$ and $\varepsilon_{\boldsymbol{E}} : B \to G_1(\boldsymbol{E})$ by $\sigma_{\boldsymbol{E}}(\varphi) = x$, $\tau_{\boldsymbol{E}}(\varphi) = y$, $\iota_{\boldsymbol{E}}(\varphi) = \varphi^{-1}$ if $\varphi \in G_1(\boldsymbol{E})(x, y)$ and $\varepsilon_{\boldsymbol{E}}(x) = id_{\pi^{-1}(x)}$.

Supppse that the following diagram is cartesian.

$$\begin{array}{ccc} G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E}) & \xrightarrow{\mathrm{pr}_2} & G_1(\boldsymbol{E}) \\ \Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \Big\downarrow{\scriptstyle \sigma_{\boldsymbol{E}}} \\ G_1(\boldsymbol{E}) & \xrightarrow{\tau_{\boldsymbol{E}}} & B \end{array}$$

As a set, $G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E})$ is given by

$$G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E}) = \{(\varphi,\psi)\in G_1(\boldsymbol{E})\times G_1(\boldsymbol{E}) \mid \tau_{\boldsymbol{E}}(\varphi)=\sigma_{\boldsymbol{E}}(\psi)\}.$$

We define a map $\mu_{\boldsymbol{E}}: G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E}) \to G_1(\boldsymbol{E})$ by $\mu_{\boldsymbol{E}}(\varphi,\psi)=\psi\varphi$.

We consider the following cartesian squares.

$$\begin{array}{ccc} E\times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) & \xrightarrow{\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})}} & G_1(\boldsymbol{E}) \\ \Big\downarrow{\scriptstyle \mathrm{pr}^{\sigma}_E} & & \Big\downarrow{\scriptstyle \sigma_{\boldsymbol{E}}} \\ E & \xrightarrow{\pi} & B \end{array} \qquad \begin{array}{ccc} E\times_B^{\tau_{\boldsymbol{E}}} G_1(\boldsymbol{E}) & \xrightarrow{\mathrm{pr}^{\tau}_{G_1(\boldsymbol{E})}} & G_1(\boldsymbol{E}) \\ \Big\downarrow{\scriptstyle \mathrm{pr}^{\tau}_E} & & \Big\downarrow{\scriptstyle \tau_{\boldsymbol{E}}} \\ E & \xrightarrow{\pi} & B \end{array}$$

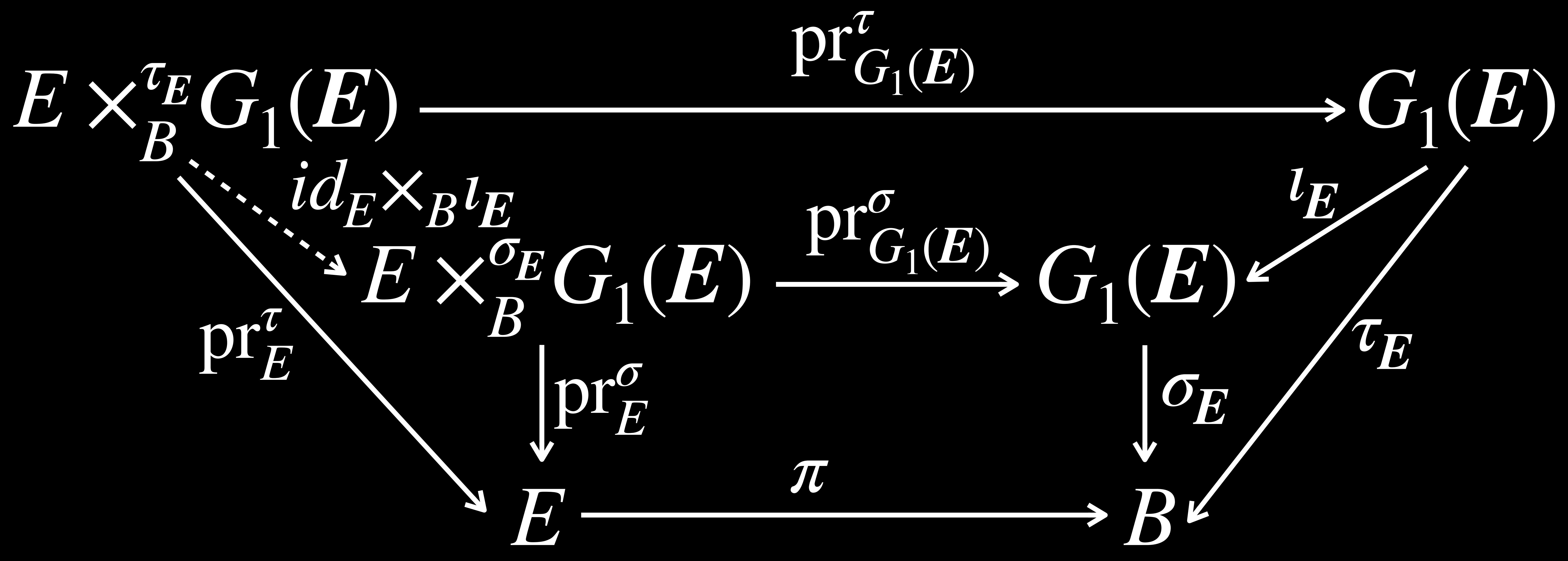
Hence $E \times_B^{\sigma_E} G_1(\boldsymbol{E})$ and $E \times_B^{\tau_E} G_1(\boldsymbol{E})$ are given as follows as sets.

$$E \times_B^{\sigma_E} G_1(\boldsymbol{E}) = \{(e,\varphi) \in E \times G_1(\boldsymbol{E}) \mid \pi(e) = \sigma_{\boldsymbol{E}}(\varphi)\},$$

$$E \times_B^{\tau_E} G_1(\boldsymbol{E}) = \{(e,\varphi) \in E \times G_1(\boldsymbol{E}) \mid \pi(e) = \tau_{\boldsymbol{E}}(\varphi)\}$$

There exists unique map $id_E \times_B \iota_{\boldsymbol{E}} : E \times_B^{\tau_E} G_1(\boldsymbol{E}) \to E \times_B^{\sigma_E} G_1(\boldsymbol{E})$

that makes the following diagram commute.



We define a map $\hat{\xi}_{\boldsymbol{E}}: E \times_B^{\sigma_E} G_1(\boldsymbol{E}) \to E$ by $\hat{\xi}_{\boldsymbol{E}}(e,\varphi) = i_{\tau_E(\varphi)}\varphi(e)$.
Let $\Sigma_{\boldsymbol{E}}$ the set of all the-ologies $\mathscr{L}$ on $G_1(\boldsymbol{E})$ which satisfy
$\mathscr{E}^{\mathrm{pr}_E^\sigma} \cap \mathscr{L}^{\mathrm{pr}_{G_1(\boldsymbol{E})}^\sigma} \subset \mathscr{E}^{\hat{\xi}_{\boldsymbol{E}}}$, $\mathscr{E}^{\mathrm{pr}_E^\tau} \cap \mathscr{L}^{\mathrm{pr}_{G_1(\boldsymbol{E})}^\tau} \subset \mathscr{E}^{\hat{\xi}_{\boldsymbol{E}}(id_E \times_B \iota_{\boldsymbol{E}})}$ and $\mathscr{L} \subset \mathscr{B}^{\sigma_{\boldsymbol{E}}} \cap \mathscr{B}^{\tau_{\boldsymbol{E}}}$.

We note that the $\mathscr{L} \in \Sigma_{\boldsymbol{E}}$ if and only if following maps are morphisms in $\mathscr{P}_F(\mathscr{C}, J)$.

$$\hat{\xi}_{\boldsymbol{E}}: \left(E \times_B^{\sigma_E} G_1(\boldsymbol{E}),\ \mathscr{E}^{\mathrm{pr}_E^\sigma} \cap \mathscr{L}^{\mathrm{pr}_{G_1(\boldsymbol{E})}^\sigma}\right) \to (E, \mathscr{E})$$
$$\hat{\xi}_{\boldsymbol{E}}(id_E \times_B \iota_{\boldsymbol{E}}): \left(E \times_B^{\tau_E} G_1(\boldsymbol{E}),\ \mathscr{E}^{\mathrm{pr}_E^\tau} \cap \mathscr{L}^{\mathrm{pr}_{G_1(\boldsymbol{E})}^\tau}\right) \to (E, \mathscr{E})$$
$$\sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}}: (G_1(\boldsymbol{E}), \mathscr{L}) \to (B, \mathscr{B})$$

Proposition 7.1
$\Sigma_{\boldsymbol{E}}$ is not empty. In fact $(G_1(\boldsymbol{E}), \mathscr{D}_{disc,G_1(\boldsymbol{E})}) \in \Sigma_{\boldsymbol{E}}$.

For $U \in \mathrm{Ob}\mathscr{C}$, we consider the following conditions (G1), (G2), (G3) on an element γ of $F_{G_1(\boldsymbol{E})}(U)$.

(G1) If $V, W \in \mathrm{Ob}\mathscr{C}$, $f \in \mathscr{C}(W, U)$, $g \in \mathscr{C}(W, V)$ and $\lambda \in \mathscr{E} \cap F_E(V)$ satisfy $\pi\lambda F(g) = \sigma_{\boldsymbol{E}}\gamma F(f)$, a composition

$$F(W) \xrightarrow{(\lambda F(g), \gamma F(f))} E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$$

belongs to $\mathscr{E} \cap F_E(W)$.

(G2) If $V, W \in \mathrm{Ob}\mathscr{C}$, $f \in \mathscr{C}(W, U)$, $g \in \mathscr{C}(W, V)$ and $\lambda \in \mathscr{E} \cap F_E(V)$ satisfy $\pi\lambda F(g) = \tau_{\boldsymbol{E}}\gamma F(f)$, a composition

$$F(W) \xrightarrow{(\lambda F(g), \iota_{\boldsymbol{E}}\gamma F(f))} E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$$

belongs to $\mathscr{E} \cap F_E(W)$.

(G3) Compositions $F(U) \xrightarrow{\gamma} G_1(\boldsymbol{E}) \xrightarrow{\sigma_{\boldsymbol{E}}} B$ and $F(U) \xrightarrow{\gamma} G_1(\boldsymbol{E}) \xrightarrow{\tau_{\boldsymbol{E}}} B$ belong to $\mathscr{B} \cap F_B(U)$.

Define a set $\mathscr{G}_{\boldsymbol{E}}$ of F-parametrizations of a set $G_1(\boldsymbol{E})$ so that $\mathscr{G}_{\boldsymbol{E}} \cap F_{G_1(\boldsymbol{E})}(U)$ is a subset of $F_{G_1(\boldsymbol{E})}(U)$ consisting of elements which satisfy the above conditions (G1), (G2) and (G3) for any $U \in \mathrm{Ob}\,\mathscr{C}$.

Remark 7.2

The conditions (G1), (G2) and (G3) on $\gamma \in F_{G_1(\boldsymbol{E})}(U)$ above are equivalent to the following conditions (G1′), (G2′) and (G3′), respectively.

