Representations of groupoids in the category of plots

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§1. Fibered category of morphisms

Let $\wp : \mathscr{F} \to \mathscr{C}$ be a functor. For an object X of $\mathscr{C}$, we denote by $\mathscr{F}_X$ a subcategory of $\mathscr{F}_X$ given as follows.

$$\mathrm{Ob}\mathscr{F}_X = \{\boldsymbol{E} \in \mathrm{Ob}\,\mathscr{F} \mid \wp(\boldsymbol{E}) = X\}$$

$$\mathrm{Mor}\,\mathscr{F}_X = \{\boldsymbol{\varphi} \in \mathrm{Mor}\,\mathscr{F} \mid \wp(\boldsymbol{\varphi}) = id_X\}$$

For a morphism $f : X \to Y$ in $\mathscr{C}$ and objects $\boldsymbol{E}$, $\boldsymbol{F}$ of $\mathscr{F}_X$, $\mathscr{F}_Y$, respectively, we denote by $\mathscr{F}_f(\boldsymbol{E}, \boldsymbol{F})$ a subset of $\mathscr{F}(\boldsymbol{E}, \boldsymbol{F})$ defined by

$$\mathscr{F}_f(\boldsymbol{E}, \boldsymbol{F}) = \{\boldsymbol{\varphi} \in \mathscr{F}(\boldsymbol{E}, \boldsymbol{F}) \mid \wp(\boldsymbol{\varphi}) = f\}.$$

Definition 1.1 ([1])

Let $\boldsymbol{\alpha} : \boldsymbol{E} \to \boldsymbol{F}$ be a morphism in $\mathscr{F}$ and set $\wp(\boldsymbol{E}) = X$, $\wp(\boldsymbol{\alpha}) = f$. We call α a cartesian morphism if, for any $\boldsymbol{G} \in \mathrm{Ob}\mathscr{F}_X$, the map $\mathscr{F}_X(\boldsymbol{G}, \boldsymbol{E}) \to \mathscr{F}_f(\boldsymbol{G}, \boldsymbol{F})$ defined by $\boldsymbol{\varphi} \mapsto \boldsymbol{\alpha}\boldsymbol{\varphi}$ is bijective.

Definition 1.2 ([1])

A functor $\wp : \mathscr{F} \to \mathscr{C}$ is called a **fibered category** if $\wp$ satisfies the following conditions.

(i) For any morphism $f : X \to Y$ in $\mathscr{C}$ and any object $\boldsymbol{F}$ of $\mathscr{F}_Y$, there exist an object $\boldsymbol{E}$ of $\mathscr{F}_X$ and a cartesian morphism $\boldsymbol{\alpha} : \boldsymbol{E} \to \boldsymbol{F}$ which maps to $f : X \to Y$ by $\wp$.

(ii) The composition of cartesian morphisms is cartesian.

Proposition 1.3 ([1])

Let $f: X \to Y$ be a morphism in $\mathscr{C}$ and $\boldsymbol{\alpha}_i : \boldsymbol{E}_i \to \boldsymbol{F}_i$ $(i = 1, 2)$ morphisms in $\mathscr{F}$ such that $\wp(\boldsymbol{\alpha}_1) = \wp(\boldsymbol{\alpha}_2) = f$, hence $\wp(\boldsymbol{E}_1) = \wp(\boldsymbol{E}_2) = X$ and $\wp(\boldsymbol{F}_1) = \wp(\boldsymbol{F}_2) = Y$.

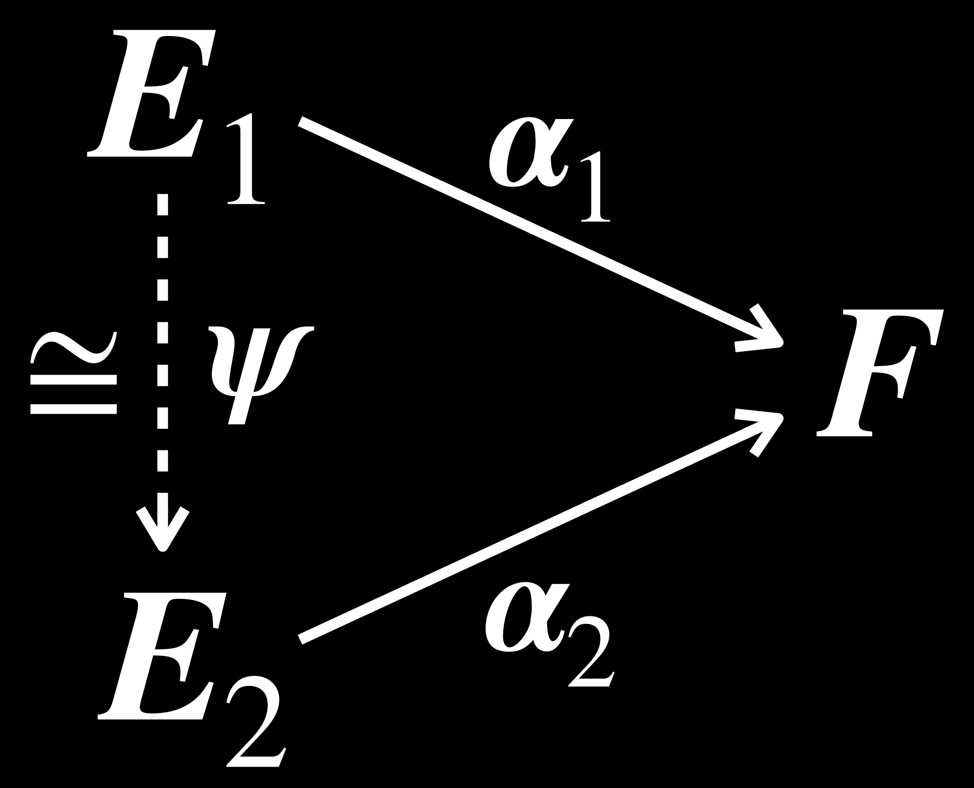
Assume that $\boldsymbol{\alpha}_2$ is a cartesian morphism.

For a morphism $\boldsymbol{\varphi} : \boldsymbol{F}_1 \to \boldsymbol{F}_2$ in $\mathscr{F}_Y$, there exists unique morphism $\boldsymbol{\psi} : \boldsymbol{E}_1 \to \boldsymbol{E}_2$ in $\mathscr{F}_X$ that makes the following diagram commute.

$$
\begin{array}{ccc}
\boldsymbol{E}_1 & \xrightarrow{\boldsymbol{\alpha}_1} & \boldsymbol{F}_1 \\
{\scriptstyle \boldsymbol{\psi}} \downarrow \vdots & & \downarrow {\scriptstyle \boldsymbol{\varphi}} \\
\boldsymbol{E}_2 & \xrightarrow{\boldsymbol{\alpha}_2} & \boldsymbol{F}_2
\end{array}
$$

Corollary 1.4 ([1])

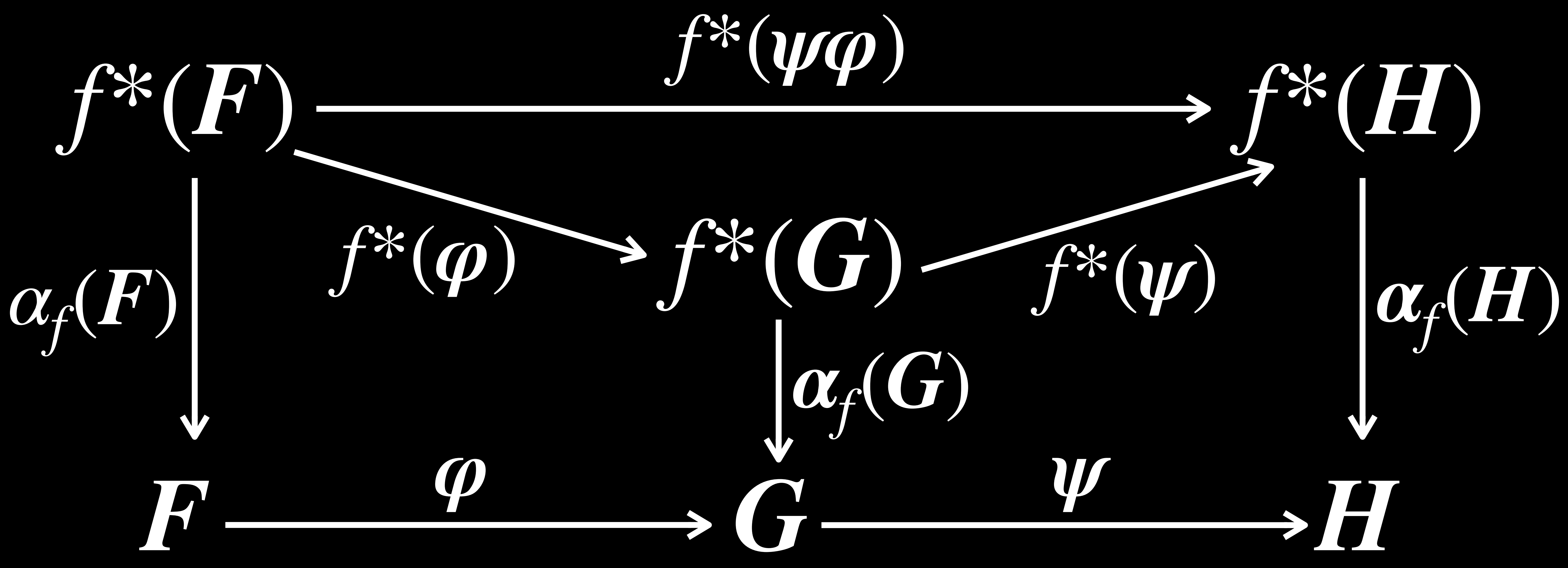
If $\alpha_i : E_i \to F_i$ $(i = 1, 2)$ are cartesian morphisms in $\mathscr{F}$ such that $\wp(E_1) = \wp(E_2) = X$ and $\wp(\alpha_1) = \wp(\alpha_2)$, there exists unique morphism $\psi : E_1 \to E_2$ in $\mathscr{F}_X$ that makes the following diagram commute. Moreover, ψ is an isomorphism.



Let $\wp : \mathscr{F} \to \mathscr{C}$ be a fibered category and $f : X \to Y$ a morphism in $\mathscr{C}$.
For an object F of $\mathscr{F}_Y$, there exist an object E of $\mathscr{F}_X$ and a cartesian morphism $\alpha : E \to F$ which maps to $f : X \to Y$ by $\wp$.
It follows from (1.4) that E is unique up to isomorphism.
We choose such E and $\alpha : E \to F$ for each object F of $\mathscr{F}_Y$ and denote E by $f^*(F)$ and $\alpha : E \to F$ by $\alpha_f(F) : f^*(F) \to F$.
For a morphism $\varphi : F \to G$ in $\mathscr{F}_Y$, there exist unique morphism $\varphi_f : f^*(F) \to f^*(G)$ in $\mathscr{F}_X$ that makes the right diagram commute by (1.3).
We denote $\varphi_f : f^*(F) \to f^*(G)$ by $f^*(\varphi) : f^*(F) \to f^*(G)$ below.

$$\begin{array}{ccc} f^*(F) & \xrightarrow{\alpha_f(F)} & F \\ \Big\downarrow{\scriptstyle \varphi_f} & & \Big\downarrow{\scriptstyle \varphi} \\ f^*(G) & \xrightarrow{\alpha_f(G)} & G \end{array}$$

For a morphism $\psi : G \to H$ in $\mathscr{F}_Y$, we have the following diagram. It follows from (1.3) that $f^*(\psi\varphi) = f^*(\psi) f^*(\varphi)$ holds.



It also follows from (1.3) that $f^*(id_F) : f^*(F) \to f^*(F)$ is the identity morphism of $f^*(F)$.

Thus we have a functor $f^* : \mathscr{F}_Y \to \mathscr{F}_X$ which is called the inverse image functor associated with $f : X \to Y$.

Remark 1.5

For an object X of $\mathscr{C}$ and an object E of $\mathscr{F}_X$, since the identity morphism $id_E : E \to E$ of E is a cartesian morphism which is mapped to the identity morphism $id_X : X \to X$ of X by $\wp : \mathscr{F} \to \mathscr{C}$, it follows from (1.4) that $\alpha_{id_X}(E) : id_X^*(E) \to E$ is an isomorphism in $\mathscr{F}_X$.

We usually choose E as an inverse image $id_X^*(E)$ of E by $id_X : X \to X$ unless otherwise stated.
In this case, $\alpha_{id_X}(E) : id_X^*(E) \to E$ is the identity morphism of E.

For morphisms $f\colon X\to Y$, $g\colon Z\to X$ in $\mathscr{C}$ and $\boldsymbol{F}$ an object of $\mathscr{F}_Y$, there exists unique morphism $\boldsymbol{c}_{f,g}(\boldsymbol{F})\colon g^*f^*(\boldsymbol{F})\to(fg)^*(\boldsymbol{F})$ in $\mathscr{F}_Z$ that makes the following diagram commute by (1.3).

$$\begin{array}{ccc} g^*f^*(\boldsymbol{F}) & \xrightarrow{\boldsymbol{\alpha}_g(f^*(\boldsymbol{F}))} & f^*(\boldsymbol{F}) \\ \downarrow{\scriptstyle \boldsymbol{c}_{f,g}(\boldsymbol{F})} & & \downarrow{\scriptstyle \boldsymbol{\alpha}_f(\boldsymbol{F})} \\ (fg)^*(\boldsymbol{F}) & \xrightarrow{\boldsymbol{\alpha}_{fg}(\boldsymbol{F})} & \boldsymbol{F} \end{array}$$

We remark that, since composition $g^*f^*(\boldsymbol{F})\xrightarrow{\boldsymbol{\alpha}_g(f^*(\boldsymbol{F}))} f^*(\boldsymbol{F})\xrightarrow{\boldsymbol{\alpha}_f(\boldsymbol{F})}\boldsymbol{F}$ is cartesian, it follows from (1.4) that $\boldsymbol{c}_{f,g}(\boldsymbol{F})\colon g^*f^*(\boldsymbol{F})\to(fg)^*(\boldsymbol{F})$ is an isomorphism.

Proposition 1.6 ([1])

For a morphism $\varphi : F \to G$ in $\mathscr{F}_Y$, the following diagram commutes.

$$\begin{array}{ccc} g^*f^*(F) & \xrightarrow{c_{f,g}(F)} & (fg)^*(F) \\ \downarrow{\scriptstyle g^*f^*(\varphi)} & & \downarrow{\scriptstyle (fg)^*(\varphi)} \\ g^*f^*(G) & \xrightarrow{c_{f,g}(G)} & (fg)^*(G) \end{array}$$

Thus we have $c_{f,g}$ a natural equivalence $c_{f,g} : g^*f^* \to (fg)^*$.

For morphisms $f : X \to Y$, $g : X \to Z$ in $\mathscr{C}$, we define a functor $F_{f,g} : \mathscr{F}_Y^{op} \times \mathscr{F}_Z \to \mathscr{S}et$ as follows.
Put $F_{f,g}(\boldsymbol{E}, \boldsymbol{F}) = \mathscr{F}_X(f^*(\boldsymbol{E}), g^*(\boldsymbol{F}))$ for $\boldsymbol{E} \in \mathrm{Ob}\mathscr{F}_Y$ and $\boldsymbol{F} \in \mathrm{Ob}\mathscr{F}_Z$.
For morphisms $\boldsymbol{\varphi} : \boldsymbol{E} \to \boldsymbol{G}$, $\boldsymbol{\psi} : \boldsymbol{H} \to \boldsymbol{F}$ in $\mathscr{F}_Y$, $\mathscr{F}_Z$, respectively, $F_{f,g}(\boldsymbol{\varphi}, \boldsymbol{\psi}) : \mathscr{F}_X(f^*(\boldsymbol{G}), g^*(\boldsymbol{H})) \to \mathscr{F}_X(f^*(\boldsymbol{E}), g^*(\boldsymbol{F}))$ maps $\zeta : f^*(\boldsymbol{G}) \to g^*(\boldsymbol{H})$ to $g^*(\boldsymbol{\psi})\zeta f^*(\boldsymbol{\varphi}) : f^*(\boldsymbol{E}) \to g^*(\boldsymbol{F})$.

Let $f : X \to Y$, $g : X \to Z$, $k : V \to X$ be morphisms in $\mathscr{C}$ and $\boldsymbol{E}$, $\boldsymbol{F}$ objects of $\mathscr{F}_Y$, $\mathscr{F}_Z$, respectively.
For a morphism $\xi : f^*(\boldsymbol{E}) \to g^*(\boldsymbol{F})$ in $\mathscr{F}_X$, we define a morphism $\xi_k : (fk)^*(\boldsymbol{E}) \to (gk)^*(\boldsymbol{F})$ in $\mathscr{F}_V$ to be the following composition.

$$(fk)^*(\boldsymbol{E}) \xrightarrow{\boldsymbol{c}_{f,k}(\boldsymbol{E})^{-1}} k^*f^*(\boldsymbol{E}) \xrightarrow{k^*(\xi)} k^*g^*(\boldsymbol{F}) \xrightarrow{\boldsymbol{c}_{g,k}(\boldsymbol{F})} (gk)^*(\boldsymbol{F})$$

Hence a correspondence $\xi \mapsto \xi_k$ defines a map

$$k^\sharp_{\boldsymbol{E},\boldsymbol{F}} : F_{f,g}(\boldsymbol{E},\boldsymbol{F}) \to F_{fk,gk}(\boldsymbol{E},\boldsymbol{F}).$$

It follows from (1.6) that the above map is natural in $\boldsymbol{E}$ and $\boldsymbol{F}$.

