Representations of groupoids and generalized homology theory

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§1. Internal categories and Hopf algebroids

Let $\mathbf{C}$ be a category with finite limits.

Definition 1.1

An internal category in $\mathbf{C}$ consists of the following data.

(1) A pair (C_0, C_1) of objects of $\mathbf{C}$.

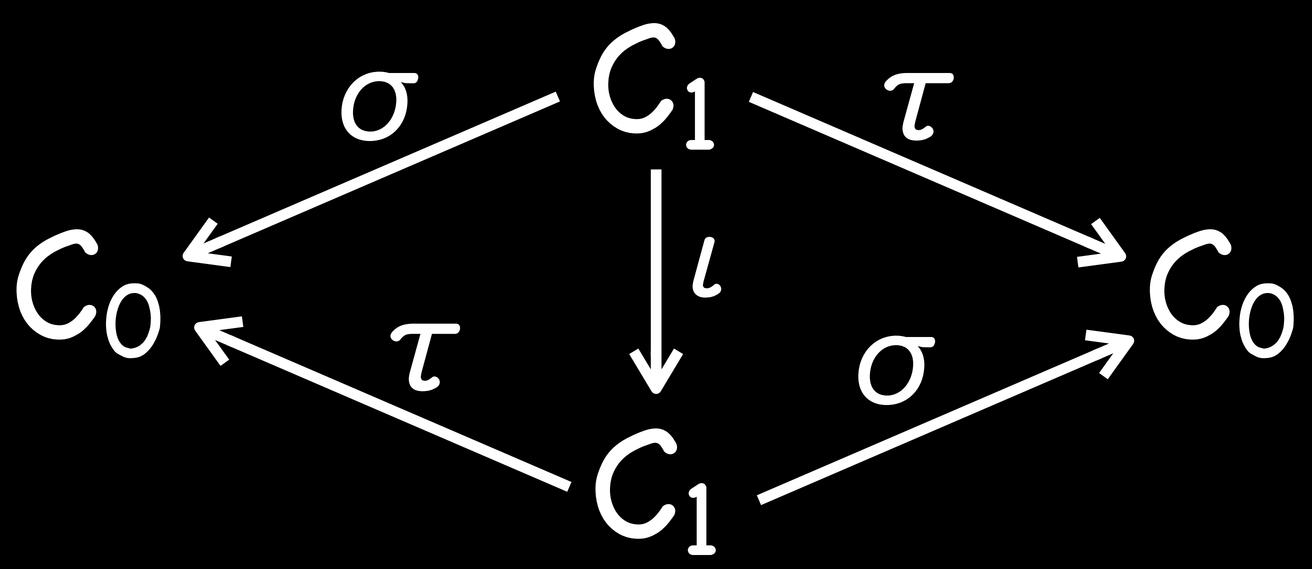
(2) Four morphisms $\sigma, \tau : C_1 \to C_0$, $\varepsilon : C_0 \to C_1$, $\mu : C_1 \times_{C_0} C_1 \to C_1$ in $\mathbf{C}$, where $C_1 \xleftarrow{pr_1} C_1 \times_{C_0} C_1 \xrightarrow{pr_2} C_1$ is a limit of $C_1 \xrightarrow{\tau} C_0 \xleftarrow{\sigma} C_1$, such that $\sigma\varepsilon = \tau\varepsilon = id_{C_0}$ and the following diagrams commute. Here $C_1 \times_{C_0} C_1 \times_{C_0} C_1$ is a limit of a diagram $C_1 \xrightarrow{\tau} C_0 \xleftarrow{\sigma} C_1 \xrightarrow{\tau} C_0 \xleftarrow{\sigma} C_1$.

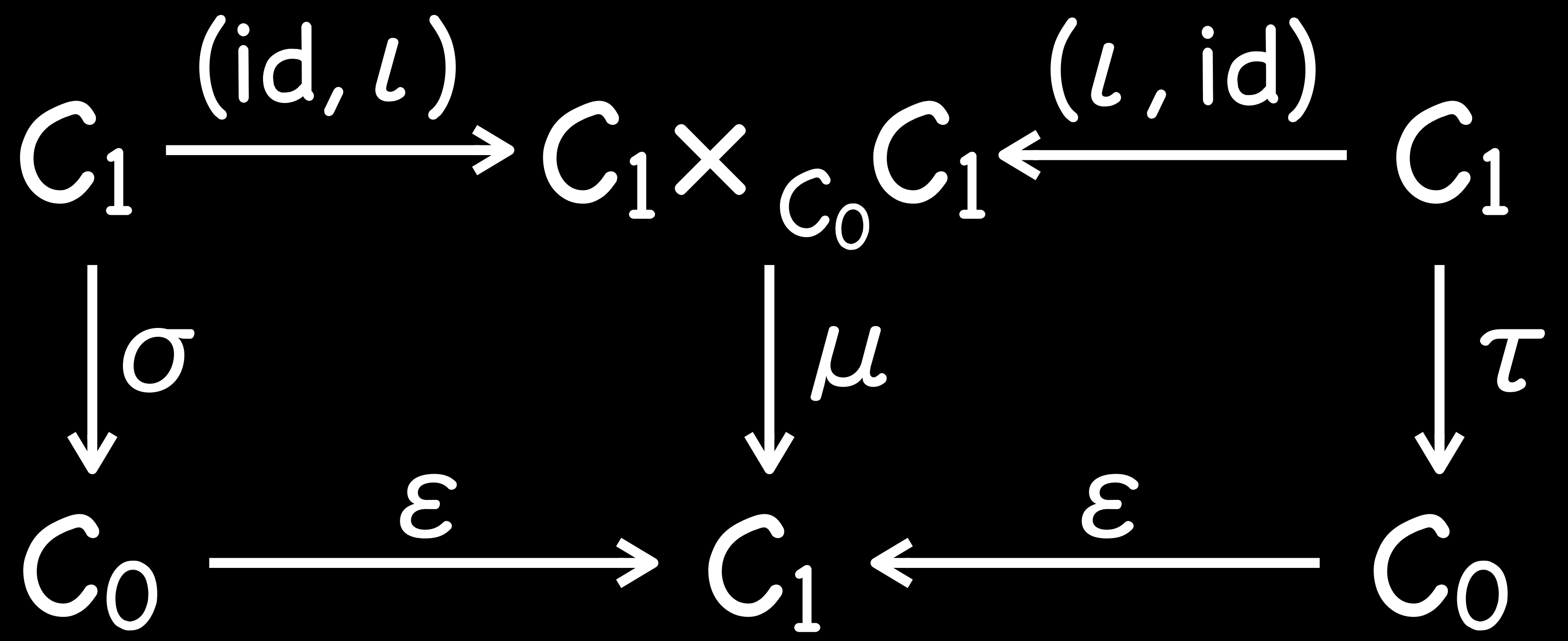
$$\begin{array}{ccccc} C_1 & \xleftarrow{pr_1} & C_1 \times_{C_0} C_1 & \xrightarrow{pr_2} & C_1 \\ \downarrow \sigma & & \downarrow \mu & & \downarrow \tau \\ C_0 & \xleftarrow{\sigma} & C_1 & \xrightarrow{\tau} & C_0 \end{array}$$

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 \times_{C_0} C_1 & \xrightarrow{\mu \times id} & C_1 \times_{C_0} C_1 \\ \downarrow id \times \mu & & \downarrow \mu \\ C_1 \times_{C_0} C_1 & \xrightarrow{\mu} & C_1 \end{array}$$

$$\begin{array}{ccccc} & \nearrow^{(id, \varepsilon\tau)} & C_1 \times_{C_0} C_1 & ^{(\varepsilon\sigma, id)}\nwarrow & \\ & & \downarrow \mu & & \\ C_1 & \xrightarrow{id} & C_1 & \xleftarrow{id} & C_1 \end{array}$$

We denote by $(C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ the internal category defined above. Moreover, if there exists a morphism $\iota : C_1 \to C_1$, which makes the following diagrams commute, we call $(C_0, C_1; \sigma, \tau, \varepsilon, \mu, \iota)$ an internal groupoid in **C**.





We also have a notion of internal functors between internal categories.

Definition 1.2

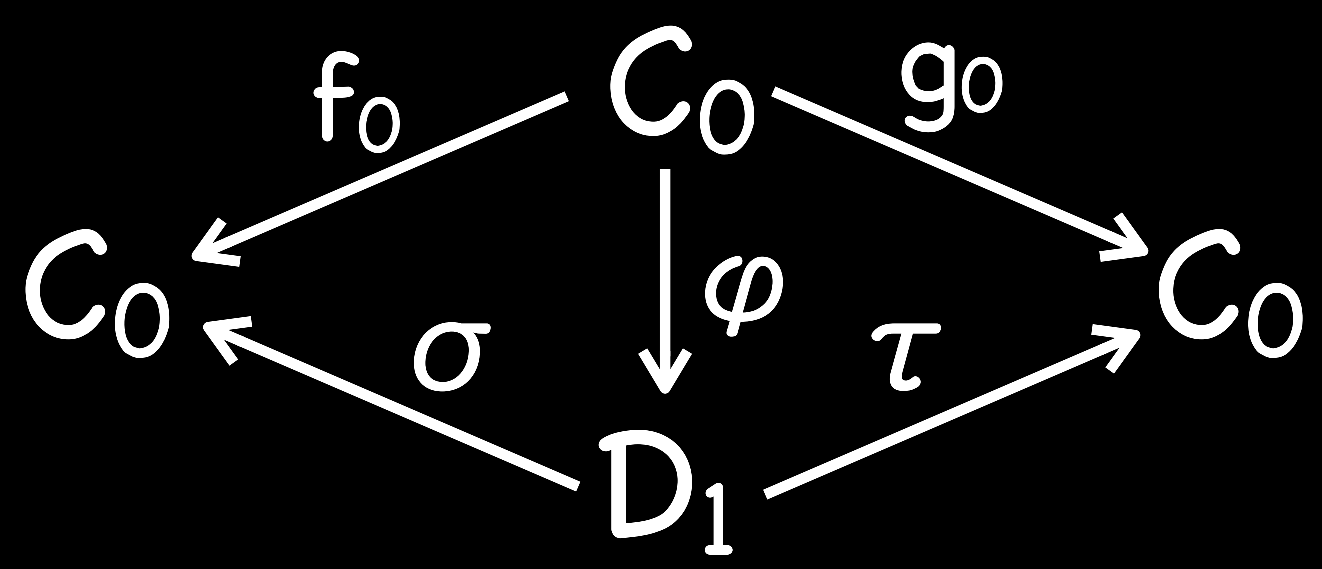
Let $C=(C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ and $D=(D_0, D_1; \sigma', \tau', \varepsilon', \mu')$ be internal categories in $\mathbf{C}$. An internal functor from C to D is a pair (f_0, f_1) of morphisms $f_0 : C_0 \to D_0$ and $f_1 : C_1 \to D_1$ which make the following diagrams commute.