(G1′) If $V, W \in \mathrm{Ob}\,\mathscr{C}$, $f \in \mathscr{C}(W, U)$, $g \in \mathscr{C}(W, V)$ and $\lambda \in \mathscr{E} \cap F_E(V)$ satisfy $\pi\lambda F(g) = \sigma_{\boldsymbol{E}}\gamma F(f)$, then γ satisfies

$$((\lambda F(g), \gamma F(f)) : F(W) \to E \times_B^{\sigma_E} G_1(\boldsymbol{E})) \in \mathscr{E}^{\hat{\xi}_E} \cap F_{E \times_B^{\sigma_E} G_1(\boldsymbol{E})}(W).$$

(G2′) If $V, W \in \mathrm{Ob}\,\mathscr{C}$, $f \in \mathscr{C}(W, U)$, $g \in \mathscr{C}(W, V)$ and $\lambda \in \mathscr{E} \cap F_E(V)$ satisfy $\pi\lambda F(g) = \tau_{\boldsymbol{E}}\gamma F(f)$, then γ satisfies

$$((\lambda F(g), \gamma F(f)) : F(W) \to E \times_B^{\tau_E} G_1(\boldsymbol{E})) \in \mathscr{E}^{\hat{\xi}_E(id_E \times_B \iota_E)} \cap F_{E \times_B^{\tau_E} G_1(\boldsymbol{E})}(W).$$

(G3′) $\gamma \in \mathscr{B}^{\sigma_E} \cap \mathscr{B}^{\tau_E} \cap F_{G_1(\boldsymbol{E})}(U)$

Proposition 7.3

$\mathscr{G}_{\boldsymbol{E}}$ is a the-ology on $G_1(\boldsymbol{E})$.

Proposition 7.4

$\mathscr{G}_{\boldsymbol{E}}$ is maximum element of $\Sigma_{\boldsymbol{E}}$.

We consider the following cartesian square.

$$\begin{array}{ccc} E\times_B^{\sigma_E}G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E}) & \xrightarrow{\mathrm{pr}_{12}} & E\times_B^{\sigma_E}G_1(\boldsymbol{E}) \\ \Big\downarrow \mathrm{pr}_3 & & \Big\downarrow \tau_{\boldsymbol{E}}\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})} \\ G_1(\boldsymbol{E}) & \xrightarrow{\sigma_{\boldsymbol{E}}} & B \end{array} \qquad \text{(i)}$$

That is, $E\times_B^{\sigma_E}G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E})$ is the following set.

$$\{(e,\varphi,\psi)\in E\times G_1(\boldsymbol{E})\times G_1(\boldsymbol{E})\mid \pi(e)=\sigma_{\boldsymbol{E}}(\varphi),\tau_{\boldsymbol{E}}(\varphi)=\sigma_{\boldsymbol{E}}(\psi)\}$$

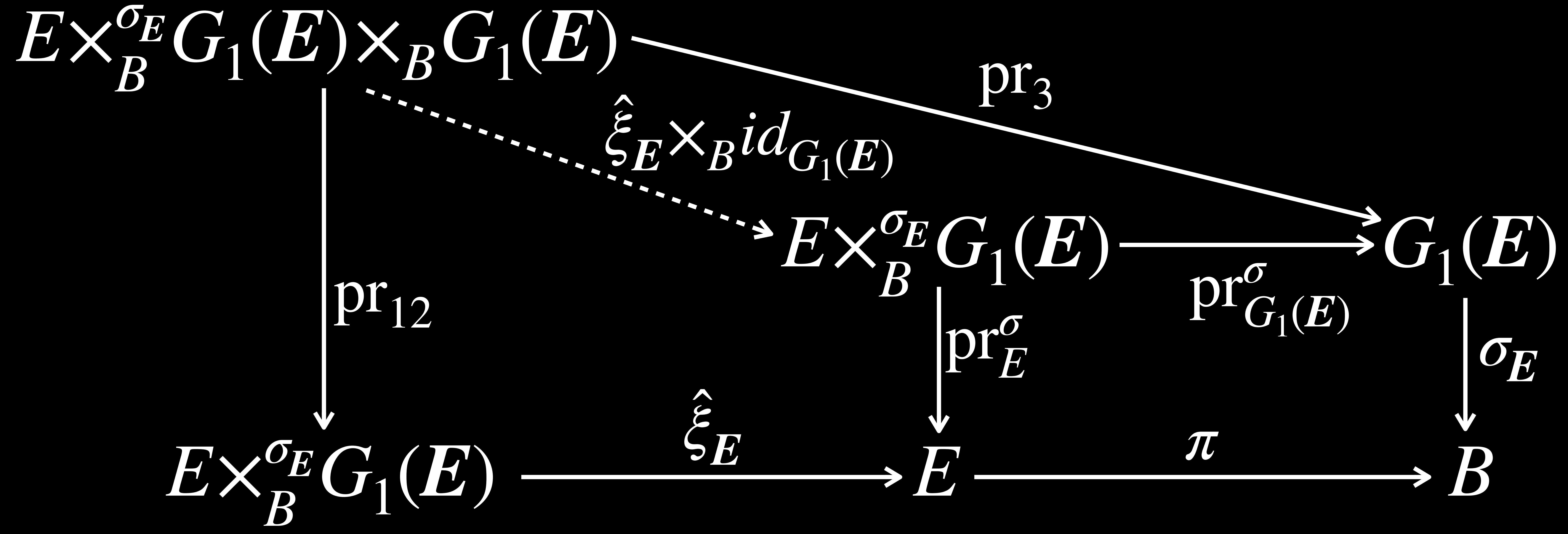
It follows from the definition of $\hat{\xi}_{\boldsymbol{E}}$ that the following diagram is commutative.

$$\begin{array}{ccc} E\times_B^{\sigma_E}G_1(\boldsymbol{E}) & \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} & E \\ \Big\downarrow \mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})} & & \Big\downarrow \pi \\ G_1(\boldsymbol{E}) & \xrightarrow{\tau_{\boldsymbol{E}}} & B \end{array} \qquad \text{(ii)}$$

There exists unique map

$$\hat{\xi}_{\boldsymbol{E}} \times_B id_{G_1(\boldsymbol{E})} : E \times_B^{\sigma_E} G_1(\boldsymbol{E}) \times_B G_1(\boldsymbol{E}) \to E \times_B^{\sigma_E} G_1(\boldsymbol{E})$$

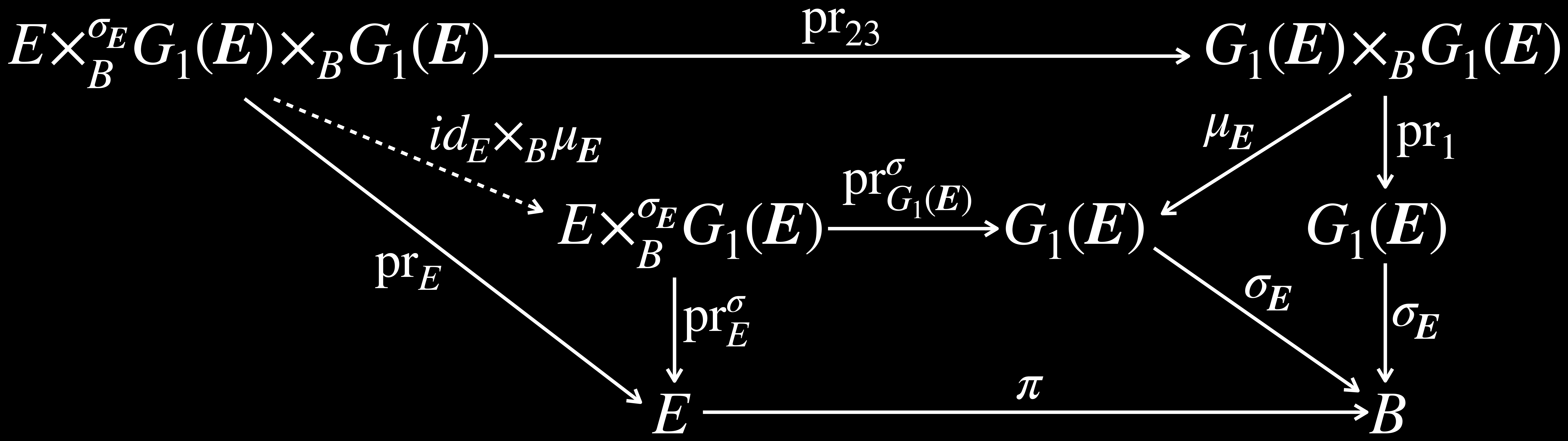
that makes the following diagram commute by the commutativity of diagrams (i) and (ii) above.



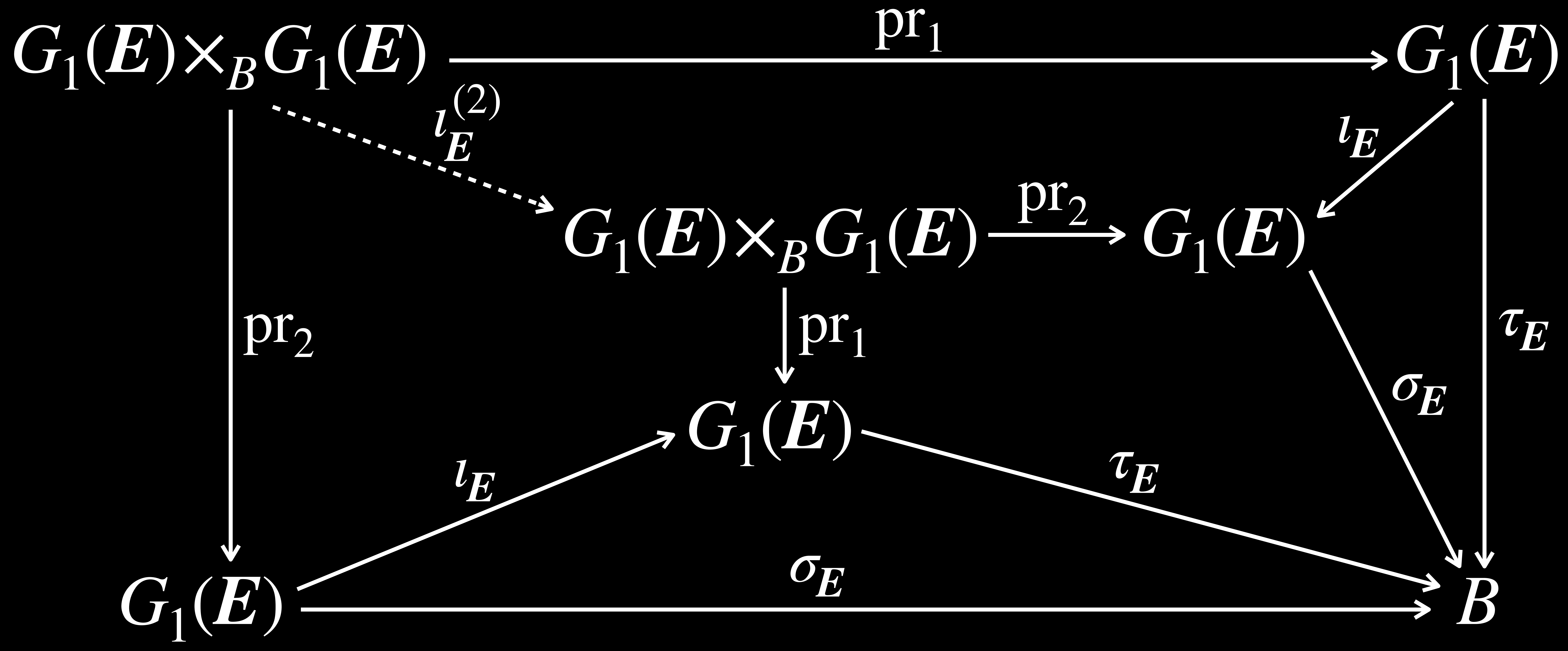
We define maps $\mathrm{pr}_{23} : E \times_B^{\sigma_E} G_1(\boldsymbol{E}) \times_B G_1(\boldsymbol{E}) \to G_1(\boldsymbol{E}) \times_B G_1(\boldsymbol{E})$ and $\mathrm{pr}_E : E \times_B^{\sigma_E} G_1(\boldsymbol{E}) \times_B G_1(\boldsymbol{E}) \to E$ by $\mathrm{pr}_{23}(e, \varphi, \psi) = (\varphi, \psi)$ and $\mathrm{pr}_E(e, \varphi, \psi) = e$, respectively. Then, there exists unique map

$$id_E \times_B \mu_{\boldsymbol{E}} : E \times_B^{\sigma_E} G_1(\boldsymbol{E}) \times_B G_1(\boldsymbol{E}) \to E \times_B^{\sigma_E} G_1(\boldsymbol{E})$$

that makes the following diagram commute.