Thus we have a natural transformation $k^\sharp : F_{f,g} \to F_{fk,gk}$.

Proposition 1.7 ([10] Proposition 1.1.15)

Let $f : X \to Y$, $g : X \to Z$, $h : X \to W$, $k : V \to X$ be morphisms in $\mathscr{C}$.

For objects $\boldsymbol{E}, \boldsymbol{F}, \boldsymbol{G}$ of $\mathscr{F}_Y, \mathscr{F}_Z, \mathscr{F}_W$, respectively and morphisms

$\xi : f^*(\boldsymbol{E}) \to g^*(\boldsymbol{F})$, $\zeta : g^*(\boldsymbol{F}) \to h^*(\boldsymbol{G})$ in $\mathscr{F}_X$,

$$k^\sharp_{\boldsymbol{E},\boldsymbol{G}}(\zeta\xi) = (\zeta\xi)_k : (fk)^*(\boldsymbol{E}) \to (hk)^*(\boldsymbol{F})$$

coincides with a composition

$$(fk)^*(\boldsymbol{E}) \xrightarrow{k^\sharp_{\boldsymbol{E},\boldsymbol{F}}(\xi)} (gk)^*(\boldsymbol{F}) \xrightarrow{k^\sharp_{\boldsymbol{F},\boldsymbol{G}}(\zeta)} (hk)^*(\boldsymbol{G}).$$

Proposition 1.8 ([10] Proposition 1.1.16)

Let $f : X \to Y$, $g : X \to Z$, $k : V \to X$, $j : U \to V$ be morphisms in $\mathscr{C}$. For objects $\boldsymbol{E}, \boldsymbol{F}$ of $\mathscr{F}_Y$, $\mathscr{F}_Z$, respectively, the following diagram is commutative. Hence $(kj)^\sharp = j^\sharp k^\sharp$ holds.

$$\begin{array}{ccc} \mathscr{F}_X(f^*(\boldsymbol{E}), g^*(\boldsymbol{F})) & \xrightarrow{(kj)^\sharp_{\boldsymbol{E},\boldsymbol{F}}} & \mathscr{F}_U((fkj)^*(\boldsymbol{E}), (gkj)^*(\boldsymbol{F})) \\ {\scriptstyle k^\sharp_{\boldsymbol{E},\boldsymbol{F}}} \searrow & & \nearrow {\scriptstyle j^\sharp_{\boldsymbol{E},\boldsymbol{F}}} \\ & \mathscr{F}_V((fk)^*(\boldsymbol{E}), (gk)^*(\boldsymbol{F})) & \end{array}$$

Definition 1.9 ([1])

A functor $\wp : \mathscr{F} \to \mathscr{C}$ is called a **cofibered category** if the functor $\wp^{op} : \mathscr{F}^{op} \to \mathscr{C}^{op}$ defined from $\wp : \mathscr{F} \to \mathscr{C}$ is a fibered category.

If $\wp : \mathscr{F} \to \mathscr{C}$ is a fibered category and a cofibered category, $\wp : \mathscr{F} \to \mathscr{C}$ is called a **bifibered category**.

Proposition 1.10 ([1])

A fibered category $\wp : \mathscr{F} \to \mathscr{C}$ is a bifibered category if and only if the inverse image functor $f^* : \mathscr{F}_Y \to \mathscr{F}_X$ has a left adjoint for any morphism $f : X \to Y$ in $\mathscr{C}$.

For a category $\mathscr{C}$, let $\mathscr{C}^{(2)}$ be the category of morphisms in $\mathscr{C}$ defined as follows.

Put $\mathrm{Ob}\mathscr{C}^{(2)} = \mathrm{Mor}\,\mathscr{C}$ and a morphism from $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ to $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ is a pair $\langle \xi : E \to F, f : X \to Y \rangle$ of morphisms in $\mathscr{C}$ which satisfies $\rho\xi = f\pi$.

The composition of morphisms $\langle \xi, f \rangle : \boldsymbol{E} \to \boldsymbol{F}$ and $\langle \zeta, g \rangle : \boldsymbol{F} \to \boldsymbol{G}$ is defined to be $\langle \zeta\xi, gf \rangle : \boldsymbol{E} \to \boldsymbol{G}$.

$$\begin{array}{ccc} E & \xrightarrow{\ \xi\ } & F \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} \\ X & \xrightarrow{\ f\ } & Y \end{array} \qquad\qquad \begin{array}{ccccc} E & \xrightarrow{\ \xi\ } & F & \xrightarrow{\ \zeta\ } & G \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \chi} \\ X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & Z \end{array}$$

Define a functor $\wp : \mathscr{C}^{(2)} \to \mathscr{C}$ by $\wp(E \xrightarrow{\pi} X) = X$ and $\wp(\langle \xi, f \rangle) = f$.

If $\mathscr{C}$ has finite limits, $\wp:\mathscr{C}^{(2)}\to\mathscr{C}$ is a fibered category as we explain below.
For a morphism $f:X\to Y$ in $\mathscr{C}$ and an object $\boldsymbol{F}=(F\xrightarrow{\rho}Y)$ of $\mathscr{C}^{(2)}_Y$, consider the following cartesian square in $\mathscr{C}$.

$$\begin{array}{ccc} F\times_Y X & \xrightarrow{f_\rho} & F \\ \downarrow{\scriptstyle\rho_f} & & \downarrow{\scriptstyle\rho} \\ X & \xrightarrow{f} & Y \end{array}$$

We put $f^*(\boldsymbol{F})=(F\times_Y X\xrightarrow{\rho_f}X)$ and $\boldsymbol{\alpha}_f(\boldsymbol{F})=\langle f_\rho,f\rangle:f^*(\boldsymbol{F})\to\boldsymbol{F}$.

Proposition 1.11

$\boldsymbol{\alpha}_f(\boldsymbol{F})$ is a cartesian morphism, that is, for any object $\boldsymbol{E}$ of $\mathscr{C}^{(2)}_X$ the map $\boldsymbol{\alpha}_f(\boldsymbol{F})_*:\mathscr{C}^{(2)}_X(\boldsymbol{E},f^*(\boldsymbol{F}))\to\mathscr{C}^{(2)}_f(\boldsymbol{E},\boldsymbol{F})$ defined by $\boldsymbol{\alpha}_f(\boldsymbol{F})_*(\xi)=\boldsymbol{\alpha}_f(\boldsymbol{F})\xi$ is bijective.

For morphisms $f:X\to Y$, $g:Z\to X$ in $\mathscr{C}$ and an object $\boldsymbol{F}=(F\xrightarrow{\rho}Y)$ of $\mathscr{C}_Y^{(2)}$, suppose that the left and right rectangles of the following diagram are cartesian. Equivalently, $\langle f_\rho,f\rangle:f^*(\boldsymbol{F})\to\boldsymbol{F}$ and $\langle g_{\rho_f},g\rangle:g^*f^*(\boldsymbol{F})\to f^*(\boldsymbol{F})$ are cartesian morphisms in $\mathscr{C}^{(2)}$.

$$
\begin{array}{ccccc}
(F\times_Y X)\times_X Z & \xrightarrow{g_{\rho_f}} & F\times_Y X & \xrightarrow{f_\rho} & F \\
\downarrow{\scriptstyle (\rho_f)_g} & & \downarrow{\scriptstyle \rho_f} & & \downarrow{\scriptstyle \rho} \\
Z & \xrightarrow{g} & X & \xrightarrow{f} & Y
\end{array}
$$

Then, the outer rectangle of the above diagram is also cartesian.

This shows that the composition $g^*f^*(\boldsymbol{F})\xrightarrow{\langle g_{\rho_f},g\rangle}f^*(\boldsymbol{F})\xrightarrow{\langle f_\rho,f\rangle}\boldsymbol{F}$ is cartesian. Thus we have the following result.

Proposition 1.12 ([1])

$\wp:\mathscr{C}^{(2)}\to\mathscr{C}$ is a fibered category.

For a morphism $f: X \to Y$ in $\mathscr{C}$, define a functor $f_*: \mathscr{C}_X^{(2)} \to \mathscr{C}_Y^{(2)}$ by $f_*(\boldsymbol{E}) = (E \xrightarrow{f\rho} Y)$ and $f_*(\langle \xi, id_X \rangle) = \langle \xi, id_Y \rangle : f_*(\boldsymbol{E}) \to f_*(\boldsymbol{F})$ for an object $\boldsymbol{E} = (E \xrightarrow{\rho} X)$ of $\mathscr{C}_X^{(2)}$ and a morphism $\langle \xi, id_X \rangle : \boldsymbol{E} \to \boldsymbol{F}$ in $\mathscr{C}_X^{(2)}$.

Proposition 1.13 ([1])

$f_*: \mathscr{C}_X^{(2)} \to \mathscr{C}_Y^{(2)}$ is a left adjoint of $f^*: \mathscr{C}_Y^{(2)} \to \mathscr{C}_X^{(2)}$.

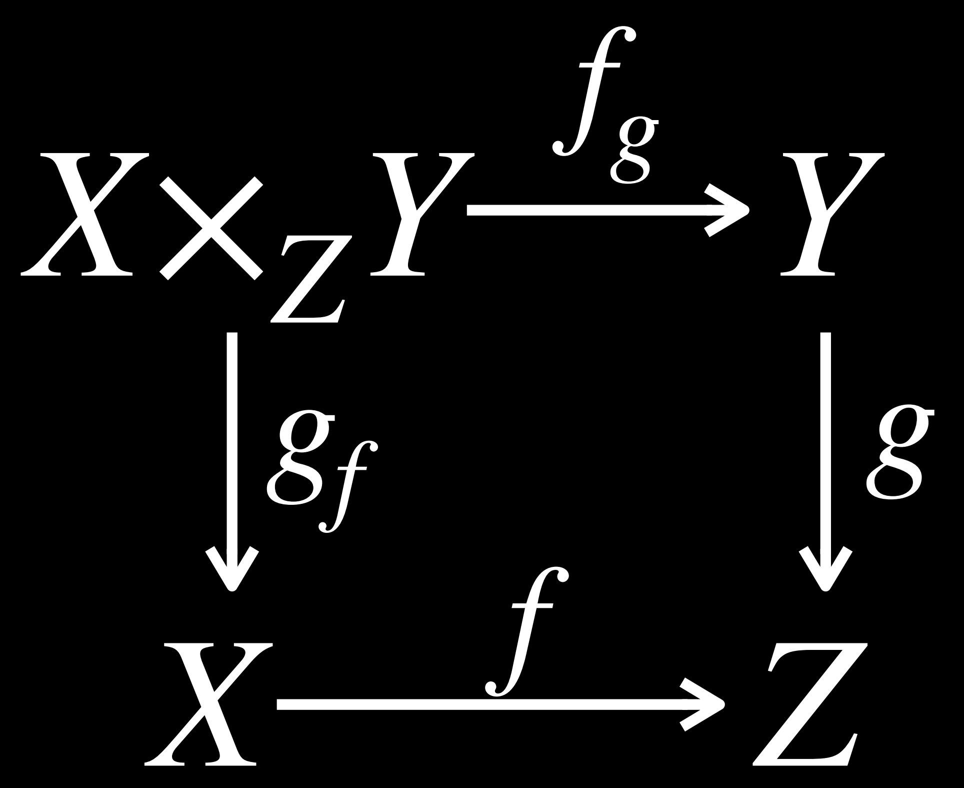
Hence $\wp: \mathscr{C}^{(2)} \to \mathscr{C}$ is a bifibered category.

For an object $\boldsymbol{E}$ of $\mathscr{C}_X^{(2)}$ and an object $\boldsymbol{F}$ of $\mathscr{C}_Y^{(2)}$, we define a map $\Phi_{\boldsymbol{E},\boldsymbol{F}}: \mathscr{C}_f^{(2)}(\boldsymbol{E}, \boldsymbol{F}) \to \mathscr{C}_Y^{(2)}(f_*(\boldsymbol{E}), \boldsymbol{F})$ by $\Phi_{\boldsymbol{E},\boldsymbol{F}}(\langle \xi, f \rangle) = \langle \xi, id_Y \rangle$, which is a natural bijection. It follows from (1.11) that we have a natural bijection $\Phi_{\boldsymbol{E},\boldsymbol{F}}\, \alpha_f(\boldsymbol{F})_*: \mathscr{C}_X^{(2)}(\boldsymbol{E}, f^*(\boldsymbol{F})) \to \mathscr{C}_Y^{(2)}(f_*(\boldsymbol{E}), \boldsymbol{F})$.

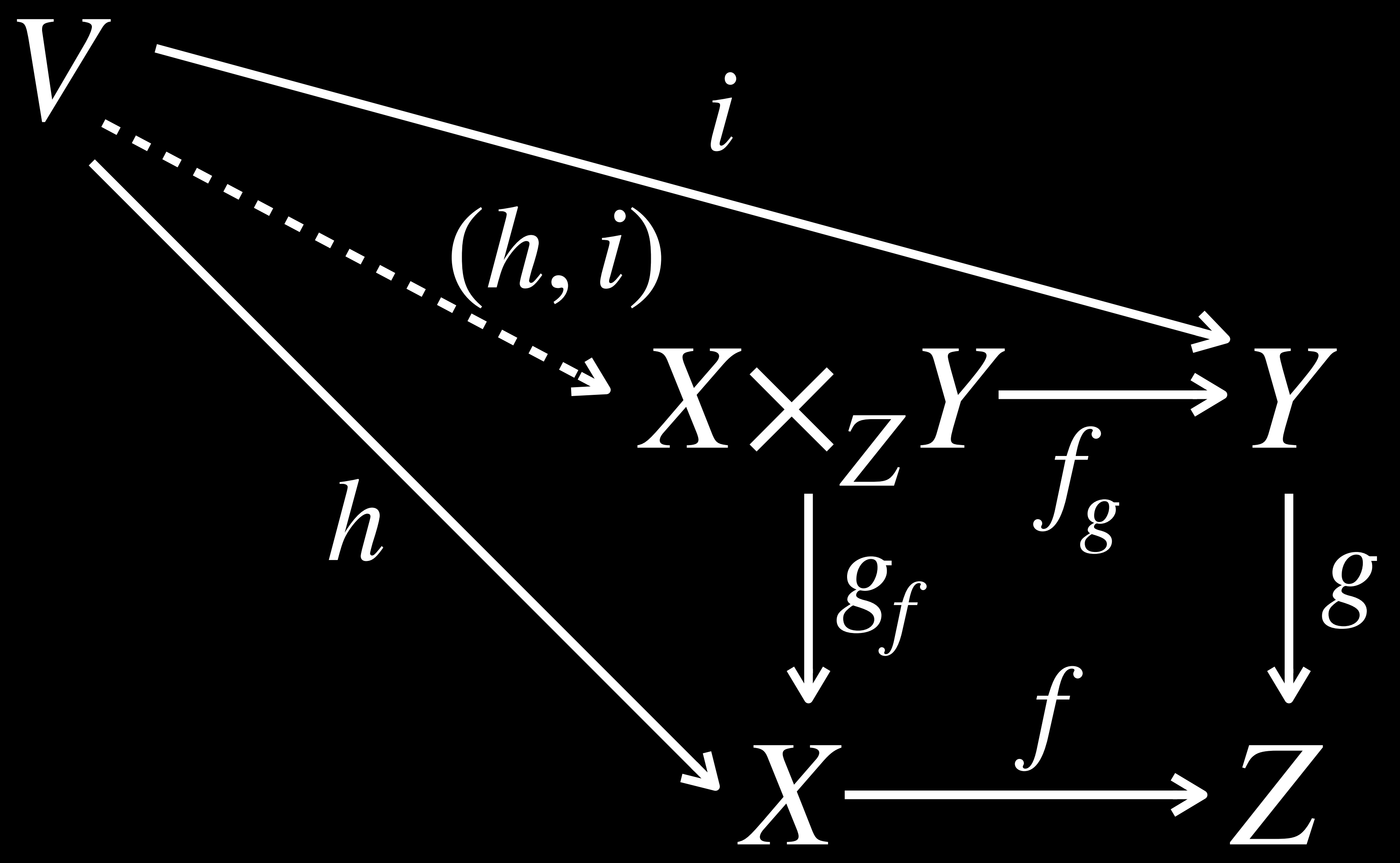
§2. Representations of groupoids

Let $\mathscr{C}$ be a category with finite limits.

