Unstable modules as representations of Steenrod groups

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Introduction

The ring of cohomology operations on the mod p ordinary cohomology theory is called the Steenrod algebra. J. Lannes developed an elegant theory of unstable modules over the Steenrod algebra which has an application to Sullivan's conjecture. Since the Steenrod algebra is not commutative, it is difficult to apply knowledge of commutative algebras.

Under certain finiteness conditions, a left module over the Steenrod algebra has a structure of a right comodule over the dual Steenrod algebra. Hence, roughly speaking, the category of left module over the Steenrod algebra is equivalent to the category of representations of an affine group scheme represented by the dual Steenrod algebra. The aim of this talk is a trial of reconstruction of Lanne's theory from the viewpoint of representation theory.

A very brief review on fibered category

Let $p : \mathcal{F} \to \mathcal{C}$ be a functor. For an object X of $\mathcal{C}$, we denote by $\mathcal{F}_X$ the subcategory of $\mathcal{F}$ consiting of objects M of $\mathcal{F}$ satisfying $p(M) = X$ and morphisms φ satisfying $p(\varphi) = id_X$.
For a morphism $f : X \to Y$ in $\mathcal{C}$ and $M \in \mathrm{Ob}\,\mathcal{F}_X$, $N \in \mathrm{Ob}\,\mathcal{F}_Y$, we put $\mathcal{F}_f(M, N) = \{\varphi \in \mathcal{F}(M, N) |\, p(\varphi) = f\}$.

Definition (Cartesian morphism)

Let $\alpha : M \to N$ be a morphism in $\mathcal{F}$ and set $X = p(M)$, $Y = p(N)$, $f = p(\alpha)$. We call α a cartesian morphism if, for any $M' \in \mathrm{Ob}\,\mathcal{F}_X$, the map $\mathcal{F}_X(M', M) \to \mathcal{F}_f(M', N)$ defined by $\varphi \mapsto \alpha\varphi$ is bijective.

Definition (Inverse image)

Let $f : X \to Y$ be a morphism in $\mathcal{C}$ and $N \in \mathrm{Ob}\,\mathcal{F}_Y$. If there exists a cartesian morphism $\alpha : M \to N$ such that $p(\alpha) = f$, M is called an inverse image of N by f. We denote M by $f^*(N)$ and α by $\alpha_f(N) : f^*(N) \to N$. $f^*(N)$ is unique up to isomorphism.

Proposition

Let $f : X \to Y$ be a morphism in $\mathcal{C}$. If, for any $N \in \operatorname{Ob} \mathcal{F}_Y$, there exists a cartesian morphism $\alpha_f(N) : f^*(N) \to N$, $N \mapsto f^*(N)$ defines a functor $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ such that, for any morphism $\varphi : N \to N'$ in $\mathcal{F}_Y$, the following square commutes.

$$\begin{array}{ccc} f^*(N) & \xrightarrow{\alpha_f(N)} & N \\ \Big\downarrow{\scriptstyle f^*(\varphi)} & & \Big\downarrow{\scriptstyle \varphi} \\ f^*(N') & \xrightarrow{\alpha_f(N')} & N' \end{array}$$

Definition (Inverse image functor)

For a morphism $f : X \to Y$ in $\mathcal{C}$, if there exists a cartesian morphism $\alpha_f(N) : f^*(N) \to N$ for any object N of $\mathcal{F}_Y$, we say that the functor of the inverse image by f exists.

Definition (Fibered category)

If a functor $p : \mathcal{F} \to \mathcal{C}$ satisfies the following condition (i), p is called a prefibered category and if p satisfies both (i) and (ii), p is called a fibered category or p is fibrant.

(i) For any morphism f in $\mathcal{C}$, the functor of the inverse image by f exists.
(ii) The composition of cartesian morphisms is cartesian.

Definition (Cleavage)

Let $p : \mathcal{F} \to \mathcal{C}$ be a functor. The following map κ is called a cleavage if $\kappa(f)$ is an inverse image functor $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ for $(f : X \to Y) \in \mathrm{Mor}\,\mathcal{C}$.

$$\kappa : \mathrm{Mor}\,\mathcal{C} \to \coprod_{X,Y \in \mathrm{Ob}\,\mathcal{C}} \mathrm{Funct}(\mathcal{F}_Y, \mathcal{F}_X)$$

A cleavage κ is said to be normalized if $\kappa(id_X) = id_{\mathcal{F}_X}$ for any $X \in \mathrm{Ob}\,\mathcal{C}$.
A functor $p : \mathcal{F} \to \mathcal{C}$ is called a cloven prefibered category (resp. normalized cloven prefibered category) if a cleavage (resp. normalized cleavage) is given.

Let $f : X \to Y$, $g : Z \to X$ be morphisms in $\mathcal{C}$ and N an object of $\mathcal{F}_Y$. If $p : \mathcal{F} \to \mathcal{C}$ is a prefibered category, there is a unique morphism $c_{f,g}(N) : g^*f^*(N) \to (fg)^*(N)$ such that the following square commutes and $p(c_{f,g}(N)) = id_Z$.

$$\begin{array}{ccc} g^*f^*(N) & \xrightarrow{\alpha_g(f^*(N))} & f^*(N) \\ \downarrow{\scriptstyle c_{f,g}(N)} & & \downarrow{\scriptstyle \alpha_f(N)} \\ (fg)^*(N) & \xrightarrow{\alpha_{fg}(N)} & N \end{array}$$

We note that $c_{f,g}(N)$ is an isomorphism if and only if $p : \mathcal{F} \to \mathcal{C}$ is a fibered category.

Proposition

For a morphism $\varphi : M \to N$ in $\mathcal{F}_Y$, the following square commutes.

$$\begin{array}{ccc} g^*f^*(M) & \xrightarrow{c_{f,g}(M)} & (fg)^*(M) \\ \downarrow{\scriptstyle g^*f^*(\varphi)} & & \downarrow{\scriptstyle (fg)^*(\varphi)} \\ g^*f^*(N) & \xrightarrow{c_{f,g}(N)} & (fg)^*(N) \end{array}$$

If $\mathcal{C}$ is a category with a terminal object 1 we denote by $o_X : X \to 1$ the unique morphism for an object X of $\mathcal{C}$.

For a morphism $f : X \to Y$ in $\mathcal{C}$ and objects M, N of $\mathcal{F}_1$, we define a map

$$f^\sharp_{M,N} : \mathcal{F}_Y(o_Y^*(M), o_Y^*(N)) \to \mathcal{F}_X(o_X^*(M), o_X^*(N))$$

to be the following composition. We note that $f^\sharp_{M,N}$ is natural in M and N.

$$\mathcal{F}_Y(o_Y^*(M), o_Y^*(N)) \xrightarrow{f^*} \mathcal{F}_X(f^*(o_Y^*(M)), f^*(o_Y^*(N))) \xrightarrow[\cong]{c_{o_Y,f}(M)^{*-1}}$$
$$\mathcal{F}_X((o_Y f)^*(M)), f^*(o_Y^*(N))) \xrightarrow[\cong]{c_{o_Y,f}(N)_*} \mathcal{F}_X((o_Y f)^*(M)), (o_Y f)^*(N)))$$
$$= \mathcal{F}_X(o_X^*(M), o_X^*(N))$$

Proposition

Let $f : Y \to X$, $g : Z \to Y$ be morphisms of $\mathcal{C}$ and L, M, N objects of $\mathcal{F}_1$.

(1) For $\zeta \in \mathcal{F}_X(o_X^*(L), o_X^*(M))$ and $\xi \in \mathcal{F}_X(o_X^*(M), o_X^*(N))$, we have $f^\sharp_{L,N}(\xi\zeta) = f^\sharp_{M,N}(\xi) f^\sharp_{L,M}(\zeta)$.

(2) $g^\sharp_{M,N} f^\sharp_{M,N} = (fg)^\sharp_{M,N}$ holds.

Fibered category with products

Let $\mathcal{C}$ be a category with a terminal object 1 and $p : \mathcal{F} \to \mathcal{C}$ a cloven fibered category.

For $X \in \mathrm{Ob}\,\mathcal{C}$ and $M \in \mathrm{Ob}\,\mathcal{F}_1$, define a presheaf $F_{X,M} : \mathcal{F}_1 \to \mathcal{S}et$ on $\mathcal{F}_1^{op}$ by

$$F_{X,M}(N) = \mathcal{F}_X(o_X^*(M), o_X^*(N))$$
$$F_{X,M}(\varphi : N \to N') = (o_X^*(\varphi)_* : F_{X,M}(N) \to F_{X,M}(N'))$$

for $N \in \mathrm{Ob}\,\mathcal{F}_1$ and $\varphi \in \mathcal{F}_1(N, N')$.

Suppose that $F_{X,M}$ is representable for $X \in \mathrm{Ob}\,\mathcal{C}$ and $M \in \mathrm{Ob}\,\mathcal{F}_1$. We choose an object $X \times M$ of $\mathcal{F}_1$ such that there exists a bijection

$$P_X(M)_N : \mathcal{F}_X(o_X^*(M), o_X^*(N)) \to \mathcal{F}_1(X \times M, N)$$

which is natural in N.

Definition (Fibered category with products)

Let $p : \mathcal{F} \to \mathcal{C}$ be a normalized cloven fibered category. We say that $p : \mathcal{F} \to \mathcal{C}$ is a fibered category with products if the presheaf $F_{X,M}$ on $\mathcal{F}_1^{op}$ is representable for any $X \in \mathrm{Ob}\,\mathcal{C}$ and $M \in \mathrm{Ob}\,\mathcal{F}_1$.

