Students' Mental Representations of Geometric Vectors and Sets Related to Linear Algebra

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Difficulty in Linear Algebra

"a cognitively and conceptually difficult subject" (Dorier&Sierpinska, 2001)

The use of embodied notions may help students to understand concepts in the formal theory of linear algebra. (cf. Stewart&Thomas, 2007)

Embodied world
(2 or 3-dim space)

Objects:

Symbolic world
(formal theory)

$$\boldsymbol{v}, \boldsymbol{w} \in V$$

$$\{s\boldsymbol{v} + t\boldsymbol{w} \mid s, t \in \mathbb{R}\}$$

$$\varphi : V \to W$$

$$\operatorname{Ker} \varphi$$

Research Questions

- **Topic:** The notions of linear independence and subspace

- **Research Questions**

  1. Do students have rich mental representations of geometric vectors related to the notion of linear independence?

  2. Do students have rich mental representations of sets related to the notion of subspace?

  3. Is students' difficulty in learning linear algebra related to the lack of mental representations of the above two?

Method

- Participants:
  107 first-year university engineering students
- A test with 5 problems (10 minutes):
  The test was conducted in linear algebra classes. Students' mental representations of geometric vectors and sets related to the notions of linear independence and subspace were assessed.
- Method of Analysis:
  Qualitative analysis of students' answers

Test Items

P1. Give 4 examples of elements of the following set.

$$\left\{ x \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

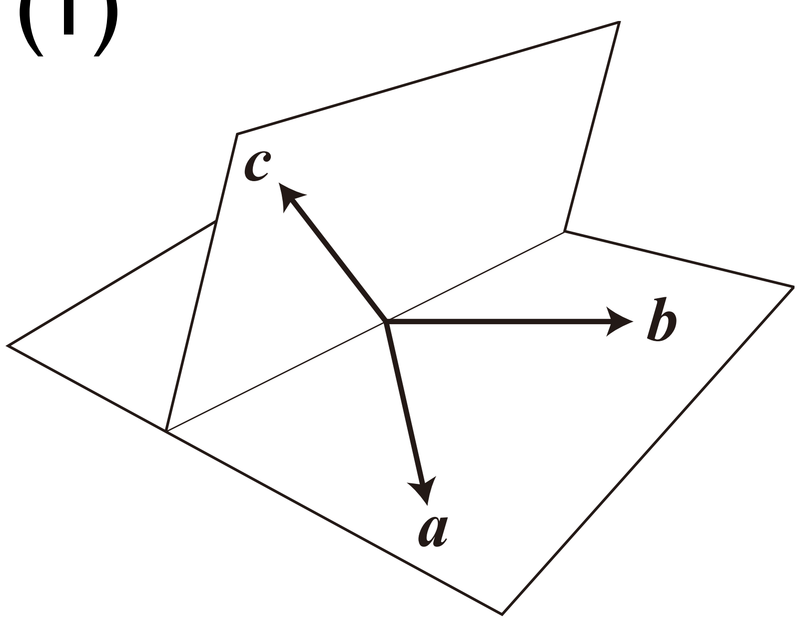
P2. Give 3 examples of elements of the following set.