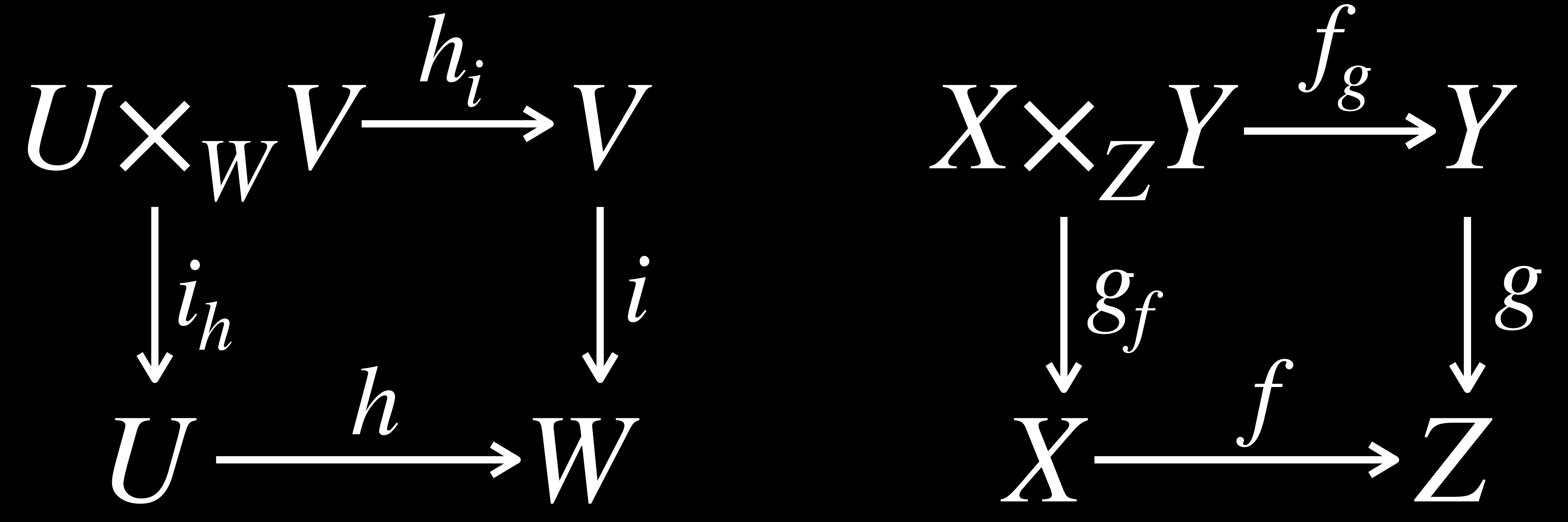
Consider the right cartesian square in $\mathscr{C}$.



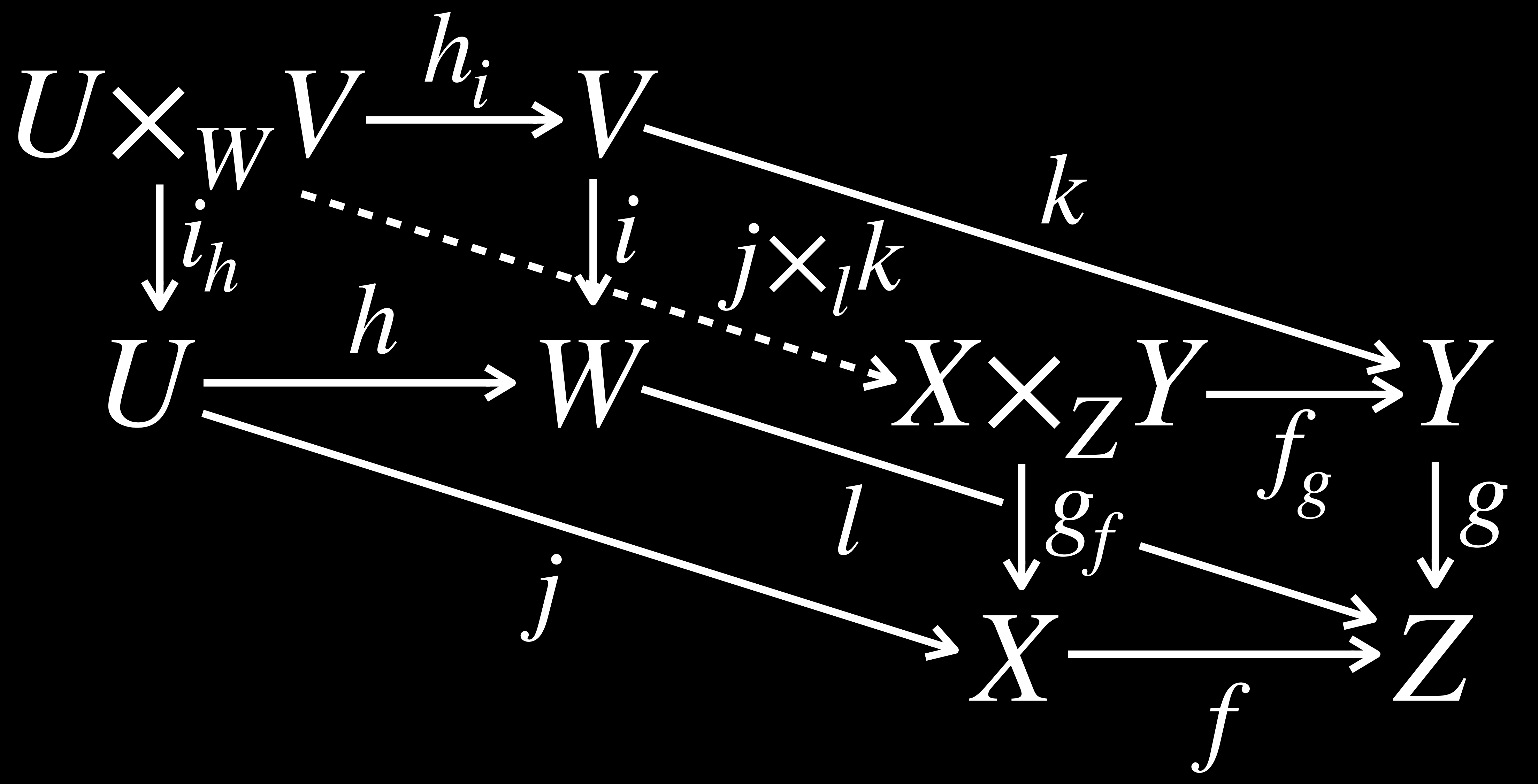
If morphisms $h:V\to X$ and $i:V\to Y$ in $\mathscr{C}$ satisfy $fh=gi$, we denote by $(h,i):V\to X\times_Z Y$ unique morphism that make the following diagram commute.



Suppose that the following diagrams are cartesian.



If morphisms $j:U\to X$, $k:V\to Y$ and $l:W\to Z$ in $\mathscr{C}$ satisfies $fj=lh$ and $gk=li$, we denote (ji_h, kh_i) by $j\times_l k:U\times_W V\to X\times_Z Y$.



If $W=Z$ and $l=id_Z$, we denote $j\times_{id_Z} k$ by $j\times_Z k$.

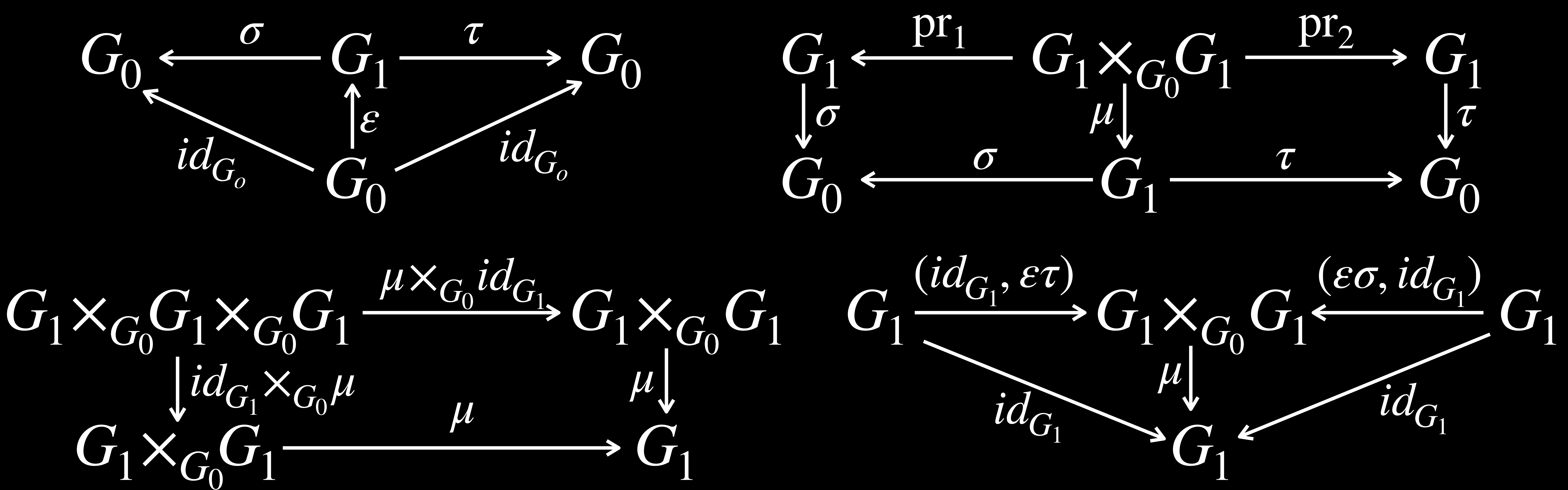
Suppose that a pair (G_0, G_1) of objects of $\mathscr{C}$ and four morphisms $\sigma, \tau : G_1 \to G_0$, $\varepsilon : G_0 \to G_1$ and $\mu : G_1 \times_{G_0} G_1 \to G_1$ in $\mathscr{C}$ are given.

Here the following diagram is cartesian.

$$\begin{array}{ccc} G_1 \times_{G_0} G_1 & \xrightarrow{\mathrm{pr}_2} & G_1 \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \sigma} \\ G_1 & \xrightarrow{\tau} & G_0 \end{array}$$

Definition 2.1

We say that $(G_0, G_1; \sigma, \tau, \varepsilon, \mu)$ is an **internal category** in $\mathscr{C}$ if the following diagrams are commutative.



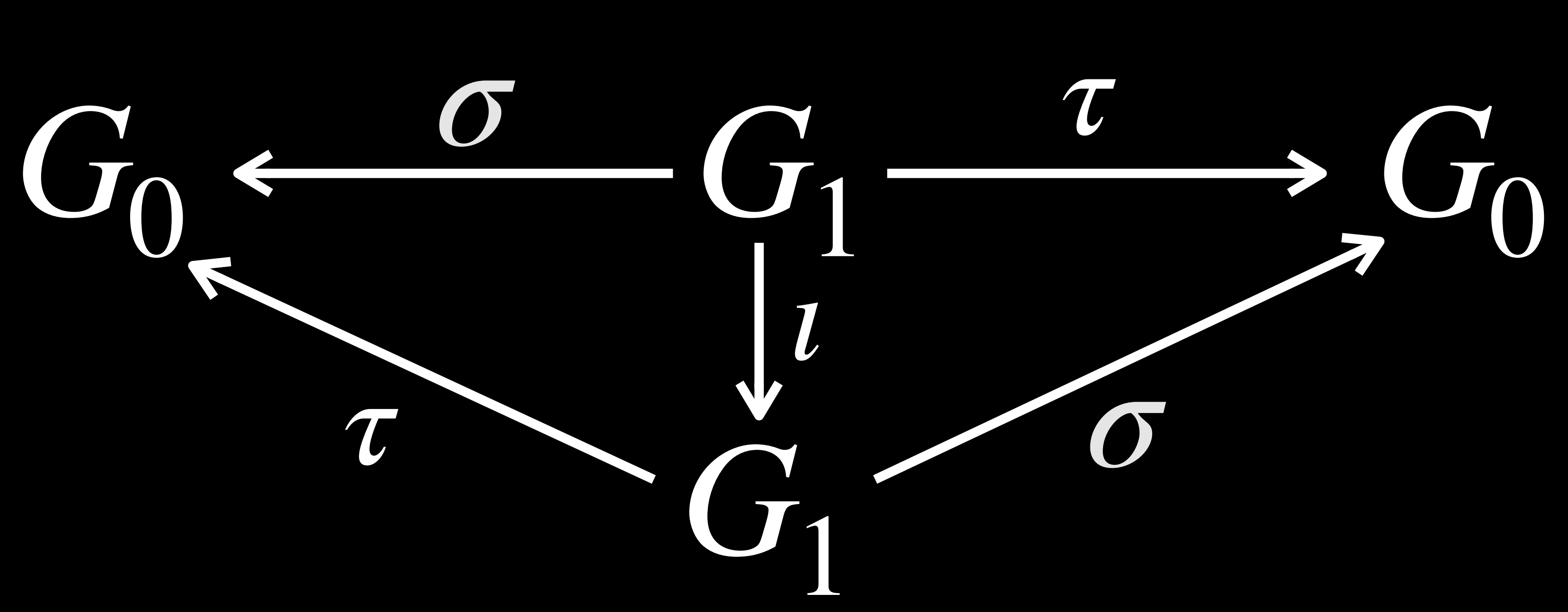
Here $G_1 \times_{G_0} G_1 \times_{G_0} G_1$ is a limit of the following diagram.

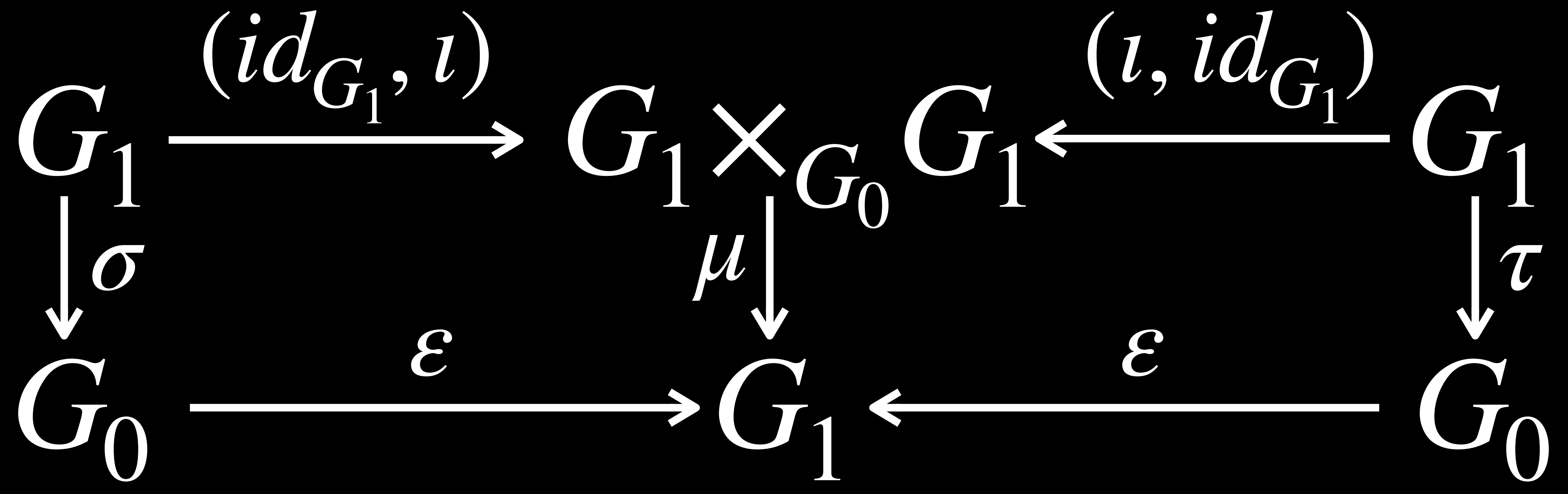
$$G_1 \xrightarrow{\tau} G_0 \xleftarrow{\sigma} G_1 \xrightarrow{\tau} G_0 \xleftarrow{\sigma} G_1$$

Definition 2.2

Let $(G_0, G_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathscr{C}$.

If a morphism $\iota : G_1 \to G_1$ in $\mathscr{C}$ makes the following diagrams commute, we call $(G_0, G_1; \sigma, \tau, \varepsilon, \mu, \iota)$ an internal groupoid in $\mathscr{C}$ or a groupoid in $\mathscr{C}$ for short.





We also have a notion of internal functors between internal categories.

Definition 2.3

Let $\boldsymbol{G}=(G_0, G_1; \sigma, \tau, \varepsilon, \mu)$ and $\boldsymbol{H}=(H_0, H_1; \sigma', \tau', \varepsilon', \mu')$ be internal categories in $\mathscr{C}$. An internal functor from $\boldsymbol{G}$ to $\boldsymbol{H}$ is a pair (f_0, f_1) of morphisms $f_0 : G_0 \to G_0$ and $f_1 : G_1 \to G_1$ in $\mathscr{C}$ which make the following diagrams commute.

$$
\begin{array}{ccccc}
G_0 & \xleftarrow{\ \sigma\ } & G_1 & \xrightarrow{\ \tau\ } & G_0 \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
H_0 & \xleftarrow{\ \sigma'\ } & H_1 & \xrightarrow{\ \tau'\ } & H_0
\end{array}
\qquad
\begin{array}{ccccc}
G_1 \times_{G_0} G_1 & \xrightarrow{\ \mu\ } & G_1 & \xleftarrow{\ \varepsilon\ } & G_0 \\
\downarrow{\scriptstyle f_1 \times_{f_0} f_1} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
H_1 \times_{H_0} H_1 & \xrightarrow{\ \mu'\ } & H_1 & \xleftarrow{\ \varepsilon'\ } & H_0
\end{array}
$$

Definition 2.4

Let $\boldsymbol{f}=(f_0,f_1),\boldsymbol{g}=(g_0,g_1):\boldsymbol{G}\to\boldsymbol{H}$ be internal functors.