We assume $p : \mathcal{F} \to \mathcal{C}$ is a fibered category with products in this section. We have a functor $\mathcal{C} \times \mathcal{F}_1 \to \mathcal{F}_1$ which assigns $(X, M) \in \mathrm{Ob}(\mathcal{C} \times \mathcal{F}_1)$ to $X \times M$. For a morphism $f : X \to Y$ in $\mathcal{C}$, $f \times id_M : X \times M \to Y \times N$ is a morphism in $\mathcal{F}_1$ such that the following diagram is commutative for any object N of $\mathcal{F}_1$.

$$\begin{array}{ccc}
\mathcal{F}_X(o_Y^*(M), o_Y^*(N)) & \xrightarrow{P_Y(M)_N} & \mathcal{F}_1(Y \times M, N) \\
\Big\downarrow f^\sharp_{M,N} & & \Big\downarrow (f \times id_M)^* \\
\mathcal{F}_Y(o_X^*(M), o_X^*(N)) & \xrightarrow{P_X(M)_N} & \mathcal{F}_1(X \times M, N)
\end{array}$$

For a morphism $\varphi : M \to L$ in $\mathcal{F}_1$, $id_X \times \varphi : X \times M \to X \times L$ is a morphisms in $\mathcal{F}_1$ such that the following diagram is commutative for any object N of $\mathcal{F}_1$.

$$\begin{array}{ccc}
\mathcal{F}_X(o_X^*(L), o_X^*(N)) & \xrightarrow{P_X(L)_N} & \mathcal{F}_1(X \times L, N) \\
\Big\downarrow o_X^*(\varphi)^* & & \Big\downarrow (id_X \times \varphi)^* \\
\mathcal{F}_Y(o_X^*(M), o_X^*(N)) & \xrightarrow{P_X(M)_N} & \mathcal{F}_1(X \times M, N)
\end{array}$$

We denote by $\iota_X(M) : o_X^*(M) \to o_X^*(X \times M)$ the morphism in $\mathcal{F}_X$ which is mapped to the identity morphism of $X \times M$ by

$$P_X(M)_{X\times M} : \mathcal{F}_X(o_X^*(M), o_X^*(X \times M)) \to \mathcal{F}_1(X \times M, X \times M).$$

Define a "diagonal morphism" $\delta_{X,M} : X \times M \to X \times (X \times M)$ to be the image of $\iota_X(X \times M)\iota_X(M) \in \mathcal{F}_X(o_X^*(M), o_X^*(X \times (X \times M)))$ by

$$P_X(M)_{X\times(X\times M)} : \mathcal{F}_X(o_X^*(M), o_X^*(X\times(X\times M))) \to \mathcal{F}_1(X\times M, X\times(X\times M)).$$

It can be verified that $\delta_{X,M}$ is natural in X and M.

Assume that $\mathcal{C}$ has a finite products. For objects X and Y of $\mathcal{C}$, let $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ be projections. For an object M of $\mathcal{F}_1$, we define a morphism $\theta_{X,Y}(M) : (X \times Y) \times M \to X \times (Y \times M)$ of $\mathcal{F}_1$ to be the following composition.

$$(X\times Y)\times M \xrightarrow{\delta_{X\times Y,M}} (X\times Y)\times((X\times Y)\times M) \xrightarrow{\mathrm{pr}_X\times(\mathrm{pr}_Y\times id_M)} X\times(Y\times M)$$

Definition (Associative fibered category with products)

If $\theta_{X,Y}(M) : (X \times Y) \times M \to X \times (Y \times M)$ is an isomorphism for any $X, Y \in \mathrm{Ob}\,\mathcal{C}$ and $M \in \mathrm{Ob}\,\mathcal{F}_1$, we say that $p : \mathcal{F} \to \mathcal{C}$ is an associative fibered category with products.

We say that a graded topological vector space M^* is “profinite” if M^* is complete Hausdorff and M^*/N^* is finite dimensional for every open graded subspace N^* of M^*.

Let K^* be a linearly topologized graded commutative algebra.

We denote by $\mathcal{T}op\mathcal{A}lg_{K^*}$ a category whose objects are linearly topologized profinite graded commutative K^*-algebras and whose morphisms are continuous K^*-algebra homomorphisms which preserve degrees.
Note that K^* is a terminal object of $\mathcal{T}op\mathcal{A}lg_{K^*}^{op}$.

We also denote by $\mathcal{M}_{K^*}$ a category whose objects are linearly topologized profinite graded K^*-modules whose morphisms are continuous K^*-module homomorphisms which preserve degrees.

Definition (Fibered category of topological modules)

We define a category $\mathcal{MOD}_{K^*}$ as follows.

Objects : (R^*, M^*, α) where $R^* \in \mathrm{Ob}\,\mathcal{T}op\mathcal{A}lg_{K^*}$, $M^* \in \mathrm{Ob}\,\mathcal{M}_{K^*}$ and $\alpha : M^* \otimes_K R^* \to M^*$ is a right R^*-module structure on M^*.

Morphisms : $(f, \varphi) : (R^*, M^*, \alpha) \to (S^*, N^*, \beta)$ where $(f : R^* \to S^*) \in \mathrm{Mor}\,\mathcal{T}op\mathcal{A}lg_{K^*}$, $(\varphi : M^* \to N^*) \in \mathrm{Mor}\,\mathcal{M}_{K^*}$ such that the following diagram is commutative.

$$\begin{array}{ccc} M^* \otimes_{K^*} R^* & \xrightarrow{\alpha} & M^* \\ \downarrow{\scriptstyle \varphi \otimes_{K^*} f} & & \downarrow{\scriptstyle \varphi} \\ N^* \otimes_{K^*} S^* & \xrightarrow{\beta} & N^* \end{array}$$

Composition of $(f, \varphi) : (R^*, M^*, \alpha) \to (S^*, N^*, \beta)$ and $(g, \psi) : (S^*, N^*, \beta) \to (T^*, L^*, \gamma)$ is defined to be $(gf, \psi\varphi)$. We define a functor $p : \mathcal{MOD}_{K^*} \to \mathcal{T}op\mathcal{A}lg_{K^*}$ by $p(R^*, M^*, \alpha) = R^*$ and $p(f, \varphi) = f$.

Proposition

$p^{op} : \mathcal{MOD}_{K^*}^{op} \to \mathcal{T}op\mathcal{A}lg_{K^*}^{op}$ is an associative fibered category with products.

Inverse image funtor :

For a morphism $f : S^* \to R^*$ in $\mathcal{T}op\mathcal{A}lg_{K^*}$ and an object $\boldsymbol{N} = (S^*, N^*, \beta)$ of $(\mathcal{MOD}_{K^*})_{S^*}$, we have $f^*(\boldsymbol{N}) = (R^*, N^* \widehat{\otimes}_{S^*} R^*, \beta_f)$ where β_f is the following composition.

$$(N^* \widehat{\otimes}_{S^*} R^*) \otimes_{K^*} R^* \xrightarrow{completion} (N^* \widehat{\otimes}_{S^*} R^*) \widehat{\otimes}_{K^*} R^* \xrightarrow{\cong}$$

$$N^* \widehat{\otimes}_{S^*} (R^* \widehat{\otimes}_{K^*} R^*) \xrightarrow{id_{N^*} \widehat{\otimes}_{S^*} \hat{\mu}} N^* \widehat{\otimes}_{S^*} \hat{R}^* \xrightarrow{\cong} N^* \widehat{\otimes}_{S^*} R^*$$

Here $\hat{\mu} : R^* \widehat{\otimes}_{K^*} R^* \to \hat{R}^*$ is the map induced by the multiplication of R^*.

Product :

For an object R^* of $\mathcal{T}op\mathcal{A}lg_{K^*}$ and an object $\boldsymbol{M} = (K^*, M^*, \alpha)$ of $(\mathcal{MOD}_{K^*})_{K^*}$, we have $R^* \times \boldsymbol{M} = (K^*, M^* \widehat{\otimes}_{K^*} R^*, \alpha_{R^*})$, where α_{R^*} is the right K^*-module structure of $M^* \widehat{\otimes}_{K^*} R^*$ obtained from the right K^*-module structure of R^*.

Fibered category with exponents

Let $\mathcal{C}$ be a category with a terminal object 1 and $p : \mathcal{F} \to \mathcal{C}$ a cloven fibered category.

For $X \in \mathrm{Ob}\,\mathcal{C}$ and $M \in \mathrm{Ob}\,\mathcal{F}_1$, define a presheaf $F_N^X : \mathcal{F}_1^{op} \to \mathcal{S}et$ on $\mathcal{F}_1$ by

$$F_N^X(M) = \mathcal{F}_X(o_X^*(M), o_X^*(N))$$
$$F_N^X(\varphi : M \to M') = (o_X^*(\varphi)^* : F_N^X(M') \to F_N^X(M))$$

for $M \in \mathrm{Ob}\,\mathcal{F}_1$ and $\varphi \in \mathcal{F}_1(M, M')$.

Suppose that F_N^X is representable for $X \in \mathrm{Ob}\,\mathcal{C}$ and $N \in \mathrm{Ob}\,\mathcal{F}_1$. We choose an object N^X of $\mathcal{F}_1$ such that there exists a bijection

$$E_X(N)_M : \mathcal{F}_X(o_X^*(M), o_X^*(N)) \to \mathcal{F}_1(M, N^X)$$

which is natural in M.

Definition (Fibered category with exponents)

Let $p : \mathcal{F} \to \mathcal{C}$ be a normalized cloven fibered category. We say that $p : \mathcal{F} \to \mathcal{C}$ is a fibered category with exponents if the presheaf F_N^X on $\mathcal{F}_1$ is representable for any $X \in \mathrm{Ob}\,\mathcal{C}$ and $N \in \mathrm{Ob}\,\mathcal{F}_1$.