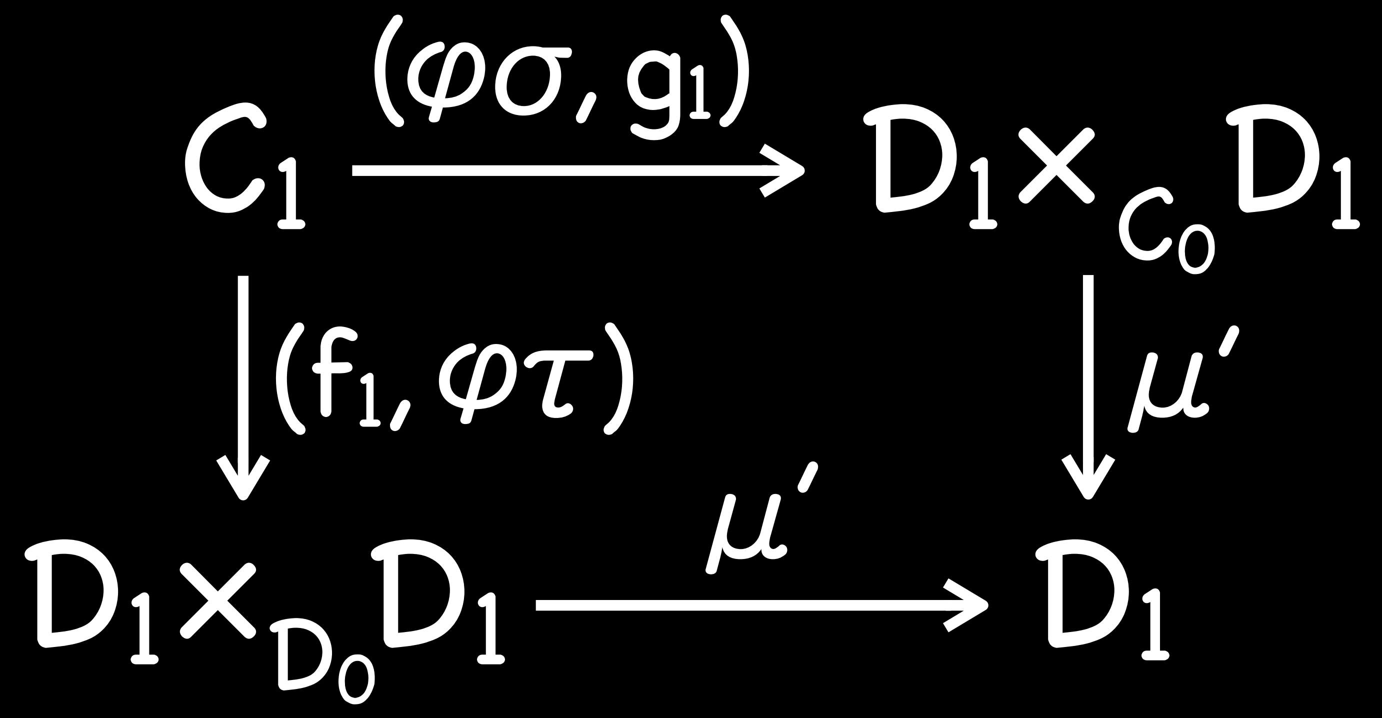
$$
\begin{array}{ccccc}
C_0 & \xleftarrow{\ \sigma\ } & C_1 & \xrightarrow{\ \tau\ } & C_0 \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
D_0 & \xleftarrow{\ \sigma'\ } & D_1 & \xrightarrow{\ \tau'\ } & D_0
\end{array}
\qquad
\begin{array}{ccccc}
C_1 \times_{C_0} C_1 & \xrightarrow{\ \mu\ } & C_1 & \xleftarrow{\ \varepsilon\ } & C_0 \\
\downarrow{\scriptstyle f_1 \times_{f_0} f_1} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
D_1 \times_{D_0} D_1 & \xrightarrow{\ \mu'\ } & D_1 & \xleftarrow{\ \varepsilon'\ } & D_0
\end{array}
$$

Definition 1.3

Let $f=(f_0,f_1),\ g=(g_0,g_1):C\to D$ be internal functors.

An internal natural transformation $\varphi:f\to g$ from f to g is a morphism $\varphi:C_0\to D_1$ in **C** which makes the following diagrams commute.





Let k be a commutative ring. We denote by $\mathbf{Alg}_k$ the category of commutative graded k-algebras and homomorphisms between them.

For objects A_* and B_* of $\mathbf{Alg}_k$, we define maps

$$i_1 : A_* \to A_* \otimes_k B_* \text{ and } i_2 : B_* \to A_* \otimes_k B_*$$

by $i_1(x) = x \otimes 1$ and $i_2(y) = 1 \otimes y$, respectively. Then, a diagram

$$A_* \xrightarrow{i_1} A_* \otimes_k B_* \xleftarrow{i_2} B_*$$

is a coproduct of A_* and B_* in $\mathbf{Alg}_k$.

For morphisms $f, g : A_* \to B_*$ in $\mathbf{Alg}_k$, let I be the ideal of B generated by $\{f(x) - g(x) \mid x \in A_*\}$. Then, the quotient map $p : B_* \to B_*/I$ is a coequalizer of f and g.

Hence $\mathbf{Alg}_k$ is a category with finite colimits, in other words, the opposite category $\mathbf{Alg}_k^{op}$ of $\mathbf{Alg}_k$ is a category with finite limits. Thus we can consider the notion of internal categories in $\mathbf{Alg}_k^{op}$.

Definition 1.4

We call an internal groupoid in $\mathbf{Alg}_k^{op}$ a Hopf algebroid.

§2. A brief review on fibered category

Let $p:\mathbf{F}\to\mathbf{C}$ be a functor.

For an object X of $\mathbf{C}$, we denote by $\mathbf{F}_X$ the subcategory of $\mathbf{F}$ consisting of objects M of $\mathbf{F}$ satisfying $p(M)=X$ and morphisms φ satisfying $p(\varphi)=\mathrm{id}_X$.

For a morphism $f:X\to Y$ of $\mathbf{C}$ and $M\in\mathrm{Ob}\,\mathbf{F}_X$, $N\in\mathrm{Ob}\,\mathbf{F}_Y$, we put

$$\mathbf{F}_f(M,N)=\{\varphi\in\mathbf{F}(M,N)\,|\,p(\varphi)=f\}.$$

Definition 2.1

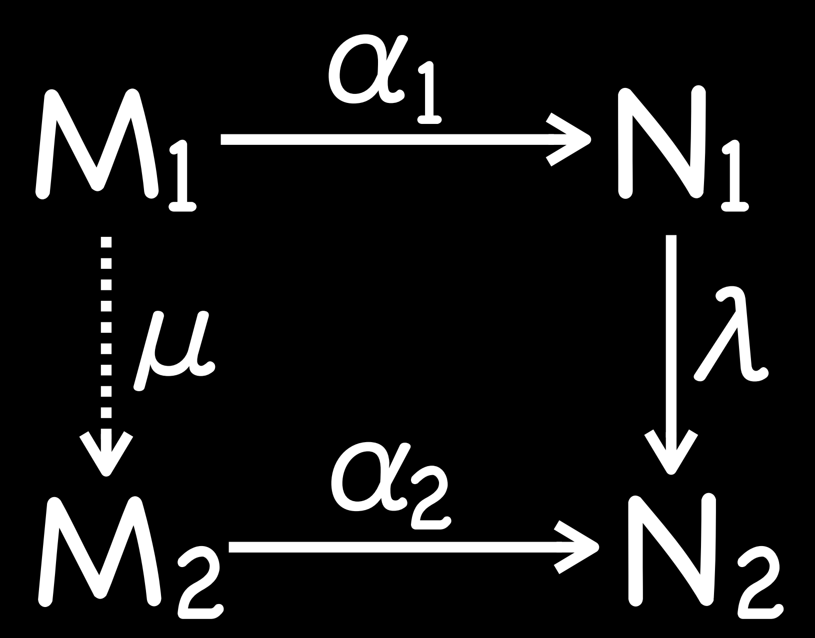
Let $\alpha:M\to N$ be a morphism in $\mathbf{F}$ and set $X=p(M)$, $f=p(\alpha)$.

We call α a **cartesian morphism** if, for any $L\in\mathrm{Ob}\,\mathbf{F}_X$, the map $\mathbf{F}_X(L,M)\to\mathbf{F}_f(L,N)$ defined by $\varphi\mapsto\alpha\varphi$ is bijective.

Proposition 2.2

Let $\alpha_i : M_i \to N_i$ $(i=1,2)$ be morphisms in $\mathbf{F}$ such that $p(M_1)=p(M_2)$, $p(N_1)=p(N_2)$, $p(\alpha_1)=p(\alpha_2)$ and $\lambda : N_1 \to N_2$ a morphism in $\mathbf{F}_{p(N_1)}$.

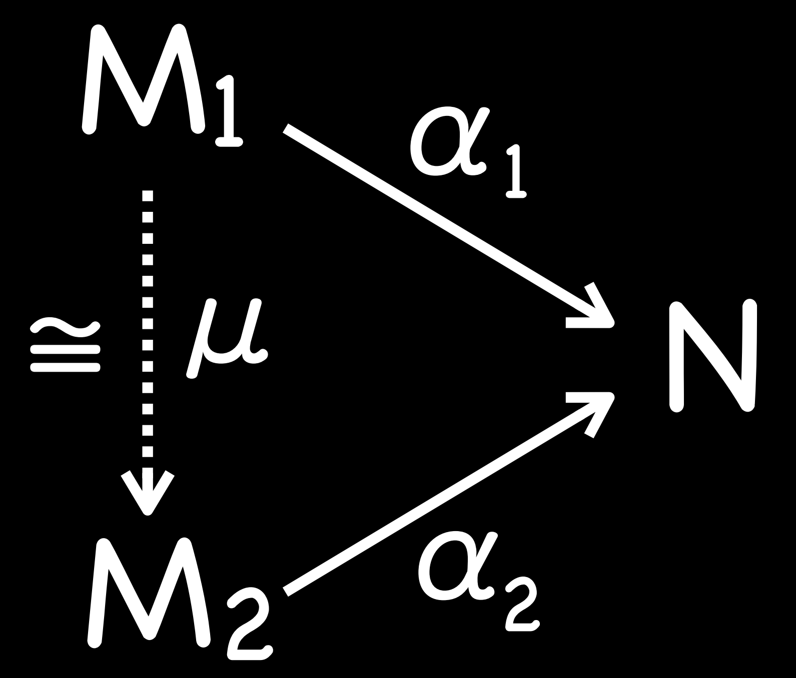
If α_2 is cartesian, there exists unique morphism $\mu : M_1 \to M_2$ in $\mathbf{F}_{p(M_1)}$ that satisfies $\alpha_2\mu = \lambda\alpha_1$.



Corollary 2.3

If $\alpha_i : M_i \to N$ $(i=1,2)$ are cartesian morphisms in $\mathbf{F}$ such that $p(M_1)=p(M_2)$ and $p(\alpha_1)=p(\alpha_2)$, there is unique morphism $\mu : M_1 \to M_2$ such that $p(\mu)=\mathrm{id}_{p(M_1)}$ and $\alpha_2\mu=\alpha_1$.