Let $\iota_E^{(2)}: G_1(E)\times_B G_1(E) \to G_1(E)\times_B G_1(E)$ be unique map that makes the following diagram commute.



We note that $\iota_E^{(2)}$ maps $(\varphi,\psi)\in G_1(E)\times_B G_1(E)$ to $(\iota_E(\psi),\iota_E(\varphi))$.

It is easy to verify the following fact.

Lemma 7.5

The following diagrams are commutative.

$$\begin{array}{ccc} E\times_B^{\sigma_E}G_1(E)\times_B G_1(E) & \xrightarrow{id_E\times_B\mu_E} & E\times_B^{\sigma_E}G_1(E) \\ \Big\downarrow{\scriptstyle \hat{\xi}_E\times_B id_{G_1(E)}} & & \Big\downarrow{\scriptstyle \hat{\xi}_E} \\ E\times_B^{\sigma_E}G_1(E) & \xrightarrow{\hat{\xi}_E} & E \end{array}$$

$$\begin{array}{ccc} G_1(E)\times_B G_1(E) & \xrightarrow{\mu_E} & G_1(E) \\ \Big\downarrow{\scriptstyle \iota_E^{(2)}} & & \Big\downarrow{\scriptstyle \iota_E} \\ G_1(E)\times_B G_1(E) & \xrightarrow{\mu_E} & G_1(E) \end{array} \qquad \begin{array}{ccc} & E & \\ {\scriptstyle (id_E,\,\varepsilon_E\pi)}\swarrow & & \searrow{\scriptstyle id_E} \\ E\times_B^{\sigma_E}G_1(E) & \xrightarrow{\hat{\xi}_E} & E \end{array}$$

Proposition 7.6
The structure maps

$$\sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}} : (G_1(\boldsymbol{E}), \mathscr{G}_{\boldsymbol{E}}) \to (B, \mathscr{B})$$
$$\varepsilon_{\boldsymbol{E}} : (B, \mathscr{B}) \to (G_1(\boldsymbol{E}), \mathscr{G}_{\boldsymbol{E}})$$
$$\mu_{\boldsymbol{E}} : \left(G_1(\boldsymbol{E}) \times_B G_1(\boldsymbol{E}), \mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}_1} \cap \mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}_2}\right) \to (G_1(\boldsymbol{E}), \mathscr{G}_{\boldsymbol{E}})$$
$$\iota_{\boldsymbol{E}} : (G_1(\boldsymbol{E}), \mathscr{G}_{\boldsymbol{E}}) \to (G_1(\boldsymbol{E}), \mathscr{G}_{\boldsymbol{E}})$$

of the groupoid $(B, G_1(\boldsymbol{E}))$ are morphisms in $\mathscr{P}_F(\mathscr{C}, J)$.

Definition 7.7
Let $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (B, \mathscr{B}))$ be an object of $\mathscr{P}_F(\mathscr{C}, J)^{(2)}_{(B,\mathscr{B})}$ such that π is an epimorphism. We call the groupoid $((B, \mathscr{B}), (G_1(\boldsymbol{E}), \mathscr{G}_{\boldsymbol{E}}); \sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}}, \varepsilon_{\boldsymbol{E}}, \mu_{\boldsymbol{E}}, \iota_{\boldsymbol{E}})$ in $\mathscr{P}_F(\mathscr{C}, J)$ the groupoid associated with $\boldsymbol{E}$ and denote this groupoid by $\boldsymbol{G}(\boldsymbol{E})$.

Example 7.8

We denote by $o_X:(X,\mathscr{X})\to(\{1\},\mathscr{D}_{coarse,\{1\}})$ the unique morphism in $\mathscr{P}_F(\mathscr{C},J)$ for an object $(X,\mathscr{X})$ of $\mathscr{P}_F(\mathscr{C},J)$. Since o_X is an epimorphism, we can consider the groupoid $\boldsymbol{G}(\boldsymbol{O}_X)$ associated with $\boldsymbol{O}_X=((X,\mathscr{X})\xrightarrow{o_X}(\{1\},\mathscr{D}_{coarse,\{1\}})$. This groupoid

$$\boldsymbol{G}(\boldsymbol{O}_X)=((\{1\},\mathscr{D}_{coarse,\{1\}}),(G_1(\boldsymbol{O}_X),\mathscr{G}_{\boldsymbol{O}_X});\sigma_{\boldsymbol{O}_X},\tau_{\boldsymbol{O}_X},\varepsilon_{\boldsymbol{O}_X},\mu_{\boldsymbol{O}_X},\iota_{\boldsymbol{O}_X})$$

is described as follows. Put $\mathrm{End}(X,\mathscr{X})=\mathscr{P}_F(\mathscr{C},J)((X,\mathscr{X}),(X,\mathscr{X}))$ and define a subset $\mathrm{Aut}(X,\mathscr{X})$ of $\mathrm{End}(X,\mathscr{X})$ by

$$\mathrm{Aut}(X,\mathscr{X})=\{\varphi\in\mathrm{End}(X,\mathscr{X})\,|\,\varphi\ \textit{is an isomorphism.}\}.$$

Then, $G_1(\boldsymbol{O}_X)$ is identified with $\mathrm{Aut}(X,\mathscr{X})$ as a set.

The source $\sigma_{\boldsymbol{O}_X}$ and the target $\tau_{\boldsymbol{O}_X}$ are the unique map $G_1(\boldsymbol{O}_X)\to\{1\}$.

The unit $\varepsilon_{\boldsymbol{O}_X}:\{1\}\to G_1(\boldsymbol{O}_X)$ maps 1 to id_X.

The composition $\mu_{\boldsymbol{O}_X} : G_1(\boldsymbol{O}_X) \times G_1(\boldsymbol{O}_X) \to G_1(\boldsymbol{O}_X)$ maps (φ, ψ) to $\psi\varphi$ and the inverse $\iota_{\boldsymbol{O}_X} : G_1(\boldsymbol{O}_X) \to G_1(\boldsymbol{O}_X)$ maps φ to φ^{-1}.

We define a map $\alpha_X : X \times G_1(\boldsymbol{O}_X) \to X$ by $\alpha_X(x, \varphi) = \varphi(x)$, then the the-ology $\mathscr{G}_{\boldsymbol{O}_X}$ on $G_1(\boldsymbol{O}_X) = \mathrm{Aut}(X, \mathscr{X})$ is given as follows.

For $U \in \mathrm{Ob}\mathscr{C}$, $\mathscr{G}_{\boldsymbol{O}_X} \cap F_{G_1(\boldsymbol{O}_X)}(U)$ is a subset of $F_{G_1(\boldsymbol{O}_X)}(U)$ consisting of elements γ which satisfy the following condition (G).

(G) For $V, W \in \mathrm{Ob}\mathscr{C}$, $f \in \mathscr{C}(W, U)$, $g \in \mathscr{C}(W, V)$ and $\lambda \in \mathscr{X} \cap F_X(V)$, the following compositions belong to $\mathscr{X} \cap F_X(W)$.

$$F(W) \xrightarrow{(\lambda F(g),\, \gamma F(f))} X \times G_1(\boldsymbol{O}_X) \xrightarrow{\alpha_X} X$$

$$F(W) \xrightarrow{\left(\lambda F(g),\, \iota_{\boldsymbol{O}_X}\gamma F(f)\right)} X \times G_1(\boldsymbol{O}_X) \xrightarrow{\alpha_X} X$$

Let $((G,\mathscr{G});\varepsilon,\mu,\iota)$ be a group object in $\mathscr{P}_F(\mathscr{C},J)$ with structure morphisms $\varepsilon:(\{1\},\mathscr{D}_{disc,\{1\}})\to(G,\mathscr{G})$, $\iota:(G,\mathscr{G})\to(G,\mathscr{G})$ and $\mu:(G\times G,\mathscr{G}^{p_1}\cap\mathscr{G}^{p_2})\to(G,\mathscr{G})$ in $\mathscr{P}_F(\mathscr{C},J)$ which make the following diagrams commute. Here, $p_i:G\times G\to G$ denotes the projection onto the i-th component for $i=1,2$.

$$\begin{array}{ccc} G\times G\times G & \xrightarrow{\mu\times id_G} & G\times G \\ \Big\downarrow{\scriptstyle id_X\times\mu} & & \Big\downarrow{\scriptstyle \mu} \\ G\times G & \xrightarrow{\mu} & G \end{array} \qquad \begin{array}{ccccc} G\times\{1\} & \xrightarrow{id_G\times\varepsilon} & G\times G & \xleftarrow{\varepsilon\times id_G} & \{1\}\times G \\ \Big\uparrow{\scriptstyle (id_G,o_G)} & & \Big\downarrow{\scriptstyle \mu} & & \Big\uparrow{\scriptstyle (o_G,id_G)} \\ G & \xrightarrow{id_G} & G & \xleftarrow{id_G} & G \end{array}$$

$$\begin{array}{ccccc} G & \xrightarrow{(id_G,\iota)} & G\times G & \xleftarrow{(\iota,id_G)} & G \\ \Big\downarrow{\scriptstyle o_G} & & \Big\downarrow{\scriptstyle \mu} & & \Big\downarrow{\scriptstyle o_G} \\ \{1\} & \xrightarrow{\varepsilon} & G & \xleftarrow{\varepsilon} & \{1\} \end{array}$$

For an object $(B, \mathscr{B})$ of $\mathscr{P}_F(\mathscr{C}, J)$, we define a groupoid $\mathbf{G}_{G,B}$ in $\mathscr{P}_F(\mathscr{C}, J)$ as follows.
Put $G_1 = B \times G \times B$ and let $\sigma_{G,B}, \tau_{G,B} : G_1 \to B$ and $\mathrm{pr}_G : G_1 \to G$ be the projections given by $\sigma_{G,B}(x, g, y) = x$, $\tau_{G,B}(x, g, y) = y$ and $\mathrm{pr}_G(x, g, y) = g$. Define maps $\varepsilon_{G,B} : B \to G_1$ by $\varepsilon_{G,B}(x) = (x, \varepsilon(1), x)$.
Consider the following cartesian square.

$$\begin{array}{ccc} G_1 \times_B G_1 & \xrightarrow{\mathrm{pr}_2} & G_1 \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \sigma_{G,B}} \\ G_1 & \xrightarrow{\tau_{G,B}} & B \end{array}$$

Then $G_1 \times_B G_1 = \{((x, g, y), (z, h, w)) \in G_1 \times G_1 \mid y = z\}$ holds as a set.
Define maps $\mu_{G,B} : G_1 \times_B G_1 \to G_1$ and $\iota_{G,B} : G_1 \to G_1$ by $\mu_{G,B}((x, g, y), (z, h, w)) = (x, \mu(g, h), w)$ and $\iota_{G,B}(x, g, y) = (y, \iota(g), x)$.