We assume $p : \mathcal{F} \to \mathcal{C}$ is a fibered category with exponents in this section. We have a functor $\mathcal{C} \times \mathcal{F}_1 \to \mathcal{F}_1$ which assigns $(X, N) \in \mathrm{Ob}\,(\mathcal{C} \times \mathcal{F}_1)$ to N^X. For a morphism $f : X \to Y$ in $\mathcal{C}$, $N^f : N^Y \to N^X$ is a morphism in $\mathcal{F}_1$ such that the following diagram is commutative for any object M of $\mathcal{F}_1$.

$$\begin{array}{ccc} \mathcal{F}_X(o_Y^*(M), o_Y^*(N)) & \xrightarrow{E_Y(N)_M} & \mathcal{F}_1(M, N^Y) \\ \Big\downarrow f^\sharp_{M,N} & & \Big\downarrow N^f_* \\ \mathcal{F}_Y(o_X^*(M), o_X^*(N)) & \xrightarrow{E_X(N)_M} & \mathcal{F}_1(M, N^X) \end{array}$$

For a morphism $\varphi : N \to L$ in $\mathcal{F}_1$, $\varphi^X : N^X \to L^X$ is a morphisms in $\mathcal{F}_1$ such that the following diagram is commutative for any object N of $\mathcal{F}_1$.

$$\begin{array}{ccc} \mathcal{F}_X(o_X^*(M), o_X^*(N)) & \xrightarrow{E_X(N)_M} & \mathcal{F}_1(M, N^X) \\ \Big\downarrow o_X^*(\varphi)_* & & \Big\downarrow \varphi^X_* \\ \mathcal{F}_Y(o_X^*(M), o_X^*(L)) & \xrightarrow{E_X(L)_M} & \mathcal{F}_1(M, L^X) \end{array}$$

We denote by $\pi_X(N) : o_X^*(N^X) \to o_X^*(N)$ the morphism in $\mathcal{F}_X$ which is mapped to the identity morphism of N^X by

$$E_X(N)_{N^X} : \mathcal{F}_X(o_X^*(N^X), o_X^*(N)) \to \mathcal{F}_1(N^X, N^X).$$

Define a "codiagonal morphism" $\epsilon_N^X : (N^X)^X \to N^X$ to be the image of $\pi_X(N)\pi_X(N^X) \in \mathcal{F}_X(o_X^*((N^X)^X), o_X^*(N))$ by

$$E_X(N)_{(N^X)^X} : \mathcal{F}_X(o_X^*((N^X)^X), o_X^*(N)) \to \mathcal{F}_1((N^X)^X, N^X).$$

It can be verified that ϵ_N^X is natural in X and N.
Assume that $\mathcal{C}$ has a finite products. For objects X and Y of $\mathcal{C}$, let $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ be projections. For an object N of $\mathcal{F}_1$, we define a morphism $\theta^{X,Y}(N) : (N^X)^Y \to N^{X\times Y}$ of $\mathcal{F}_1$ to be a composition $(N^X)^Y \xrightarrow{(N^{\mathrm{pr}_X})^{\mathrm{pr}_Y}} (N^{X\times Y})^{X\times Y} \xrightarrow{\epsilon_N^{X\times Y}} N^{X\times Y}$.

Remark

Let $p : \mathcal{F} \to \mathcal{C}$ be a fibered category and $X \in \mathrm{Ob}\,\mathcal{C}$. If, for $M, N \in \mathrm{Ob}\,\mathcal{F}_1$, both $X \times M$ and N^X exist, then there are the following bijections.

$$\mathcal{F}_1(X \times M, N) \xrightarrow[\cong]{P_X(M)_N^{-1}} \mathcal{F}_X(o_X^*(M), o_X^*(N)) \xrightarrow[\cong]{E_X(N)_M} \mathcal{F}_1(M, N^X)$$

Definition (Associative fibered category with exponents)

If $\theta^{X,Y}(N) : (N^X)^Y \to N^{X\times Y}$ is an isomorphism for any $X, Y \in \mathrm{Ob}\,\mathcal{C}$ and $N \in \mathrm{Ob}\,\mathcal{F}_1$, we say that $p : \mathcal{F} \to \mathcal{C}$ is an associative fibered category with exponents.

Definition (Skeletal topology)

For a graded vector space M^* over a field K and a non-negative integer n, we put $M^*[n] = \sum_{|i|\geq n} M^i$. A topology on M^* is called a "skeletal topology" if $\{M^*[n] \,|\, n = 0, 1, 2, \dots\}$ is a fundamental system of the neighborhood of 0 of its topology.

Definition (Connectiveness and coconnectiveness)

Let M^* be a graded vector space over a field K. If there exists an integer l such that $M^i = \{0\}$ if $i < l$ (resp.$i > l$), we say that M^* is connective (resp. coconnective).

Let K^* be a field such that $K^i = \{0\}$ for $i \neq 0$. For a graded K^*-module M^* with skeletal topology, the dual $\mathcal{H}om^*(M^*, K^*)$ of M^* is a graded K^*-module such that $\mathcal{H}om^n(M^*, K^*)$ is the set of all linear maps $M^{-n} \to K^0$. We denote $\mathcal{H}om^*(M^*, K^*)$ by M^{**} for short and give M^{**} the skeletal topology.

Proposition

Let K^* be a field such that $K^i = \{0\}$ for $i \neq 0$. Assume that an object R^* of $\mathcal{T}op\mathcal{A}lg_{K^*}$ and an object $\mathbf{N} = (K^*, N^*, \beta)$ of $\mathcal{MOD}_{K^*}$ satisfies the following conditions.

(i) R^* is finite type, connective and has skeletal topology.
(ii) N^* is finite type, coconnective and has skeletal topology.

Then, the presheaf $F_{\mathbf{N}}^{R^*}$ on $\mathcal{MOD}_{K^*}^{op}$ is representable and $\mathbf{N}^{R^*}$ is given by $(K^*, R^{**} \otimes_{K^*} N^*, \beta')$ where β' is the K^*-module structure obtained from the K^*-module structure β of N^*.

Representation of group objects

Definition (Group object in a category)

A group object in a category $\mathcal{C}$ with finite products is an object G of $\mathcal{C}$ with morphisms $\mu : G \times G \to G$ (product), $\varepsilon : 1 \to G$ (unit) and $\iota : G \to G$ (inverse) which make the following diagrams commute.

$$
\begin{array}{ccc}
G \times G \times G & \xrightarrow{\mu \times id_G} & G \times G \\
\Big\downarrow{\scriptstyle id_G \times \mu} & & \Big\downarrow{\scriptstyle \mu} \\
G \times G & \xrightarrow{\quad\mu\quad} & G
\end{array}
\qquad
\begin{array}{ccc}
1 \times G & \xleftarrow{(o_G, id_G)} & G \\
\Big\downarrow{\scriptstyle \varepsilon \times id_G} & & \Big\downarrow{\scriptstyle id_G} \\
G \times G & \xrightarrow{\quad\mu\quad} & G \\
\Big\uparrow{\scriptstyle id_G \times \varepsilon} & & \Big\uparrow{\scriptstyle id_G} \\
G \times 1 & \xleftarrow{(id_G, o_G)} & G
\end{array}
\qquad
\begin{array}{ccc}
G & \xrightarrow{\quad o_G\quad} & 1 \\
\Big\downarrow{\scriptstyle (\iota, id_G)} & & \Big\downarrow{\scriptstyle \varepsilon} \\
G \times G & \xrightarrow{\quad\mu\quad} & G \\
\Big\uparrow{\scriptstyle (id_G, \iota)} & & \Big\uparrow{\scriptstyle \varepsilon} \\
G & \xrightarrow{\quad o_G\quad} & 1
\end{array}
$$

We denote the above group object by $(G, \mu, \varepsilon, \iota)$.

Definition (Homomorphism between group objects)

Let $(H, \mu', \varepsilon', \iota')$ be a group object in $\mathcal{C}$. If a morphism $f : G \to H$ makes the following diagram commute, we call f a morphism of group objects and denote this by $f : (G, \mu, \varepsilon, \iota) \to (H, \mu', \varepsilon', \iota')$.

$$\begin{array}{ccc} G \times G & \xrightarrow{f \times f} & H \times H \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu'} \\ G & \xrightarrow{\quad f \quad} & H \end{array}$$

Let $p : \mathcal{F} \to \mathcal{C}$ be a cloven fibered category and $f : Y \to X$ a morphism in $\mathcal{C}$. For objects M and N of $\mathcal{F}_1$ and a morphism $\xi : o_X^*(M) \to o_X^*(N)$ of $\mathcal{F}_X$, we denote $f^\sharp_{M,N}(\xi) : o_Y^*(M) \to o_Y^*(N)$ by ξ_f for short. That is, ξ_f is the following composition.

$$o_Y^*(M) = (o_X f)^*(M) \xrightarrow{c_{o_X,f}(M)^{-1}} f^* o_X^*(M) \xrightarrow{f^*(\xi)} f^* o_X^*(N)$$
$$\xrightarrow{c_{o_X,f}(N)} (o_X f)^*(N) = o_Y^*(N)$$

We denote by $\mathrm{pr}_1, \mathrm{pr}_2 : G \times G \to G$ the projections below.