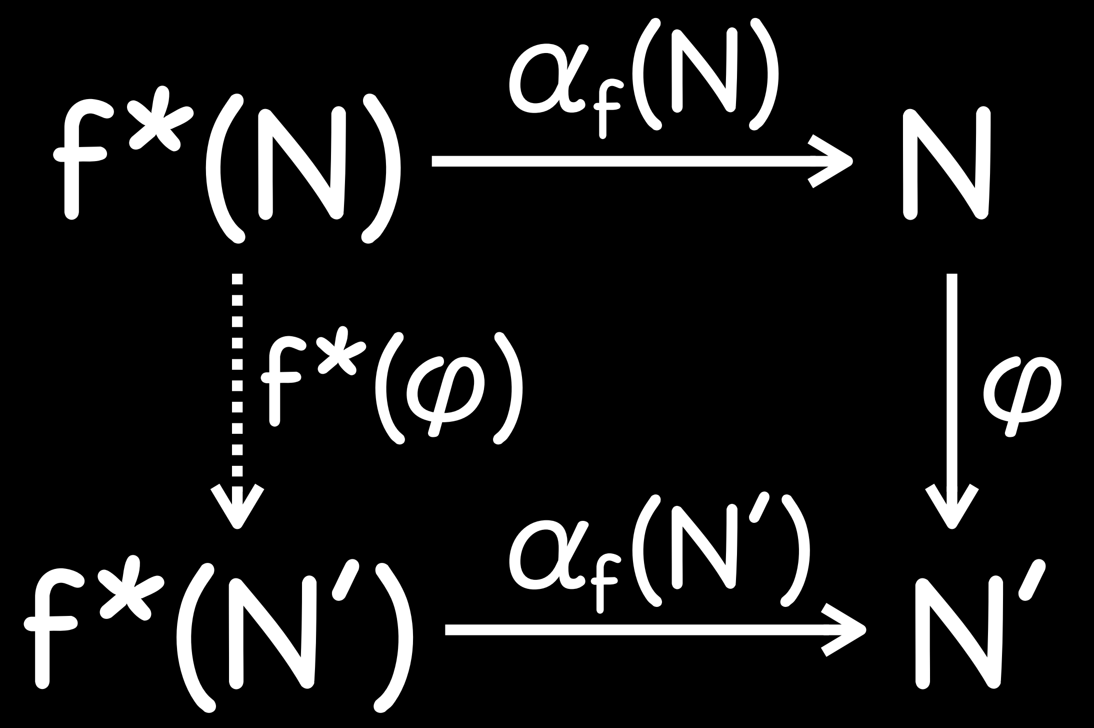
Moreover, μ is an isomorphism.



Definition 2.4
Let $f: X \to Y$ be a morphism in $\mathbf{C}$ and $N \in \mathrm{Ob}\,\mathbf{F}_Y$. If there exists a cartesian morphism $\alpha: M \to N$ such that $p(\alpha) = f$, M is called an **inverse image** of N by f.
We denote M by $f^*(N)$ and α by $\alpha_f(N): f^*(N) \to N$.
By (2.3), $f^*(N)$ is unique up to isomorphism.

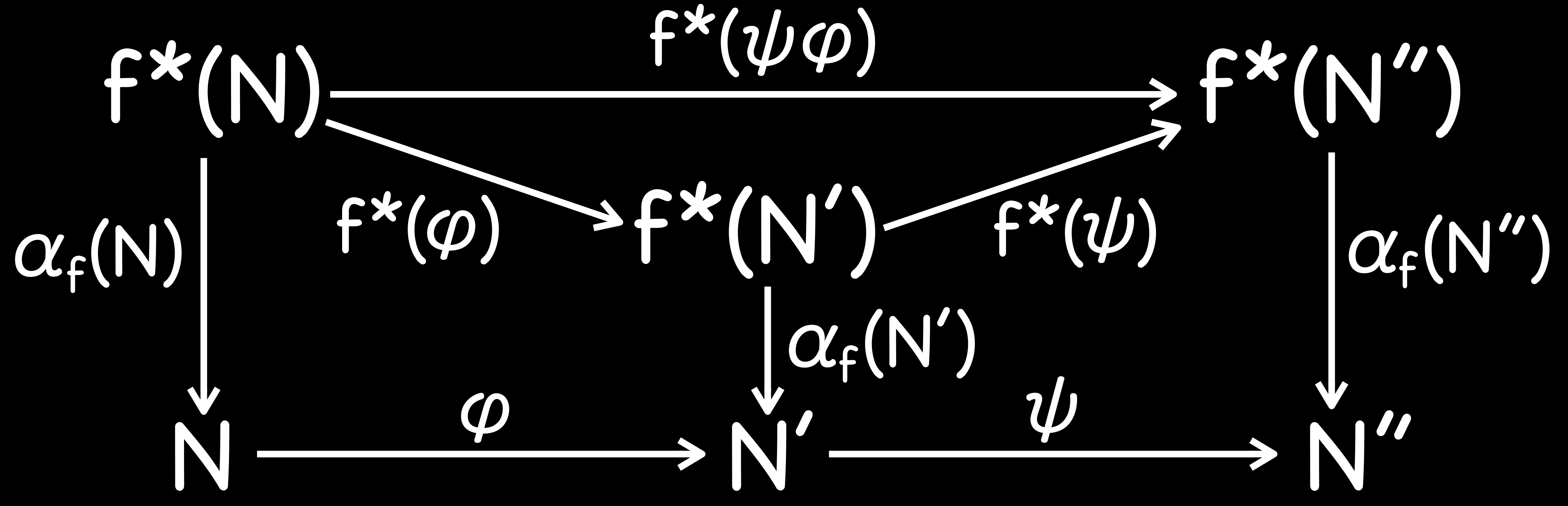
Remark 2.5
For $X \in \mathrm{Ob}\,\mathbf{C}$ and $N \in \mathrm{Ob}\,\mathbf{F}_X$, since the identity morphism id_N of N is obviously cartesian, the inverse image of N by the identity morphism id_X of X always exists and $\alpha_{id_X}(N): id_X^*(N) \to N$ can be chosen as the identity morphism of N.
By the uniqueness of $id_X^*(N)$ up to isomorphism, $\alpha_{id_X}(N): id_X^*(N) \to N$ is an isomorphism for any choice of $id_X^*(N)$.

Let $f: X \to Y$ be a morphism in **C**. Assume that cartesian morphisms $\alpha_f(N): f^*(N) \to N$ and $\alpha_f(N'): f^*(N') \to N'$ which satisfy $p(\alpha_f(N)) = p(\alpha_f(N')) = f$ exist. Then, for a morphism $\varphi: N \to N'$ in $\mathbf{F}_Y$, there exists unique morphism $f^*(\varphi): f^*(N) \to f^*(N')$ that makes the right diagram commute.



Moreover, for a morphism $\psi: N' \to N''$ in $\mathbf{F}_Y$, if an inverse image $f^*(N'')$ of N'' by f exists, we have the following diagram.

It follows from (2.2) that $f^*(\psi\varphi) = f^*(\psi) f^*(\varphi)$ holds.



Proposition 2.6

Let $f: X \to Y$ be a morphism in $\mathbf{C}$. Assume that there exists a cartesian morphism $\alpha_f(N): f^*(N) \to N$ for any $N \in \mathrm{Ob}\,\mathbf{F}_Y$. Then a correspondence $N \mapsto f^*(N)$ defines a functor $f^*: \mathbf{F}_Y \to \mathbf{F}_X$ such that, for any morphism $\varphi: N \to N'$ in $\mathbf{F}_Y$, the following diagram commutes.

$$\begin{array}{ccc} f^*(N) & \xrightarrow{\alpha_f(N)} & N \\ \downarrow{\scriptstyle f^*(\varphi)} & & \downarrow{\scriptstyle \varphi} \\ f^*(N') & \xrightarrow{\alpha_f(N')} & N' \end{array}$$

Definition 2.7

If the assumption of (2.6) is satisfied, we say that the functor of the inverse image by f exists.

Definition 2.8

If a functor $p:\mathbf{F}\to\mathbf{C}$ satisfies the following condition (i), p is called a prefibered category and if p satisfies both (i) and (ii), p is called a fibered category.

(i) For any morphism f in **C**, the functor of the inverse image by f exists.
(ii) The composition of cartesian morphisms is cartesian.

For categories **C** and **D**, we denote by Funct(**C**,**D**) the category of functors from **C** to **D** and natural transformations between them.

Definition 2.9

Let $p:\mathbf{F}\to\mathbf{C}$ be a functor. A map

$$\kappa:\mathrm{Mor}\,\mathbf{C}\to\coprod_{X,Y\in\mathrm{Ob}\,\mathbf{C}}\mathrm{Funct}(\mathbf{F}_Y,\mathbf{F}_X)$$

is called a **cleavage** if $\kappa(f)$ is an inverse image functor $f^*:\mathbf{F}_Y\to\mathbf{F}_X$ for $(f:X\to Y)\in\mathrm{Mor}\,\mathbf{C}$.

A cleavage κ is said to be **normalized** if $\kappa(\mathrm{id}_X)=\mathrm{id}_{\mathbf{F}_X}$ for any $X\in\mathrm{Ob}\,\mathbf{C}$.

A functor $p:\mathbf{F}\to\mathbf{C}$ is called a **cloven prefibered category** (resp. **normalized cloven prefibered category**) if a cleavage (resp. normalized cleavage) is given.

We assume that all fibered categories below are normalized and cloven fibered categories.

Let $f: X \to Y$, $g: Z \to X$ be morphisms in $\mathbf{C}$ and N an object of $\mathbf{F}_Y$. If $p: \mathbf{F} \to \mathbf{C}$ is a prefibered category, there exists unique morphism $c_{f,g}(N): g^*f^*(N) \to (fg)^*(N)$ of $\mathbf{F}_Z$ which makes the right diagram commute.

$$\begin{array}{ccc} g^*f^*(N) & \xrightarrow{\alpha_g(f^*(N))} & f^*(N) \\ \downarrow{\scriptstyle c_{f,g}(N)} & & \downarrow{\scriptstyle \alpha_f(N)} \\ (fg)^*(N) & \xrightarrow{\alpha_{fg}(N)} & N \end{array}$$

Then, we see the following.

Proposition 2.10

For a morphism $\varphi: M \to N$ in $\mathbf{F}_Y$, the right diagram commutes. In other words, $c_{f,g}$ gives a natural transformation $g^*f^* \to (fg)^*$ of functors from $\mathbf{F}_Y$ to $\mathbf{F}_Z$.

$$\begin{array}{ccc} g^*f^*(M) & \xrightarrow{c_{f,g}(M)} & (fg)^*(M) \\ \downarrow{\scriptstyle g^*f^*(\varphi)} & & \downarrow{\scriptstyle (fg)^*(\varphi)} \\ g^*f^*(N) & \xrightarrow{c_{f,g}(N)} & (fg)^*(N) \end{array}$$

Proposition 2.11

Let $p:\mathbf{F}\to\mathbf{C}$ is a prefibered category. Then, p is a fibered category if and only if $c_{f,g}(N)$ is an isomorphism for any diagram $Z\xrightarrow{g}X\xrightarrow{f}Y$ in $\mathbf{C}$ and $N\in\mathrm{Ob}\,\mathbf{F}_Y$.

Proposition 2.12

Let $p:\mathbf{F}\to\mathbf{C}$ be a cloven prefibered category. For a diagram $Z\xrightarrow{g}X\xrightarrow{f}Y\xrightarrow{h}W$ in $\mathbf{C}$ and an object M of $\mathbf{F}_W$, we have

$$c_{h,id_Y}(M)=\alpha_{id_Y}(id_Y^*h^*(M)),\quad c_{id_W,h}(M)=h^*(\alpha_{id_W}(M))$$

and the following diagram commutes.

$$\begin{array}{ccccc}
(f^*g^*)h^*(M) & \xrightarrow{c_{g,f}(h^*(M))} & (gf)^*h^*(M) & \xrightarrow{c_{h,gf}(M)} & (h(gf))^*(M) \\
\| & & & & \| \\
f^*(g^*h^*)(M) & \xrightarrow{f^*(c_{h,g}(M))} & f^*(hg)^*(M) & \xrightarrow{c_{hg,f}(M)} & ((hg)f)^*(M)
\end{array}$$

Example 2.13

For a commutative ring k, we denote by $\mathbf{Mod}_k$ the category of graded right k-modules and homomorphisms preserving degrees. We define a category **MOD** as follows.