It is clear that $\sigma_{G,B}, \tau_{G,B} : (G_1, \mathscr{B}^{\sigma_{G,B}} \cap \mathscr{G}^{\mathrm{pr}_G} \cap \mathscr{B}^{\tau_{G,B}}) \to (B, \mathscr{B})$ and $\mathrm{pr}_G : (G_1, \mathscr{B}^{\sigma_{G,B}} \cap \mathscr{G}^{\mathrm{pr}_G} \cap \mathscr{B}^{\tau_{G,B}}) \to (G, \mathscr{G})$ are morphisms in $\mathscr{P}_F(\mathscr{C}, J)$. Since $\sigma_{G,B}\varepsilon_{G,B} = \tau_{G,B}\varepsilon_{G,B} = id_X$ and the following diagram is commutative, it follows that $\varepsilon_{G,B} : (B, \mathscr{B}) \to (G_1, \mathscr{B}^{\sigma_{G,B}} \cap \mathscr{G}^{\mathrm{pr}_G} \cap \mathscr{B}^{\tau_{G,B}})$ is also a morphism in $\mathscr{P}_F(\mathscr{C}, J)$.

$$
\begin{array}{ccc}
(B, \mathscr{B}) & \xrightarrow{\varepsilon_{G,B}} & (G_1, \mathscr{B}^{\sigma_{G,B}} \cap \mathscr{G}^{\mathrm{pr}_G} \cap \mathscr{B}^{\tau_{G,B}}) \\
\downarrow o_B & & \downarrow \mathrm{pr}_G \\
(\{1\}, \mathscr{D}_{disc,\{1\}}) & \xrightarrow{\varepsilon} & (G, \mathscr{G})
\end{array}
$$

We note that $\sigma_{G,B}\mu_{G,B} = \sigma_{G,B}\mathrm{pr}_1$ and $\tau_{G,B}\mu_{G,B} = \tau_{G,B}\mathrm{pr}_2$ hold and that the following diagram commutes.

$$\begin{array}{ccc} G_1 \times_B G_1 & \xrightarrow{(\mathrm{pr}_G,\, \mathrm{pr}_G)} & G \times G \\ \Big\downarrow \mu_{G,B} & & \Big\downarrow \mu \\ G_1 & \xrightarrow{\mathrm{pr}_G} & G \end{array}$$

Since $\sigma_{G,B}$, $\tau_{G,B}$, $(\mathrm{pr}_G, \mathrm{pr}_G)$ and μ are morphisms in $\mathscr{P}_F(\mathscr{C}, J)$, it follows that

$$\mu_{G,B} : (G_1 \times_B G_1, (\mathscr{B}^{\sigma_{G,B}} \cap \mathscr{G}^{\mathrm{pr}_G} \cap \mathscr{B}^{\tau_{G,B}})^{\mathrm{pr}_1} \cap (\mathscr{B}^{\sigma_{G,B}} \cap \mathscr{G}^{\mathrm{pr}_G} \cap \mathscr{B}^{\tau_{G,B}})^{\mathrm{pr}_2}) \to (G_1, \mathscr{B}^{\sigma_{G,B}} \cap \mathscr{G}^{\mathrm{pr}_G} \cap \mathscr{B}^{\tau_{G,B}})$$

is a morphism in $\mathscr{P}_F(\mathscr{C}, J)$.

We also have $\sigma_{G,B}\iota_{G,B}=\tau_{G,B}$, $\tau_{G,B}\iota_{G,B}=\sigma_{G,B}$ and $\mathrm{pr}_G\iota_{G,B}=\iota\mathrm{pr}_G$ which imply that

$$\iota_{G,B}:(G_1,\mathscr{B}^{\sigma_{G,B}}\cap\mathscr{G}^{\mathrm{pr}_G}\cap\mathscr{B}^{\tau_{G,B}})\to(G_1,\mathscr{B}^{\sigma_{G,B}}\cap\mathscr{G}^{\mathrm{pr}_G}\cap\mathscr{B}^{\tau_{G,B}})$$

is a morphism in $\mathscr{P}_F(\mathscr{C},J)$. It is easy to verify that

$$((B,\mathscr{B}),(B\times G\times B,\mathscr{B}^{\sigma_{G,B}}\cap\mathscr{G}^{\mathrm{pr}_G}\cap\mathscr{B}^{\tau_{G,B}});\sigma_{G,B},\tau_{G,B},\varepsilon_{G,B},\mu_{G,B},\iota_{G,B})$$

is a groupoid in $\mathscr{P}_F(\mathscr{C},J)$.

Definition 7.9

The groupoid

$$((B,\mathscr{B}),(B\times G\times B,\mathscr{B}^{\sigma_{G,B}}\cap\mathscr{G}^{\mathrm{pr}_G}\cap\mathscr{B}^{\tau_{G,B}});\sigma_{G,B},\tau_{G,B},\varepsilon_{G,B},\mu_{G,B},\iota_{G,B})$$

in $\mathscr{P}_F(\mathscr{C},J)$ constructed above is called the trivial groupoid associated with $((G,\mathscr{G});\varepsilon,\mu,\iota)$ and $(B,\mathscr{B})$.

Let $(X, \mathscr{X})$ and $(B, \mathscr{B})$ be objects of $\mathscr{P}_F(\mathscr{C}, J)$.
Let us denote by $\mathrm{pr}_X : X \times B \to X$ and $\mathrm{pr}_B : X \times B \to B$ the projections.
Then we have an object $\boldsymbol{X} = ((X \times B, \mathscr{X}^{\mathrm{pr}_X} \cap \mathscr{B}^{\mathrm{pr}_B}) \xrightarrow{\mathrm{pr}_B} (B, \mathscr{B}))$ of $\mathrm{Epi}_c(\mathscr{P}_F(\mathscr{C}, J))$.
We also have a group object $G_1(\boldsymbol{O}_X) = \mathrm{Aut}(X, \mathscr{X})$ in $\mathscr{P}_F(\mathscr{C}, J)$ with unit $\varepsilon_{\boldsymbol{O}_X} : \{1\} \to G_1(\boldsymbol{O}_X)$, product $\mu_{\boldsymbol{O}_X} : G_1(\boldsymbol{O}_X) \times G_1(\boldsymbol{O}_X) \to G_1(\boldsymbol{O}_X)$ and inverse $\iota_{\boldsymbol{O}_X} : G_1(\boldsymbol{O}_X) \to G_1(\boldsymbol{O}_X)$ as we considered in (7.8).

Proposition 7.10

The groupoid $\boldsymbol{G}(\boldsymbol{X}) = ((B, \mathscr{B}), (G_1(\boldsymbol{X}), \mathscr{G}_{\boldsymbol{X}}); \sigma_{\boldsymbol{X}}, \tau_{\boldsymbol{X}}, \varepsilon_{\boldsymbol{X}}, \mu_{\boldsymbol{X}}, \iota_{\boldsymbol{X}})$ in $\mathscr{P}_F(\mathscr{C}, J)$ associated with $\boldsymbol{X}$ is isomorphic to the trivial groupoid associated with $((G_1(\boldsymbol{O}_X), \mathscr{G}_{\boldsymbol{O}_X}); \varepsilon_{\boldsymbol{O}_X}, \mu_{\boldsymbol{O}_X}, \iota_{\boldsymbol{O}_X})$ and $(B, \mathscr{B})$.

Let us denote by $\mathrm{Epi}_c(\mathscr{P}_F(\mathscr{C},J))$ a subcategory of $\mathscr{P}_F(\mathscr{C},J)^{(2)}$ whose objects are epimorphisms in $\mathscr{P}_F(\mathscr{C},J)$ and morphisms are cartesian morphisms in the fibered category $\wp : \mathscr{P}_F(\mathscr{C},J)^{(2)} \to \mathscr{P}_F(\mathscr{C},J)$ of morphisms in $\mathscr{P}_F(\mathscr{C},J)$.

Let $\boldsymbol{D} = ((D,\mathscr{D}) \xrightarrow{\rho} (A,\mathscr{A}))$, $\boldsymbol{E} = ((E,\mathscr{E}) \xrightarrow{\pi} (B,\mathscr{B}))$ be objects of $\mathrm{Epi}_c(\mathscr{P}_F(\mathscr{C},J))$ and $\boldsymbol{\xi} = \langle \xi, f \rangle : \boldsymbol{D} \to \boldsymbol{E}$ a morphism in $\mathrm{Epi}_c(\mathscr{P}_F(\mathscr{C},J))$. For $x \in A$ and $y \in B$, we denote by $j_x : \rho^{-1}(x) \to D$ and $i_y : \pi^{-1}(y) \to E$ the inclusion maps, respectively.

Then, we have unique map $\xi_x : \rho^{-1}(x) \to \pi^{-1}(f(x))$ that makes the right diagram commute.

$$\begin{array}{ccc} \rho^{-1}(x) & \xrightarrow{\xi_x} & \pi^{-1}(f(x)) \\ \downarrow{\scriptstyle j_x} & & \downarrow{\scriptstyle i_{f(x)}} \\ D & \xrightarrow{\xi} & E \end{array}$$

Lemma 7.11

$\xi_x : (\rho^{-1}(x), \mathscr{D}^{j_x}) \to (\pi^{-1}(f(x)), \mathscr{E}^{i_{f(x)}})$ is an isomorphism in $\mathscr{P}_F(\mathscr{C}, J)$.

Remark 7.12.

We consider the following cartesian square.

$$\begin{array}{ccc} A\times_B E & \xrightarrow{f_\pi} & E \\ \downarrow{\scriptstyle \pi_f} & & \downarrow{\scriptstyle \pi} \\ A & \xrightarrow{f} & B \end{array}$$

Since ξ is cartesian, $(\rho, \xi) : (D, \mathscr{D}) \to (A\times_B E, \mathscr{A}^{\pi_f} \cap \mathscr{E}^{f_\pi})$ is an isomorphism in $\mathscr{P}_F(\mathscr{C}, J)$. Put $\xi_f = (\rho, \xi)$ then ξ_f satisfies $\pi_f \xi_f = \rho$ and $f_\pi \xi_f = \xi$. Thus we have

$$\mathscr{D} = (\mathscr{A}^{\pi_f} \cap \mathscr{E}^{f_\pi})^{\xi_f} = \mathscr{A}^{\pi_f \xi_f} \cap \mathscr{E}^{f_\pi \xi_f} = \mathscr{A}^{\rho} \cap \mathscr{E}^{\xi}.$$

By (7.11), we can define a bijection

$$\xi_{x,y}: G_1(\boldsymbol{D})(x, y) \to G_1(\boldsymbol{E})(f(x), f(y))$$

by $\xi_{x,y}(\varphi) = \xi_y \varphi\, \xi_x^{-1}$ for $x, y \in A$.

We also define a map $\xi_1 : G_1(\boldsymbol{D}) \to G_1(\boldsymbol{E})$ by $\xi_1(\varphi) = \xi_{x,y}(\varphi)$ where $x = \sigma_{\boldsymbol{D}}(\varphi)$ and $y = \tau_{\boldsymbol{D}}(\varphi)$.