Definition (Representation of group object)

Let $(G, \mu, \varepsilon, \iota)$ be a group object in $\mathcal{C}$.
A pair (M, ξ) of an object M of $\mathcal{F}_1$ and a morphism $\xi : o_G^*(M) \to o_G^*(M)$ in $\mathcal{F}_G$ is called a left (resp. right) representation of G on M if the following conditions (i) (resp. (i')) and (ii) are satisfied.

$$(i)\ \ \xi_\mu = \xi_{\mathrm{pr}_1}\xi_{\mathrm{pr}_2} \qquad (i')\ \ \xi_\mu = \xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1} \qquad (ii)\ \ \xi_\varepsilon = id_M$$

If we say simply "a representation", this means a left representation.
Let $\xi : o_G^*(M) \to o_G^*(M)$ and $\zeta : o_G^*(N) \to o_G^*(N)$ be representations of G on M and N, respectively. A morphism $\varphi : M \to N$ of $\mathcal{F}_1$ is a called a morphism of representations of G from (M, ξ) to (N, ζ) if the following diagram commutes.

$$\begin{array}{ccc} o_G^*(M) & \xrightarrow{\ \xi\ } & o_G^*(M) \\ \downarrow{\scriptstyle o_G^*(\varphi)} & & \downarrow{\scriptstyle o_G^*(\varphi)} \\ o_G^*(N) & \xrightarrow{\ \zeta\ } & o_G^*(N) \end{array}$$

We denote by $\mathrm{Rep}(G\,;\mathcal{F})$ the category of representations of G and morphisms between them.

Proposition

Let $f : (G, \mu, \varepsilon, \iota) \to (H, \mu', \varepsilon', \iota')$ *be a morphism of group objects in* $\mathcal{C}$*.*

(1) If (M, ξ) *is a representation of* H *on* M*,* (M, ξ_f) *is a representation of* G *on* M*.*

(2) If $\varphi : (M, \xi) \to (N, \zeta)$ *is a morphism of representations of* H*,* $\varphi : (M, \xi_f) \to (N, \zeta_f)$ *is a morphism of representations of* G*.*

Thus we have a functor $f^* : \mathrm{Rep}(H; \mathcal{F}) \to \mathrm{Rep}(G; \mathcal{F})$ given by $f^*(M, \xi) = (M, \xi_f)$ and $f^*(\varphi : (M, \xi) \to (N, \zeta)) = (\varphi : (M, \xi_f) \to (N, \zeta_f))$.

Proposition

For $M \in \operatorname{Ob} \mathcal{F}_1$ and $\xi \in \mathcal{F}_G(o_G^*(M), o_G^*(M))$, we put $\hat{\xi} = P_G(M)_M(\xi) : G \times M \to M$. Then, (M, ξ) is a representation of G on M if and only if the following diagrams commute.

$$
\begin{array}{ccccc}
(G \times G) \times M & \xrightarrow{\mu \times M} & G \times M & \xrightarrow{\hat{\xi}} & M \\
\big\downarrow{\scriptstyle \theta_{G,G}(M)} & & & \nearrow{\scriptstyle \hat{\xi}} & \\
G \times (G \times M) & \xrightarrow{G \times \hat{\xi}} & G \times M & &
\end{array}
\qquad
\begin{array}{ccc}
1 \times M & & \\
\big\downarrow{\scriptstyle \varepsilon \times M} & \searrow{\scriptstyle id_M} & \\
G \times M & \xrightarrow{\hat{\xi}} & M
\end{array}
$$

Proposition

For $M \in \operatorname{Ob} \mathcal{F}_1$ and $\xi \in \mathcal{F}_G(o_G^*(M), o_G^*(M))$, we put $\check{\xi} = E_G(M)_M(\xi) : M \to M^G$. Then, (M, ξ) is a representation of G on M if and only if the following diagrams commute.

$$
\begin{array}{ccccc}
M & \xrightarrow{\check{\xi}} & M^G & \xrightarrow{(\check{\xi})^G} & (M^G)^G \\
& \searrow{\scriptstyle \check{\xi}} & & & \big\downarrow{\scriptstyle \theta^{G,G}(M)} \\
& & M^G & \xrightarrow{M^{\mu}} & M^{G \times G}
\end{array}
\qquad
\begin{array}{ccc}
M & \xrightarrow{\check{\xi}} & M^G \\
& \searrow{\scriptstyle id_M} & \big\downarrow{\scriptstyle M^{\varepsilon}} \\
& & M^1
\end{array}
$$

Let A^* be a graded commutative Hopf algebra over a field K^* which satisfies $K^i = \{0\}$ if $i \neq 0$. We assume that A^* is finite type, connective and has skeletal topology. We denote coproduct, counit and conjugation of A^* by $\mu : A^* \to A^* \otimes_{K^*} A*$, $\varepsilon : A^* \to K^*$ and $\iota : A^* \to A^*$, respectively. Then, $(A^*, \mu, \varepsilon, \iota)$ is a group object in $\mathcal{T}op\mathcal{A}lg^{op}_{K^*}$.

Let M^* be a K^*-module with skeletal topology. We assume that M^* is finite type and coconnective. We denote by α the K^*-module structure of M^* and consider an object $\boldsymbol{M} = (K^*, M^*, \alpha)$ of $\mathcal{MOD}_{K^*}$.

Since A^* and M^* are profinite, both $A^* \times \boldsymbol{M}$ and $\boldsymbol{M}^{A^*}$ exist. There are the following bijections.

$$P_{A^*}(\boldsymbol{M})_{\boldsymbol{M}} : (\mathcal{MOD}^{op}_{K^*})_{A^*}(o^*_{A^*}(\boldsymbol{M}), o^*_{A^*}(\boldsymbol{M})) \to (\mathcal{MOD}^{op}_{K^*})_{K^*}(A^* \times \boldsymbol{M}, \boldsymbol{M})$$
$$E_{A^*}(\boldsymbol{M})_{\boldsymbol{M}} : (\mathcal{MOD}^{op}_{K^*})_{A^*}(o^*_{A^*}(\boldsymbol{M}), o^*_{A^*}(\boldsymbol{M})) \to (\mathcal{MOD}^{op}_{K^*})_{K^*}(\boldsymbol{M}, \boldsymbol{M}^{A^*})$$

Since we have $(\mathcal{MOD}^{op}_{K^*})_{K^*}(\boldsymbol{M}, \boldsymbol{M}^{A^*}) = \mathrm{Hom}_{K^*}(A^{**} \otimes_{K^*} M^*, M^*)$ and $(\mathcal{MOD}^{op}_{K^*})_{K^*}(A^* \times \boldsymbol{M}, \boldsymbol{M}) = \mathrm{Hom}_{K^*}(M^*, M^* \widehat{\otimes}_{K^*} A^*)$, there is a bijection

$$\mathrm{Hom}_{K^*}(A^{**} \otimes_{K^*} M^*, M^*) \to \mathrm{Hom}_{K^*}(M^*, M^* \widehat{\otimes}_{K^*} A^*)$$

which maps the set of all left A^{**}-module structures on M^* onto the set of all right A^*-comodule structures on M^*.

Filtered algebras and unstable modules

We denote by $\mathcal{A}_p^*$ the Steenrod algebra over a prime field $\boldsymbol{F}_p$.

Let Seq be the set of all infinite sequences $(i_1, i_2, \dots, i_n, \dots)$ of non-negative integers such that $i_n = 0$ for all but finite number of n's.

Let Seq^o be a subset of Seq consisting of sequences $(i_1, i_2, \dots, i_n, \dots)$ such that $i_k = 0, 1$ if k is odd.

If $i_s = 0$ for $s > n$, we denote $(i_1, i_2, \dots, i_s, \dots) \in \mathrm{Seq}$ by $(i_1, i_2, \dots, i_n)$.

For an odd prime p and $I = (\varepsilon_0, i_1, \varepsilon_1, \dots, i_n, \varepsilon_n) \in \mathrm{Seq}^o$, we put

$$d_p(I) = -2(p-1)\sum_{s=1}^{n} i_s - \sum_{s=0}^{n} \varepsilon_s, \;\; e_p(I) = -\sum_{s=0}^{n} \varepsilon_s - 2\sum_{s=1}^{n}(i_s - p i_{s+1} - \varepsilon_s).$$

Then, we have $\wp^I = \beta^{\varepsilon_0}\wp^{i_1}\beta^{\varepsilon_1}\wp^{i_2}\beta^{\varepsilon_2}\cdots\wp^{i_n}\beta^{\varepsilon_n} \in \mathcal{A}_p^{d_p(I)}$.

For $I = (i_1, i_2, \dots, i_n) \in \mathrm{Seq}$, we put

$$d_2(I) = -\sum_{s=1}^{n} i_s, \quad e_2(I) = -\sum_{s=1}^{n}(i_s - 2i_{s+1}).$$

Then, we have $Sq^I = Sq^{i_1}Sq^{i_2}\cdots Sq^{i_n} \in \mathcal{A}_2^{d_2(I)}$.

We call $d_p(I)$ the degree of I and $e_p(I)$ the excess of I.

For an odd prime p, we say that $I = (\varepsilon_0, i_1, \varepsilon_1, \dots, i_n, \varepsilon_n) \in \mathrm{Seq}^o$ is p-admissible if p is and $i_s \geq p i_{s+1} + \varepsilon_s$ for $s = 1, 2, \dots, n$.
We say that $I = (i_1, i_2, \dots, i_n) \in \mathrm{Seq}$ is 2-admissible if $i_s \geq 2 i_{s+1}$ for $s = 1, 2, \dots, n$.
We denote by Seq_p the subset of Seq consisting of p-admissible sequences.