$\mathrm{Ob}\,\mathbf{MOD}$ consists of triples (R_*, M_*, α) where $R_* \in \mathrm{Ob}\,\mathbf{Alg}_k$, $M_* \in \mathrm{Ob}\,\mathbf{Mod}_k$ and $\alpha : M_* \otimes_k R_* \to M_*$ is a right R_*-module structure of M_*. A morphism from (R_*, M_*, α) to (S_*, N_*, β) is a pair (λ, φ) of morphisms $\lambda \in \mathbf{Alg}_k(R_*, S_*)$ and $\varphi \in \mathbf{Mod}_k(M_*, N_*)$ such that the right diagram commutes.

$$\begin{array}{ccc} M_* \otimes_k R_* & \xrightarrow{\alpha} & M_* \\ \downarrow{\scriptstyle \varphi \otimes_k \lambda} & & \downarrow{\scriptstyle \varphi} \\ N_* \otimes_k S_* & \xrightarrow{\beta} & N_* \end{array}$$

Composition of $(\lambda, \varphi) : (R_*, M_*, \alpha) \to (S_*, N_*, \beta)$ and $(\nu, \psi) : (S_*, N_*, \beta) \to (T_*, L_*, \gamma)$ is defined to be $(\nu\lambda, \psi\varphi) : (R_*, M_*, \alpha) \to (T_*, L_*, \gamma)$.

Define a functor $p:\mathbf{MOD}\to\mathbf{Alg}_K$ by $p(R_*,M_*,\alpha)=R_*$ and $p(\lambda,\varphi)=\lambda$.
For a morphism $\lambda:S_*\to R_*$ in $\mathbf{Alg}_K$ and an object (S_*,N_*,β) of $\mathbf{MOD}$, let $\beta_\lambda:(N_*\otimes_{S_*}R_*)\otimes_K R_*\to N_*\otimes_{S_*}R_*$ be the following composition.

$$(N_*\otimes_{S_*}R_*)\otimes_K R_*\cong N_*\otimes_{S_*}(R_*\otimes_K R_*)\xrightarrow{id_{N_*}\otimes_{S_*}m}N_*\otimes_{S_*}R_*$$

Here m denotes the multiplication of R_*.
Let $i_{N_*}:N_*\to N_*\otimes_{S_*}R_*$ be the map defined by $i_{N_*}(x)=x\otimes 1$.
Then, $(\lambda,i_{N_*}):(R_*,N_*\otimes_{S_*}R_*,\beta_\lambda)\to(S_*,N_*,\beta)$ is a cartesian morphism in $\mathbf{MOD}^{op}$ and the inverse image functor $\lambda^*:\mathbf{MOD}^{op}_{S_*}\to\mathbf{MOD}^{op}_{R_*}$ is given by $\lambda^*(S_*,N_*,\beta)=(R_*,N_*\otimes_{S_*}R_*,\beta_\lambda)$ and $\lambda^*(id_{S_*},\varphi)=(id_{R_*},\varphi\otimes_{S_*}id_{R_*})$.
It can be verified that the composition of cartesian morphisms is cartesian. Hence $p^{op}:\mathbf{MOD}^{op}\to\mathbf{Alg}_K^{op}$ is a fibered category.

For a morphism $\lambda: S_* \to R_*$ in $\mathbf{Alg}_k$, we define a functor $\lambda_*: \mathbf{MOD}_{R_*} \to \mathbf{MOD}_{S_*}$ as follows.

For $(R_*, M_*, \alpha) \in \mathrm{Ob}\,\mathbf{MOD}$, we put $\lambda_*(R_*, M_*, \alpha) = (S_*, M_*, \alpha(id_{M_*} \otimes_k \lambda))$.

For a morphism $(id_{R_*}, \varphi): (R_*, M_*, \alpha) \to (R_*, N_*, \beta)$ in $\mathbf{MOD}_{R_*}$, we put $\lambda_*(id_{R_*}, \varphi) = (id_{S_*}, \varphi)$. Then, it is easy to verify that $\lambda_*: \mathbf{MOD}_{R_*} \to \mathbf{MOD}_{S_*}$ is a right adjoint of $\lambda^*: \mathbf{MOD}_{S_*} \to \mathbf{MOD}_{R_*}$.

Proposition 2.14

For any morphism $\lambda: R_* \to S_*$ in $\mathbf{Alg}_k^{op}$, the inverse image functor $\lambda^*: \mathbf{MOD}_{S_*}^{op} \to \mathbf{MOD}_{R_*}^{op}$ has a left adjoint $\lambda_*: \mathbf{MOD}_{R_*}^{op} \to \mathbf{MOD}_{S_*}^{op}$.

§3. Representations of internal categories

Let $p:\mathbf{F}\to\mathbf{C}$ be a fibered category. For a diagram $Y\xleftarrow{f}X\xrightarrow{g}Z$ in $\mathbf{C}$, we define a functor $F_{f,g}:\mathbf{F}_Y^{op}\times\mathbf{F}_Z\to\mathbf{Set}$ by $F_{f,g}(M,N)=\mathbf{F}_X(f^*(M),g^*(N))$ for $M\in\mathrm{Ob}\,\mathbf{F}_Y$, $N\in\mathrm{Ob}\,\mathbf{F}_Z$ and $F_{f,g}(\varphi,\psi):F_{f,g}(M,N)\to F_{f,g}(K,L)$ is defined to be the following composition for $(\varphi:K\to M)\in\mathrm{Mor}\,\mathbf{F}_Y$ and $(\psi:N\to L)\in\mathrm{Mor}\,\mathbf{F}_Z$.

$$\mathbf{F}_X(f^*(M),g^*(N))\xrightarrow{g^*(\psi)_*}\mathbf{F}_X(f^*(M),g^*(L))\xrightarrow{f^*(\varphi)^*}\mathbf{F}_X(f^*(K),g^*(N))$$

For a morphism $k:V\to X$ in $\mathbf{C}$, $M\in\mathrm{Ob}\,\mathbf{F}_Y$ and $N\in\mathrm{Ob}\,\mathbf{F}_Z$, let us define a map $k^\#_{M,N}:F_{f,g}(M,N)\to F_{fk,gk}(M,N)$ to be the following composition.

$$F_{f,g}(M,N)=\mathbf{F}_X(f^*(M),g^*(N))\xrightarrow{k^*}\mathbf{F}_V(k^*f^*(M),k^*g^*(N))\xrightarrow{(c_{f,k}(M)^{-1})^*}$$

$$\mathbf{F}_V((fk)^*(M),k^*g^*(N))\xrightarrow{c_{g,k}(N)_*}\mathbf{F}_V((fk)^*(M),(gk)^*(N))=F_{fk,gk}(M,N)$$

Proposition 3.1

Let $\varphi : M \to L$ and $\psi : P \to N$ be morphisms in $\mathbf{F}_Y$ and $\mathbf{F}_Z$, respectively. Then, the following diagram is commutative.

$$\begin{array}{ccc} \mathbf{F}_X(f^*(L), g^*(P)) & \xrightarrow{k^{\#}_{L,P}} & \mathbf{F}_V((fk)^*(L), (gk)^*(P)) \\ \downarrow f^*(\varphi)^* g^*(\psi)_* & & \downarrow (fk)^*(\varphi)^*(gk)^*(\psi)_* \\ \mathbf{F}_X(f^*(M), g^*(N)) & \xrightarrow{k^{\#}_{M,N}} & \mathbf{F}_V((fk)^*(M), (gk)^*(N)) \end{array}$$

Hence we have a natural transformation $k^{\#} : F_{f,g} \to F_{fk,gk}$.

Proposition 3.2

For morphisms $f : X \to Y$, $g : X \to Z$, $k : V \to X$, $j : W \to V$ in $\mathbf{C}$ and $M \in \operatorname{Ob} \mathbf{F}_Y$, $N \in \operatorname{Ob} \mathbf{F}_Z$, the following diagram is commutative.

$$\begin{array}{ccc} \mathbf{F}_X(f^*(M), g^*(N)) & \xrightarrow{(kj)^{\#}_{M,N}} & \mathbf{F}_V((fkj)^*(M), (gkj)^*(N)) \\ & \searrow k^{\#}_{M,N} \qquad j^{\#}_{M,N} \nearrow & \\ & \mathbf{F}_V((fk)^*(M), (gk)^*(N)) & \end{array}$$

Proposition 3.3

Let $f: X \to Y$, $g: X \to Z$, $h: X \to W$, $k: V \to X$ be morphisms in $\mathbf{C}$. For objects L, M, N of $\mathbf{F}_Y$, $\mathbf{F}_Z$, $\mathbf{F}_W$, respectively, the following diagram is commutative. Here, the horizontal maps "comp" are compositions of morphisms.

$$\begin{array}{ccc}
\mathbf{F}_X(f^*(L), g^*(M)) \times \mathbf{F}_X(g^*(M), h^*(N)) & \xrightarrow{\text{comp}} & \mathbf{F}_X(f^*(L), g^*(N)) \\
\downarrow k^{\#}_{L,M} \times k^{\#}_{M,N} & & \downarrow k^{\#}_{L,N} \\
\mathbf{F}_X((fk)^*(L), (gk)^*(M)) \times \mathbf{F}_X((gk)^*(M), (hk)^*(N)) & \xrightarrow{\text{comp}} & \mathbf{F}_X((fk)^*(L), (gk)^*(N))
\end{array}$$

For $\xi \in F_{f,g}(M,N)$, we denote $k^{\#}_{M,N}(\xi)$ by ξ_k for short below.

Definition 3.4

Let $C=(C_0,C_1;\sigma,\tau,\varepsilon,\mu)$ be an internal category in $\mathbf{C}$.
A pair (M,ξ) of an object M of $\mathbf{F}_{C_0}$ and a morphism $\xi:\sigma^*(M)\to\tau^*(M)$ in $\mathbf{F}_{C_1}$ is called a representation of C on M if the following conditions are satisfied.

(A) Let $C_1 \xleftarrow{pr_1} C_1\times_{C_0}C_1 \xrightarrow{pr_2} C_1$ be a limit of $C_1 \xrightarrow{\tau} C_0 \xleftarrow{\sigma} C_1$.
Then, the following diagram is commutative.

$$\begin{array}{ccc} (\sigma pr_1)^*(M)=(\sigma\mu)^*(M) & \xrightarrow{\xi_\mu} & (\tau\mu)^*(M)=(\tau pr_2)^*(M) \\ & {\scriptstyle \xi_{pr_1}} \searrow \qquad \nearrow {\scriptstyle \xi_{pr_2}} & \\ & (\tau pr_1)^*(M)=(\sigma pr_2)^*(M) & \end{array}$$

(U) $\xi_\varepsilon:M=(\sigma\varepsilon)^*(M)\to(\tau\varepsilon)^*(M)=M$ coincides with the identity morphism of M.

Let (M,ξ) and (N,ζ) be representations of C on M and N, respectively. A morphism $\varphi : M \to N$ in $\mathbf{F}_{C_0}$ is called a morphism of representations of C if φ makes the following diagram commute.

$$\begin{array}{ccc} \sigma^*(M) & \xrightarrow{\xi} & \tau^*(M) \\ \downarrow{\scriptstyle \sigma^*(\varphi)} & & \downarrow{\scriptstyle \tau^*(\varphi)} \\ \sigma^*(N) & \xrightarrow{\zeta} & \tau^*(N) \end{array}$$

Thus we have the category of the representations of C, which we denote by $\mathrm{Rep}(C;\mathbf{F})$.

Let $C=(C_0,C_1;\sigma,\tau,\varepsilon,\mu)$, $D=(D_0,D_1;\sigma',\tau',\varepsilon',\mu')$ be internal categories in $\mathbf{C}$ and $\mathbf{f}=(f_0,f_1):D\to C$ an internal functor.