An internal natural transformation $\chi:\boldsymbol{f}\to\boldsymbol{g}$ from $\boldsymbol{f}$ to $\boldsymbol{g}$ is a morphism $\chi:G_0\to H_1$ in $\mathscr{C}$ which makes the following diagrams commute.

$$
\begin{array}{ccccc}
H_0 & \xleftarrow{\;f_0\;} & G_0 & \xrightarrow{\;g_0\;} & H_0 \\
 & \nwarrow{\scriptstyle \sigma'} & \downarrow{\scriptstyle \chi} & \nearrow{\scriptstyle \tau'} & \\
 & & H_1 & &
\end{array}
\qquad
\begin{array}{ccc}
G_1 & \xrightarrow{(\chi\sigma,\, g_1)} & H_1\times_{H_0}H_1 \\
\downarrow{\scriptstyle (f_1,\chi\tau)} & & \downarrow{\scriptstyle \mu'} \\
H_1\times_{H_0}H_1 & \xrightarrow{\;\mu'\;} & H_1
\end{array}
$$

Let $\wp:\mathscr{F}\to\mathscr{C}$ be a fibered category and assume that $\mathscr{C}$ is a category with finite limits below.

Definition 2.5 ([9], [10] Definition 3.1.2)

Let $\boldsymbol{G}=(G_0, G_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathscr{C}$.

A pair $(\boldsymbol{E}, \xi)$ of an object $\boldsymbol{E}$ of $\mathscr{F}_{G_0}$ and a morphism

$$\xi : \sigma^*(\boldsymbol{E}) \to \tau^*(\boldsymbol{E})$$

in $\mathscr{F}_{G_1}$ is called a representation of $\boldsymbol{G}$ on $\boldsymbol{E}$ if the following diagrams are commutative.

$$\begin{array}{ccc}
(\sigma \mathrm{pr}_1)^*(\boldsymbol{E}) = (\sigma\mu)^*(\boldsymbol{E}) & \xrightarrow{\ \xi_\mu\ } & (\tau\mu)^*(\boldsymbol{E}) = (\tau \mathrm{pr}_2)^*(\boldsymbol{E}) \\
{}_{\xi_{\mathrm{pr}_1}} \searrow & & \nearrow {}_{\xi_{\mathrm{pr}_2}} \\
& (\tau \mathrm{pr}_1)^*(\boldsymbol{E}) = (\sigma \mathrm{pr}_2)^*(\boldsymbol{E}) &
\end{array}$$

$$\begin{array}{ccc}
(id_{G_0})^*(\boldsymbol{E}) = (\sigma\varepsilon)^*(\boldsymbol{E}) & \xrightarrow{\ \xi_\varepsilon\ } & (\tau\varepsilon)^*(\boldsymbol{E}) = (id_{G_0})^*(\boldsymbol{E}) \\
{}_{\alpha_{id_{G_0}}(\boldsymbol{E})} \searrow & & \swarrow {}_{\alpha_{id_{G_0}}(\boldsymbol{E})} \\
& \boldsymbol{E} &
\end{array}$$

Definition 2.6 ([9], [10] Definition 3.1.2)

Let (E,ξ) and (F,ζ) be representations of G on E and F.
A morphism $\varphi : E \to F$ in $\mathscr{F}_{G_0}$ is called a morphism of representations of G if $\varphi : E \to F$ makes the following diagram commute.

$$\begin{array}{ccc} \sigma^*(E) & \xrightarrow{\xi} & \tau^*(E) \\ \downarrow{\scriptstyle \sigma^*(\varphi)} & & \downarrow{\scriptstyle \tau^*(\varphi)} \\ \sigma^*(F) & \xrightarrow{\zeta} & \tau^*(F) \end{array}$$

Thus we have the category of representations of G, which we denote by $\mathrm{Rep}(G)$.

Let $\boldsymbol{G}=(G_0, G_1; \sigma, \tau, \varepsilon, \mu)$ and $\boldsymbol{H}=(H_0, H_1; \sigma', \tau', \varepsilon', \mu')$ be internal categories in $\mathscr{C}$ and $\boldsymbol{f}=(f_0, f_1):\boldsymbol{H}\to\boldsymbol{G}$ be an internal functor.

For a representation $(\boldsymbol{E}, \boldsymbol{\xi})$ of $\boldsymbol{G}$ on $\boldsymbol{E}$, we define

$$\boldsymbol{\xi}_{\boldsymbol{f}} : \sigma'^{*}f_0^{*}(\boldsymbol{E}) \to \tau'^{*}f_0^{*}(\boldsymbol{E})$$

to be the following composition.

$$\sigma'^{*}f_0^{*}(\boldsymbol{E}) \xrightarrow{c_{f_0,\sigma'}(\boldsymbol{E})} (f_0\sigma')^{*}(\boldsymbol{E}) = (\sigma f_1)^{*}(\boldsymbol{E}) \xrightarrow{\boldsymbol{\xi}_{f_1}} (\tau f_1)^{*}(\boldsymbol{E}) = (f_0\tau')^{*}(\boldsymbol{E})$$

$$\xrightarrow{c_{f_0,\tau'}(\boldsymbol{E})^{-1}} (f_0\tau')^{*}(\boldsymbol{E}) = \tau'^{*}f_0^{*}(\boldsymbol{E})$$

Proposition 2.7 ([9], [10] Proposition 3.2.1)
$(f_0^*(E), \xi_f)$ is a representation of H on $f_0^*(E)$.
If $\varphi : (E, \xi) \to (F, \zeta)$ is a morphism of representations of G, then $f_0^*(\varphi) : f_0^*(E) \to f_0^*(F)$ gives a morphism

$$f_0^*(\varphi) : (f_0^*(E), \xi_f) \to (f_0^*(F), \zeta_f)$$

of representations of H.

Definition 2.8 ([9], [10] Definition 3.2.3)
Define a functor $f^\bullet : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$ by $f^\bullet(E, \xi) = (f_0^*(E), \xi_f)$ for an object (E, ξ) of $\mathrm{Rep}(G)$ and $f^\bullet(\varphi) = f_0^*(\varphi)$ for a morphism $\varphi : (E, \xi) \to (F, \zeta)$ in $\mathrm{Rep}(G)$.
We call this functor the restriction functor along $f : H \to G$.

Let $\boldsymbol{G}=(G_0, G_1; \sigma, \tau, \varepsilon, \mu)$ and $\boldsymbol{H}=(H_0, H_1; \sigma', \tau', \varepsilon', \mu')$ be internal categories in $\mathscr{C}$, $\boldsymbol{f}=(f_0, f_1), \boldsymbol{g}=(g_0, g_1): \boldsymbol{H} \to \boldsymbol{G}$ be internal functors and $\chi: \boldsymbol{f} \to \boldsymbol{g}$ an internal natural transformation.

For a representation $(\boldsymbol{E}, \boldsymbol{\xi})$ of $\boldsymbol{G}$ on $\boldsymbol{E}$, we define a morphism $\chi^{\bullet}_{(\boldsymbol{E},\boldsymbol{\xi})}: f_0^*(\boldsymbol{E}) \to g_0^*(\boldsymbol{E})$ in $\mathscr{F}_{H_0}$ to be

$$\chi^{\sharp}_{\boldsymbol{E},\boldsymbol{E}}(\boldsymbol{E}) = \boldsymbol{\xi}_{\chi}: f_0^*(\boldsymbol{E}) = (\sigma\chi)^*(\boldsymbol{E}) \to (\tau\chi)^*(\boldsymbol{E}) = g_0^*(\boldsymbol{E}).$$

Proposition 2.9 ([10] Proposition 3.2.5)

$\chi^{\bullet}_{(E,\xi)}$ is a morphism of representations from $f^{\bullet}(E,\xi)=(f_0^*(E),\xi_f)$ to $g^{\bullet}(E,\xi)=(g_0^*(E),\xi_g)$ and the following diagram in $\mathrm{Rep}(H)$ is commutative for a morphism $\varphi:(E,\xi)\to(F,\zeta)$ of representations of G.

$$\begin{array}{ccc} (f_0^*(E),\xi_f) & \xrightarrow{f^{\bullet}(\varphi)} & (f_0^*(F),\zeta_f) \\ \downarrow{\scriptstyle \chi^{\bullet}_{(E,\xi)}} & & \downarrow{\scriptstyle \chi^{\bullet}_{(F,\zeta)}} \\ (g_0^*(E),\xi_g) & \xrightarrow{g^{\bullet}(\varphi)} & (g_0^*(F),\zeta_g) \end{array}$$

Thus we have a natural transformation $\chi^{\bullet}:f^{\bullet}\to g^{\bullet}$.

§3. Recollections on Grothendieck site

We denote by $\mathcal{S}et$ the category of sets and maps.

For a category $\mathscr{C}$, we call a functor $\mathscr{C}^{op} \to \mathcal{S}et$ presheaf on $\mathscr{C}$.

For an object X of $\mathscr{C}$, let $h_X : \mathscr{C}^{op} \to \mathcal{S}et$ be a functor defined by $h_X(U) = \mathscr{C}(U, X)$ for an object U of $\mathscr{C}$ and

$$h_X(f : U \to V) = (f^* : \mathscr{C}(V, X) \to \mathscr{C}(U, X))$$

for a morphism $f : U \to V$ in $\mathscr{C}$.

Here, $\mathscr{C}(U, X)$ denotes the set of morphisms in $\mathscr{C}$ from U to X.

We call $h_X : \mathscr{C}^{op} \to \mathcal{S}et$ the presheaf on $\mathscr{C}$ represented by X.

For a morphism $\varphi : X \to Y$ in $\mathscr{C}$, let $h_\varphi : h_X \to h_Y$ be a natural transformation defined by $(h_\varphi)_U = \varphi_* : \mathscr{C}(U, X) \to \mathscr{C}(U, Y)$.

For a morphism f in a category $\mathscr{C}$, let us denote by $\mathrm{dom}(f)$ the source of f and $\mathrm{codom}(f)$ the target of f.

For set valued functors $F, G : \mathscr{C} \to \mathscr{S}et$, if $F(U)$ is a subset of $G(U)$ for any object U of $\mathscr{C}$ and the inclusion map $i_U : F(U) \to G(U)$ defines a natural transformation $i : F \to G$, we call F a subfunctor of G. If F is a subfunctor of G, we denote this by $F \subset G$.

For an object X of a category $\mathscr{C}$, we call a subfunctor of h_X a sieve on X.

Definition 3.1

Let $\mathscr{C}$ be a category. For each $X \in \mathrm{Ob}\,\mathscr{C}$, a set $J(X)$ of sieves on X is given. If the following conditions are satisfied, a correspondence $J : X \mapsto J(X)$ is called a (Grothendieck) topology on $\mathscr{C}$. A category $\mathscr{C}$ with a topology J is called a site which we denote by $(\mathscr{C}, J)$.

(T1) For any $X \in \mathrm{Ob}\,\mathscr{C}$, $h_X \in J(X)$.

(T2) For any $X \in \mathrm{Ob}\,\mathscr{C}$, $R \in J(X)$ and morphism $f : Y \to X$ of $\mathscr{C}$, a subfunctor $h_f^{-1}(R)$ of h_Y defined below belongs to $J(Y)$.

$$h_f^{-1}(R)(Z) = \{ g : Z \to Y \mid fg \in R(Z) \}$$

(T3) A sieve S on X belongs to $J(X)$, if there exists $R \in J(X)$ such that $h_f^{-1}(S) \in J(\mathrm{dom}(f))$ for any $f \in \mathrm{Ob}\, R$.

Proposition 3.2

Consider the following conditions on J.

(T3') A sieve S on X belongs to $J(X)$, if there exists $R \in J(X)$ such that S is a subfunctor of R and $h_f^{-1}(S) \in J(\mathrm{dom}(f))$ for $f \in \mathrm{Ob}\, R$.

(T4) A sieve S on X belongs to $J(X)$ if it has a subfunctor which belongs to $J(X)$.

(T5) Suppose that $R \in J(X)$ and that $R_f \in J(\mathrm{dom}(f))$ is given for each $f \in \mathrm{Ob}\, R$. Then, $\{fg \mid f \in \mathrm{Ob}\, R, g \in \mathrm{Ob}\, R_f\} \in J(X)$.

(1) (T2) and (T3) imply (T4). (T1) and (T3) imply (T5).

(2) (T4) and (T5) imply (T3). (T3') and (T4) imply (T3).

Proposition 3.3

For a set R of morphisms in $\mathscr{C}$ with target X, we put

$$\bar{R} = \bigcup_{f \in R} \mathrm{Im}(h_f : h_{\mathrm{dom}(f)} \to h_X).$$

In other words, $\bar{R}$ is the set of all morphisms of the form fg such that $f \in R$, $g \in \mathrm{Mor}\,\mathscr{C}$ and $\mathrm{codom}(g) = \mathrm{dom}(f)$.

Then, $\bar{R}$ is the smallest sieve containing R.

Definition 3.4

Let $(\mathscr{C}, J)$ be a site.

(1) For a set R of morphisms in $\mathscr{C}$ with target X, we call $\bar{R}$ the sieve generated by R.

(2) A family of morphisms $(f_i : X_i \to X)_{i \in I}$ is called a covering of X if the sieve generated by f_i's belongs to $J(X)$.

§4. Plots on a set

Definition 4.1 ([11] Definition 1.1)
Let $\mathscr{C}$ be a category and $F: \mathscr{C} \to \mathscr{S}et$ a functor.
For a set X, we define a presheaf F_X on $\mathscr{C}$ to be a composition

$$\mathscr{C}^{op} \xrightarrow{F^{op}} \mathscr{S}et^{op} \xrightarrow{h_X} \mathscr{S}et.$$

Here we denote by $F^{op}: \mathscr{C}^{op} \to \mathscr{S}et^{op}$ a functor defined by
$F^{op}(U) = F(U)$ for $U \in \mathrm{Ob}\,\mathscr{C}$ and $F^{op}(f) = F(f)$ for $f \in \mathrm{Mor}\,\mathscr{C}$.
An element of $\coprod_{U \in \mathrm{Ob}\mathscr{C}} F_X(U)$ is called an F-parametrization of X.

We note that F_X is given by $F_X(U) = \mathscr{S}et(F(U), X)$ for $U \in \mathrm{Ob}\mathscr{C}$
and $F_X(f)(\alpha) = \alpha F(f)$ for $(f: U \to V) \in \mathrm{Mor}\,\mathscr{C}$ and $\alpha \in F_X(V)$.

Definition 4.2 ([11] Definition 1.2)

Let $(\mathscr{C}, J)$ be a site, X a set and $F: \mathscr{C} \to \mathscr{S}et$ a functor. Assume that $\mathscr{C}$ has a terminal object $1_{\mathscr{C}}$ and that $F(1_{\mathscr{C}})$ consists of a single element. If a subset $\mathscr{D}$ of $\coprod_{U \in \mathrm{Ob}\mathscr{C}} F_X(U)$ satisfies the following conditions, we call $\mathscr{D}$ a the-ology on X.

(i) $\mathscr{D} \supset F_X(1_{\mathscr{C}})$

(ii) For a morphism $f: U \to V$ in $\mathscr{C}$, the map $F_X(f): F_X(V) \to F_X(U)$ induced by f maps $\mathscr{D} \cap F_X(V)$ into $\mathscr{D} \cap F_X(U)$.

(iii) For an object U of $\mathscr{C}$, an element x of $F_X(U)$ belongs to $\mathscr{D} \cap F_X(U)$ if there exists a covering $(f_i: U_i \to U)_{i \in I}$ such that $F_X(f_i): F_X(U) \to F_X(U_i)$ maps x into $\mathscr{D} \cap F_X(U_i)$ for any $i \in I$.

We call a pair $(X, \mathscr{D})$ a the-ological object and call an element of $\mathscr{D}$ an F-plot of $(X, \mathscr{D})$.

Proposition 4.3 ([11] Proposition 1.4)

Condition (iii) is of (4.2) is equivalent to the following condition if we assume condition (ii).

(iii') For an object U of $\mathscr{C}$, an element x of $F_X(U)$ belongs to $\mathscr{D} \cap F_X(U)$ if there exists $R \in J(U)$ such that $F_X(f) : F_X(U) \to F_X(\mathrm{dom}(f))$ maps x into $\mathscr{D} \cap F_X(\mathrm{dom}(f))$ for any $f \in R$.