Note that a pair (f, ξ_1) of maps is a morphism $\boldsymbol{G}(\boldsymbol{D}) \to \boldsymbol{G}(\boldsymbol{E})$ of groupoids, that is, the following diagrams are commutative.

Here, $\xi_1 \times_f \xi_1 : G_1(\boldsymbol{D}) \times_A G_1(\boldsymbol{D}) \to G_1(\boldsymbol{E}) \times_B G_1(\boldsymbol{E})$ maps (φ, ψ) to $(\xi_1(\varphi), \xi_1(\psi))$.

$$
\begin{array}{ccccccccc}
A & \xleftarrow{\sigma_D} & G_1(\boldsymbol{D}) & \xrightarrow{\tau_D} & A & \xrightarrow{\varepsilon_D} & G_1(\boldsymbol{D}) & \xrightarrow{l_D} & G_1(\boldsymbol{D}) \\
\downarrow f & & \downarrow \xi_1 & & \downarrow f & & \downarrow \xi_1 & & \downarrow \xi_1 \\
B & \xleftarrow{\sigma_E} & G_1(\boldsymbol{E}) & \xrightarrow{\tau_E} & B & \xrightarrow{\varepsilon_E} & G_1(\boldsymbol{E}) & \xrightarrow{l_E} & G_1(\boldsymbol{E})
\end{array}
\qquad
\begin{array}{ccc}
G_1(\boldsymbol{D})\times_A G_1(\boldsymbol{D}) & \xrightarrow{\mu_D} & G_1(\boldsymbol{D}) \\
\downarrow \xi_1\times_f \xi_1 & & \downarrow \xi_1 \\
G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E}) & \xrightarrow{\mu_E} & G_1(\boldsymbol{E})
\end{array}
$$

Define a map $\xi\times_f\xi_1 : D\times_A^{\sigma_D} G_1(\boldsymbol{D}) \to E\times_B^{\sigma_E} G_1(\boldsymbol{E})$ by

$$(\xi\times_f\xi_1)(e,\varphi)=(\xi(e),\xi_1(\varphi)).$$

Then, the following diagram is commutative.

$$
\begin{array}{ccc}
D\times_A^{\sigma_D} G_1(\boldsymbol{D}) & \xrightarrow{\hat{\xi}_D} & D \\
\downarrow \xi\times_f\xi_1 & & \downarrow \xi \\
E\times_B^{\sigma_E} G_1(\boldsymbol{E}) & \xrightarrow{\hat{\xi}_E} & E
\end{array}
$$

Lemma 7.13

$\xi_1 : (G_1(\boldsymbol{D}), \mathscr{G}_{\boldsymbol{D}}) \to (G_1(\boldsymbol{E}), \mathscr{G}_{\boldsymbol{E}})$ is a morphism in $\mathscr{P}_F(\mathscr{C}, J)$.
It follows that a pair of morphisms $(f, \xi_1) : \boldsymbol{G}(\boldsymbol{D}) \to \boldsymbol{G}(\boldsymbol{E})$ is a morphism of groupoids in $\mathscr{P}_F(\mathscr{C}, J)$.

We denote by $\mathrm{Grp}(\mathscr{P}_F(\mathscr{C}, J))$ the category of groupopids in $\mathscr{P}_F(\mathscr{C}, J)$. That is, objects of $\mathrm{Grp}(\mathscr{P}_F(\mathscr{C}, J))$ are groupopids in $\mathscr{P}_F(\mathscr{C}, J)$ and morphisms of $\mathrm{Grp}(\mathscr{P}_F(\mathscr{C}, J))$ are morphisms of groupopids.

Define a functor $\mathbf{Gr}: \mathrm{Epi}_c(\mathscr{P}_F(\mathscr{C}, J)) \to \mathrm{Grp}(\mathscr{P}_F(\mathscr{C}, J))$ as follows.

For an object $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (B, \mathscr{B}))$ of $\mathrm{Epi}_c(\mathscr{P}_F(\mathscr{C}, J))$, let $\mathbf{Gr}(\boldsymbol{E})$ be the groupoid $\boldsymbol{G}(\boldsymbol{E})$ associated with $\boldsymbol{E}$ as we defined in (7.7).

For a morphism $\boldsymbol{\xi} = \langle \xi, f \rangle : \boldsymbol{D} \to \boldsymbol{E}$ in $\mathrm{Epi}_c(\mathscr{P}_F(\mathscr{C}, J))$, we put $\mathbf{Gr}(\boldsymbol{\xi}) = (f, \xi_1) : \boldsymbol{G}(\boldsymbol{D}) \to \boldsymbol{G}(\boldsymbol{E})$.

Then $\mathbf{Gr}(\boldsymbol{\xi})$ is a morphism in $\mathrm{Grp}(\mathscr{P}_F(\mathscr{C}, J))$ by (7.13).

§8. Fibrations

Definition 8.1 ([4], 8.4, 8.8)

Let $\boldsymbol{G}=((G_0,\mathscr{G}_0),(G_1,\mathscr{G}_1);\sigma,\tau,\varepsilon,\mu,\iota)$ be a groupoid in $\mathscr{P}_F(\mathscr{C},J)$. We denote by $\mathrm{pr}_\sigma, \mathrm{pr}_\tau : G_0\times G_0\to G_0$ the projections given by $\mathrm{pr}_\sigma(x,y)=x$ and $\mathrm{pr}_\tau(x,y)=y$.

If a map $(\sigma,\tau):G_1\to G_0\times G_0$ given by $(\sigma,\tau)(\varphi)=(\sigma(\varphi),\tau(\varphi))$ is an epimorphism and the the-ology $(\mathscr{G}_1)_{(\sigma,\tau)}$ on $G_0\times G_0$ coincides with $\mathscr{G}_0^{\mathrm{pr}_\sigma}\cap\mathscr{G}_0^{\mathrm{pr}_\tau}$, we say that $\boldsymbol{G}$ is fibrating.

Let $\boldsymbol{E}$ be an object of $\mathrm{Epi}_c(\mathscr{P}_F(\mathscr{C},J))$. If the groupoid $\boldsymbol{G}(\boldsymbol{E})$ associated with $\boldsymbol{E}$ (7.7) is fibrating, we call $\boldsymbol{E}$ a fibration.

Remark 8.2

If $\boldsymbol{E}=((E,\mathscr{E})\xrightarrow{\pi}(B,\mathscr{B}))$ is a fibration, since $(\sigma_{\boldsymbol{E}},\tau_{\boldsymbol{E}}):G_1(\boldsymbol{E})\to B\times B$ is surjective, $G_1(\boldsymbol{E})(x,y)$ is not empty for any $x,y\in B$.
Hence fibers $(\pi^{-1}(x),\mathscr{E}^{i_x})$ of π are all isomorphic.

Lemma 8.3

Let $(X,\mathscr{X})$ and $(B,\mathscr{B})$ be objects of $\mathscr{P}_F(\mathscr{C},J)$.
We denote the projections by $\mathrm{pr}_X:X\times B\to X$ and $\mathrm{pr}_B:X\times B\to B$.
Then $\mathscr{B}$ coincides with $(\mathscr{X}^{\mathrm{pr}_X}\cap\mathscr{B}^{\mathrm{pr}_B})_{\mathrm{pr}_B}$.

Proposition 8.4

Let $\xi:\boldsymbol{D}\to\boldsymbol{E}$ be a morphism in $\mathrm{Epi}_c(\mathscr{P}_F(\mathscr{C},J))$.
If $\boldsymbol{E}$ is a fibration, so is $\boldsymbol{D}$.

Example 8.5

Let $((G,\mathscr{G});\varepsilon,\mu,\iota)$ be a group in $\mathscr{P}_F(\mathscr{C},J)$ and $(B,\mathscr{B})$ an object of $\mathscr{P}_F(\mathscr{C},J)$. Consider the trivial groupoid

$$((B,\mathscr{B}),(B\times G\times B,\mathscr{B}^{\sigma_{G,B}}\cap\mathscr{B}^{\tau_{G,B}}\cap\mathscr{G}^{\mathrm{pr}_G});\sigma_{G,B},\tau_{G,B},\varepsilon_{G,B},\mu_{G,B},\iota_{G,B})$$

in $\mathscr{P}_F(\mathscr{C},J)$ associated with $((G,\mathscr{G});\varepsilon,\mu,\iota)$ and $(B,\mathscr{B})$.

We denote this groupoid by $\boldsymbol{G}_{G,B}$.

Since $(\sigma_{G,B},\tau_{G,B}):B\times G\times B\to B\times B$ is a projection, it follows from (8.3) that $\boldsymbol{G}_{G,B}$ is fibrating.

Hence $\boldsymbol{X}=((X\times B,\mathscr{X}^{\mathrm{pr}_X}\cap\mathscr{B}^{\mathrm{pr}_B})\xrightarrow{\mathrm{pr}_B}(B,\mathscr{B}))$ is a fibration by (7.10).

We call $\boldsymbol{X}$ a product fibration.

Definition 8.6

Let $\mathscr{C}$ be a category with a terminal object $1_{\mathscr{C}}$.
For an object U of $\mathscr{C}$, we say that a functor $F:\mathscr{C}\to\mathscr{S}et$ is U-pointed if $F:\mathscr{C}(1_{\mathscr{C}},U)\to\mathscr{S}et(F(1_{\mathscr{C}}),F(U))$ is surjective.
If F is U-pointed for any object U of $\mathscr{C}$, we say that F is pointed.

Proposition 8.7

If a category $\mathscr{C}$ has a terminal object $1_{\mathscr{C}}$, then the functor $h^{1_{\mathscr{C}}}:\mathscr{C}\to\mathscr{S}et$ defined by $h^{1_{\mathscr{C}}}(U)=\mathscr{C}(1_{\mathscr{C}},U)$ and $h^{1_{\mathscr{C}}}(f:U\to V)=(f_*:\mathscr{C}(1_{\mathscr{C}},U)\to\mathscr{C}(1_{\mathscr{C}},V))$ is pointed.

Definition 8.8

Let $(\mathscr{C}, J)$ be a site. For an object U of $\mathscr{C}$, we say that a functor $F:\mathscr{C}\to\mathscr{S}et$ is U-local if F satisfies the following condition (L). If F is U-local for any object U of $\mathscr{C}$, we say that F is local.

(L) For an object V of $\mathscr{C}$ and a map $\alpha: F(V)\to F(U)$, if there exists a covering $(V_i \xrightarrow{f_i} V)_{i\in I}$ of V such that

$$F(f_i)^*:\mathscr{S}et(F(V),F(U))\to\mathscr{S}et(F(V_i),F(U))$$

maps α into the image of $F:\mathscr{C}(V_i,U)\to\mathscr{S}et(F(V_i),F(U))$ for any $i\in I$, then α belongs to the image of

$$F:\mathscr{C}(V,U)\to\mathscr{S}et(F(V),F(U)).$$

Remark 8.9

Let $\mathscr{C}$ be a category and $F:\mathscr{C}\to\mathscr{S}et$ a functor. For an object U of $\mathscr{C}$, we define a subset $\mathscr{F}_U$ of $\coprod_{V\in\mathrm{Ob}\,\mathscr{C}} F_{F(U)}(V)$ by

$$\mathscr{F}_U=\coprod_{V\in\mathrm{Ob}\,\mathscr{C}}\mathrm{Im}(F:\mathscr{C}(V,U)\to\mathscr{S}et(F(V),F(U))=F_{F(U)}(V)).$$

Then, it is easy to verify that $\mathscr{F}_U$ satisfies condition (ii) of (2.2).