For a representation (M,ξ) of C on M, we define

$$\xi_{\mathbf{f}}:\sigma'^*(f_0^*(M))\to\tau'^*(f_0^*(M))$$

to be the following composition.

$$\sigma'^*(f_0^*(M))\xrightarrow{c_{f_0,\sigma'}(M)}(f_0^*\sigma')^*(M)=(\sigma f_1)^*(M)\xrightarrow{\xi_{f_1}}(\tau f_1)^*(M)=(f_0^*\tau')(M)$$
$$\xrightarrow{c_{f_0,\tau'}(M)^{-1}}\tau'^*(f_0^*(M))$$

Proposition 3.5

$(f_0^*(M),\xi_{\mathbf{f}})$ is a representation of D on $f_0^*(M)$. If $\varphi:(M,\xi)\to(N,\zeta)$ is a morphism of representations of C, then $f_0^*(\varphi):f_0^*(M)\to f_0^*(N)$ gives a morphism $f_0^*(\varphi):(f_0^*(M),\xi_{\mathbf{f}})\to(f_0^*(N),\zeta_{\mathbf{f}})$ of representations of D.

Definition 3.6

We call $(f_0^*(M), \xi_{\mathbf{f}})$ the restriction of (M, ξ) along $\mathbf{f}$. It follows from (3.5) that we have a functor $\mathbf{f}^{\cdot} : \mathrm{Rep}(C; \mathbf{F}) \to \mathrm{Rep}(D; \mathbf{F})$ given by

$$\mathbf{f}^{\cdot}(M, \xi) = (f_0^*(M), \xi_{\mathbf{f}}) \text{ and } \mathbf{f}^{\cdot}(\varphi) = f_0^*(\varphi).$$

Let $C=(C_0,C_1;\sigma,\tau,\varepsilon,\mu)$, $D=(D_0,D_1;\sigma',\tau',\varepsilon',\mu')$ be internal categories in $\mathbf{C}$, $\mathbf{f}=(f_0,f_1)$, $\mathbf{g}=(g_0,g_1):D\to C$ internal functors and χ an internal natural transformation from f to g. For a representation (M,ξ) of C on M, we define a morphism $\chi^{\cdot}_{(M,\xi)}:f_0^*(M)\to g_0^*(M)$ in $\mathbf{F}_{D_0}$ to be

$$\chi^{\#}_{M,M}(\xi):f_0^*(M)=(\sigma\chi)^*(M)\to(\tau\chi)^*(M)=g_0^*(M).$$

Proposition 3.7

$\chi^{\cdot}_{(M,\xi)}$ is a morphism of representations from $\mathbf{f}^{\cdot}(M,\xi)=(f_0^*(M),\xi_{\mathbf{f}})$ to $\mathbf{g}^{\cdot}(M,\xi)=(g_0^*(M),\xi_{\mathbf{g}})$ and the right diagram in $\mathrm{Rep}(D;\mathbf{F})$ commutes for a morphism $\varphi:(M,\xi)\to(N,\zeta)$ of representations of C. Thus we have a natural transformation $\chi^{\cdot}:\mathbf{f}^{\cdot}\to\mathbf{g}^{\cdot}$.

$$\begin{array}{ccc}
(f_0^*(M),\xi_{\mathbf{f}}) & \xrightarrow{f^*(\varphi)} & (f_0^*(N),\zeta_{\mathbf{f}}) \\
\downarrow{\scriptstyle \chi^{\cdot}_{(M,\xi)}} & & \downarrow{\scriptstyle \chi^{\cdot}_{(N,\zeta)}} \\
(g_0^*(M),\xi_{\mathbf{g}}) & \xrightarrow{g^*(\varphi)} & (g_0^*(N),\zeta_{\mathbf{g}})
\end{array}$$

§4. Notion of fibered representable pair

Let $p:\mathbf{F}\to\mathbf{C}$ be a fibered category and $Y\xleftarrow{f}X\xrightarrow{g}Z$ a diagram in $\mathbf{C}$.

For $M\in\mathrm{Ob}\,\mathbf{F}_Y$, we define a functor $F_{f,g,M}:\mathbf{F}_Z\to\mathbf{Set}$ by

$$F_{f,g,M}(N)=\mathbf{F}_X(f^*(M),g^*(N)) \text{ and } F_{f,g,M}(\varphi)=g^*(\varphi)_*.$$

For $N\in\mathrm{Ob}\,\mathbf{F}_Z$, we define a functor $F^N_{f,g}:\mathbf{F}_Y^{op}\to\mathbf{Set}$ by

$$F^N_{f,g}(M)=\mathbf{F}_X(f^*(M),g^*(N)) \text{ and } F^N_{f,g}(\psi)=f^*(\psi)^*.$$

Definition 4.1

If $F_{f,g,M}$ (resp. $F^N_{f,g}$) is representable, we call (f,g) a left (resp. right) **fibered representable pair** with respect to M (resp. N).

We say that (f,g) is a left (resp. right) fibered representable pair if (f,g) is a left (resp. right) fibered representable pair with respect to any $M\in\mathrm{Ob}\,\mathbf{F}_Y$ (resp. $N\in\mathrm{Ob}\,\mathbf{F}_Z$).

Remark 4.2

If $g^*:\mathbf{F}_Z\to\mathbf{F}_X$ (resp. $f^*:\mathbf{F}_Y\to\mathbf{F}_X$) has a left (resp. right) adjoint $g_*:\mathbf{F}_X\to\mathbf{F}_Z$ (resp. $f_!:\mathbf{F}_X\to\mathbf{F}_Y$), (f,g) is a left (resp. right) fibered representable pair for any morphism $f:X\to Y$ (resp. $g:X\to Z$) in $\mathbf{C}$. It follows from (2.14) that (λ,ν) is a left fibered representable pair for any diagram $S_*\xleftarrow{\lambda}R_*\xrightarrow{\nu}T_*$ in $\mathbf{Alg}_k^{op}$.

If (f,g) is a left fibered representable pair with respect to $M \in \mathrm{Ob}\,\mathbf{F}_Y$, we choose an object $M_{[f,g]}$ of $\mathbf{F}_Z$ and denote by

$$P_{f,g}(M)_N : \mathbf{F}_X(f^*(M), g^*(N)) \to \mathbf{F}_Z(M_{[f,g]}, N)$$

a bijection which is natural in $N \in \mathrm{Ob}\,\mathbf{F}_Z$.

We denote by $\iota_{f,g}(M) : f^*(M) \to g^*(M_{[f,g]})$ the morphism in $\mathbf{F}_X$ which is mapped to the identity morphism of $M_{[f,g]}$ by

$$P_{f,g}(M)_{M_{[f,g]}} : \mathbf{F}_X(f^*(M), g^*(M_{[f,g]})) \to \mathbf{F}_Z(M_{[f,g]}, M_{[f,g]}).$$

We note that, if $g^* : \mathbf{F}_Z \to \mathbf{F}_X$ has a left adjoint $g_* : \mathbf{F}_X \to \mathbf{F}_Z$, we can choose $g_*(f^*(M))$ as $M_{[f,g]}$. We denote by $\eta : \mathrm{id}_{\mathbf{F}_X} \to g^* g_*$ the unit of the adjunction $g_* \dashv g^*$. Then, we have

$$\iota_{f,g}(M) = \eta_{f^*(M)} : f^*(M) \to g^*(g_*(f^*(M))) = g^*(M_{[f,g]}).$$

Proposition 4.3

Let $\varphi : L \to M$ be a morphism in $\mathbf{F}_Y$. Suppose that (f,g) is a left fibered representable pair with respect to L and M.

Define a morphism $\varphi_{[f,g]} : L_{[f,g]} \to M_{[f,g]}$ of $\mathbf{F}_Z$ to be the image of $\iota_{f,g}(M)f^*(\varphi)$ by the following map.

$$P_{f,g}(L)_{M_{[f,g]}} : \mathbf{F}_X(f^*(L), g^*(M_{[f,g]})) \to \mathbf{F}_Z(L_{[f,g]}, M_{[f,g]})$$

Then, the following diagram commutes for any $N \in \mathrm{Ob}\,\mathbf{F}_Z$.

$$\begin{array}{ccc} \mathbf{F}_X(f^*(M), g^*(N)) & \xrightarrow{f^*(\varphi)^*} & \mathbf{F}_X(f^*(L), g^*(N)) \\ \downarrow{\scriptstyle P_{f,g}(M)_N} & & \downarrow{\scriptstyle P_{f,g}(L)_N} \\ \mathbf{F}_Z(M_{[f,g]}, N) & \xrightarrow{\varphi^*_{[f,g]}} & \mathbf{F}_Z(L_{[f,g]}, N) \end{array}$$

If (f,g) is a left fibered representable pair with respect to $N \in \mathrm{Ob}\,\mathbf{F}_Y$ and $\psi : M \to N$ is a morphism in $\mathbf{F}_Y$, we have $(\psi\varphi)_{[f,g]} = \psi_{[f,g]}\varphi_{[f,g]}$.

Proposition 4.4

Let $k : V \to X$ be a morphism in $\mathbf{C}$. Suppose that (f, g) and (fk, gk) are left fibered representable pairs with respect to $M \in \mathrm{Ob}\,\mathbf{F}_Y$. Define a morphism $M_k : M_{[fk,gk]} \to M_{[f,g]}$ of $\mathbf{F}_Z$ to be the image of $k^{\#}_{M,M_{[f,g]}}(\iota_{f,g}(M))$ by the following map.

$$p_{fk,gk}(M)_{M_{[f,g]}} : \mathbf{F}_X((fk)^*(M), (gk)^*(M_{[f,g]})) \to \mathbf{F}_Z(M_{[fk,gk]}, M_{[f,g]})$$

Then, the following diagram commutes for any $N \in \mathrm{Ob}\,\mathbf{F}_Z$.

$$\begin{array}{ccc} \mathbf{F}_X(f^*(M), g^*(N)) & \xrightarrow{k^{\#}_{M,N}} & \mathbf{F}_V((fk)^*(M), (gk)^*(N)) \\ \downarrow{\scriptstyle p_{f,g}(M)_N} & & \downarrow{\scriptstyle p_{fk,gk}(M)_N} \\ \mathbf{F}_Z(M_{[f,g]}, N) & \xrightarrow{M_k^*} & \mathbf{F}_Z(M_{[fk,gk]}, N) \end{array}$$

If (fkh, gkh) is a left fibered representable pair with respect to M for a morphism $h : U \to V$, $M_{kh} : M_{[fkh,gkh]} \to M_{[f,g]}$ coincides with a composition $M_{[fkh,gkh]} \xrightarrow{M_h} M_{[fk,gk]} \xrightarrow{M_k} M_{[f,g]}$.

From now on, we assume left fibered representability if necessary.

Proposition 4.5

Under the assumptions of (4.3) and (4.4), the right diagram is commutative.

$$\begin{array}{ccc} L_{[fk,gk]} & \xrightarrow{L_k} & L_{[f,g]} \\ \downarrow{\scriptstyle \varphi_{[fk,gk]}} & & \downarrow{\scriptstyle \varphi_{[f,g]}} \\ M_{[fk,gk]} & \xrightarrow{M_k} & M_{[f,g]} \end{array}$$

Remark 4.6

For morphisms $f:X\to Y$, $g:X\to Z$, $k:V\to X$, $i:W\to Z$, $j:W\to T$, $h:U\to W$ in $\mathbf{C}$ and $M\in \mathrm{Ob}\,\mathbf{F}_Y$, it follows from the above result that the following diagram is commutative.

$$\begin{array}{ccc} (M_{[fk,gk]})_{[ih,jh]} & \xrightarrow{(M_{[fk,gk]})_h} & (M_{[fk,gk]})_{[i,j]} \\ \downarrow{\scriptstyle (M_k)_{[ih,jh]}} & & \downarrow{\scriptstyle (M_k)_{[i,j]}} \\ (M_{[f,g]})_{[ih,jh]} & \xrightarrow{(M_{[f,g]})_h} & (M_{[f,g]})_{[i,j]} \end{array}$$

We denote $(M_k)_{[i,j]}(M_{[fk,gk]})_h=(M_{[fk,gk]})_h(M_k)_{[ih,jh]}$ by $(M_k)_h$.