For a map $\varphi : X \to Y$ and a functor $F : \mathscr{C} \to \mathscr{S}et$, we define a morphism $F_\varphi : F_X \to F_Y$ of presheaves by

$$(F_\varphi)_U = \varphi_* : F_X(U) = \mathscr{S}et(F(U), X) \to \mathscr{S}et(F(U), Y) = F_Y(U).$$

Definition 4.4 ([11] Definition 1.7)

Let $(\mathscr{C}, J)$ be a site, X a set and $F:\mathscr{C}\to\mathscr{S}et$ a functor.

(1) Let $(X,\mathscr{D})$ and $(Y,\mathscr{E})$ be the-ological objects.
If the map $(F_\varphi)_U : F_X(U)\to F_Y(U)$ induced by a map $\varphi : X\to Y$ maps $\mathscr{D}\cap F_X(U)$ into $\mathscr{E}\cap F_Y(U)$ for each $U\in \mathrm{Ob}\mathscr{C}$, we call φ a morphism of the-ological objects.
We denote this by $\varphi : (X,\mathscr{D})\to(Y,\mathscr{E})$.

(2) We define a category $\mathscr{P}_F(\mathscr{C}, J)$ of the-ological objects as follows. Objects of $\mathscr{P}_F(\mathscr{C}, J)$ are the-ological objects and morphisms of $\mathscr{P}_F(\mathscr{C}, J)$ are morphism of the-ological objects.

For a the-ological object $(X, \mathscr{D})$ and $U \in \mathrm{Ob}\mathscr{C}$, we put $F_{\mathscr{D}}(U) = \mathscr{D} \cap F_X(U)$. Then $U \mapsto F_{\mathscr{D}}(U)$ defines a presheaf $F_{\mathscr{D}}$ on $\mathscr{E}$.

Remark 4.5 ([11] Remark 1.8)

Let $\varphi : (X, \mathscr{D}) \to (Y, \mathscr{E})$ be a morphism of the-ological objects. It follows from the definition of a morphism of the-ological objects that $(F_\varphi)_U : F_X(U) \to F_Y(U)$ defines a map $(F_\varphi)_U : F_{\mathscr{D}}(U) \to F_{\mathscr{E}}(U)$ which is natural in $U \in \mathrm{Ob}\mathscr{C}$. Thus we have a morphism $F_\varphi : F_{\mathscr{D}} \to F_{\mathscr{E}}$ of presheaves.

Definition 4.6 ([11] Definition 1.9)

For the-ologies $\mathscr{D}$ and $\mathscr{E}$ on X, we say that $\mathscr{D}$ is finer than $\mathscr{E}$ and that $\mathscr{E}$ is coarser than $\mathscr{D}$ if $\mathscr{D} \subset \mathscr{E}$.

Remark 4.7 ([11] Remark 1.10)

We put $\mathscr{D}_{coarse,X} = \coprod_{U \in \mathrm{Ob}\mathscr{C}} F_X(U)$. It is clear that $\mathscr{D}_{coarse,X}$ is the coarsest the-ology on X. For a map $f: Y \to X$ and a the-ology $\mathscr{E}$ on Y, $f: (Y, \mathscr{E}) \to (X, \mathscr{D}_{coarse,X})$ is a morphism of the-ologies.

Proposition 4.8 ([11] Proposition 1.11)

Let $(\mathscr{D}_i)_{i \in I}$ be a family of the-ologies on a set X. Then, $\bigcap_{i \in I} \mathscr{D}_i$ is a the-ology on X that is the finest the-ology among the-ologies on X which are coarser than $\mathscr{D}_i$ for any $i \in I$.

For a set X, we denote by $\mathscr{P}_F(\mathscr{C}, J)_X$ a subcategory of $\mathscr{P}_F(\mathscr{C}, J)$ consisting of objects of the form $(X, \mathscr{D})$ and morphisms of the form $id_X : (X, \mathscr{D}) \to (X, \mathscr{E})$. Then, $\mathscr{P}_F(\mathscr{C}, J)_X$ is regarded as an ordered set of the-ologies on X.

We often denote by $\mathscr{D}$ an object $(X, \mathscr{D})$ of $\mathscr{P}_F(\mathscr{C}, J)_X$ for short.

It follows from (4.7) that $(X, \mathscr{D}_{coarse, X})$ is the maximum (terminal) object of $\mathscr{P}_F(\mathscr{C}, J)_X$.

Corollary 4.9 ([11] Corollary 1.12)

$\mathscr{P}_F(\mathscr{C}, J)_X$ is complete as an ordered set.

Proposition 4.10 ([11] Proposition 1.13)

Let $\mathscr{S}$ be a subset of $\coprod_{U\in \mathrm{Ob}\mathscr{C}} F_X(U)$ which contains $F_X(1_{\mathscr{C}})$.

For $f\in \mathrm{Mor}\mathscr{C}$, define a subset $\mathscr{S}_f$ of $F_X(\mathrm{dom}(f))$ by

$$\mathscr{S}_f = F_X(f)(\mathscr{S}\cap F_X(\mathrm{codom}(f))).$$

For $U\in \mathrm{Ob}\mathscr{C}$, we define a subset $\mathscr{S}(U)$ of $F_X(U)$ by

$$\mathscr{S}(U)=\left\{ x\in F_X(U) \,\middle|\, \begin{array}{l} \textit{There exists } R\in J(U) \textit{ such that} \\ \qquad F_X(g)(x)\in \bigcup_{f\in \mathrm{Mor}\mathscr{C}} \mathscr{S}_f \textit{ for all } g\in R. \end{array} \right\}.$$

If we put $\mathscr{G}(\mathscr{S})=\coprod_{U\in \mathrm{Ob}\,\mathscr{C}} \mathscr{S}(U)$ and $\Sigma=\{\mathscr{D}\in \mathscr{P}_F(\mathscr{C},J)_X \,|\, \mathscr{D}\supset \mathscr{S}\}$,

then we have $\mathscr{G}(\mathscr{S})=\inf\Sigma\in \mathscr{P}_F(\mathscr{C},J)_X$.

Remark 4.11 ([11] Remark 1.14)

(1) For $U \in \mathrm{Ob}\mathscr{C}$, the subset $\mathcal{S}(U)$ of $F_X(U)$ defined in (4.10) coincides with the following set.

$$\left\{ x \in F_X(U) \,\middle|\, \begin{array}{r} \textit{There exists a covering } (U_i \xrightarrow{g_i} U)_{i \in I} \textit{ such that} \\ F_X(g_i)(x) \in \bigcup_{f \in \mathrm{Mor}\mathscr{C}} \mathcal{S}_f \textit{ for all } i \in I. \end{array} \right\}$$

(2) Let Σ be a non-empty subset of $\mathscr{P}_F(\mathscr{C}, J)_X$ and put $\mathcal{S}(\Sigma) = \bigcup_{\mathscr{D} \in \Sigma} \mathscr{D}$. Then $\mathcal{S}(\Sigma)(U)$ coincides with the following set.

$$\left\{ x \in F_X(U) \,\middle|\, \begin{array}{r} \textit{There exists a covering } (U_i \xrightarrow{g_i} U)_{i \in I} \textit{ such that} \\ F_X(g_i)(x) \in \bigcup_{\mathscr{D} \in \Sigma} \mathscr{D} \textit{ for all } i \in I. \end{array} \right\}$$

Hence $\sup \Sigma = \mathscr{G}(\mathcal{S}(\Sigma)) = \bigcup_{U \in \mathscr{C}} \mathcal{S}(\Sigma)(U)$ holds.

Definition 4.12 ([11] Definition 1.15)
For a subset $\mathscr{S}$ of $\coprod_{U \in \mathrm{Ob}\mathscr{C}} F_X(U)$ containing $F_X(1_{\mathscr{C}})$, we call $\mathscr{G}(\mathscr{S})$ defined in (4.10) the the-ology generated by $\mathscr{S}$.

Definition 4.13 ([11] Definition 1.16)
Let $(\mathscr{C}, J)$ be a site and X a set. We put $\mathscr{D}_{disc,X} = \bigcap_{\mathscr{D} \in \mathrm{Ob}\mathscr{P}_F(\mathscr{C},J)_X} \mathscr{D}$ and call this the discrete the-ology on X. $\mathscr{D}_{disc,X}$ is the finest the-ology on X.

Remark 4.14 ([11] Remark 1.17)
For any map $f: X \to Y$ and a the-ology $\mathscr{E}$ on Y, $f: (X, \mathscr{D}_{disc,X}) \to (Y, \mathscr{E})$ is a morphism of the-ologies.

Remark 4.15 ([11] Remark 1.17)

(1) Since $\mathscr{D}_{disc,X} \supset F_X(1_{\mathscr{C}})$, $\mathscr{D}_{disc,X}$ contains the image of the map $F_X(o_U): F_X(1_{\mathscr{C}}) \to F_X(U)$ induced by the unique map $o_U: U \to 1_{\mathscr{C}}$ for any $U \in \mathrm{Ob}\,\mathscr{C}$. Hence every constant map in $F_X(U)$ belongs to $\mathscr{D}_{disc,X}$.

(2) Let $\mathscr{S}_{const}$ be the set of all constant maps in $\coprod_{U \in \mathrm{Ob}\,\mathscr{C}} F_X(U)$. Then $\mathscr{S}_{const} = \bigcup_{f \in \mathrm{Mor}\,\mathscr{C}} (\mathscr{S}_{const})_f$. Thus $\mathscr{D}_{disc,X} \cap F_X(U) = \mathscr{D}(\mathscr{S}_{const}) \cap F_X(U)$ coincides with the following set.

$$\left\{ x \in F_X(U) \mid \textit{There exists a covering } (U_i \xrightarrow{g_i} U)_{i \in I} \textit{ such that } F_X(g_i)(x) \textit{ is a contant map for all } i \in I. \right\}$$

§5. Category of F-plots

For a map $f: X \to Y$ and $(Y, \mathscr{E}) \in \mathrm{Ob}\,\mathscr{P}_F(\mathscr{C}, J)$, we define a the-ology $\mathscr{E}^f$ on X to be the coarsest the-ology such that $f: (X, \mathscr{E}^f) \to (Y, \mathscr{E})$ is a morphism of the-ologies.

Proposition 5.1 ([11] Proposition 2.1)

For a map $f: X \to Y$ and $(Y, \mathscr{E}) \in \mathrm{Ob}\,\mathscr{P}_F(\mathscr{C}, J)$, $\mathscr{E}^f$ is as follows.

$$\mathscr{E}^f = \coprod_{U \in \mathrm{Ob}\mathscr{C}} (F_f)^{-1}(\mathscr{E} \cap F_Y(U)) = \coprod_{U \in \mathrm{Ob}\mathscr{C}} \left\{ \varphi \in F_X(U) \mid f\varphi \in \mathscr{E} \cap F_Y(U) \right\}$$

Proposition 5.2 ([11] Proposition 2.2)

Let $(\mathscr{E}_i)_{i \in I}$ a family of the-ologies on a set Y. For a map $f: X \to Y$, $\left(\bigcap_{i \in I} \mathscr{E}_i\right)^f = \bigcap_{i \in I} \mathscr{E}_i^f$ holds.

We define a forgetful functor $\Gamma:\mathscr{P}_F(\mathscr{C},J)\to\mathscr{S}et$ by $\Gamma(X,\mathscr{D})=X$ for $(X,\mathscr{D})\in\mathrm{Ob}\mathscr{P}_F(\mathscr{C},J)$ and $\Gamma(\varphi:(X,\mathscr{D})\to(Y,\mathscr{E}))=(\varphi:X\to Y)$ for a morphism $\varphi:(X,\mathscr{D})\to(Y,\mathscr{E})$ in $\mathscr{P}_F(\mathscr{C},J)$.

It is clear that Γ is faithful. In other words, if we put

$$\mathscr{P}_F(\mathscr{C},J)_f((X,\mathscr{D}),(Y,\mathscr{E}))=\Gamma^{-1}(f)\cap\mathscr{P}_F(\mathscr{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))$$

for a map $f:X\to Y$ and $(X,\mathscr{D}),(Y,\mathscr{E})\in\mathrm{Ob}\,\mathscr{P}_F(\mathscr{C},J)$, $\mathscr{P}_F(\mathscr{C},J)_f((X,\mathscr{D}),(Y,\mathscr{E}))$ has at most one element.

$\mathscr{P}_F(\mathscr{C},J)_f((X,\mathscr{D}),(Y,\mathscr{E}))$ is not empty if and only if $\mathscr{D}\subset\mathscr{E}^f$ which is equivalent that $\mathscr{P}_F(\mathscr{C},J)_X((X,\mathscr{D}),(X,\mathscr{E}^f))$ is not empty.

Proposition 5.3 ([11] Proposition 2.3)
For maps $f: X \to Y$, $g: W \to X$ and an object $(Y, \mathscr{E})$ of $\mathscr{P}_F(\mathscr{C}, J)_Y$, $\mathscr{E}^{fg} = (\mathscr{E}^f)^g$ holds and $\Gamma: \mathscr{P}_F(\mathscr{C}, J) \to \mathscr{S}et$ is a fibered category.

In fact, $f: (X, \mathscr{E}^f) \to (Y, \mathscr{E})$ is unique cartesian morphism over a map $f: X \to Y$ whose target is $(Y, \mathscr{E})$. Hence the inverse image functor

$$f^*: \mathscr{P}_F(\mathscr{C}, J)_Y \to \mathscr{P}_F(\mathscr{C}, J)_X$$

associated with f is given by $f^*(Y, \mathscr{E}) = (X, \mathscr{E}^f)$ and

$$f^*(id_Y: (Y, \mathscr{E}) \to (Y, \mathscr{G})) = (id_X: (X, \mathscr{E}^f) \to (X, \mathscr{G}^f)).$$

It is clear that $\mathscr{E}^{fg} = (\mathscr{E}^f)^g$ holds, which implies $(fg)^* = g^* f^*$.

For a map $f\colon X \to Y$ and $(X, \mathscr{D}) \in \operatorname{Ob} \mathscr{P}_F(\mathscr{C}, J)$, we define a the-ology $\mathscr{D}_f$ on Y to be the finest the-ology such that $f\colon (X, \mathscr{D}) \to (Y, \mathscr{D}_f)$ is a morphism of the-ologies, that is, $\mathscr{D}_f = \bigcap_{\mathscr{E} \in \Sigma} \mathscr{E}$, where

$$\Sigma = \left\{ \mathscr{E} \in \operatorname{Ob} \mathscr{P}_F(\mathscr{C}, J)_Y \,\middle|\, \mathscr{E} \supset \coprod_{U \in \operatorname{Ob} \mathscr{C}} (F_f)_U(\mathscr{D} \cap F_X(U)) \right\}.$$

Remark 5.4 ([11] Remark 2.4)

For $U \in \mathrm{Ob}\mathscr{C}$, the subset $\mathcal{S}(U)$ of $F_X(U)$ defined in (4.9) is the set of elements x of $F_X(U)$ which satisfy the following condition (∗) if $f: X \to Y$ is surjective.

(∗) There exists $R \in J(U)$ such that, for each $h \in R$, there exists $y \in \mathscr{D} \cap F_X(\mathrm{dom}(h))$ which satisfies $F_Y(h)(x) = (F_f)_{\mathrm{dom}(h)}(y)$.

If we put $\mathscr{G}(\mathcal{S}) = \coprod_{U \in \mathrm{Ob}\mathscr{C}} \mathcal{S}(U)$, we have $\mathscr{D}_f = \mathscr{G}(\mathcal{S})$.

Proposition 5.5 ([11] Proposition 2.5)
$\Gamma : \mathscr{P}_F(\mathscr{C}, J) \to \mathscr{S}et$ is a bifibered category.

For a map $f : X \to Y$, define a functor $f_* : \mathscr{P}_F(\mathscr{C}, J)_X \to \mathscr{P}_F(\mathscr{C}, J)_Y$ as follows. For $(X, \mathscr{D}) \in \mathrm{Ob}\mathscr{P}_F(\mathscr{C}, J)_X$, we put $f_*(X, \mathscr{D}) = (Y, \mathscr{D}_f)$.
If $(X, \mathscr{D}), (X, \mathscr{D}') \in \mathrm{Ob}\mathscr{P}_F(\mathscr{C}, J)_X$ satisfies $\mathscr{D} \subset \mathscr{D}'$, then $\mathscr{D}_f \subset \mathscr{D}'_f$ holds. Hence, for a morphism $id_X : (X, \mathscr{D}) \to (X, \mathscr{D}')$ in $\mathscr{P}_F(\mathscr{C}, J)_X$, we put $f_*(id_X : (X, \mathscr{D}) \to (X, \mathscr{D}')) = (id_Y : (Y, \mathscr{D}_f) \to (Y, \mathscr{D}'_f))$.
It can be verified that $\mathscr{P}_F(\mathscr{C}, J)_Y(f_*(X, \mathscr{D}), (Y, \mathscr{E}))$ is not empty if and only if $\mathscr{P}_F(\mathscr{C}, J)_Y((X, \mathscr{D}), f^*(Y, \mathscr{E}))$ is not empty.
This shows that f_* is a left adjoint of f^*.

Proposition 5.6 ([11] Proposition 2.6)
Let $p: \mathscr{F} \to \mathscr{C}$ be a prefibered category. If $\mathscr{F}_X$ has an initial object for any object X of $\mathscr{C}$, then p has a left adjoint.