Definition (Excess filtration on the Steenrod algebra)

Let $F_i \mathcal{A}_p^*$ be a subspace of $\mathcal{A}_p^*$ spanned by the following set of monomials.

$$\left\{ \wp^I \,\middle|\, I \in \mathrm{Seq}_p,\ e_p(I) \leq i \right\} \text{ if } p \neq 2, \quad \left\{ Sq^I \,\middle|\, I \in \mathrm{Seq}_2,\ e_2(I) \leq i \right\} \text{ if } p = 2$$

Thus we have an increasing filtration $\mathfrak{F}_p = (F_i \mathcal{A}_p^*)_{i \in \boldsymbol{Z}}$ which is called the excess filtration on $\mathcal{A}_p^*$.

Proposition

Let X be a topological space and $H^*(X)$ the mod p ordinary cohomology group of X. If $\theta \in F_{n-1} \mathcal{A}_p^*$ and $x \in H^n(X)$, then $\theta x = 0$ holds.

We assume that K^* is a field such that $K^i = \{0\}$ if $i \neq 0$ in this section. Let A^* be a graded K^*-algebra with an increasing filtration $(F_i A^*)_{i \in \boldsymbol{Z}}$. We denote by $\mathcal{M}od(A^*)$ the category of left A^*-modules and homomorphisms.

Definition (Unstable modules)

Let M^* be a left A^*-module with a multiplication $\alpha : A^* \otimes_{K^*} M^* \to M^*$. M^* is called an unstable A^*-module if $\alpha(F_{n-1}A^* \otimes_{K^*} M^n) = \{0\}$ for $n \in \boldsymbol{Z}$. We denote by $\mathcal{UM}od(A^*)$ the full subcategory of $\mathcal{M}od(A^*)$.

Remark

If $(F_i A^*)_{i \in \boldsymbol{Z}}$ satisfies $F_0 A^* = A^*$, an unstable A^*-module M^* satisfies $M^n = \{0\}$ for $n \geq 0$. In fact, if $n \geq 1$, then $A^* = F_{n-1}A^*$, hence $M^n \subset \alpha(A^* \otimes_{K^*} M^n) = \alpha(F_{n-1}A^* \otimes_{K^*} M^n) = \{0\}$.

It is clear that submodules and quotient modules of an unstable module are also unstable and that the sum and the product of unstable modules are unstable. Hence $\mathcal{UM}od(A^*)$ is complete and cocomplete and the inclusion functor $I_{A^*} : \mathcal{UM}od(A^*) \to \mathcal{M}od(A^*)$ preserves limits and colimits.

Proposition

The inclusion functor $I_{A^*} : \mathcal{U}\mathcal{M}od(A^*) \to \mathcal{M}od(A^*)$ has a right adjoint.

Proof: For an object M^* of $\mathcal{M}od(A^*)$, since sums of unstable submodules of M^* are also unstable submodules, there exists the largest unstable submodule M_u^* of M^*. For a morphism $f : M^* \to N^*$ of $\mathcal{M}od(A^*)$, since $f(M_u^*)$ is an unstable submodule of N^*, $f(M_u^*) \subset N_u^*$ holds and f induces $f_u : M_u^* \to N_u^*$. Thus we have a functor $R_{A^*} : \mathcal{M}od(A^*) \to \mathcal{M}od(A^*)$ given by $R_{A^*}(M^*) = M_u^*$ and $R_{A^*}(f) = f_u$. It can be verified that R_{A^*} is a right adjoint of I_{A^*}.

Notations

Let M^* be a non-trivial graded K^*-module.
For an increasing filtration $\mathfrak{F} = (F_i M^*)_{i \in \boldsymbol{Z}}$ of M^*, we define $E_i^* M^*$, $E_i^j M^*$, a subset $S(\mathfrak{F})$ of $\boldsymbol{Z}$ and a map $c_{\mathfrak{F}} : S(\mathfrak{F}) \to \boldsymbol{Z}$ as follows.

$E_i^* M^* = F_i M^*/F_{i-1} M^*$ *and* $E_i^j M^* = (F_i M^*/F_{i-1} M^*)^j$.

$S(\mathfrak{F}) = \{ i \in \boldsymbol{Z} \mid E_i^* M^* \neq \{0\} \}$.

If M^* is coconnective, put $c_{\mathfrak{F}}(i) = \max\{ j \in \boldsymbol{Z} \mid E_i^j M^* \neq \{0\} \}$ for $i \in S(\mathfrak{F})$.

Let A^* be a graded K^*-algebra with product $\mu : A^* \otimes_{K^*} A^* \to A^*$.
For an increasing filtration $\mathfrak{F} = (F_i A^*)_{i \in \boldsymbol{Z}}$ of A^*, we give the following list of conditions on $\mathfrak{F}$.

Conditions

(*f*1) $\bigcap_{i \in \boldsymbol{Z}} F_i A^* = \{0\}$

(*f*2) $F_0 A^* = A^*$

(*f*3) $E_i^j A^* = \{0\}$ *if* $i + j \neq k + c_{\mathfrak{F}}(k)$ *for any* $k \in S(\mathfrak{F})$.

(*f*4) *A map* $S(\mathfrak{F}) \to \boldsymbol{Z}$ *which assigns* $i \in S(\mathfrak{F})$ *to* $i + c_{\mathfrak{F}}(i)$ *is injective.*

(*f*5) $\mu(A^* \otimes_{K^*} F_i A^*) \subset F_i A^*$ *for* $i \in \boldsymbol{Z}$.

(*f*6) $\mu(F_i A^* \otimes_{K^*} A^j) \subset F_{i-j} A^*$ *for* $i, j \in \boldsymbol{Z}$.

(*f*7) $\mu((F_i A^*)^{c_{\mathfrak{F}}(i)} \otimes_{K^*} A^j) + (F_{i-j-1} A^*)^{j + c_{\mathfrak{F}}(i)} = (F_{i-j} A^*)^{j + c_{\mathfrak{F}}(i)}$
for $i \in S(\mathfrak{F})$, $j \in \boldsymbol{Z}$.

(*f*8) *Under* (*f*6), μ *defines* $\mu_i^{k,j} : (F_i A^*)^k \otimes_{K^*} A^j \to (F_{i-j} A^*)^{j+k}$.
For $i \in S(\mathfrak{F})$ *and* $j \in \boldsymbol{Z}$, $(\mu_i^{c_{\mathfrak{F}}(i), j})^{-1}((F_{i-j-1} A^*)^{j + c_{\mathfrak{F}}(i)})$
$= (F_{i-1} A^*)^{c_{\mathfrak{F}}(i)} \otimes_{K^*} A^j + (F_i A^*)^{c_{\mathfrak{F}}(i)} \otimes_{K^*} (F_{i-j-1} A^*)^j$

Theorem (The case of the Steenrod algebra)

The excess filtration $\mathfrak{F}_p = (F_i\mathcal{A}_p^*)_{i\in\boldsymbol{Z}}$ on $\mathcal{A}_p^*$ satisfies all of the conditions $(f1) \sim (f8)$ above. Here $S(\mathfrak{F}_p)$ is the set of all non-positive integers and $c_{\mathfrak{F}_p} : S(\mathfrak{F}_p) \to \boldsymbol{Z}$ is given by $c_{\mathfrak{F}_p}(2i-\varepsilon) = 2i(p-1)-\varepsilon$ $(\varepsilon = 0,1)$.

Definition (Suspension)

Let M^* be a graded K^*-module. For an integer n, we define a graded K^*-module $\Sigma^n M^*$ by $(\Sigma^n M^*)^i = \{[n]\} \times M^{i-n}$ such that the projection $\{[n]\} \times M^{i-n} \to M^{i-n}$ is an isomorphism of K^*-modules.
If $f : M^* \to N^*$ is a homomorphism of graded K^*-module, we define $\Sigma^n f : \Sigma^n M^* \to \Sigma^n N^*$ by $\Sigma^n f([n], x) = ([n], f(x))$.

Remark

Suppose that $\mathfrak{F} = (F_iA^*)_{i\in\boldsymbol{Z}}$ satisfies $(f5)$ above. Then $\mathfrak{F}$ satisfies $(f6)$ if and only if $\Sigma^n(A^*/F_{n-1}A^*)$ is an unstable A^*-module for any $n \in \boldsymbol{Z}$.

Proposition

For a left A^*-module M^* with structure map $\alpha : A^* \otimes_{K^*} M^* \to M^*$, define a subspace $\bar{M}^n$ of M^n by

$$\bar{M}^n = \{x \in M^n | \, \alpha(a \otimes x) = 0 \text{ for any } a \in F_{n-1}M^*\}$$

and put $\bar{M}^* = \sum_{n \in \boldsymbol{Z}} \bar{M}^n$. If $(F_iA^*)_{i\in \boldsymbol{Z}}$ satisfies $(f6)$, $\bar{M}^*$ is the largest unstable submodule of M^*. Hence we have $R_{A^*}(M^*) = \bar{M}^*$.

Proof: For $x \in \bar{M}^n$, $b \in A^m$ and $a \in F_{m+n-1}A^*$, since $\mu(a \otimes b) \in F_{n-1}A^*$ holds by $(f6)$, we have an equality $\alpha(a \otimes \alpha(b \otimes x)) = \alpha(\mu(a \otimes b) \otimes x) = 0$ which shows $\alpha(b \otimes x) \in \bar{M}^{m+n}$. Hence $\bar{M}^*$ is an unstable submodule of M^*. It is clear that $\bar{M}^*$ is the largest submodule among unstable submodules of M^*.