Proposition 4.7

For morphisms $f:X\to Y$, $g:X\to Z$, $h:X\to W$ of $\mathbf{C}$ and $M\in \mathrm{Ob}\,\mathbf{F}_Y$, we define a morphism $\delta_{f,g,h,M}:M_{[f,h]}\to (M_{[f,g]})_{[g,h]}$ in $\mathbf{F}_W$ to be the image of $\iota_{g,h}(M_{[f,g]})\iota_{f,g}(M)$ by

$$P_{f,h}(M)_{(M_{[f,g]})_{[g,h]}}:\mathbf{F}_X(f^*(M),h^*((M_{[f,g]})_{[g,h]}))\to \mathbf{F}_W(M_{[f,h]},(M_{[f,g]})_{[g,h]}).$$

Then, the following diagram commutes for any $N\in \mathrm{Ob}\,\mathbf{F}_W$.

$$\begin{array}{ccc}
\mathbf{F}_X(g^*(M_{[f,g]}),h^*(N)) & \xrightarrow{\iota_{f,g}(M)^*} & \mathbf{F}_X(f^*(M),h^*(N)) \\
\downarrow{\scriptstyle P_{g,h}(M_{[f,g]})_N} & & \downarrow{\scriptstyle P_{f,h}(M)_N} \\
\mathbf{F}_W((M_{[f,g]})_{[g,h]},N) & \xrightarrow{\delta^*_{f,g,h,M}} & \mathbf{F}_W(M_{[f,h]},N)
\end{array}$$

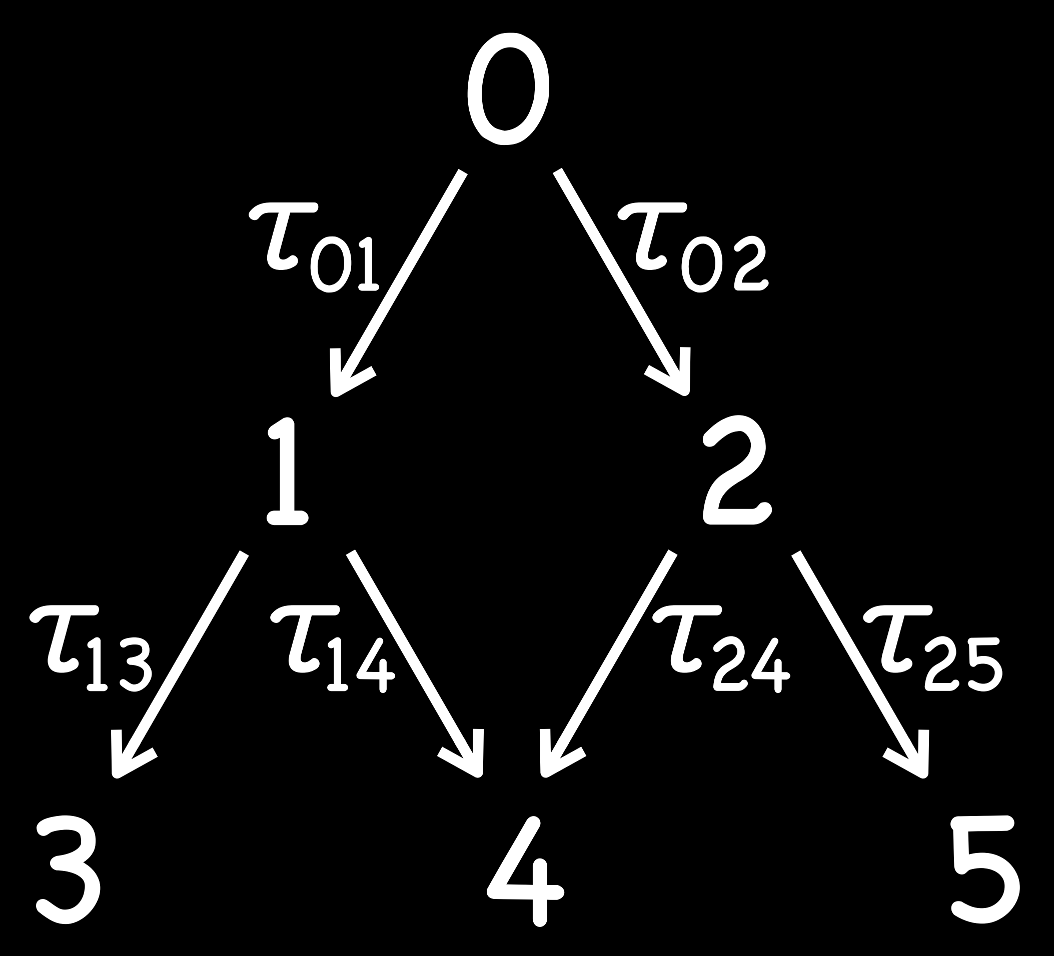
Proposition 4.8

For morphisms $f:X\to Y$, $g:X\to Z$, $h:X\to W$, $i:X\to V$, $k:V\to X$ in $\mathbf{C}$, $M, L\in \mathrm{Ob}\,\mathbf{F}_Y$ and a morphism $\varphi:L\to M$ in $\mathbf{F}_Y$, the following diagrams are commutative.

$$\begin{array}{ccc} L_{[f,h]} & \xrightarrow{\delta_{f,g,h,L}} & (L_{[f,g]})_{[g,h]} \\ \Big\downarrow{\scriptstyle \varphi_{[f,h]}} & & \Big\downarrow{\scriptstyle (\varphi_{[f,g]})_{[g,h]}} \\ M_{[f,h]} & \xrightarrow{\delta_{f,g,h,M}} & (M_{[f,g]})_{[g,h]} \end{array} \qquad \begin{array}{ccc} M_{[fk,hk]} & \xrightarrow{\delta_{fk,gk,hk,M}} & (M_{[fk,gk]})_{[gk,hk]} \\ \Big\downarrow{\scriptstyle M_k} & & \Big\downarrow{\scriptstyle (M_k)_k} \\ M_{[f,h]} & \xrightarrow{\delta_{f,g,h,M}} & (M_{[f,g]})_{[g,h]} \end{array}$$

$$\begin{array}{ccc} M_{[f,i]} & \xrightarrow{\delta_{f,g,i,M}} & (M_{[f,g]})_{[g,i]} \\ \Big\downarrow{\scriptstyle \delta_{f,h,i,M}} & & \Big\downarrow{\scriptstyle \delta_{g,h,i,M_{[f,g]}}} \\ (M_{[f,h]})_{[h,i]} & \xrightarrow{(\delta_{f,g,h,M})_{[h,i]}} & ((M_{[f,g]})_{[g,h]})_{[h,i]} \end{array}$$

Let **P** be a poset defined as follows.
$\mathrm{Ob}\,\mathbf{P}=\{0,1,2,3,4,5\}$ and $\mathbf{P}(i,j)$ is not empty if and only if $i=j$ or $i=0$ or $(i,j)=(1,3),(1,4),(2,4),(2,5)$.
We put $\mathbf{P}(i,j)=\{\tau_{ij}\}$ if $\mathbf{P}(i,j)$ is not empty.



For a functor $D:\mathbf{P}\to\mathbf{C}$ and $M\in\mathrm{Ob}\,\mathbf{F}_{D(3)}$, we put $D(\tau_{ij})=f_{ij}$ and define a morphism

$$\theta_D(M):M_{[f_{13}f_{01},\,f_{25}f_{02}]}\to(M_{[f_{13},\,f_{14}]})_{[f_{24},\,f_{25}]}$$

in $\mathbf{F}_{D(5)}$ to be the following composition.

$$M_{[f_{13}f_{01},\,f_{25}f_{02}]}\xrightarrow{\delta_{f_{13}f_{01},\,f_{14}f_{01},\,f_{25}f_{02},\,M}}(M_{[f_{13}f_{01},\,f_{14}f_{01}]})_{[f_{24}f_{02},\,f_{25}f_{02}]}\xrightarrow{(M_{f_{01}})_{f_{02}}}(M_{[f_{13},\,f_{14}]})_{[f_{24},\,f_{25}]}$$

Proposition 4.9

For a morphism $\varphi : L \to M$ of $\mathbf{F}_Y$, the following diagram is commutative.

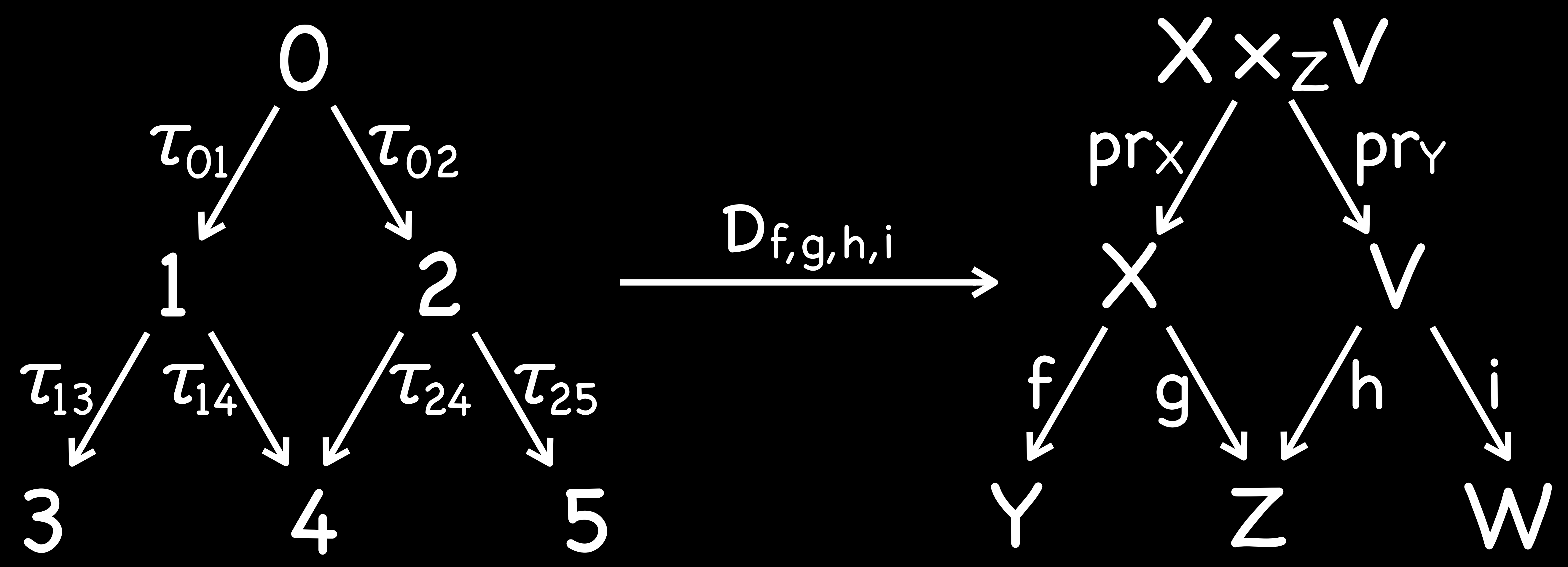
$$\begin{array}{ccc} L_{[f_{13}f_{01},\, f_{25}f_{02}]} & \xrightarrow{\theta_D(L)} & (L_{[f_{13},\, f_{14}]})_{[f_{24},\, f_{25}]} \\ \downarrow{\scriptstyle \varphi_{[f_{13}f_{01},\, f_{25}f_{02}]}} & & \downarrow{\scriptstyle (\varphi_{[f_{13},\, f_{14}]})_{[f_{24},\, f_{25}]}} \\ M_{[f_{13}f_{01},\, f_{25}f_{02}]} & \xrightarrow{\theta_D(M)} & (M_{[f_{13},\, f_{14}]})_{[f_{24},\, f_{25}]} \end{array}$$