Corollary 5.7 ([11] Corollary 2.7)
Let $p: \mathscr{F} \to \mathscr{C}$ be a bifibered category. If $\mathscr{F}_X$ has a terminal object for any object X of $\mathscr{C}$, then p has a right adjoint.

Corollary 5.8 ([11] Corollary 2.9)
$\Gamma: \mathscr{P}_F(\mathscr{C}, J) \to \mathscr{S}et$ has left and right adjoints.

Let $\{(X_i, \mathscr{D}_i)\}_{i\in I}$ be a family of objects of $\mathscr{P}_F(\mathscr{C}, J)$.
We denote by $\mathrm{pr}_i : \prod_{j\in I} X_j \to X_i$ the projection to the i-th component and $\iota_i : X_i \to \coprod_{j\in I} X_j$ the inclusion to the i-th summand.
Put $\mathscr{D}^I = \bigcap_{j\in I} \mathscr{D}_i^{\mathrm{pr}_i}$. Then, $\mathscr{D}^I$ is the finest the-ology such that $\mathrm{pr}_i : \left(\prod_{j\in I} X_j, \mathscr{D}^I\right) \to (X_i, \mathscr{D}_i)$ is a morphism in $\mathscr{P}_F(\mathscr{C}, J)$ for any $i \in I$.
Let $\mathscr{D}_I$ be the coarsest the-ology on $\coprod_{j\in I} X_j$ such that $\iota_i : (X_i, \mathscr{D}_i) \to \left(\coprod_{j\in I} X_j, \mathscr{D}_I\right)$ is a morphism in $\mathscr{P}_F(\mathscr{C}, J)$ for any $i \in I$.
If we put $\mathscr{S}_I = \left\{ \mathscr{E} \in \mathrm{Ob}\, \mathscr{P}_F(\mathscr{C}, J)_{\coprod_{j\in I} X_j} \,\middle|\, \mathscr{E} \supset \bigcup_{j\in I} (\mathscr{D}_j)_{\iota_j} \right\}$, then
$\mathscr{D}_I = \bigcap_{\mathscr{E}\in\mathscr{S}_I} \mathscr{E}$.

Proposition 5.9 ([11] Proposition 2.11)

(1) $\left(\left(\prod_{j\in I} X_j, \mathscr{D}^I\right) \xrightarrow{\mathrm{pr}_i} (X_i, \mathscr{D}_i)\right)_{i\in I}$ is a product of $\{(X_i, \mathscr{D}_i)\}_{i\in I}$.

(2) $\left((X_i, \mathscr{D}_i) \xrightarrow{\iota_i} \left(\coprod_{j\in I} X_j, \mathscr{D}_I\right)\right)_{i\in I}$ is a coproduct of $\{(X_i, \mathscr{D}_i)\}_{i\in I}$.

Proposition 5.10 ([11] Proposition 2.14)

Let $f, g : (X, \mathscr{D}) \to (Y, \mathscr{E})$ be morphisms in $\mathscr{P}_F(\mathscr{C}, J)$. Then, equalizers and coequalizers of f and g exist.

In fact, if $Z \xrightarrow{i} X$ is an equalizer of f and g in the category of sets, then $(Z, \mathscr{D}^i) \xrightarrow{i} (X, \mathscr{D})$ is an equalizer of f and g in $\mathscr{P}_F(\mathscr{C}, J)$.
If $Y \xrightarrow{q} W$ is a coequalizer of f and g in the category of sets, then $(Y, \mathscr{E}) \xrightarrow{q} (W, \mathscr{E}_q)$ is a coequalizer of f and g in $\mathscr{P}_F(\mathscr{C}, J)$.

Since $\mathscr{P}_F(\mathscr{C}, J)$ has finite limits by (5.9) and (5.10), we can consider the fibered category $\wp : \mathscr{P}_F(\mathscr{C}, J)^{(2)} \to \mathscr{P}_F(\mathscr{C}, J)$ of morphisms in $\mathscr{P}_F(\mathscr{C}, J)$ by (1.12).

It follows from (1.13) that the inverse image functors of this fibered category have left adjoints.

The inverse image functors also have right adjoints, namely we can show the following fact.

Proposition 5.11 ([11] Proposition 3.18)
For a morphism $\varphi : (X, \mathscr{D}) \to (Y, \mathscr{F})$ in $\mathscr{P}_F(\mathscr{C}, J)$, the inverse functor $\varphi^* : \mathscr{P}_F(\mathscr{C}, J)^{(2)}_{(Y,\mathscr{D})} \to \mathscr{P}_F(\mathscr{C}, J)^{(2)}_{(X,\mathscr{E})}$ has a right adjoint $\varphi_! : \mathscr{P}_F(\mathscr{C}, J)^{(2)}_{(X,\mathscr{D})} \to \mathscr{P}_F(\mathscr{C}, J)^{(2)}_{(Y,\mathscr{E})}$.

In fact, $\varphi_! : \mathscr{P}_F(\mathscr{C}, J)^{(2)}_{(X,\mathscr{D})} \to \mathscr{P}_F(\mathscr{C}, J)^{(2)}_{(Y,\mathscr{E})}$ is constructed as follows.

For $y \in Y$, we denote by $\iota_y : \varphi^{-1}(y) \to X$ the inclusion map and consider a the-ology $\mathscr{D}^{\iota_y}$ on $\varphi^{-1}(y)$.

For an object $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (X, \mathscr{D}))$ of $\mathscr{P}_F(\mathscr{C}, J)^{(2)}$, we define a subset $E(\varphi; y)$ of $\mathscr{P}_F(\mathscr{C}, J)((\varphi^{-1}(y), \mathscr{D}^{\iota_y}), (E, \mathscr{E}))$ by

$$E(\varphi; y) = \{\alpha \in \mathscr{P}_F(\mathscr{C}, J)((\varphi^{-1}(y), \mathscr{D}^{\iota_y}), (E, \mathscr{E})) \mid \pi\alpha = \iota_y\}$$

if $\varphi^{-1}(y) \neq \emptyset$ and $E(\varphi; y) = \emptyset$ if $\varphi^{-1}(y) = \emptyset$.

Put $E(\varphi) = \coprod_{y \in Y} E(\varphi; y)$ and define map $\varphi_{!\boldsymbol{E}} : E(\varphi) \to Y$ by $\varphi_{!\boldsymbol{E}}(\alpha) = y$ if $\alpha \in E(\varphi; y)$.

We consider the following cartesian square (*) in $\mathscr{S}et$.

$$
(*) \qquad \begin{array}{ccc}
E(\varphi)\times_Y X & \xrightarrow{\tilde{\varphi}_E} & E(\varphi) \\
\downarrow{\scriptstyle \widetilde{\varphi_{!E}}} & & \downarrow{\scriptstyle \varphi_{!E}} \\
X & \xrightarrow{\varphi} & Y
\end{array}
$$

Define a map $\varepsilon_E^{\varphi} : E(\varphi)\times_Y X \to E$ by $\varepsilon_E^{\varphi}(\alpha, x) = \alpha(x)$ if $\alpha \in E(\varphi; y)$ and $x \in \varphi^{-1}(y)$ for $y \in Y$.

Let $\Sigma_{E,\varphi}$ the set of all the-ologies $\mathscr{L}$ on $E(\varphi)$ such that $\mathscr{L} \subset \mathscr{F}^{\varphi_{!E}}$ and $\mathscr{D}^{\widetilde{\varphi_{!E}}} \cap \mathscr{L}^{\tilde{\varphi}_E} \subset \mathscr{E}^{\varepsilon_E^{\varphi}}$ hold.

Note that $\mathscr{L} \in \Sigma_{E,\varphi}$ if and only if $\varphi_{!E} : (E(\varphi), \mathscr{L}) \to (Y, \mathscr{F})$ and $\varepsilon_E^{\varphi} : (E(\varphi)\times_Y X, \mathscr{D}^{\widetilde{\varphi_{!E}}} \cap \mathscr{L}^{\tilde{\varphi}_E}) \to (E, \mathscr{E})$ are morphisms in $\mathscr{P}_F(\mathscr{C}, J)$.

For $U \in \mathrm{Ob}\,\mathscr{C}$, we consider the following condition (LE) on an element γ of $F_{E(\varphi)}(U)$.

(LE) If $V, W \in \mathrm{Ob}\,\mathscr{C}$, $f \in \mathscr{C}(W, U)$, $g \in \mathscr{C}(W, V)$ and $\psi \in \mathscr{D} \cap F_X(V)$ satisfy $\varphi\psi F(g) = \varphi_{!E}\gamma F(f)$, a composition

$$F(W) \xrightarrow{(\gamma F(f), \psi F(g))} E(\varphi) \times_Y X \xrightarrow{\varepsilon_E^\varphi} E$$

belongs to $\mathscr{E} \cap F_E(W)$ and a composition $F(U) \xrightarrow{\gamma} E(\varphi) \xrightarrow{\varphi_{!E}} Y$ belongs to $\mathscr{F} \cap F_Y(U)$.

Define a set $\mathscr{D}_{E,\varphi}$ of F-parametrizations of a set $E(\varphi)$ so that $\mathscr{D}_{E,\varphi} \cap F_{E(\varphi)}(U)$ is a subset of $F_{E(\varphi)}(U)$ consisting of elements which satisfy the above condition (LE) for any $U \in \mathrm{Ob}\,\mathscr{C}$.

Proposition 5.12 ([11] Proposition 3.11)
$\mathscr{D}_{\boldsymbol{E},\varphi}$ is maximum element of $\Sigma_{\boldsymbol{E},\varphi}$.

Let $\boldsymbol{E}=((E,\mathscr{E})\xrightarrow{\pi}(X,\mathscr{D}))$, $\boldsymbol{G}=((G,\mathscr{G})\xrightarrow{\rho}(X,\mathscr{D}))$ be objects of $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})}$ and $\varphi:(X,\mathscr{D})\to(Y,\mathscr{F})$ a morphism in $\mathscr{P}_F(\mathscr{C},J)$.

Let $\boldsymbol{\xi}=\langle\xi,id_X\rangle:\boldsymbol{E}\to\boldsymbol{G}$ be a morphism in $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{D})}$.

If $\alpha\in E(\varphi;y)$ for $y\in Y$, we have $\rho\xi\alpha=\pi\alpha=\iota_y$, hence $\xi\alpha\in G(\varphi;y)$.

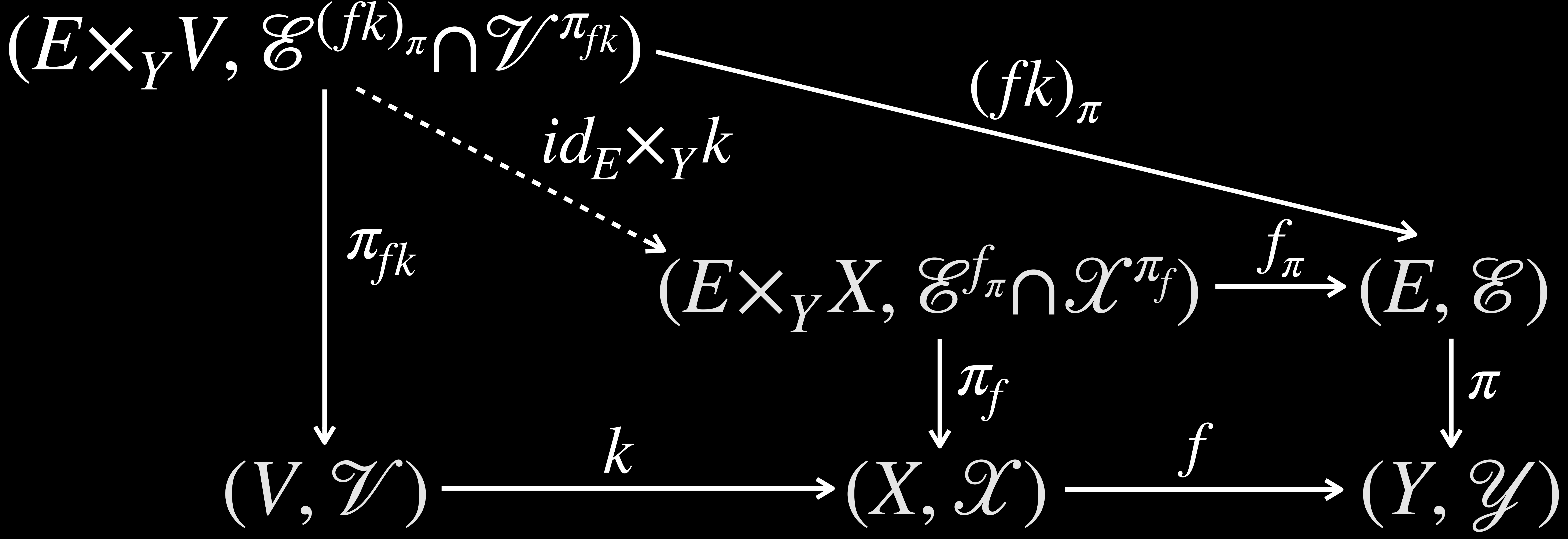
Thus we can define a map $\xi_\varphi:E(\varphi)\to G(\varphi)$ by $\varphi(\xi)(\alpha)=\xi\alpha$.

We define by $\varphi_!(\boldsymbol{E})=((E(\varphi),\mathscr{D}_{\boldsymbol{E},\varphi})\xrightarrow{\varphi_{!\boldsymbol{E}}}(Y,\mathscr{F}))$ and

$\varphi_!(\boldsymbol{\xi})=\langle\xi_\varphi,id_Y\rangle:\varphi(\boldsymbol{E})\to\varphi(\boldsymbol{G})$.

§6. Representations of groupoids in the category of plots

Let $f:(X,\mathscr{X})\to(Y,\mathscr{Y})$, $g:(X,\mathscr{X})\to(Z,\mathscr{Z})$ and $k:(V,\mathscr{V})\to(X,\mathscr{X})$ be morphisms in $\mathscr{P}_F(\mathscr{C},J)$ and $\boldsymbol{E}=((E,\mathscr{E})\xrightarrow{\pi}(Y,\mathscr{Y}))$ an object of $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(Y,\mathscr{Y})}$. We consider the following commutative diagram in $\mathscr{P}_F(\mathscr{C},J)$ whose outer trapezoid and lower rectangle are cartesian.



There exists unique morphism

$$id_E \times_Y k : (E \times_Y V, \mathscr{E}^{(fk)_\pi} \cap \mathscr{V}^{\pi_{fk}}) \to (E \times_Y X, \mathscr{E}^{f_\pi} \cap \mathscr{X}^{\pi_f})$$

that makes the above diagram commute. Since objects $(gk)_*(fk)^*(\boldsymbol{E})$ and $g_* f^*(\boldsymbol{E})$ of $\mathscr{P}_F(\mathscr{C}, J)^{(2)}_{(\mathrm{Z}, \mathscr{Z})}$ are given by

$$(gk)_*(fk)^*(\boldsymbol{E}) = ((E \times_Y V, \mathscr{E}^{(fk)_\pi} \cap \mathscr{V}^{\pi_{fk}}) \xrightarrow{gk\pi_{fk}} (\mathrm{Z}, \mathscr{Z}))$$

$$g_* f^*(\boldsymbol{E}) = ((E \times_Y X, \mathscr{E}^{f_\pi} \cap \mathscr{X}^{\pi_f}) \xrightarrow{g\pi_f} (\mathrm{Z}, \mathscr{Z})),$$

we define a morphism $\boldsymbol{E}_k : (gk)_*(fk)^*(\boldsymbol{E}) \to g_* f^*(\boldsymbol{E})$ in $\mathscr{P}_F(\mathscr{C}, J)^{(2)}_{(\mathrm{Z}, \mathscr{Z})}$ by $\boldsymbol{E}_k = \langle id_E \times_Y k, id_Z \rangle$.

Let $f:(X,\mathscr{X})\to(Y,\mathscr{Y})$ be a morphism in $\mathscr{P}_F(\mathscr{C},J)$.
We denote by $\eta^f : id_{\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(X,\mathscr{X})}} \to f^*f_!$ and $\varepsilon^f : f_!f^* \to id_{\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(Y,\mathscr{Y})}}$ the unit and the counit of the adjunction $f^* \dashv f_!$, respectively.
Let $f:(X,\mathscr{X})\to(Y,\mathscr{Y})$, $g:(X,\mathscr{X})\to(Z,\mathscr{Z})$ and $k:(V,\mathscr{V})\to(X,\mathscr{X})$ be morphisms in $\mathscr{P}_F(\mathscr{C},J)$ and $\boldsymbol{E}$ an object of $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(Z,\mathscr{Z})}$.
Define a morphism $\boldsymbol{E}^k : f_!g^*(\boldsymbol{E}) \to (fk)_!(gk)^*(\boldsymbol{E})^*$ in $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(Y,\mathscr{Y})}$ to be the following composition.

$$f_!g^*(\boldsymbol{E}) \xrightarrow{\eta^{fk}_{f_!g^*(\boldsymbol{E})}} (fk)_!(fk)^*(f_!g^*(\boldsymbol{E})) \xrightarrow{(fk)_!(\boldsymbol{c}_{f,k}(f_!g^*(\boldsymbol{E})))^{-1}} (fk)_!k^*f^*f_!g^*(\boldsymbol{E})$$

$$\xrightarrow{(fk)_!k^*\left(\varepsilon^f_{g^*(\boldsymbol{E})}\right)} (fk)_!k^*g^*(\boldsymbol{E}) \xrightarrow{(fk)_!(\boldsymbol{c}_{g,k}(\boldsymbol{E}))} (fk)_!(gk)^*(\boldsymbol{E})$$

Let $f\colon(X,\mathscr{X})\to(Y,\mathscr{Y})$, $g\colon(X,\mathscr{X})\to(Z,\mathscr{Z})$, $h\colon(V,\mathscr{V})\to(Z,\mathscr{Z})$ and $i\colon(V,\mathscr{V})\to(W,\mathscr{W})$ be morphisms in $\mathscr{P}_F(\mathscr{C},J)$.