(1) Assume that $\mathscr{C}$ has a terminal object $1_{\mathscr{C}}$. Since

$$\mathscr{F}_U\cap F_{F(U)}(1_{\mathscr{C}})=\mathrm{Im}(F:\mathscr{C}(1_{\mathscr{C}},U)\to F_{F(U)}(1_{\mathscr{C}})),$$

F is U-pointed if and only if $\mathscr{F}_U$ satisfies condition (i) of (2.2).

(2) For a site $(\mathscr{C},J)$, F is U-local if and only if $\mathscr{F}_U$ satisfies condition (iii) of (2.2).

Thus $\mathscr{F}_U$ is a the-ology on $F(U)$ if and only if F is U-pointed and U-local. Assume that F is pointed and local below.

For an object V, a morphism $f\colon U\to W$ in $\mathscr{C}$ and $\varphi\in\mathscr{F}_U\cap F_{F(U)}(V)$, since there exists $g\in\mathscr{C}(V,U)$ such that $F(g)=\varphi$, we have

$$(F_{F(f)})_V(\varphi)=F(f)\varphi=F(f)F(g)=F(fg)\in\mathscr{F}_U\cap F_{F(W)}(V).$$

It follows that $(F_{F(f)})_V\colon F_{F(U)}(V)\to F_{F(W)}(V)$ maps $\mathscr{F}_U\cap F_{F(U)}(V)$ into $\mathscr{F}_W\cap F_{F(W)}(V)$.

Define a functor $\check{F}\colon\mathscr{C}\to\mathscr{P}_F(\mathscr{C},J)$ by $\check{F}(U)=(F(U),\mathscr{F}_U)$ for $U\in\mathrm{Ob}\mathscr{C}$ and $\check{F}(f\colon U\to W)=(F(f)\colon(F(U),\mathscr{F}_U)\to(F(W),\mathscr{F}_W))$ for a morphism $f\colon U\to W$ in $\mathscr{C}$. Then $\Gamma\check{F}=F$ holds.

Example 8.10

Define a category $\mathscr{C}^\infty$ as follows. Objects of $\mathscr{C}^\infty$ are open sets of n dimensional Euclidean space $\boldsymbol{R}^n$ for some $n \geqq 0$. Morphisms of $\mathscr{C}^\infty$ are C^∞-maps. For $U \in \mathrm{Ob}\,\mathscr{C}^\infty$, let $P_\infty(U)$ be the set of families $(U_i \xrightarrow{f_i} U)_{i\in I}$ of open embeddings such that $U = \bigcup_{i\in I} f_i(U_i)$. It is easy to verify that P_∞ is a pretopology on $\mathscr{C}^\infty$.

We give a Grothendieck topology J_∞ on $\mathscr{C}^\infty$ generated by P_∞.

Then, the forgetful functor $F : \mathscr{C}^\infty \to \mathscr{S}et$ is pointed and local.

For a set X, a the-ology on X is usually called a diffeology on X and a the-ological object is called a diffeological space.

Example 8.11

Let k be an algebraically closed field. We denote by $\mathscr{A}\!f\!f_k$ the category of affine varieties over k. For $V \in \operatorname{Ob} \mathscr{A}\!f\!f_k$, let $P_{\mathscr{A}\!f\!f_k}(V)$ be the set of families $(V_i \xrightarrow{f_i} V)_{i \in I}$ of Zariski open embeddings such that $V = \bigcup_{i \in I} f_i(V_i)$. It is easy to verify that $P_{\mathscr{A}\!f\!f_k}(V)$ is a pretopology on $\mathscr{A}\!f\!f_k$. We give a Grothendieck topology $J_{\mathscr{A}\!f\!f_k}$ on $\mathscr{A}\!f\!f_k$ generated by $P_{\mathscr{A}\!f\!f_k}(V)$.

Then, the forgetful functor $F : \mathscr{A}\!f\!f_k \to \mathscr{S}et$ is pointed and local.

Proposition 8.12

Let $(X,\mathscr{X})$ be an object of $\mathscr{P}_F(\mathscr{C},J)$. Suppose that $F:\mathscr{C}\to\mathscr{S}et$ is U-pointed and U-local for an object U of $\mathscr{C}$.

Then, a map $\varphi:F(U)\to X$ is an F-plot if and only if $\varphi:(F(U),\mathscr{F}_U)\to(X,\mathscr{X})$ is a morphism in $\mathscr{P}_F(\mathscr{C},J)$.

Lemma 8.13

For an object $\boldsymbol{E}=((E,\mathscr{E})\xrightarrow{\pi}(B,\mathscr{B}))$ of $\mathscr{P}_F(\mathscr{C},J)$, the following diagram in $\mathscr{P}_F(\mathscr{C},J)$ is cartesian.

$$\begin{array}{ccc}
\left(E\times_B^{\sigma_E}G_1(\boldsymbol{E}),\ \mathscr{E}^{\mathrm{pr}_E^{\sigma}}\cap\mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}_{G_1(\boldsymbol{E})}^{\sigma}}\right) & \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} & (E,\mathscr{E}) \\
\Big\downarrow \mathrm{pr}_{G_1(\boldsymbol{E})}^{\sigma} & & \Big\downarrow \pi \\
(G_1(\boldsymbol{E}),\mathscr{G}_{\boldsymbol{E}}) & \xrightarrow{\tau_{\boldsymbol{E}}} & (B,\mathscr{B})
\end{array}$$

Let $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (B, \mathscr{B}))$ be a fibration. For $b \in B$, define a map $\iota_b : B \to B \times B$ by $\iota_b(x) = (b, x)$. We denote by $\mathrm{pr}_{Bi} : B \times B \to B$ the projection onto the i-th component for $i = 1, 2$.

Since $\mathrm{pr}_{B1}\iota_b$ is a constant map and $\mathrm{pr}_{B2}\iota_b$ is the identity map of B, $\iota_b : (B, \mathscr{B}) \to (B \times B, \mathscr{B}^{\mathrm{pr}_{B1}} \cap \mathscr{B}^{\mathrm{pr}_{B2}})$ is a morphism in $\mathscr{P}_F(\mathscr{C}, J)$.

For $U \in \mathrm{Ob}\,\mathscr{C}$ and $\gamma \in \mathscr{B} \cap F_B(U)$, since

$$(F_{\iota_b})_U(\gamma) \in \mathscr{B}^{\mathrm{pr}_{B1}} \cap \mathscr{B}^{\mathrm{pr}_{B2}} = (\mathscr{G}_{\boldsymbol{E}})_{(\sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}})},$$

it follows from (3.4) that there exists $R \in J(U)$ such that, for each $h \in R$, there exists $\gamma_h \in \mathscr{G}_{\boldsymbol{E}} \cap F_{G_1(\boldsymbol{E})}(\mathrm{dom}(h))$ which satisfies

$$F_{B \times B}(h)((F_{\iota_b})_U(\gamma)) = (F_{(\sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}})})_{\mathrm{dom}(h)}(\gamma_h).$$

For $u \in F(\mathrm{dom}(h))$, since $\gamma_h(u)$ belongs to $G_1(\boldsymbol{E})(b, \gamma(F(h)(u)))$ by the commutativity of the following diagram, we have $\pi((\gamma_h(u))(e)) = \gamma(F(h)(u))$ for $e \in \pi^{-1}(b)$.

$$
\begin{array}{ccccc}
F(\mathrm{dom}(h)) & & \xrightarrow{\quad \gamma_h \quad} & & G_1(\boldsymbol{E}) \\
\downarrow{\scriptstyle F(h)} & & & & \downarrow{\scriptstyle (\sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}})} \\
F(U) & \xrightarrow{\ \gamma\ } & B & \xrightarrow{\ \iota_b\ } & B \times B
\end{array}
$$

We denote by $\mathrm{pr}_{\pi^{-1}(b)} : \pi^{-1}(b) \times F(\mathrm{dom}(h)) \to \pi^{-1}(b)$ and $\mathrm{pr}_{F(\mathrm{dom}(h))} : \pi^{-1}(b) \times F(\mathrm{dom}(h)) \to F(\mathrm{dom}(h))$ the projections onto the first and second components, respectively.

We also denote by $i_b : \pi^{-1}(b) \to E$ the inclusion map.

For $(e,u)\in\pi^{-1}(b)\times F(\mathrm{dom}(h))$, since $\pi(e)=b=\sigma_{\boldsymbol{E}}\gamma_h(u)$ by the commutativity of the above diagram, we have a map

$$(i_b\mathrm{pr}_{\pi^{-1}(b)},\gamma_h\mathrm{pr}_{F(\mathrm{dom}(h))}):\pi^{-1}(b)\times F(\mathrm{dom}(h))\to E\times_B^{\sigma_{\boldsymbol{E}}}G_1(\boldsymbol{E}).$$

Let us denote by $\bar{\gamma}_h:\pi^{-1}(b)\times F(\mathrm{dom}(h))\to E$ a composition

$$\pi^{-1}(b)\times F(\mathrm{dom}(h))\xrightarrow{(i_b\mathrm{pr}_{\pi^{-1}(b)},\gamma_h\mathrm{pr}_{F(\mathrm{dom}(h))})}E\times_B^{\sigma_{\boldsymbol{E}}}G_1(\boldsymbol{E})\overset{\hat{\xi}_{\boldsymbol{E}}}{\to}E.$$

Then $\bar{\gamma}_h(e,u)=(\gamma_h(u))(e)$ holds for $(e,u)\in\pi^{-1}(b)\times F(\mathrm{dom}(h))$.

Lemma 8.14

The following diagram is cartesian in the category of sets.

$$\begin{array}{ccc}
\pi^{-1}(b)\times F(\mathrm{dom}(h)) & \xrightarrow{\bar{\gamma}_h} & E \\
\downarrow{\scriptstyle \mathrm{pr}_{F(\mathrm{dom}(h))}} & & \downarrow{\scriptstyle \pi} \\
F(\mathrm{dom}(h)) & \xrightarrow{\gamma F(h)} & B
\end{array}$$

Lemma 8.15

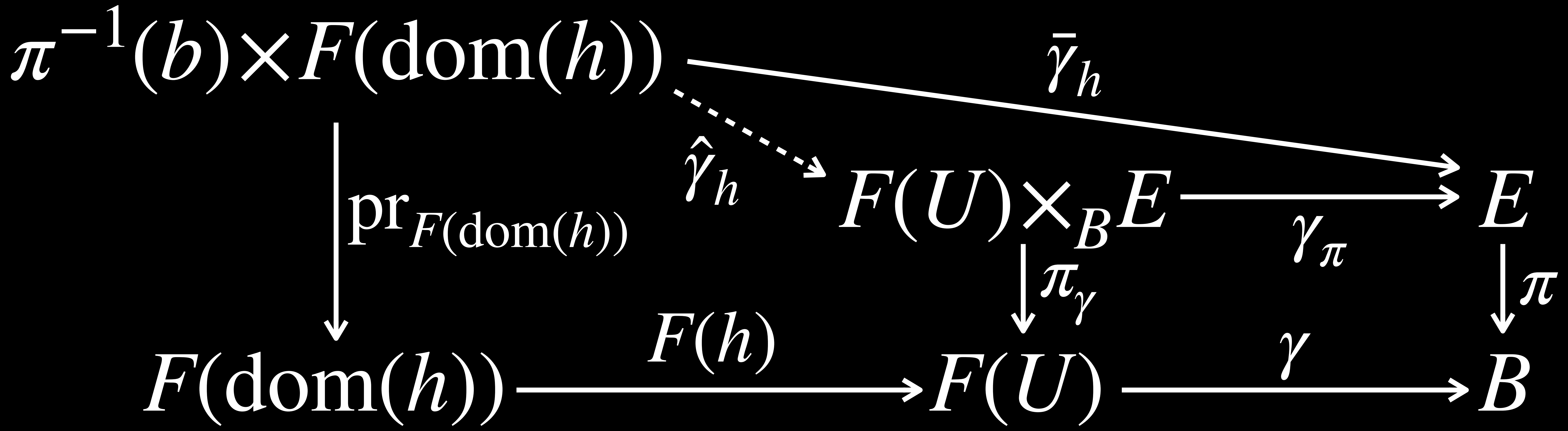
If $F : \mathscr{C} \to \mathscr{S}et$ is pointed and local, the following diagram is cartesian in $\mathscr{P}_F(\mathscr{C}, J)$.