Proposition 4.10

Let $D, E : \mathbf{P} \to \mathbf{C}$ be functors which satisfies $D(i) = E(i)$ for $i = 3,4,5$ and $\lambda : D \to E$ a natural transformation which satisfies $\lambda_i = id_{D(i)}$ for $i = 3,4,5$. Put $D(\tau_{ij}) = f_{ij}$ and $E(\tau_{ij}) = g_{ij}$. The following diagram is commutative for $M \in \mathrm{Ob}\, \mathbf{F}_{D(3)}$.

$$\begin{array}{ccc} M_{[f_{13}f_{01},\, f_{25}f_{02}]} & \xrightarrow{\theta_D(M)} & (M_{[f_{13},\, f_{14}]})_{[f_{24},\, f_{25}]} \\ \downarrow{\scriptstyle M_{\lambda_0}} & & \downarrow{\scriptstyle (M_{\lambda_1})_{\lambda_2}} \\ M_{[g_{13}g_{01},\, g_{25}g_{02}]} & \xrightarrow{\theta_E(M)} & (M_{[g_{13},\, g_{14}]})_{[g_{24},\, g_{25}]} \end{array}$$

For a diagram $Y \xleftarrow{f} X \xrightarrow{g} Z \xleftarrow{h} V \xrightarrow{i} W$ in $\mathbf{C}$, let $X \xleftarrow{pr_X} X \times_Z V \xrightarrow{pr_V} V$ be a limit of $X \xrightarrow{g} Z \xleftarrow{h} V$. We define a functor $D_{f,g,h,i} : \mathbf{P} \to \mathbf{C}$ by $D_{f,g,h,i}(0) = X \times_Z V$, $D_{f,g,h,i}(1) = X$, $D_{f,g,h,i}(2) = V$, $D_{f,g,h,i}(3) = Y$, $D_{f,g,h,i}(4) = Z$, $D_{f,g,h,i}(5) = W$ and $D_{f,g,h,i}(\tau_{01}) = pr_X$, $D_{f,g,h,i}(\tau_{02}) = pr_V$, $D_{f,g,h,i}(\tau_{13}) = f$, $D_{f,g,h,i}(\tau_{14}) = g$, $D_{f,g,h,i}(\tau_{24}) = h$, $D_{f,g,h,i}(\tau_{25}) = i$.



We denote $\theta_{D_{f,g,h,i}}(M) : M_{[f pr_X, i pr_Y]} \to (M_{[f,g]})_{[h,i]}$ by $\theta_{f,g,h,i}(M)$.

§5. Existence of induced representations

Definition 5.1

For a fibered category $p : \mathbf{F} \to \mathbf{C}$, we say that an internal category $C = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ in $\mathbf{C}$ is left (resp. right) fibered representable if (σ, τ) and $(\sigma pr_1, \tau pr_2)$ are left (resp. right) fibered representable pairs.

We assume that internal categories below are left fibered representable unless otherwise stated.

Proposition 5.2

Suppose that $C=(C_0,C_1;\sigma,\tau,\varepsilon,\mu)$ is a left fibered representable internal category. For $M\in \mathrm{Ob}\,\mathbf{F}_{C_0}$ and $\xi\in\mathbf{F}_{C_1}(\sigma^*(M),\tau^*(M))$, we put

$$\hat{\xi}=P_{\sigma,\tau}(M)_M(\xi):M_{[\sigma,\tau]}\to M.$$

Then, (M,ξ) is a representation of C on M if and only if a composition $M=M_{[\sigma\varepsilon,\tau\varepsilon]}\xrightarrow{M_\varepsilon}M_{[\sigma,\tau]}\xrightarrow{\hat{\xi}}M$ coincides with the identity morphism of M and the following diagram is commutative.

$$\begin{array}{ccccc}
M_{[\sigma \mathrm{pr}_1,\tau \mathrm{pr}_2]}=M_{[\sigma\mu,\tau\mu]} & \xrightarrow{M_\mu} & M_{[\sigma,\tau]} & & \\
 & & & \searrow^{\hat{\xi}} & \\
\Big\downarrow{\scriptstyle \theta_{\sigma,\tau,\sigma,\tau}(M)} & & & & M \\
 & & & \nearrow_{\hat{\xi}} & \\
(M_{[\sigma,\tau]})_{[\sigma,\tau]} & \xrightarrow{\hat{\xi}_{[\sigma,\tau]}} & M_{[\sigma,\tau]} & &
\end{array}$$

Proposition 5.3

Let (M,ξ) and (N,ζ) be representations of C on M and N, respectively and $\varphi : M \to N$ a morphism in $\mathbf{F}_{C_0}$. We put

$$\hat{\xi} = P_{\sigma,\tau}(M)_M(\xi) : M_{[\sigma,\tau]} \to M \text{ and } \hat{\zeta} = P_{\sigma,\tau}(N)_N(\zeta) : N_{[\sigma,\tau]} \to N.$$

Then, φ defines a morphism $\varphi : (M,\xi) \to (N,\zeta)$ of representations if and only if the following diagram is commutative.

$$\begin{array}{ccc} M_{[\sigma,\tau]} & \xrightarrow{\hat{\xi}} & M \\ \downarrow{\scriptstyle \varphi_{[\sigma,\tau]}} & & \downarrow{\scriptstyle \varphi} \\ N_{[\sigma,\tau]} & \xrightarrow{\hat{\zeta}} & N \end{array}$$

Example 5.4

Consider the fibered category $p^{op}:\mathbf{MOD}^{op}\to\mathbf{Alg}_K^{op}$.

Let $\Gamma=(A_*,\Gamma_*;\sigma,\tau,\varepsilon,\mu)$ be a Hopf algebroid in $\mathbf{Alg}_K$ and $M=(A_*,M_*,\alpha)$ an object of $\mathbf{MOD}_{A_*}$. Then, we have

$$M_{[\sigma,\tau]}=\tau_*(\sigma^*(M))=(A_*,M_*\otimes^{\sigma}_{A_*}\Gamma_*,\alpha_\sigma(id_{M_*\otimes^{\sigma}_{A_*}\Gamma_*}\otimes_K\tau)).$$

Define a map $i_{\Gamma_*}:M_*\to M_*\otimes^{\tau}_{A_*}\Gamma_*$ by $i_{\Gamma_*}(x)=x\otimes 1$. For a morphism

$$\xi=(id_{\Gamma_*},\tilde{\xi}):\sigma^*(M)=(\Gamma_*,M_*\otimes^{\sigma}_{A_*}\Gamma_*,\alpha_\sigma)\to(\Gamma_*,M_*\otimes^{\tau}_{A_*}\Gamma_*,\alpha_\tau)=\tau^*(M)$$

in $\mathbf{MOD}^{op}_{\Gamma_*}$, we denote by $\bar{\xi}:M_*\to M_*\otimes^{\sigma}_{A_*}\Gamma_*$ the following composition.

$$M_*\xrightarrow{i_{\Gamma_*}}M_*\otimes^{\tau}_{A_*}\Gamma_*\xrightarrow{\tilde{\xi}}M_*\otimes^{\sigma}_{A_*}\Gamma_*$$

Then, $(id_{A_*},\bar{\xi}):M\to M_{[\sigma,\tau]}$ is a morphism in $\mathbf{MOD}_{A_*}$ and this coincides with a morphism $\hat{\xi}=P_{\sigma,\tau}(M)_M(\xi):M_{[\sigma,\tau]}\to M$ in $\mathbf{MOD}^{op}_{A_*}$.

Put $\beta = \alpha_\sigma(id_{M_* \otimes^\sigma_{A_*} \Gamma_*} \otimes_K \tau) : (M_* \otimes^\sigma_{A_*} \Gamma_*) \otimes_K A_* \to M_* \otimes^\sigma_{A_*} \Gamma_*$. Then we have the following equalities.

$$M_{[\sigma pr_1, \tau pr_2]} = M_{[\sigma\mu, \tau\mu]} = (A_*, M_* \otimes^{\mu\sigma}_{A_*} (\Gamma_* \otimes_{A_*} \Gamma_*), \alpha_{\mu\sigma}(id_{M_* \otimes^{\mu\sigma}_{A_*}(\Gamma_* \otimes_{A_*} \Gamma_*)} \otimes_K \mu\tau))$$

$$(M_{[\sigma,\tau]})_{[\sigma,\tau]} = (A_*, (M_* \otimes^\sigma_{A_*} \Gamma_*) \otimes^\sigma_{A_*} \Gamma_*, \beta_\sigma(id_{(M_* \otimes^\sigma_{A_*} \Gamma_*) \otimes^\sigma_{A_*} \Gamma_*} \otimes_K \tau))$$

Let $\bar\theta_{\sigma,\tau,\sigma,\tau}(M) : (M_* \otimes^\sigma_{A_*} \Gamma_*) \otimes_{A_*} \Gamma_* \to M_* \otimes^{\mu\sigma}_{A_*} (\Gamma_* \otimes_{A_*} \Gamma_*)$ be a map defined by $\bar\theta_{\sigma,\tau,\sigma,\tau}(M)((x \otimes g) \otimes h) = x \otimes (g \otimes h)$. $\bar\theta_{\sigma,\tau,\sigma,\tau}(M)$ is an isomorphism of right A_*-modules and $\theta_{\sigma,\tau,\sigma,\tau}(M) = (id_{A_*}, \bar\theta_{\sigma,\tau,\sigma,\tau}(M))$ holds.

Let $i_{M_*} : M_* \to M_* \otimes_{A_*} A_*$ be the isomorphism given by $i_{M_*}(x) = x \otimes 1$. Morphisms $M_\varepsilon : M = M_{[\sigma\varepsilon, \tau\varepsilon]} \to M_{[\sigma,\tau]}$ and $M_\mu : M_{[\sigma\mu, \tau\mu]} \to M_{[\sigma,\tau]}$ in $\mathbf{MOD}^{op}_{A_*}$ are given by $M_\varepsilon = (id_{A_*}, i^{-1}_{M_*}(id_{M_*} \otimes_{A_*} \varepsilon))$ and $M_\mu = (id_{A_*}, id_{M_*} \otimes_{A_*} \mu)$.

$\hat\xi_{[\sigma,\tau]} : (M_{[\sigma,\tau]})_{[\sigma,\tau]} \to M_{[\sigma,\tau]}$ is given by $\hat\xi_{[\sigma,\tau]} = (id_{A_*}, \bar\xi \otimes_{A^*} id_{\Gamma^*})$.

It follows from (5.2) that a morphism

$$\xi=(\mathrm{id}_{\Gamma_*},\tilde{\xi}):\sigma^*(M)=(\Gamma_*,M_*\otimes^{\sigma}_{A_*}\Gamma_*,\alpha_\sigma)\to(\Gamma_*,M_*\otimes^{\tau}_{A_*}\Gamma_*,\alpha_\tau)=\tau^*(M)$$

in $\mathbf{MOD}^{op}_{\Gamma_*}$ is a representation of Γ on $M=(A_*,M_*,\alpha)$ if and only if the following diagrams in the category of right A_*-modules are commutative.

$$\begin{array}{ccccc}
 & & M_*\otimes^{\sigma}_{A_*}\Gamma_* & \xrightarrow{\bar{\xi}\otimes_{A_*}\mathrm{id}_{\Gamma_*}} & (M_*\otimes^{\sigma}_{A_*}\Gamma_*)\otimes^{\sigma}_{A_*}\Gamma_* \\
 & \overset{\bar{\xi}}{\nearrow} & & & \downarrow{\scriptstyle \bar{\theta}_{\sigma,\tau,\sigma,\tau}(M)} \\
M_* & & & & \\
 & \underset{\bar{\xi}}{\searrow} & & & \\
 & & M_*\otimes^{\sigma}_{A_*}\Gamma_* & \xrightarrow{\mathrm{id}_{M_*}\otimes_{A_*}\mu} & M_*\otimes^{\mu\sigma}_{A_*}(\Gamma_*\otimes_{A_*}\Gamma_*)
\end{array}
\qquad
\begin{array}{ccc}
M_* & \xrightarrow{\bar{\xi}} & M_*\otimes^{\sigma}_{A_*}\Gamma_* \\
 & \underset{\cong}{\overset{i_{M_*}}{\searrow}} & \downarrow{\scriptstyle \mathrm{id}_{M_*}\otimes_{A_*}\varepsilon} \\
 & & M_*\otimes_{A_*}A_*
\end{array}$$

We call a pair $(M_*,\bar{\xi}:M_*\to M_*\otimes^{\sigma}_{A_*}\Gamma_*)$ of a right A_*-module and a homomorphism of right A_*-modules which makes the above diagrams commute a right Γ_*-comodule.