We consider the following cartesian square in $\mathscr{P}_F(\mathscr{C},J)$.

$$\begin{array}{ccc}(X\times_Z V,\mathscr{X}^{h_g}\cap\mathscr{V}^{g_h}) & \xrightarrow{g_h} & (V,\mathscr{V}) \\ \downarrow h_g & & \downarrow h \\ (X,\mathscr{X}) & \xrightarrow{g} & (Z,\mathscr{Z})\end{array}$$

For an object $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (Y, \mathscr{Y}))$ of $\mathscr{P}_F(\mathscr{C}, J)^{(2)}_{(Y,\mathscr{Y})}$, we consider the following commutative diagrams in $\mathscr{P}_F(\mathscr{C}, J)$ whose rectangles are all cartesian.

$$\begin{array}{ccccc}
(E\times_Y(X\times_Z V), \mathscr{E}^{(fh_g)_\pi}\cap(\mathscr{X}^{h_g}\cap\mathscr{V}^{g_h})^{\pi_{fh_g}}) & & \xrightarrow{(fh_g)_\pi} & & (E, \mathscr{E}) \\
\downarrow{\scriptstyle \pi_{fh_g}} & & & & \downarrow{\scriptstyle \pi} \\
(X\times_Z V, \mathscr{X}^{h_g}\cap\mathscr{V}^{g_h}) & \xrightarrow{h_g} & (X, \mathscr{X}) & \xrightarrow{f} & (Y, \mathscr{Y})
\end{array}$$

$$\begin{array}{ccccc}
((E\times_Y X)\times_Z V, (\mathscr{E}^{f_\pi}\cap\mathscr{X}^{\pi_f})^{h_{g\pi_f}}\cap\mathscr{V}^{(g\pi_f)_h}) & \xrightarrow{h_{g\pi_f}} & (E\times_Y X, \mathscr{E}^{f_\pi}\cap\mathscr{X}^{\pi_f}) & \xrightarrow{f_\pi} & (E, \mathscr{E}) \\
\Big\downarrow{\scriptstyle (g\pi_f)_h} & & \downarrow{\scriptstyle \pi_f} & & \downarrow{\scriptstyle \pi} \\
 & & (X, \mathscr{X}) & \xrightarrow{f} & (Y, \mathscr{Y}) \\
 & & \downarrow{\scriptstyle g} & & \\
(V, \mathscr{V}) & \xrightarrow{h} & (Z, \mathscr{Z}) & &
\end{array}$$

Thus we have the following equalities.

$$(ig_h)_*(fh_g)^*(\boldsymbol{E}) = ((E\times_Y(X\times_Z V), \mathscr{E}^{(fh_g)_\pi}\cap(\mathscr{X}^{h_g}\cap\mathscr{V}^{g_h})^{\pi_{fh_g}}) \xrightarrow{ig_h\pi_{fh_g}} (W,\mathscr{W}))$$

$$i_*h^*g_*f^*(\boldsymbol{E}) = (((E\times_Y X)\times_Z V, (\mathscr{E}^{f_\pi}\cap\mathscr{X}^{\pi_f})^{h_{g\pi_f}}\cap\mathscr{V}^{(g\pi_f)_h}) \xrightarrow{i(g\pi_f)_h} (W,\mathscr{W}))$$

We define a morphism $\theta_{f,g,h,i}(\boldsymbol{E}) : (ig_h)_*(fh_g)^*(\boldsymbol{E}) \to i_*h^*g_*f^*(\boldsymbol{E})$ in $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(W,\mathscr{W})}$ by $\theta_{f,g,h,i}(\boldsymbol{E}) = \langle(id_E\times_Y h_g, g_h\pi_{fh_g}), id_W\rangle$.

Proposition 6.1

$\theta_{f,g,h,i}(\boldsymbol{E}) : (ig_h)_*(fh_g)^*(\boldsymbol{E}) \to i_*h^*g_*f^*(\boldsymbol{E})$ is an isomorphism which is natural in $\boldsymbol{E}$.

In fact, $(id_E\times_Y h_g, g_h\pi_{fh_g}) : E\times_Y(X\times_Z V) \to (E\times_Y X)\times_Z V$ maps $(u,(x,v)) \in E\times_Y(X\times_Z V)$ to $((u,x),v) \in (E\times_Y X)\times_Z V$.

Let $f:(X,\mathscr{X})\to(Y,\mathscr{Y})$, $g:(X,\mathscr{X})\to(Z,\mathscr{Z})$, $h:(V,\mathscr{V})\to(Z,\mathscr{Z})$ and $i:(V,\mathscr{V})\to(W,\mathscr{W})$ be morphisms in $\mathscr{P}_F(\mathscr{C},J)$.

We consider the following cartesian square in $\mathscr{P}_F(\mathscr{C},J)$.

$$\begin{array}{ccc}
(X\times_Z V, \mathscr{X}^{h_g}\cap\mathscr{V}^{g_h}) & \xrightarrow{\quad g_h \quad} & (V,\mathscr{V}) \\
\downarrow{\scriptstyle h_g} & & \downarrow{\scriptstyle h} \\
(X,\mathscr{X}) & \xrightarrow{\quad g \quad} & (Z,\mathscr{Z})
\end{array}$$

For an object $\boldsymbol{E}$ of $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(W,\mathscr{W})}$, we define a morphism

$$\theta^{f,g,h,i}(\boldsymbol{E}): f_! g^* h_! i^*(\boldsymbol{E}) \to (fh_g)_!(ig_h)^*(\boldsymbol{E})$$

in $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(Y,\mathscr{Y})}$ to be the adjoint of the following composition with respect to the adjunction $(fh_g)^* \dashv (fh_g)_!$.

$$(fh_g)^*(f_! g^* h_! i^*(\boldsymbol{E})) \xrightarrow{\boldsymbol{c}_{f,h_g}(f_! g^* h_! i^*(\boldsymbol{E}))^{-1}} h_g^* f^* f_! g^* h_! i^*(\boldsymbol{E})$$

$$\xrightarrow{h_g^*\left(\varepsilon^f_{g^* h_! i^*(\boldsymbol{E})}\right)} h_g^* g^* h_! i^*(\boldsymbol{E}) \xrightarrow{\boldsymbol{c}_{g,h_g}(h_! i^*(\boldsymbol{E}))} (gh_g)^* h_! i^*(\boldsymbol{E}) = (hg_h)^* h_! i^*(\boldsymbol{E})$$

$$\xrightarrow{\boldsymbol{c}_{h,g_h}(h_! i^*(\boldsymbol{E}))^{-1}} g_h^* h^* h_! i^*(\boldsymbol{E}) \xrightarrow{g_h^*\left(\varepsilon^h_{i^*(\boldsymbol{E})}\right)} g_h^* i^*(\boldsymbol{E}) \xrightarrow{\boldsymbol{c}_{i,g_h}(\boldsymbol{E})} (ig_h)^*(\boldsymbol{E})$$

Proposition 6.2 ([10] Remark 1.5.5)

$\theta^{f,g,h,i}(\boldsymbol{E}): f_! g^* h_! i^*(\boldsymbol{E}) \to (fh_g)_!(ig_h)^*(\boldsymbol{E})$ is an isomorphism which is natural in $\boldsymbol{E}$.

Let $\boldsymbol{G} = ((G_0, \mathscr{G}_0), (G_1, \mathscr{G}_1); \sigma, \tau, \varepsilon, \mu, \iota)$ be a groupoid in $\mathscr{P}_F(\mathscr{C}, J)$, $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (G_0, \mathscr{G}_0))$ an object of $\mathscr{P}_F(\mathscr{C}, J)^{(2)}_{(G_0, \mathscr{G}_0)}$ and $\boldsymbol{\xi} : \sigma^*(\boldsymbol{E}) \to \tau^*(\boldsymbol{E})$ a morphism in $\mathscr{P}_F(\mathscr{C}, J)^{(2)}_{(G_1, \mathscr{G}_1)}$.

We denote by $\hat{\boldsymbol{\xi}} : \tau_* \sigma^*(\boldsymbol{E}) \to \boldsymbol{E}$ and $\check{\boldsymbol{\xi}} : \boldsymbol{E} \to \sigma_! \tau^*(\boldsymbol{E})$ the adjoint of $\boldsymbol{\xi}$ with respect to the adjunctions $\tau_* \dashv \tau^*$ and $\sigma^* \dashv \sigma_!$, respectively.

Proposition 6.3 ([10] Proposition 3.3.2)

ξ makes the upper diagram of (2.5) commute if and only if $\hat{\xi}$ makes the following diagram commute.

$$\begin{array}{ccccc} (\tau \mathrm{pr}_2)_*(\sigma \mathrm{pr}_1)^*(E) & \xrightarrow{\theta_{\sigma,\tau,\sigma,\tau}(E)} & \tau_*\sigma^*\tau_*\sigma^*(E) & \xrightarrow{\tau_*\sigma^*(\hat{\xi})} & \tau_*\sigma^*(E) \\ \| & & & & \downarrow \hat{\xi} \\ (\tau\mu)_*(\sigma\mu)^*(E) & \xrightarrow{E_\mu} & \tau_*\sigma^*(E) & \xrightarrow{\hat{\xi}} & E \end{array}$$

ξ makes the lower diagram of (2.5) commute if and only if a composition $E = (\tau\varepsilon)_*(\sigma\varepsilon)^*(E) \xrightarrow{E_\varepsilon} \tau_*\sigma^*(E) \xrightarrow{\hat{\xi}} E$ coincides with the identity morphism of E.

Proposition 6.4 ([10] Proposition 3.4.2)

ξ makes the upper diagram of (2.5) commute if and only if $\check{\xi}$ makes the following diagram commute.

$$\begin{array}{ccccc} E & \xrightarrow{\check{\xi}} & \sigma_!\tau^*(E) & \xrightarrow{\sigma_!\tau^*(\check{\xi})} & \sigma_!\tau^*\sigma_!\tau^*(E) \\ \downarrow{\scriptstyle \check{\xi}} & & & & \downarrow{\scriptstyle \theta^{\sigma,\tau,\sigma,\tau}(E)} \\ \sigma_!\tau^*(E) & \xrightarrow{E^{\mu}} & (\sigma\mu)_!(\tau\mu)^*(E) & = & (\sigma\mathrm{pr}_1)_!(\tau\mathrm{pr}_2)^*(E) \end{array}$$

ξ makes the lower diagram of (2.5) commute if and only if a composition $E \xrightarrow{\check{\xi}} \sigma_!\tau^*(E) \xrightarrow{E^{\varepsilon}} (\sigma\varepsilon)_!(\tau\varepsilon)^*(E) = E$ coincides with the identity morphism of E.

The next result follows from the naturality of the adjointness.

Proposition 6.5

Let (E,ξ) and (F,ζ) be representations of G.

If a morphism $\varphi : E \to F$ in $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(G_0,\mathscr{G}_0)}$ makes one of the following diagrams commute, the other two diagrams also commute.

$$
\begin{array}{ccc}
\sigma^*(E) & \xrightarrow{\quad \xi \quad} & \tau^*(E) \\
\big\downarrow{\scriptstyle \sigma^*(\varphi)} & & \big\downarrow{\scriptstyle \tau^*(\varphi)} \\
\sigma^*(F) & \xrightarrow{\quad \zeta \quad} & \tau^*(F)
\end{array}
\qquad
\begin{array}{ccc}
\tau_*\sigma^*(E) & \xrightarrow{\quad \hat{\xi} \quad} & E \\
\big\downarrow{\scriptstyle \tau_*\sigma^*(\varphi)} & & \big\downarrow{\scriptstyle \varphi} \\
\tau_*\sigma^*(F) & \xrightarrow{\quad \hat{\zeta} \quad} & F
\end{array}
\qquad
\begin{array}{ccc}
E & \xrightarrow{\quad \check{\xi} \quad} & \sigma_!\tau^*(E) \\
\big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle \sigma_!\tau^*(\varphi)} \\
F & \xrightarrow{\quad \check{\zeta} \quad} & \sigma_!\tau^*(F)
\end{array}
$$

Consider the following diagrams whose rectangles are all cartesian.

$$
\begin{array}{ccc}
E \times^{\sigma\mu}_{G_0}(G_1 \times_{G_0} G_1) & \xrightarrow{(\sigma\mu)_\pi} & E \\
\downarrow{\scriptstyle \pi_{\sigma\mu} = \pi_{\sigma \mathrm{pr}_1}} & & \downarrow{\scriptstyle \pi} \\
G_1 \times_{G_0} G_1 & \xrightarrow{\sigma\mu = \sigma \mathrm{pr}_1} & G_0
\end{array}
\qquad
\begin{array}{ccc}
E \times^{\tau}_{G_0} G_1 & \xrightarrow{\tau_\pi} & E \\
\downarrow{\scriptstyle \pi_\tau} & & \downarrow{\scriptstyle \pi} \\
G_1 & \xrightarrow{\tau} & G_0
\end{array}
$$

$$
\begin{array}{ccccc}
(E \times^{\sigma}_{G_0} G_1) \times^{\sigma}_{G_0} G_1 & \xrightarrow{\sigma_{\tau\pi_\sigma}} & E \times^{\sigma}_{G_0} G_1 & \xrightarrow{\sigma_\pi} & E \\
 & & \downarrow{\scriptstyle \pi_\sigma} & & \downarrow{\scriptstyle \pi} \\
\downarrow{\scriptstyle (\tau\pi_\sigma)_\sigma} & & G_1 & \xrightarrow{\sigma} & G_0 \\
 & & \downarrow{\scriptstyle \tau} & & \\
G_1 & \xrightarrow{\sigma} & G_0 & &
\end{array}
$$

Then, we have the following equalities.

$$\tau_*\sigma^*(\boldsymbol{E}) = ((E \times^{\sigma}_{G_0} G_1, \mathscr{E}^{\sigma_\pi} \cap \mathscr{G}_1^{\pi_\sigma}) \xrightarrow{\tau\pi_\sigma} (G_0, \mathscr{G}_0))$$

$$(\tau\mu)_*(\sigma\mu)^*(\boldsymbol{E}) = ((\tau \mathrm{pr}_2)_*(\sigma \mathrm{pr}_1)^*(\boldsymbol{E})$$

$$= ((E \times^{\sigma\mu}_{G_0} (G_1 \times_{G_0} G_1), \mathscr{E}^{(\sigma\mu)_\pi} \cap (\mathscr{G}_1^{\mathrm{pr}_1} \cap \mathscr{G}_1^{\mathrm{pr}_2})^{\pi_{\sigma\mu}}) \xrightarrow{\tau\mu\pi_{\sigma\mu}} (G_0, \mathscr{G}_0))$$

$$\tau_*\sigma^*\tau_*\sigma^*(\boldsymbol{E}) = ((E \times^{\sigma}_{G_0} G_1) \times^{\sigma}_{G_0} G_1, (\mathscr{E}^{\sigma_\pi} \cap \mathscr{G}_1^{\pi_\sigma})^{\sigma_{\tau\pi_\sigma}} \cap \mathscr{G}_1^{(\tau\pi_\sigma)_\sigma}) \xrightarrow{\tau(\tau\pi_\sigma)_\sigma} (G_0, \mathscr{G}_0))$$

If we put $\boldsymbol{\xi} = \langle \xi, id_{G_1} \rangle$ and $\hat{\boldsymbol{\xi}} = \langle \hat{\xi}, id_{G_0} \rangle$ for morphisms $\xi : (E \times^{\sigma}_{G_0} G_1, \mathscr{E}^{\sigma_\pi} \cap \mathscr{G}_1^{\pi_\sigma}) \to (E \times^{\tau}_{G_0} G_1, \mathscr{E}^{\tau_\pi} \cap \mathscr{G}_1^{\pi_\tau})$ and $\hat{\xi} : (E \times^{\sigma}_{G_0} G_1, \mathscr{E}^{\sigma_\pi} \cap \mathscr{G}_1^{\pi_\sigma}) \to (E, \mathscr{E})$ in $\mathscr{P}_F(\mathscr{C}, J)$, then $\hat{\xi}$ is a composition $E \times^{\sigma}_{G_0} G_1 \xrightarrow{\xi} E \times^{\tau}_{G_0} G_1 \xrightarrow{\tau_\pi} E$.