$$\begin{array}{ccc}
\left(\pi^{-1}(b)\times F(\mathrm{dom}(h)), (\mathscr{E}^{i_b})^{\mathrm{pr}_{\pi^{-1}(b)}} \cap \mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}\right) & \xrightarrow{\bar{\gamma}_h} & (E, \mathscr{E}) \\
\Big\downarrow \mathrm{pr}_{F(\mathrm{dom}(h))} & & \Big\downarrow \pi \\
(F(\mathrm{dom}(h)), \mathscr{F}_{\mathrm{dom}(h)}) & \xrightarrow{\gamma F(h)} & (B, \mathscr{B})
\end{array}$$

Assume that the lower right rectangle of the following diagram is cartesian. Then, there exists unique map

$$\hat{\gamma}_h : \pi^{-1}(b) \times F(\mathrm{dom}(h)) \to F(U) \times_B E$$

that makes the following diagram commute.



Proposition 8.16

We assume that $F\colon \mathscr{C} \to \mathscr{S}et$ is pointed and local. Consider objects

$$\gamma^*(\boldsymbol{E}) = \big((F(U) \times_B E, \mathscr{F}_U^{\pi_\gamma} \cap \mathscr{E}^{\gamma_\pi}) \xrightarrow{\pi_\gamma} (F(U), \mathscr{F}_U)\big)$$

$$\boldsymbol{G} = \Big(\big(\pi^{-1}(b) \times F(\mathrm{dom}(h)), (\mathscr{E}^{i_b})^{\mathrm{pr}_{\pi^{-1}(b)}} \cap \mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}\big) \xrightarrow{\mathrm{pr}_{F(\mathrm{dom}(h))}} (F(\mathrm{dom}(h)), \mathscr{F}_{\mathrm{dom}(h)})\Big)$$

of $\mathscr{P}_F(\mathscr{C}, J)$. Then, $\boldsymbol{\gamma}_h = \langle \hat{\gamma}_h, F(h) \rangle \colon \boldsymbol{G} \to \gamma^*(\boldsymbol{E})$ is cartesian morphism in $\mathscr{P}_F(\mathscr{C}, J)^{(2)}$.

For morphisms $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2 : \boldsymbol{D} \to \boldsymbol{E}$ in $\mathrm{Epi}_c(\mathscr{P}_F(\mathscr{C}, J))$, we put $\boldsymbol{D} = ((D, \mathscr{D}) \xrightarrow{\rho} (A, \mathscr{A}))$, $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (B, \mathscr{B}))$ and $\boldsymbol{\zeta}_k = \langle \zeta_k, f_k \rangle$ for $k = 1,2$. For $a \in A$ and $b \in B$, we denote by $j_a : \rho^{-1}(a) \to D$, $i_b : \pi^{-1}(b) \to E$ the inclusion maps. It follows from (7.11) that the morphisms $\zeta_{k,x} : (\rho^{-1}(x), \mathscr{D}^{j_x}) \to (\pi^{-1}(f_k(x)), \mathscr{E}^{i_{f_k(x)}})$ $(k = 1,2)$ obtained by restricting $\zeta_k : (D, \mathscr{D}) \to (E, \mathscr{E})$ are isomorphisms in $\mathscr{P}_F(\mathscr{C}, J)$.

Thus we have the following isomorphism in $\mathscr{P}_F(\mathscr{C}, J)$.

$$\zeta_{2,x}\zeta_{1,x}^{-1} : (\pi^{-1}(f_1(x)), \mathscr{E}^{i_{f_1(x)}}) \to (\pi^{-1}(f_2(x)), \mathscr{E}^{i_{f_2(x)}})$$

We define a map $\tilde{\zeta} : A \to G_1(\boldsymbol{E})$ by $\tilde{\zeta}(x) = \zeta_{2,x}\zeta_{1,x}^{-1}$.

Then, $\sigma_{\boldsymbol{E}}\tilde{\zeta}(x) = f_1(x)$ and $\tau_{\boldsymbol{E}}\tilde{\zeta}(x) = f_2(x)$ hold.

The following diagram is commutative.

$$
\begin{array}{ccc}
 & & G_1(\boldsymbol{E}) \\
 & \overset{\tilde{\zeta}}{\nearrow} & \downarrow (\sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}}) \\
A & \xrightarrow{(f_1, f_2)} & B \times B
\end{array}
$$

Lemma 8.17

$\tilde{\zeta} : (A, \mathscr{A}) \to (G_1(\boldsymbol{E}), \mathscr{G}_{\boldsymbol{E}})$ is a morphism in $\mathscr{P}_F(\mathscr{C}, J)$.

Proposition 8.18 ([4], 8.9)

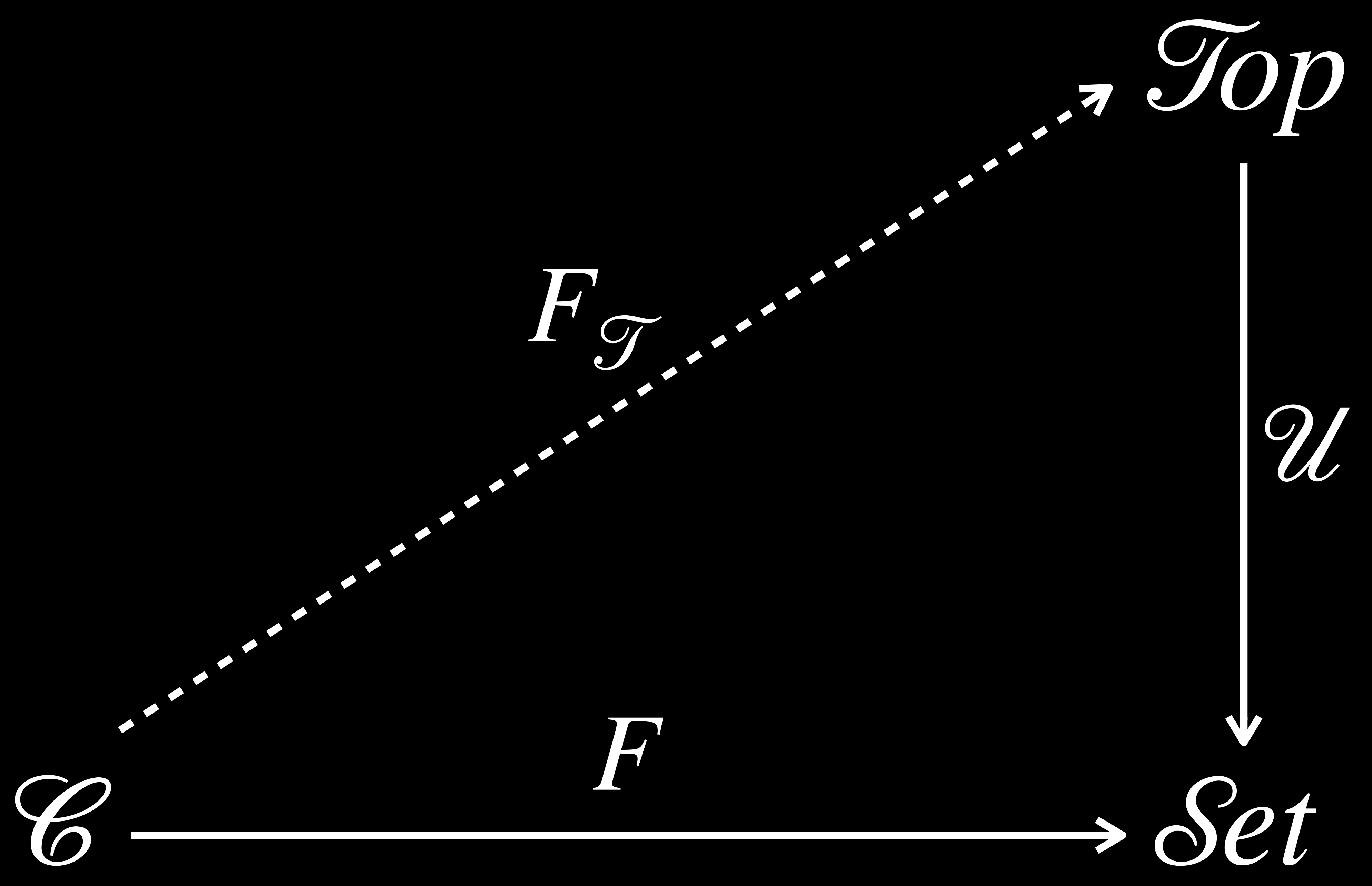
We assume that $F:\mathscr{C}\to\mathscr{S}et$ is pointed and local.
An object $\boldsymbol{E}=((E,\mathscr{E})\overset{\pi}{\to}(B,\mathscr{B}))$ of $\mathrm{Epi}_c(\mathscr{P}_F(\mathscr{C},J))$ is a fibration if and only if the following condition (P) is satisfied.

(P) There exists an object $(T,\mathscr{T})$ of $\mathscr{P}_F(\mathscr{C},J)$ such that, for any $U\in\mathrm{Ob}\mathscr{C}$ and $\gamma\in\mathscr{B}\cap F_B(U)$, there exists a covering $(U_i\overset{f_i}{\to}U)_{i\in U}$ of U such that the inverse image $(\gamma F(f_i))^*(\boldsymbol{E})$ of $\boldsymbol{E}$ by $\gamma F(f_i):F(U_i)\to B$ is isomorphic to a product fibration $(\mathrm{pr}_{F(U_i)}:(T\times F(U_i),\mathscr{T}^{\mathrm{pr}_T}\cap\mathscr{F}_{U_i}^{\mathrm{pr}_{F(U_i)}})\to(F(U_i),\mathscr{F}_{U_i})$ for any $i\in I$. Here $\mathrm{pr}_T:T\times F(U_i)\to T$ and $\mathrm{pr}_{F(U_i)}:T\times F(U_i)\to F(U_i)$ denote the projections.

§9. F-topology

Let $\mathscr{T}\!op$ be the category of topological spaces and continuous maps. We denote by $\mathscr{U}:\mathscr{T}\!op\to\mathscr{S}\!et$ the forgetful functor.

For a functor $F:\mathscr{C}\to\mathscr{S}\!et$, we assume in this section that there exists a functor $F_{\mathscr{T}}:\mathscr{C}\to\mathscr{T}\!op$ which satisfies $F=\mathscr{U}F_{\mathscr{T}}$.



We denote by $\mathscr{O}_U$ the sets of open sets of $F_{\mathscr{T}}(U)$ for $U \in \mathrm{Ob}\mathscr{C}$.