Let M^* be a left A^*-module with structure map $\alpha : A^* \otimes_{K^*} M^* \to M^*$ and put $\mathcal{N}(M^*) = \sum_{n \in \boldsymbol{Z}} \alpha(F_{n-1}A^* \otimes_{K^*} M^n)$. Then, M^* is an unstable if and only if $\mathcal{N}(M^*) = \{0\}$. Assume that $\mathfrak{F}$ satisfies $(f5)$.
Put $L_{A^*}(M^*) = M^*/\mathcal{N}(M^*)$, then $L_{A^*}(M^*)$ is an unstable A^*-module.
For a morphism $f : M^* \to N^*$ in $\mathcal{M}od(A^*)$, since $f(\mathcal{N}(M^*)) \subset \mathcal{N}(N^*)$ holds, f induces a morphism $L_{A^*}(f) : L_{A^*}(M^*) \to L_{A^*}(N^*)$ in $\mathcal{UM}od(A^*)$.
Thus we have a functor $L_{A^*} : \mathcal{M}od(A^*) \to \mathcal{UM}od(A^*)$.

Proposition

If $\mathfrak{F}$ satisfies $(f5)$, $I_{A^*} : \mathcal{UM}od(A^*) \to \mathcal{M}od(A^*)$ has a left adjoint L_{A^*}.

Proof: Since $L_{A^*} I_{A^*}$ is the identity functor of $\mathcal{UM}od(A^*)$, let $\varepsilon : L_{A^*} I_{A^*} \to id_{\mathcal{UM}od(A^*)}$ be the identity natural transformation.
For $M^* \in \mathrm{Ob}\,\mathcal{M}od(A^*)$, let $\eta_{M^*} : M^* \to I_{A^*} L_{A^*}(M^*)$ be the quotient map $M^* \to M^*/\mathcal{N}(M^*)$. Then η_{M^*} is natural in M^* and that compositions

$$L_{A^*}(M^*) \xrightarrow{L_{A^*}(\eta_{M^*})} L_{A^*} I_{A^*} L_{A^*}(M^*) \xrightarrow{\varepsilon_{L_{A^*}(M^*)}} L_{A^*}(M^*),$$
$$I_{A^*}(M^*) \xrightarrow{\eta_{I_{A^*}(M^*)}} I_{A^*} L_{A^*} I_{A^*}(M^*) \xrightarrow{I_{A^*}(\varepsilon_{M^*})} I_{A^*}(M^*)$$

are identity morphisms of $L_{A^*}(M^*)$ and $I_{A^*}(M^*)$, respectively.

We denote by $\mathcal{M}od_{K^*}$ the category of graded K^*-modules and degree preserving homomorphisms.

The forgetful functor $O : \mathcal{M}od(A^*) \to \mathcal{M}od_{K^*}$ has a left adjoint $F : \mathcal{M}od_{K^*} \to \mathcal{M}od(A^*)$ given by $F(M^*) = A^* \otimes_{K^*} M^*$, $F(f) = id_{A^*} \otimes_{K^*} f$.

Let us denote by $\mathcal{F} : \mathcal{M}od_{K^*} \to \mathcal{UM}od(A^*)$ the composition of F and L_{A^*}, by $\mathcal{O} : \mathcal{UM}od(A^*) \to \mathcal{M}od_{K^*}$ the composition of I_{A^*} and O.

Proposition

If $\mathfrak{F}$ satisfies $(f5)$, $\mathcal{F}$ is a left adjoint of $\mathcal{O}$.

Remark

If $\mathfrak{F}$ satisfies $(f5)$ and $(f6)$, $\mathcal{N}(F(M^*)) = \sum_{n \in \boldsymbol{Z}} F_{n-1}A^* \otimes_{K^*} M^n$ holds for $M^* \in \mathrm{Ob}\, \mathcal{M}od_{K^*}$. Hence $\mathcal{F}(M^*)$ is isomorphic to $\sum_{n \in \boldsymbol{Z}} A^*/F_{n-1}A^* \otimes_{K^*} M^n$ as a left A^*-module.

Let $f : A^* \to B^*$ be a homomorphism of graded K^*-algebras.

For a left B^*-module N^* with structure map $\beta : B^* \otimes_{K^*} N^* \to N^*$, we denote by $f_*(N^*)$ a left A^*-module N^* with a structure map $\beta(f \otimes_{K^*} id_{N^*}) : A^* \otimes_{K^*} N^* \to N^*$.
Define a functor $f_* : \mathcal{M}od(B^*) \to \mathcal{M}od(A^*)$ by $N^* \mapsto f_*(N^*)$ for $N^* \in \mathrm{Ob}\, \mathcal{M}od(B^*)$ and $f_*(\varphi) = \varphi$ for a morphism φ of $\mathcal{M}od(B^*)$.

$f_* : \mathcal{M}od(B^*) \to \mathcal{M}od(A^*)$ has a left adjoint $f^* : \mathcal{M}od(A^*) \to \mathcal{M}od(B^*)$ defined as follows. Put $f^*(M^*) = B^* \otimes_{A^*} M^*$ for $M^* \in \mathrm{Ob}\, \mathcal{M}od(A^*)$ and the left B^*-module structure of $f^*(M^*)$ is defined from the product of B^*. For a homomorphism $\varphi : M^* \to L^*$, put $f^*(\varphi) = id_{B^*} \otimes_{A^*} \varphi$. Then, f^* is a left adjoint of f_*.

Proposition

Let $\mathfrak{F}_{A^*} = (F_i A^*)_{i \in \boldsymbol{Z}}$ and $\mathfrak{F}_{B^*} = (F_i B^*)_{i \in \boldsymbol{Z}}$ be increasing filtrations of A^* and B^*, respectively. If $f : A^* \to B^*$ satisfies $f(F_i A^*) \subset F_i B^*$ for $i \in \boldsymbol{Z}$, $f_* : \mathcal{M}od(B^*) \to \mathcal{M}od(A^*)$ maps each object of $\mathcal{UM}od(B^*)$ to that of $\mathcal{UM}od(A^*)$. Hence $f_* : \mathcal{M}od(B^*) \to \mathcal{M}od(A^*)$ restricts to a functor $f_{u*} : \mathcal{UM}od(B^*) \to \mathcal{UM}od(A^*)$.

Proposition

If $\mathfrak{F}_{B^*}$ satisfies $(f5)$, $f_{u*} : \mathcal{UM}od(B^*) \to \mathcal{UM}od(A^*)$ has a left adjoint.

Proof :

Define $f_u^* : \mathcal{UM}od(A^*) \to \mathcal{UM}od(B^*)$ to be the following composition.

$$\mathcal{UM}od(A^*) \xrightarrow{I_{A^*}} \mathcal{M}od(A^*) \xrightarrow{f^*} \mathcal{M}od(B^*) \xrightarrow{L_{B^*}} \mathcal{UM}od(B^*)$$

Since $I_{B^*} : \mathcal{UM}od(B^*) \to \mathcal{M}od(B^*)$ has a left adjoint $L_{B^*} : \mathcal{M}od(B^*) \to \mathcal{UM}od(B^*)$ and $f^* : \mathcal{M}od(A^*) \to \mathcal{M}od(B^*)$ has a right adjoint $f_* : \mathcal{M}od(B^*) \to \mathcal{M}od(A^*)$, we have the following chain of natural bijections for $M^* \in \mathrm{Ob}\,\mathcal{UM}od(A^*)$ and $N^* \in \mathrm{Ob}\,\mathcal{UM}od(B^*)$.

$$\begin{aligned}
\mathcal{UM}od(B^*)(f_u^*(M^*), N^*) &= \mathcal{UM}od(B^*)(L_{B^*}f^*I_{A^*}(M^*), N^*) \\
&\cong \mathcal{M}od(B^*)(f^*I_{A^*}(M^*), I_{B^*}(N^*)) \\
&\cong \mathcal{M}od(A^*)(I_{A^*}(M^*), f_*I_{B^*}(N^*)) \\
&= \mathcal{M}od(A^*)(I_{A^*}(M^*), I_{A^*}f_{u*}(N^*)) \\
&\cong \mathcal{UM}od(A^*)(M^*, f_{u*}(N^*))
\end{aligned}$$

Hence $f_u^* : \mathcal{UM}od(A^*) \to \mathcal{UM}od(B^*)$ is a left adjoint of $f_{u*} : \mathcal{UM}od(B^*) \to \mathcal{UM}od(A^*)$.

Loop functor

Proposition

Assume A^* is coconnective and finite type and that $\mathfrak{F} = (F_i A^*)_{i \in \mathbf{Z}}$ satisfies $(f1)$, $(f3)$ and $(f7)$.
A left A^*-module M^* with structure map $\alpha : A^* \otimes_{K^*} M^* \to M^*$ is unstable if and only if $\alpha((F_i A^*)^{c_{\mathfrak{F}}(i)} \otimes_{K^*} M^k) = \{0\}$ for any $i \in S(\mathfrak{F})$, $k > i$.