Remark 6.6 ([11] Remark 8.14)

The diagram of (6.3) is commutative if and only if the following diagram is commutative.

$$
\begin{array}{ccccc}
E\times^{\sigma \mathrm{pr}_1}_{G_0}(G_1\times_{G_0}G_1) & \xrightarrow{(id_E\times_Y \mathrm{pr}_1,\, \mathrm{pr}_2\pi_{\sigma\mathrm{pr}_1})} & (E\times^{\sigma}_{G_0}G_1)\times^{\sigma}_{G_0}G_1 & \xrightarrow{\hat{\xi}\times_{G_0} id_{G_1}} & E\times^{\sigma}_{G_0}G_1 \\
\| & & & & \downarrow{\scriptstyle \hat{\xi}} \\
E\times^{\sigma\mu}_{G_0}(G_1\times_{G_0}G_1) & \xrightarrow{id_E\times_{G_0}\mu} & E\times^{\sigma}_{G_0}G_1 & \xrightarrow{\hat{\xi}} & E
\end{array}
$$

A composition $\boldsymbol{E}=(\tau\varepsilon)_*(\sigma\varepsilon)^*(\boldsymbol{E}) \xrightarrow{\boldsymbol{E}_\varepsilon} \tau_*\sigma^*(\boldsymbol{E}) \xrightarrow{\hat{\boldsymbol{\xi}}} \boldsymbol{E}$ coincides with the identity morphism of $\boldsymbol{E}$ if and only if a composition

$$E \xrightarrow{(id_E,\varepsilon\pi)} E\times^{\sigma}_{G_0}G_1 \xrightarrow{\hat{\xi}} E$$

coincides with the identity morphism of E.

Hence if a morphism $\hat{\boldsymbol{\xi}}: \tau_* \sigma^*(\boldsymbol{E}) \to \boldsymbol{E}$ in $\mathscr{P}_F(\mathscr{C}, J)^{(2)}_{(G_0, \mathscr{G}_0)}$ satisfies both conditions of (6.3), we may call a pair $(\boldsymbol{E}, \hat{\boldsymbol{\xi}}: \tau_* \sigma^*(\boldsymbol{E}) \to \boldsymbol{E})$ a representation of $\boldsymbol{G}$ on $\boldsymbol{E}$.

Example 6.7 ([11] Example 8.16)
For an epimorphism $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (G_0, \mathscr{G}_0))$ in $\mathscr{P}_F(\mathscr{C}, J)^{(2)}$, consider the groupoid $\boldsymbol{G}(\boldsymbol{E}) = ((G_0, \mathscr{G}_0), (G_1(\boldsymbol{E}), \mathscr{G}_{\boldsymbol{E}}); \sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}}, \varepsilon_{\boldsymbol{E}}, \mu_{\boldsymbol{E}}, \iota_{\boldsymbol{E}})$ associated with $\boldsymbol{E}$. Let $\hat{\xi}_{\boldsymbol{E}}: E \times_{G_0} G_1(\boldsymbol{E}) \to E$ be the map defined by $\hat{\xi}_{\boldsymbol{E}}(e, g) = g(e)$ and define a morphism $\hat{\boldsymbol{\xi}}_{\boldsymbol{E}}: \tau_{\boldsymbol{E}*} \sigma_{\boldsymbol{E}}^*(\boldsymbol{E}) \to \boldsymbol{E}$ in $\mathscr{P}_F(\mathscr{C}, J)^{(2)}_{(G_0, \mathscr{G}_0)}$ by $\hat{\boldsymbol{\xi}}_{\boldsymbol{E}} = \langle \hat{\xi}_{\boldsymbol{E}}, id_{G_0} \rangle$.
It can be verified that $(\boldsymbol{E}, \hat{\boldsymbol{\xi}}_{\boldsymbol{E}})$ is a representation of $\boldsymbol{G}(\boldsymbol{E})$ on $\boldsymbol{E}$.
We call $(\boldsymbol{E}, \hat{\boldsymbol{\xi}}_{\boldsymbol{E}})$ the canonical representation on $\boldsymbol{E}$.

Proposition 6.8 ([11] Proposition 8.17)

Let $\boldsymbol{E}=((E,\mathscr{E})\xrightarrow{\pi}(G_0,\mathscr{G}_0))$ be an object of $\mathscr{P}_F(\mathscr{C},J)^{(2)}_{(G_0,\mathscr{G}_0)}$ such that π is surjective and $(\boldsymbol{E},\hat{\boldsymbol{\xi}}:\tau_*\sigma^*(\boldsymbol{E})\to\boldsymbol{E})$ a representation of $\boldsymbol{G}=((G_0,\mathscr{G}_0),(G_1,\mathscr{G}_1);\sigma,\tau,\varepsilon,\mu,\iota)$ on $\boldsymbol{E}$.

There exists a morphism $f=(f_0,f_1):\boldsymbol{G}\to\boldsymbol{G}(\boldsymbol{E})$ of groupoids in $\mathscr{P}_F(\mathscr{C},J)$ such that $f_0=id_{G_0}$ and that $(\boldsymbol{E},\hat{\boldsymbol{\xi}})$ coincides with the restriction of the canonical representation $(\boldsymbol{E},\hat{\boldsymbol{\xi}}_{\boldsymbol{E}})$ along f.

Moreover, if $g=(id_{G_0},g_1):\boldsymbol{G}\to\boldsymbol{G}(\boldsymbol{E})$ is a morphism of groupoids in $\mathscr{P}_F(\mathscr{C},J)$ such that $(\boldsymbol{E},\hat{\boldsymbol{\xi}})$ coincides with the restriction of the canonical representation $(\boldsymbol{E},\hat{\boldsymbol{\xi}}_{\boldsymbol{E}})$ along g, then $g=f$ holds.

Theorem 6.9 ([10] Proposition 3.5.16, Proposition 3.6.15)
Let $f : \boldsymbol{H} \to \boldsymbol{G}$ be a morphism of groupoids in $\mathscr{P}_F(\mathscr{C}, J)$.
The restriction functor $f^\bullet : \mathrm{Rep}(\boldsymbol{G}) \to \mathrm{Rep}(\boldsymbol{H})$ along f has a left adjoint and a right adjoint.

Let $\boldsymbol{G} = (G_0, G_1; \sigma, \tau, \varepsilon, \mu)$, $\boldsymbol{H} = (H_0, H_1; \sigma', \tau', \varepsilon', \mu')$ be groupiods in $\mathscr{P}_F(\mathscr{C}, J)$ and $f : \boldsymbol{H} \to \boldsymbol{G}$ a homomorphism of groupoids.
A left adjoint $f_\bullet : \mathrm{Rep}(\boldsymbol{H}) \to \mathrm{Rep}(\boldsymbol{G})$ of the restiction functor $f^\bullet : \mathrm{Rep}(\boldsymbol{H}) \to \mathrm{Rep}(\boldsymbol{G})$ along f is consrtructed as follows.
For an object $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (H_0, \mathscr{H}_0))$ of $\mathscr{P}_F(\mathscr{C}, J)^{(2)}_{(H_0, \mathscr{H}_0)}$, we consider the following diagrams whose rectangles are cartesian.

$$
\begin{array}{ccccc}
E\times_{H_0}(H_1\times_{G_0}G_1) & \xrightarrow{id_E\times_{H_0}\tilde{\mathrm{p}}\mathrm{r}_1} & E\times^{\sigma'}_{G_0}H_1 & \xrightarrow{\sigma'_\pi} & E \\
\downarrow{\scriptstyle \pi_{\sigma'\tilde{\mathrm{p}}\mathrm{r}_1}} & & \downarrow{\scriptstyle \pi_{\sigma'}} & & \downarrow{\scriptstyle \pi} \\
H_1\times_{G_0}G_1 & \xrightarrow{\tilde{\mathrm{p}}\mathrm{r}_1} & H_1 & \xrightarrow{\sigma'} & H_0 \\
\downarrow{\scriptstyle \tilde{\mathrm{p}}\mathrm{r}_2} & & \downarrow{\scriptstyle f_0\tau'} & & \\
G_1 & \xrightarrow{\sigma} & G_0 & &
\end{array}
\qquad
\begin{array}{ccc}
E\times^{\sigma}_{G_0}G_1 & \xrightarrow{\sigma_{f_0\pi}} & E \\
\downarrow{\scriptstyle (f_0\pi)_\sigma} & & \downarrow{\scriptstyle f_0\pi} \\
G_1 & \xrightarrow{\sigma} & G_0
\end{array}
$$

$$
\begin{array}{ccc}
(E\times^{\sigma'}_{H_0}H_1)\times_{H_0}(H_0\times_{G_0}G_1) & \xrightarrow{(\sigma_{f_0})_{\tau'\pi_{\sigma'}}} & E\times^{\sigma'}_{G_0}H_1 \\
\downarrow{\scriptstyle \pi_{\sigma'}\times_{f_0}(f_0)_\sigma} & & \downarrow{\scriptstyle \pi_{\sigma'}} \\
H_1\times_{G_0}G_1 & \xrightarrow{\tilde{\mathrm{p}}\mathrm{r}_1} & H_1 \\
\downarrow{\scriptstyle \tau'\times_{G_0}id_{G_1}} & & \downarrow{\scriptstyle \tau'} \\
H_0\times_{G_0}G_1 & \xrightarrow{\sigma_{f_0}} & H_0
\end{array}
\qquad
\begin{array}{ccc}
E\times_{H_0}(H_0\times_{G_0}G_1) & \xrightarrow{(\sigma_{f_0})_\pi} & E \\
\downarrow{\scriptstyle \pi_{\sigma_{f_0}}} & & \downarrow{\scriptstyle \pi} \\
H_0\times_{G_0}G_1 & \xrightarrow{\sigma_{f_0}} & H_0 \\
\downarrow{\scriptstyle (f_0)_\sigma} & & \downarrow{\scriptstyle f_0} \\
G_1 & \xrightarrow{\sigma} & G_0
\end{array}
$$

Thus $(\tau(f_0)_\sigma(\tau' \times_{G_0} id_{G_1}))_*(\sigma'\tilde{\mathrm{pr}}_1)^*(\boldsymbol{E})$ is the following morphism

$$(E \times_{G_0} (H_1 \times_{G_0} G_1), \mathscr{E}^{\sigma'_\pi(id_E \times_{D_0} \tilde{\mathrm{pr}}_1)} \cap (\mathscr{H}_1^{\tilde{\mathrm{pr}}_1} \cap \mathscr{G}_1^{\tilde{\mathrm{pr}}_2})^{\pi_{\sigma'\tilde{\mathrm{pr}}_1}}) \xrightarrow{\tau(f_0)_\sigma(\tau' \times_{G_0} id_{G_1})\pi_{\sigma'\tilde{\mathrm{pr}}_1}} (G_0, \mathscr{G}_0)$$

in $\mathscr{P}_F(\mathscr{C}, J)$. We also have

$$\tau_*\sigma^*(f_0)_*(\boldsymbol{E}) = ((E \times_{G_0} G_1, \mathscr{E}^{\sigma_{f_0\pi}} \cap \mathscr{G}_1^{(\pi f_0)_\sigma}) \xrightarrow{\tau(f_0\pi)_\sigma} (G_0, \mathscr{G})).$$

Suppose that $(\boldsymbol{E}, \hat{\boldsymbol{\xi}} : \tau'_* \sigma'^*(\boldsymbol{E}) \to \boldsymbol{E})$ is a representation of $\boldsymbol{H}$.

We put $\hat{\boldsymbol{\xi}} = \langle \hat{\xi}, id_{H_0} \rangle$ and define morphisms

$$\boldsymbol{\alpha}, \boldsymbol{\beta} : (\tau(f_0)_\sigma(\tau' \times_{G_0} id_{G_1}))_* (\sigma' \tilde{\mathrm{pr}}_1)^*(\boldsymbol{E}) \to \tau_* \sigma^*(f_0)_*(\boldsymbol{E})$$

by $\boldsymbol{\alpha} = \langle (\hat{\xi}(id_E \times_{H_0} \tilde{\mathrm{pr}}_1), \tilde{\mathrm{pr}}_2 \pi_{\sigma' \tilde{\mathrm{pr}}_1}) : E \times_{G_0} (H_1 \times_{G_0} G_1) \to E \times_{G_0} G_1, id_{G_0} \rangle$

$\boldsymbol{\beta} = \langle id_E \times_{H_0} \mu(f_1 \times_{G_0} id_{G_1}) : E \times_{G_0} (H_1 \times_{G_0} G_1) \to E \times_{G_0} G_1, id_{G_0} \rangle$,

respectively.

Let $P^f_{(\boldsymbol{E}, \boldsymbol{\xi})} : \tau_* \sigma^*(f_0)_*(\boldsymbol{E}) \to (\boldsymbol{E}, \boldsymbol{\xi})_f$ be a coequalizer of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

We can give a structure $\hat{\boldsymbol{\xi}}_f : \tau_* \sigma^*((\boldsymbol{E}, \boldsymbol{\xi})_f) \to (\boldsymbol{E}, \boldsymbol{\xi})_f$ of a representation of $\boldsymbol{G}$ on $(\boldsymbol{E}, \boldsymbol{\xi})_f$ and $\boldsymbol{f}_\bullet(\boldsymbol{E}, \hat{\boldsymbol{\xi}})$ is defined to be $(\boldsymbol{E}, \boldsymbol{\xi})_f$.

On the other hand, a right adjoint $f_!:\mathrm{Rep}(\boldsymbol{H})\to\mathrm{Rep}(\boldsymbol{G})$ of the restriction functor is constructed as follows.

Consider the following diagrams whose rectangles are cartesian.

$$
\begin{array}{ccc}
G_1\times_{G_0}H_1 & \xrightarrow{\tilde{\mathrm{pr}}_2} & H_1 \\
\downarrow{\scriptstyle \tilde{\mathrm{pr}}_1} & & \downarrow{\scriptstyle f_0\sigma'} \\
G_1 & \xrightarrow{\tau} & G_0
\end{array}
\qquad
\begin{array}{ccc}
G_1\times_{G_0}H_1 & \xrightarrow{\tilde{\mathrm{pr}}_2} & H_1 \\
\downarrow{\scriptstyle id_{G_1}\times_{G_0}\sigma'} & & \downarrow{\scriptstyle f_0} \\
G_1\times_{G_0}H_0 & \xrightarrow{\tau_{f_0}} & H_0 \\
\downarrow{\scriptstyle (f_0)_\tau} & & \downarrow{\scriptstyle f_0} \\
G_1 & \xrightarrow{\tau} & G_0
\end{array}
$$

For a representation $(\boldsymbol{E}, \boldsymbol{\xi}: \sigma'^{*}(\boldsymbol{E}) \to \tau'^{*}(\boldsymbol{E}))$ of $\boldsymbol{H}$, we denote by $\check{\boldsymbol{\xi}}: \boldsymbol{E} \to \sigma'_{!}\tau'^{*}(\boldsymbol{E})$ the adjoint of $\boldsymbol{\xi}$ with respect to the adjunction $\sigma'^{*} \dashv \sigma'_{!}$, and by γ the following composition.

$$(\sigma(f_0)_\tau)_!\tau_{f_0}^{*}(\boldsymbol{E}) \xrightarrow{(\sigma(f_0)_\tau)_!\tau_{f_0}^{*}(\check{\boldsymbol{\xi}})} (\sigma(f_0)_\tau)_!\tau_{f_0}^{*}\sigma'_!\tau'^{*}(\boldsymbol{E})$$
$$\xrightarrow{\theta^{\sigma(f_0)_\tau, \tau_{f_0}, \sigma', \tau'}(\boldsymbol{E})} (\sigma(f_0)_\tau(id_{C_1} \times_{C_0} \sigma'))_!(\tau'\tilde{\mathrm{pr}}_2)^{*}(\boldsymbol{E})$$

Let $E^{f}_{(\boldsymbol{E},\boldsymbol{\xi})}: (\boldsymbol{E}, \boldsymbol{\xi})^{f} \to (\sigma(f_0)_\tau)_!\tau_{f_0}^{*}(\boldsymbol{E})$ be an equalizer of γ and $\boldsymbol{E}^{(\mu(id_{C_1} \times_{C_0} f_1), \tau'\tilde{\mathrm{pr}}_2)}: (\sigma(f_0)_\tau)_!\tau_{f_0}^{*}(\boldsymbol{E}) \to (\sigma(f_0)_\tau(id_{C_1} \times_{C_0} \sigma'))_!(\tau'\tilde{\mathrm{pr}}_2)^{*}(\boldsymbol{E})$.

We can give a structure $\check{\boldsymbol{\xi}}_f: (\boldsymbol{E}, \boldsymbol{\xi})^{f} \to \sigma_!\tau^{*}((\boldsymbol{E}, \boldsymbol{\xi})^{f})$ of a representation of $\boldsymbol{G}$ on $(\boldsymbol{E}, \boldsymbol{\xi})^{f}$ and $f_!(\boldsymbol{E}, \boldsymbol{\xi})$ is defined to be $(\boldsymbol{E}, \boldsymbol{\xi})^{f}$.

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