For $M \in \mathrm{Ob}\,\mathbf{F}_{C_0}$, we assume that

$$\theta_{\sigma,\tau,\sigma,\tau}(M) : M_{[\sigma \mathrm{pr}_1, \tau \mathrm{pr}_2]} \to (M_{[\sigma,\tau]})_{[\sigma,\tau]}$$

is an isomorphism. Define a morphism

$$\hat{\mu}_M : (M_{[\sigma,\tau]})_{[\sigma,\tau]} \to M_{[\sigma,\tau]}$$

in $\mathbf{F}_{C_0}$ to be the following composition.

$$(M_{[\sigma,\tau]})_{[\sigma,\tau]} \xrightarrow{\theta_{\sigma,\tau,\sigma,\tau}(M)^{-1}} M_{[\sigma \mathrm{pr}_1, \tau \mathrm{pr}_2]} = M_{[\sigma\mu, \tau\mu]} \xrightarrow{M_\mu} M_{[\sigma,\tau]}$$

We put $\mu^!_M = P_{\sigma,\tau}(M_{[\sigma,\tau]})^{-1}_{M_{[\sigma,\tau]}}(\hat{\mu}_M) : \sigma^*(M_{[\sigma,\tau]}) \to \tau^*(M_{[\sigma,\tau]})$.

Let $C_1 \times_{C_0} C_1 \xleftarrow{\mathrm{pr}_{12}} C_1 \times_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_{23}} C_1 \times_{C_0} C_1$ be a limit of a diagram $C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_2} C_1 \xleftarrow{\mathrm{pr}_1} C_1 \times_{C_0} C_1$.

Proposition 5.5.

If $\theta_{\sigma,\tau,\sigma \mathrm{pr}_1,\tau \mathrm{pr}_2}(M) : M_{[\sigma \mathrm{pr}_1 \mathrm{pr}_{12}, \tau \mathrm{pr}_2 \mathrm{pr}_{23}]} \to (M_{[\sigma,\tau]})_{[\sigma \mathrm{pr}_1, \tau \mathrm{pr}_2]}$ is an epimorphism, $(M_{[\sigma,\tau]}, \mu^!_M)$ is a representation of C.

Theorem 5.6

Let M be an object of $\mathbf{F}_{C_0}$ and (N,ζ) a representation of C. Assume that $\theta_{\sigma,\tau,\sigma,\tau}(L):L_{[\sigma pr_1,\tau pr_2]}\to(L_{[\sigma,\tau]})_{[\sigma,\tau]}$ is an isomorphism for $L=M,N$ and that $\theta_{\sigma,\tau,\sigma pr_1,\tau pr_2}(L):L_{[\sigma pr_1 pr_{12},\tau pr_2 pr_{23}]}\to(L_{[\sigma,\tau]})_{[\sigma pr_1,\tau pr_2]}$ is an epimorphism for $L=M,N$. Then a map

$$\Phi:\mathrm{Rep}(C;\mathbf{F})((M_{[\sigma,\tau]},\mu^{!}_{M}),(N,\zeta))\to\mathbf{F}_{C_0}(M,N)$$

defined by $\Phi(\varphi)=(M=M_{[\sigma\varepsilon,\tau\varepsilon]}\xrightarrow{M_\varepsilon}M_{[\sigma,\tau]}\xrightarrow{\varphi}N)$ is bijective.

Hence if $\theta_{\sigma,\tau,\sigma,\tau}(L)$ is an isomorphism and $\theta_{\sigma,\tau,\sigma pr_1,\tau pr_2}(L)$ is an epimorphism for all $L\in\mathrm{Ob}\,\mathbf{F}_{C_0}$, a functor $\mathcal{L}_C:\mathbf{F}_{C_0}\to\mathrm{Rep}(C;\mathbf{F})$ defined by $\mathcal{L}_C(M)=(M_{[\sigma,\tau]},\mu^{!}_{M})$ and $\mathcal{L}_C(\varphi)=\varphi_{[\sigma,\tau]}$ is a left adjoint of the forgetful functor $\mathcal{F}_C:\mathrm{Rep}(C;\mathbf{F})\to\mathbf{F}_{C_0}$ given by $\mathcal{F}_C(M,\xi)=M$ and $\mathcal{F}_C(\varphi)=\varphi$.

Theorem 5.7

Let C, D be internal categories in $\mathbf{C}$ and $\mathbf{f}: D \to C$ an internal functor. The functor $\mathbf{f}^{\cdot}: \mathrm{Rep}(C;\mathbf{F}) \to \mathrm{Rep}(D;\mathbf{F})$ obtained from the restrictions of representations of C along $\mathbf{f}$ has a left adjoint if the following conditions are satisfied.

(i) $\mathbf{F}_{C_0}$ has coequalizers.

(ii) A functor $\mathbf{F}_{C_0} \to \mathbf{F}_{C_0}$ which maps $M \in \mathrm{Ob}\,\mathbf{F}_{C_0}$ to $M_{[\sigma,\tau]}$ and $\varphi \in \mathrm{Mor}\,\mathbf{F}_{C_0}$ to $\varphi_{[\sigma,\tau]}$ preserves coequalizers.

(iii) $(\sigma\mu)^*: \mathbf{F}_{C_0} \to \mathbf{F}_{C_1 \times_{C_0} C_1}$ maps coequalizers to epimorphisms.

(iv) For any diagram $Y \xleftarrow{f} X \xrightarrow{g} Z \xleftarrow{h} V \xrightarrow{i} W$ in $\mathbf{C}$ and any object M of $\mathbf{F}_{C_0}$, $\theta_{f,g,h,i}(M): M_{[fpr_X, ipr_Y]} \to (M_{[f,g]})_{[h,i]}$ is an isomorphism.

Remark 5.8

The fibered category $p^{op}:\mathbf{MOD}^{op}\to\mathbf{Alg}_k^{op}$ of graded k-modules satisfies the conditions (i) and (iv) of (5.7).

Let $\Gamma=(A_*,\Gamma_*;\sigma,\tau,\varepsilon,\mu)$ be a Hopf algebroid in $\mathbf{Alg}_k$.

If $\sigma:A_*\to\Gamma_*$ is a flat morphism in $\mathbf{Alg}_k$, then the conditions (ii) and (iii) of (5.7) are satisfied.

Hence, for a morphism $\mathbf{f}:\Gamma\to\Delta$ of Hopf algebroids, the restriction functor $\mathbf{f}^{\cdot}:\mathrm{Rep}(\Gamma;\mathbf{F})\to\mathrm{Rep}(\Delta;\mathbf{F})$ has a left adjoint if $\sigma:A_*\to\Gamma_*$ is a flat morphism in $\mathbf{Alg}_k$.

§6. Hopf algebroid associated with homology theory

Let E be a commutative ring spectrum with unit $\eta : S^0 \to E$ and product $m : E \wedge E \to E$.

Suppose that the coefficient ring $E_* = \pi_*(E)$ is a k-algebra for a commutative ring k ($k = E_0$ for example) and that $E_*E = \pi_*(E \wedge E)$ is flat over E_*. Then, the functor from the category of spectra to the category of graded E_*-modules given by $X \mapsto E_*(X) \otimes_{E_*} E_*E$ is a homology theory.

We put $h_*(X) = E_*(X) \otimes_{E_*} E_*E$. The product m induces

$$h_*(X) = \pi_*(X \wedge E) \otimes_{E_*} \pi_*(E \wedge E) \xrightarrow{\wedge} \pi_*(X \wedge E \wedge E \wedge E) \xrightarrow{(id_X \wedge m \wedge id_E)_*} \pi_*(X \wedge E \wedge E)$$

a natural transformation $\psi : h_* \to (E \wedge E)_*$ of homology theories.

Since $\psi_{S^0}: h_*(S^0) \to (E\wedge E)_*(S^0)$ is an isomorphism, $\psi: h_* \to (E\wedge E)_*$ is an is an equivalence of homology theories. In other words, we see the the following fact.

Proposition 6.1.
There is an isomorphism of right E_*-modules

$$\psi_X: E_*(X) \otimes_{E_*} E_*E \to \pi_*(X\wedge E\wedge E)$$

which is natural in X.

Let $\sigma, \tau: E_* \to E_*E$, $\varepsilon: E_*E \to E_*$ and $\iota: E_*E \to E_*E$ be the maps induced by $E \simeq E\wedge S^0 \xrightarrow{id_E \wedge \eta} E\wedge E$, $E \simeq S^0 \wedge E \xrightarrow{\eta \wedge id_E} E\wedge E$, $E\wedge E \xrightarrow{m} E$ and the switching map $c: E\wedge E \to E\wedge E$, respectively.

Let

$$D_X : E_*(X) = \pi_*(X \wedge E) \to \pi_*(X \wedge E \wedge E)$$

be the map induced by $X \wedge E \simeq X \wedge S^0 \wedge E \xrightarrow{id_X \wedge \eta \wedge id_E} X \wedge E \wedge E$.
Put $\mu = \psi_E^{-1} D_E : E_*E = \pi_*(E \wedge E) \to E_*E \otimes_{E_*} E_*E$. Then, it can be verified that $(E_*, E_*E; \sigma, \tau, \varepsilon, \mu, \iota)$ is a Hopf algebroid in $\mathbf{Alg}_K$, which we call the Hopf algebroid associated with E. We denote this by H_E.
For a spectrum X, we put $\varphi_X = \psi_X^{-1} D_X : E_*(X) \to E_*(X) \otimes_{E_*} E_*E$. Then, it turns out that φ_X is a structure map of right E_*E-comodule on $E_*(X)$. Hence E-homology theory $X \mapsto E_*(X)$ takes the values in the category $\mathrm{Rep}(H_E; \mathbf{MOD}^{op})$ of representations of H_E.
That is, E-homology theory is regarded as a functor from "stable homotopy category" to $\mathrm{Rep}(H_E; \mathbf{MOD}^{op})$.

References

[1] J. F. Adams, Lectures on generalised cohomology, Lecture Notes in Mathematics, vol.99, Springer-Verlag, 1969, 1-138.

[2] J. F. Adams, Stable homotopy and generalised homology, Chicago Lectures in Mathematics, The University of Chicago Press, 1974

[3] A. Grothendieck, Catégorie fibrées et Descente, Lecture Notes in Mathematics, vol.224, Springer-Verlag, 1971, 145-194.

[4] A. Yamaguchi, The structure of the Hopf algebroid associated with the elliptic homology theory, Osaka Journal of Mathematics, vol.33, (1996), 57-68.

[5] A. Yamaguchi, Representations of internal categories, Kyushu Journal of Mathematics Vol.62, No.1, (2008) 139-169.

Thank you for listening

and your patience.