Assume that $\mathfrak{F}$ satisfies $(f6)$. Since the horizontal rows of the following diagram are exact, there exists unique map $\bar{\mu}_i^{k,j} : E_i^k A^* \otimes_{K^*} A^j \to E_{i-j}^k A^*$ that make the following diagram commute.

$$\begin{array}{ccccccc}
0 \to (F_{i-1}A^*)^k \otimes A^j & \xrightarrow{\iota_{A^*,i} \otimes id_{A^j}} & (F_i A^*)^k \otimes A^j & \xrightarrow{\rho_{A^*,i} \otimes id_{A^j}} & E_i^k A^* \otimes A^j \to 0 \\
\downarrow \mu_{i-1}^{k,j} & & \downarrow \mu_i^{k,j} & & \downarrow \bar{\mu}_i^{k,j} \\
0 \to (F_{i-j-1}A^*)^{j+k} & \xrightarrow{\iota_{A^*,\, i-j}} & (F_{i-j}A^*)^{j+k} & \xrightarrow{\rho_{A^*,\, i-j}} & E_{i-j}^{j+k} A^* \to 0
\end{array}$$

If moreover $\mathfrak{F}$ satisfies $(f5)$, since μ maps $A^* \otimes_{K*} F_{i-j-1}A^*$ into $F_{i-j-1}A^*$, $\bar{\mu}_i^{k,j}$ maps $E_i^k A^* \otimes_{K^*} (F_{i-j-1}A^*)^j$ to zero. Hence the exists unique map $\tilde{\mu}_i^{k,j} : E_i^k A^* \otimes_{K^*} (A^*/F_{i-j-1}A^*)^j \to E_{i-j}^{j+k}A^*$ that make the following diagram commute.

$$\begin{array}{ccc} E_i^k A^* \otimes_{K^*} A^j & \xrightarrow{id_{E_i^k A^*} \otimes_{K^*} \pi_{A^*,\, i-j}} & E_i^k A^* \otimes_{K^*} (A^*/F_{i-j-1}A^*)^j \\ & \searrow^{\bar{\mu}_i^{k,\,j}} & \Big\downarrow{\scriptstyle \tilde{\mu}_i^{k,\,j}} \\ & & E_{i-j}^{j+k} A^* \end{array}$$

Remark

Assume that $\mathfrak{F}$ satisfies $(f5)$ and $(f6)$. $\mathfrak{F}$ satisfies $(f7)$ if and only if

$$\tilde{\mu}_i^{c_{\mathfrak{F}}(i),j} : E_i^{c_{\mathfrak{F}}(i)} A^* \otimes_{K^*} (A^*/F_{i-j-1}A^*)^j \to E_{i-j}^{j+c_{\mathfrak{F}}(i)} A^* \quad \cdots \ (*)$$

is surjective for $i \in S(\mathfrak{F})$, $j \in \boldsymbol{Z}$. $\mathfrak{F}$ satisfies $(f8)$ if and only if $(*)$ is injective for $i \in S(\mathfrak{F})$, $j \in \boldsymbol{Z}$. Thus $\mathfrak{F}$ satisfies $(f7)$ and $(f8)$ if and only if $(*)$ is an isomorphism for $i \in S(\mathfrak{F})$, $j \in \boldsymbol{Z}$.

Let A^* be a K^*-algebra with an increasing filtration $\mathfrak{F} = (F_i A^*)_{i \in \boldsymbol{Z}}$. Suppose that $\mathfrak{F}$ satisfies $(f3)$, $(f4)$, $(f5)$, $(f6)$, $(f7)$ and $(f8)$. For an unstable A^*-module M^*, define a left A^*-module ΦM^* as follows. Put

$$\Phi M^* = \sum_{i \in S(\mathfrak{F})} E_i^{c_{\mathfrak{F}}(i)} A^* \otimes_{K^*} M^i.$$

In other words, $(\Phi M^*)^k = \{0\}$ if $k \neq i + c_{\mathfrak{F}}(i)$ for any $i \in S(\mathfrak{F})$ and $(\Phi M^*)^k = E_i^{c_{\mathfrak{F}}(i)} A^* \otimes_{K^*} M^i$ if $k = i + c_{\mathfrak{F}}(i)$ for $i \in S(\mathfrak{F})$ which is uniquely determined by $(f4)$.

Since $F_i A^*$ is a left ideal of A^* by $(f5)$, $E_i^* A^*$ is a left A^*-module and we denote by $\mu_i : A^* \otimes_{K^*} E_i^* A^* \to E_i^* A^*$ the left A^*-module structure.

Let $\alpha : A^* \otimes_{K^*} M^* \to M^*$ be the A^*-module structure map of M^*. Since M^* is unstable, α induces $\alpha_i : A^*/F_{i-1}A^* \otimes_{K^*} M^i \to M^*$.

We define maps $\alpha_{j,k} : A^j \otimes_{K^*} (\Phi M^*)^k \to (\Phi M^*)^{j+k}$ for $j, k \in \boldsymbol{Z}$ below.

Since $(\Phi M^*)^k = \{0\}$ if $k \neq i + c_{\mathfrak{F}}(i)$ for any $i \in S(\mathfrak{F})$, $\alpha_{j,k}$ should be trivial $k \neq i + c_{\mathfrak{F}}(i)$ for any $i \in S(\mathfrak{F})$ or $j + k \neq s + c_{\mathfrak{F}}(s)$ for any $s \in S(\mathfrak{F})$.

If there exist $i, s \in S(\mathfrak{F})$ which satisfy $k = i + c_{\mathfrak{F}}(i)$ and $j + k = s + c_{\mathfrak{F}}(s)$, then such i and s are unique by $(f4)$. In this case, define $\alpha_{j,k}$ to be the following composition.

$$
\begin{aligned}
A^j \otimes_{K^*} (\Phi M^*)^{i+c_{\mathfrak{F}}(i)} = A^j \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)} A^* \otimes_{K^*} M^i &\xrightarrow{\mu_i \otimes_{K^*} id_{M^i}} \\
E_i^{j+c_{\mathfrak{F}}(i)} A^* \otimes_{K^*} M^i = E_i^{s-i+c_{\mathfrak{F}}(s)} A^* \otimes_{K^*} M^i &\xrightarrow{(\tilde{\mu}_s^{c_{\mathfrak{F}}(s), s-i})^{-1} \otimes_{K^*} id_{M^i}} \\
E_s^{c_{\mathfrak{F}}(s)} A^* \otimes_{K^*} (A^*/F_{i-1}A^*)^{s-i} \otimes_{K^*} M^i &\xrightarrow{id_{E_s^{c_{\mathfrak{F}}(s)} A^*} \otimes_{K^*} \alpha_i} \\
E_s^{c_{\mathfrak{F}}(s)} A^* \otimes_{K^*} M^s &= (\Phi M^*)^{s+c_{\mathfrak{F}}(s)}
\end{aligned}
$$

Let us denote by $\alpha_\Phi : A^* \otimes_{K^*} \Phi M^* \to \Phi M^*$ the map induced by $\alpha_{j,k}$'s which gives a left A^*-module strucutre of M^*. Since μ_i maps $F_{i+c_{\mathfrak{F}}(i)-1} A^* \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)} A^*$ into $\{0\}$ by $(f6)$, ΦM^* is an unstable A^*-module.

For a homomorphism $\varphi : M^* \to N^*$ between unstable A^*-modules, let $\Phi\varphi : \Phi M^* \to \Phi N^*$ be the map induced by $id_{E_i^{c_{\mathfrak{F}}(i)} A^*} \otimes_{K^*} \varphi$. Then, $\Phi\varphi$ is a homomorphism of left A^*-modules and Φ is an endofunctor of $\mathcal{UM}od(A^*)$.

For an unstable A^*-module M^* with structure map $\alpha : A^* \otimes_{K^*} M^* \to M^*$, let $\bar{\alpha}_i^j : E_i^j A^* \otimes_{K^*} M^i \to M^{i+j}$ be a restriction of $\alpha_i : A^*/F_{i-1}A^* \otimes_{K^*} M^i \to M^*$. We define a map $\lambda_{M^*} : \Phi M^* \to M^*$ as follows.

If $k = i + c_{\mathfrak{F}}(i)$ for $i \in S(\mathfrak{F})$, we put

$$\lambda^k_{M^*} = \bar{\alpha}_i^{c_{\mathfrak{F}}(i)} : (\Phi M^*)^k = E_i^{c_{\mathfrak{F}}(i)} A^* \otimes_{K^*} M^i \to M^{i+c_{\mathfrak{F}}(i)} = M^k.$$

If $k \neq i + c_{\mathfrak{F}}(i)$ for any $i \in S(\mathfrak{F})$, $\lambda^k_{M^*} : (\Phi M^*)^k \to M^k$ is the trivial map. Let λ_{M^*} be the map induced by $\lambda^k_{M^*}$'s.

Proposition

λ_{M^*} is a homomorphism of left A^*-modules and natural in M^*.

Let us denote by $\tilde{\iota}_{A^*,n} : A^*/F_{n-1}A^* \to A^*/F_nA^*$ the quotient map. For a K^*-module M^*, we define a map $\sigma_{M^*} : \mathcal{F}(M^*) \to \Sigma^{-1}\mathcal{F}(\Sigma M^*)$ by $\sigma_{M^*}(x \otimes y) = ([-1], \tilde{\iota}_{A^*,n}(x) \otimes ([1], y))$ for $x \in A^*/F_{n-1}A^*$ and $y \in M^*$.

Proposition

Let M^* be a graded K^*-module. If $\mathfrak{F}$ satisfies $(f3)$ and $(f5) \sim (f8)$, the following is a short exact sequence.

$$0 \to \Phi\mathcal{F}(M^*) \xrightarrow{\lambda_{\mathcal{F}(M^*)}} \mathcal{F}(M^*) \xrightarrow{\sigma_{M^*}} \Sigma^{-1}\mathcal{F}(\Sigma M^*) \to 0$$

Proof: Recall that $\mathcal{F}(M^*) = \sum_{n\in \boldsymbol{Z}} A^*/F_{n-1}A^* \otimes_{K^*} M^n$. Hence we have

$$\begin{aligned}
\Phi\mathcal{F}(M^*) &= \sum_{i\in S(\mathfrak{F})} E_i^{c_{\mathfrak{F}}(i)} A^* \otimes_{K^*} \mathcal{F}(M^*)^i \\
&= \sum_{i\in S(\mathfrak{F})} \sum_{n\in \boldsymbol{Z}} E_i^{c_{\mathfrak{F}}(i)} A^* \otimes_{K^*} (A^*/F_{n-1}A^*)^{i-n} \otimes_{K^*} M^n.
\end{aligned}$$

By $(f3)$, $(f7)$ and $(f8)$, $\lambda_{\mathcal{F}(M^*)}$ is an injection onto $\sum_{n\in \boldsymbol{Z}} E_n^* A^* \otimes_{K^*} M^n$, which is the kernel of σ_{M^*}.