Definition 9.1

For an object $(X, \mathscr{D})$ of $\mathscr{P}_F(\mathscr{C}, J)$, we define a set $\mathscr{O}_{(X,\mathscr{D})}$ of subsets of X by

$$\mathscr{O}_{(X,\mathscr{D})} = \{ O \subset X \mid \alpha^{-1}(O) \in \mathscr{O}_U \text{ if } U \in \mathrm{Ob}\mathscr{C}, \alpha \in \mathscr{D} \cap F_X(U) \}.$$

It is easy to verify that $\mathscr{O}_{(X,\mathscr{D})}$ is a topology on X.

In fact, $\mathscr{O}_{(X,\mathscr{D})}$ is the coarsest topology on X such that $\alpha : F_{\mathscr{T}}(U) \to X$ is continuous for any $U \in \mathrm{Ob}\mathscr{C}$ and $\alpha \in \mathscr{D} \cap F_X(U)$.

We call $\mathscr{O}_{(X,\mathscr{D})}$ the F-topology on X associated with $\mathscr{D}$.

Let $\varphi:(X,\mathscr{D})\to(Y,\mathscr{E})$ be a morphism in $\mathscr{P}_F(\mathscr{C},J)$.
For $O\in\mathscr{O}_{(Y,\mathscr{E})}$ and $U\in\mathrm{Ob}\mathscr{C}$, $\alpha\in\mathscr{D}\cap F_X(U)$, since $\varphi\alpha=(F_\varphi)_U(\alpha)$ belongs to $\mathscr{E}\cap F_Y(U)$, $\alpha^{-1}(\varphi^{-1}(O))=(\varphi\alpha)^{-1}(O)\in\mathscr{O}_U$ holds.
Hence we have $\varphi^{-1}(O)\in\mathscr{O}_{(X,\mathscr{D})}$ and $\varphi:(X,\mathscr{O}_{(X,\mathscr{D})})\to(Y,\mathscr{O}_{(Y,\mathscr{E})})$ is a continuous map.

Define a functor $\mathscr{T}:\mathscr{P}_F(\mathscr{C},J)\to\mathscr{T}\!op$ by $\mathscr{T}((X,\mathscr{D}))=(X,\mathscr{O}_{(X,\mathscr{D})})$ and $\mathscr{T}(\varphi:(X,\mathscr{D})\to(Y,\mathscr{E}))=(\varphi:(X,\mathscr{O}_{(X,\mathscr{D})})\to(Y,\mathscr{O}_{(Y,\mathscr{E})}))$.

Definition 9.2
For a topological space $(X,\mathscr{O})$, we define a set $\mathscr{D}_{(X,\mathscr{O})}$ by

$$\mathscr{D}_{(X,\mathscr{O})}=\coprod_{U\in\mathrm{Ob}\mathscr{C}}\{\alpha\in F_X(U)\mid \alpha:F_{\mathscr{T}}(U)\to X \text{ is continuous.}\}.$$

If $\mathscr{D}_{(X,\mathscr{O})}$ is a the-ology on X, we call an element of $\mathscr{D}_{(X,\mathscr{O})}$ an F-$(X,\mathscr{O})$-plot.

The following proposition gives a sufficient condition for $\mathscr{D}_{(X,\mathscr{O})}$ being a the-ology on X.

Proposition 9.3

Let $(X, \mathscr{O})$ be a topological space. If the following condition (C) is satisfied for $(X, \mathscr{O})$, then $\mathscr{D}_{(X,\mathscr{O})}$ is a the-ology on X.

(C) For any $U \in \mathrm{Ob}\mathscr{C}$, a map $\alpha : F_{\mathscr{T}}(U) \to X$ is continuous if there exists a covering $(U_i \xrightarrow{f_i} U)_{i\in I}$ of U such that compositions $F_{\mathscr{T}}(U_i) \xrightarrow{F_{\mathscr{T}}(f_i)} F_{\mathscr{T}}(U) \xrightarrow{\alpha} X$ are continuous for any $i \in I$.

Remark 9.4

We consider the following condition (Q) on $F_{\mathscr{T}}: \mathscr{C} \to \mathscr{T}op$.

(Q) For any $U \in \mathrm{Ob}\,\mathscr{C}$, there exists a covering $(U_i \xrightarrow{f_i} U)_{i \in I}$ of U such that the map $\coprod_{i \in I} F_{\mathscr{T}}(U_i) \to F_{\mathscr{T}}(U)$ induced by the family $\left(F_{\mathscr{T}}(U_i) \xrightarrow{F_{\mathscr{T}}(f_i)} F_{\mathscr{T}}(U)\right)_{i \in I}$ of maps is a quotient map.

If the condition (Q) is satisfied, the condition (C) of (9.3) is satisfied for any topological space $(X, \mathscr{O})$.

Lemma 9.5

Let $(X, \mathscr{O}_X)$, $(Y, \mathscr{O}_Y)$ and $(Z, \mathscr{O}_Z)$ be topological spaces.
For continuous maps $f: X \to Y$ and $g: Y \to Z$, if $gf: X \to Z$ is a quotient map, so is g.

Proposition 9.6

For an object U of $\mathscr{C}$, suppose that there exists a covering R of U such that the map $\rho\colon \coprod_{f\in R} F_{\mathscr{T}}(\mathrm{dom}(f)) \to F_{\mathscr{T}}(U)$ induced by the family $\left(F_{\mathscr{T}}(\mathrm{dom}(f)) \xrightarrow{F_{\mathscr{T}}(f)} F_{\mathscr{T}}(U)\right)_{f\in R}$ of maps is a quotient map. Let $\bar{R}$ be the sieve on U generated by R. Then, the map

$$\bar{\rho}\colon \coprod_{u\in \bar{R}} F_{\mathscr{T}}(\mathrm{dom}(u)) \to F_{\mathscr{T}}(U)$$

$\left(F_{\mathscr{T}}(\mathrm{dom}(u)) \xrightarrow{F_{\mathscr{T}}(u)} F_{\mathscr{T}}(U)\right)_{u\in \bar{R}}$ of maps is a quotient map.

Thus we have the following result.

Proposition 9.7

The condition (Q) in (9.4) is equivalent to the following condition.

(Q') For any $U \in \mathrm{Ob}\mathscr{C}$, there exists $R \in J(U)$ such that the map $\coprod_{f \in R} F_{\mathscr{T}}(\mathrm{dom}(f)) \to F_{\mathscr{T}}(U)$ induced by the family $\left(F_{\mathscr{T}}(\mathrm{dom}(f)) \xrightarrow{F_{\mathscr{T}}(f)} F_{\mathscr{T}}(U)\right)_{f \in R}$ of maps is a quotient map.

Proposition 9.8

(1) For an object $(X, \mathscr{D})$ of $\mathscr{P}_F(\mathscr{C}, J)$, we have $\mathscr{D} \subset \mathscr{D}_{(X, \mathscr{O}_{(X,\mathscr{D})})}$.

(2) For a topological space $(X, \mathscr{O})$, $\mathscr{O} \subset \mathscr{O}_{(X, \mathscr{D}_{(X,\mathscr{O})})}$ holds.

Assume that $\mathscr{D}_{(X,\mathscr{O})}$ is an object of $\mathscr{P}_F(\mathscr{C}, J)$ for any topological space $(X, \mathscr{O})$. Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be topological spaces and $f: X \to Y$ a continuous map.

Then $f: (X, \mathscr{D}_{(X,\mathscr{O}_X)}) \to (Y, \mathscr{D}_{(Y,\mathscr{O}_Y)})$ is a morphism in $\mathscr{P}_F(\mathscr{C}, J)$.
In fact, for $U \in \mathrm{Ob}\mathscr{C}$ and $\alpha \in \mathscr{D} \cap F_X(U)$, since

$$(F_f)_U(\alpha) = f\alpha : F_{\mathscr{T}}(U) \to Y$$

is continuous, $(F_f)_U(\alpha) \in \mathscr{D}_{(Y,\mathscr{O}_Y)} \cap F_Y(U)$ holds.

Define a functor $P: \mathscr{T}op \to \mathscr{P}_F(\mathscr{C}, J)$ by $P((X, \mathscr{O})) = (X, \mathscr{D}_{(X,\mathscr{O})})$ for an object $(X, \mathscr{O})$ of $\mathscr{T}op$ and

$$P(f: (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)) = (f: (X, \mathscr{D}_{(X,\mathscr{O}_X)}) \to (Y, \mathscr{D}_{(Y,\mathscr{O}_Y)}))$$

for a continuous map $f: (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$.

We remark that $\Gamma P = \mathscr{U}$ and $\mathscr{U}\mathscr{T} = \Gamma$ hold and that both P and $\mathscr{T}$ are faithful.

Proposition 9.9

Suppose that $(X, \mathscr{D}_{(X,\mathscr{O})})$ is an object of $\mathscr{P}_F(\mathscr{C}, J)$ for any topological space $(X, \mathscr{O})$. Then, $P: \mathscr{T}op \to \mathscr{P}_F(\mathscr{C}, J)$ is a right adjoint of $\mathscr{T}: \mathscr{P}_F(\mathscr{C}, J) \to \mathscr{T}op$.

For a topological space $(Y, \mathscr{O}_Y)$ and a map $f: X \to Y$, we put

$$\mathscr{O}^f = \{O \subset X \mid O = f^{-1}(V) \; for\ some\ V \in \mathscr{O}_Y\}.$$

Then $\mathscr{O}^f$ is the coarsest topology on X such that $f: X \to Y$ is a continuous map.

Proposition 9.10
For a map $f\colon X\to Y$ and an object $(Y,\mathscr{E})$ of $\mathscr{P}_F(\mathscr{C},J)$, consider the F-$(\mathscr{C},J)$-ology $\mathscr{E}^f$ on X. Then, the F-topology $\mathscr{O}_{(X,\mathscr{E}^f)}$ on X associated with $\mathscr{E}^f$ is finer than $\mathscr{O}^f_{(Y,\mathscr{E})}$.

For a topological space $(X,\mathscr{O}_X)$ and a map $f\colon X\to Y$, we put

$$\mathscr{O}_f=\{O\subset Y\mid f^{-1}(O)\in\mathscr{O}_X\}.$$

Then $\mathscr{O}_f$ is the finest topology on Y such that $f\colon X\to Y$ is a continuous map.

Proposition 9.11

For a map $f: X \to Y$ and an object $(X, \mathscr{D})$ of $\mathscr{P}_F(\mathscr{C}, J)$, consider the the-ology $\mathscr{D}_f$ on Y. Then, the F-topology $\mathscr{O}_{(Y,\mathscr{D}_f)}$ on Y associated with $\mathscr{D}_f$ is coarser than $(\mathscr{O}_{(X,\mathscr{D})})_f$.

If $F_{\mathscr{T}}: \mathscr{C} \to \mathscr{T}op$ satisfies the following condition (Q''), $\mathscr{O}_{(Y,\mathscr{D}_f)}$ coincides with $(\mathscr{O}_{(X,\mathscr{D})})_f$.

(Q'') For any $U \in \mathrm{Ob}\,\mathscr{C}$ and $R \in J(U)$, the map

$$\coprod_{f \in R} F_{\mathscr{T}}(\mathrm{dom}(f)) \to F_{\mathscr{T}}(U)$$

induced by the family $\left(F_{\mathscr{T}}(\mathrm{dom}(h)) \xrightarrow{F_{\mathscr{T}}(h)} F_{\mathscr{T}}(U)\right)_{h \in R}$ of maps is a quotient map.

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