Proposition

Let M^* be an unstable A^*-module. If $\mathfrak{F}$ satisfies $(f1) \sim (f8)$, then $\Sigma \mathrm{Coker}\, \lambda_{M^*}$ is an unstable A^*-module.

Proof :
Let $\alpha : A^* \otimes_{K^*} M^* \to M^*$ the structure map of M^*. Since

$$\mathrm{Im}\, \lambda_{M^*}^{i+c_{\mathfrak{F}}(i)} = \alpha((F_i A^*)^{c_{\mathfrak{F}}(i)} \otimes_{K^*} M^i),$$

we have $(F_i A^*)^{c_{\mathfrak{F}}(i)}(\mathrm{Coker}\, \lambda_{M^*})^i = \{0\}$ for $i \in S(\mathfrak{F})$.
If $i \in S(\mathfrak{F})$ and $k > i$, the instability of M^* and the first proposition in this section imply $(F_i A^*)^{c_{\mathfrak{F}}(i)}(\mathrm{Coker}\, \lambda_{M^*})^k = \{0\}$.

Suppose that $\mathfrak{F}$ satisfies $(f1) \sim (f8)$ for the rest of this section unless otherwise stated.
Define a functor $\Omega : \mathcal{UM}od(A^*) \to \mathcal{UM}od(A^*)$ as follows.

For an objevt M^* of $\mathcal{UM}od(A^*)$, we put $\Omega M^* = \Sigma \mathrm{Coker} \lambda_{M^*}$ and denote by $\tilde{\eta}_{M^*} : M^* \to \mathrm{Coker}\, \lambda_{M^*} = \Sigma^{-1}\Omega M^*$ the quotient map.
It follows from the above proposition that ΩM^* is an object of $\mathcal{UM}od(A^*)$.

For a morphism $\varphi : M^* \to N^*$ of $\mathcal{UM}od(A^*)$, there exists unique map $\bar{\varphi} : \mathrm{Coker}\, \lambda_{M^*} \to \mathrm{Coker}\, \lambda_{N^*}$ that makes the following diagram commute.
We put $\Omega\varphi = \Sigma\bar{\varphi} : \Omega M^* \to \Omega N^*$.

$$\begin{array}{ccccccc}
\Phi M^* & \xrightarrow{\lambda_{M^*}} & M^* & \xrightarrow{\tilde{\eta}_{M^*}} & \mathrm{Coker}\, \lambda_{M^*} = \Sigma^{-1}\Omega M^* & \longrightarrow & 0 \\
\downarrow{\scriptstyle \Phi\varphi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \bar{\varphi}} & & \\
\Phi N^* & \xrightarrow{\lambda_{N^*}} & N^* & \xrightarrow{\tilde{\eta}_{N^*}} & \mathrm{Coker}\, \lambda_{N^*} = \Sigma^{-1}\Omega N^* & \longrightarrow & 0
\end{array}$$

Proposition

Ω is a left adjoint of the desuspension functor Σ^{-1}.

Remark

Since $\sigma_{M^*} : \mathcal{F}(M^*) \to \Sigma^{-1}\mathcal{F}(\Sigma M^*)$ *is a cokernel of* $\lambda_{\mathcal{F}(M^*)}$ *for a graded* K^**-module* M^**,* $\Omega\mathcal{F}(M^*)$ *is identified with* $\mathcal{F}(\Sigma M^*)$*.*

Lemma

Assume that a filtration $\mathfrak{F} = (F_i A^*)_{i\in \mathbf{Z}}$ *satisfies* $(f5)$*,* $(f6)$*,* $(f7)$ *and* $(f8)$*. If* $i, i + c_{\mathfrak{F}}(i) \in S(\mathfrak{F})$*, the following composition maps* $(F_{i+c_{\mathfrak{F}}(i)}A^*)^{c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i))} \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)}A^*$ *onto* $E_{i+c_{\mathfrak{F}}(i)}^{c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i))}A^* \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)}A^*$*.*

$$A^{c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i))} \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)}A^* \xrightarrow{\mu_i} E_i^{c_{\mathfrak{F}}(i)+c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i))}A^*$$
$$\xrightarrow{\left(\tilde{\mu}_{i+c_{\mathfrak{F}}(i)}^{c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i)),c_{\mathfrak{F}}(i)}\right)^{-1}} E_{i+c_{\mathfrak{F}}(i)}^{c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i))}A^* \otimes_{K^*} (A^*/F_{i-1}A^*)^{c_{\mathfrak{F}}(i)}$$

Proposition

Let M^* *be an unstable* A^**-module.* $\Sigma \mathrm{Ker}\, \lambda_{M^*}$ *is an unstable* A^**-module.*

Define a functor $\Omega^1 : \mathcal{UM}od(A^*) \to \mathcal{UM}od(A^*)$ as follows.

For an object M^* of $\mathcal{UM}od(A^*)$, we put $\Omega^1(M^*) = \Sigma \mathrm{Ker}\, \lambda_{M^*}$ and denote by $\iota_{M^*} : \mathrm{Ker}\, \lambda_{M^*} \to \Phi M^*$ the inclusion map.
The above proposition implies that $\Omega^1 M^*$ is an object of $\mathcal{UM}od(A^*)$.

For a morphism $\varphi : M^* \to N^*$ of $\mathcal{UM}od(A^*)$, there exists unique map $\hat{\varphi} : \mathrm{Ker}\, \lambda_{M^*} \to \mathrm{Ker}\, \lambda_{N^*}$ that makes the following diagram commute.
We put $\Omega^1\varphi = \Sigma\hat{\varphi} : \Omega^1 M^* \to \Omega^1 N^*$.

$$\begin{array}{ccccccc}
0 \longrightarrow \Sigma^{-1}\Omega^1 M^* = \mathrm{Ker}\, \lambda_{M^*} & \xrightarrow{\iota_{M^*}} & \Phi M^* & \xrightarrow{\lambda_{M^*}} & M^* \\
\downarrow \hat{\varphi} & & \downarrow \Phi\varphi & & \downarrow \varphi \\
0 \longrightarrow \Sigma^{-1}\Omega^1 N^* = \mathrm{Ker}\, \lambda_{N^*} & \xrightarrow{\iota_{N^*}} & \Phi N^* & \xrightarrow{\lambda_{N^*}} & N^*
\end{array}$$

Proposition

Ω^1 is the first left derived functor of Ω and all the higher derived functors of Ω are trivial.

Proof:
Let $M^* \xleftarrow{\varepsilon_{M^*}} B_0^* \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_{n-1}} B_{n-1}^* \xleftarrow{\partial_n} B_n^* \xleftarrow{\partial_{n+1}} \cdots$ be the bar resolusion of M^*.
Consider chain complexes $B. = (B_n^*, \partial_n)_{n \in \boldsymbol{Z}}$, $\Phi B. = (\Phi B_n^*, \Phi(\partial_n))_{n \in \boldsymbol{Z}}$ and $\Sigma^{-1}\Omega B. = (\Sigma^{-1}\Omega B_n^*, \Sigma^{-1}\Omega(\partial_n))_{n \in \boldsymbol{Z}}$. We denote by $\lambda. : \Phi B. \to B.$ and $\tilde{\eta}. : B. \to \Sigma^{-1}\Omega B.$ the chain maps given by $\lambda_{B_n^*}$'s and $\tilde{\eta}_{B_n^*}$'s, respectively.
$\Phi B_n^* \xrightarrow{\lambda_{B_n^*}} B_n^* \xrightarrow{\tilde{\eta}_{B_n^*}} \Sigma^{-1}\Omega B_n^* \to 0$ is exact by the definition of ΩB_n^*. Since $B_n^* = \mathcal{F}(M_n^*)$ for some graded K^*-module, $\lambda_{B_n^*} = \lambda_{\mathcal{F}(M_n^*)}$ is injective by the previous proposition. Hence $0 \to \Phi B_n^* \xrightarrow{\lambda_{B_n^*}} B_n^* \xrightarrow{\tilde{\eta}_{B_n^*}} \Sigma^{-1}\Omega B_n^* \to 0$ is exact. Thus we have a short exact sequence of complexes

$$0 \to \Phi B. \xrightarrow{\lambda.} B. \xrightarrow{\tilde{\eta}.} \Sigma^{-1}\Omega B. \to 0.$$

Consider the long exact sequence associated with this short exact sequence. Clearly, Φ is an exact functor. We deduce that $\Sigma^{-1}H^n(\Omega B.) = H^n(\Sigma^{-1}\Omega B.)$ is trivial and that there is an exact sequence

$$0 \to \Sigma^{-1}H^1(\Omega B.) = H^1(\Sigma^{-1}\Omega B.) \to \Phi M_* \xrightarrow{\lambda_{M^*}} M^* \xrightarrow{\tilde{\eta}_{M^*}} \Sigma^{-1}\Omega M^* \to 0.$$

Thus $\Omega^n M^* = H^n(\Omega B.)$ is trivial if $n > 1$ and Ω^1 defined above is the first left derived functor of Ω.

Thank you for your patience.