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x + y + 2z = 0 \right\}$$

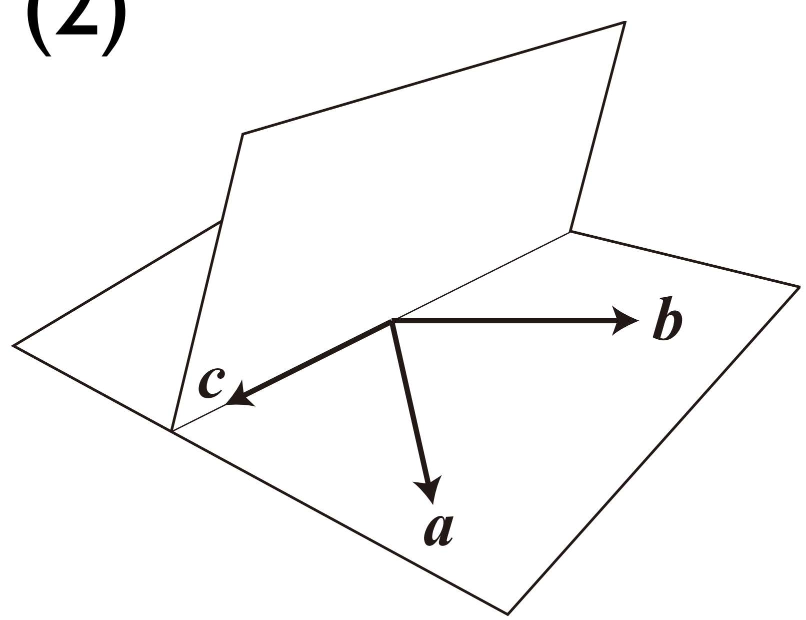
Test Items

P3. Determine whether vectors in the following pictures are linearly independent or not and give reasons. Moreover, when they are linearly dependent, give an maximal set of linearly independent vectors.

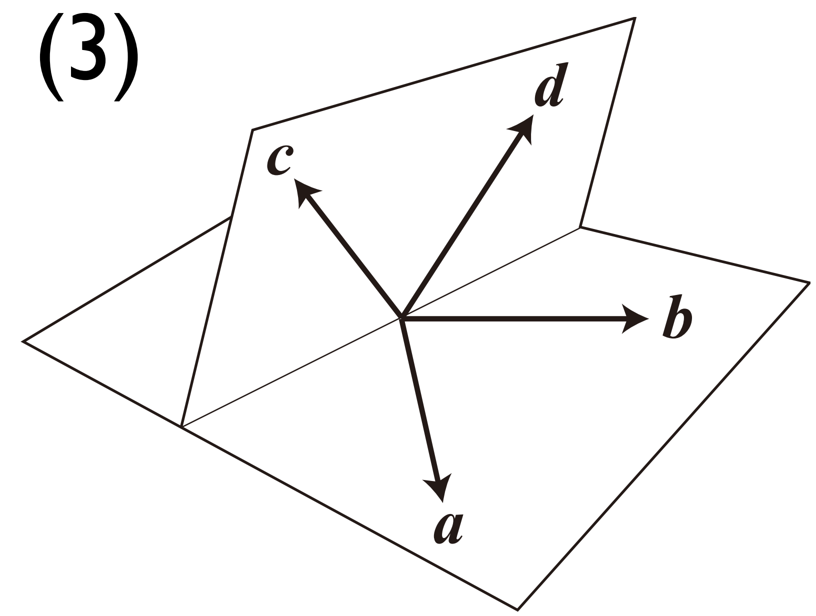
(1)



(2)



(3)



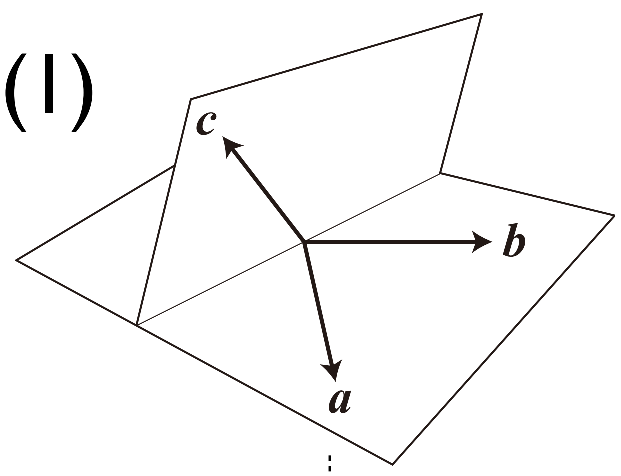
Results

P1 $$\left\{ x\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + y\begin{pmatrix}0\\0\\0\\1\end{pmatrix} + z\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \,\middle|\, x,y,z\in\mathbb{R} \right\}$$

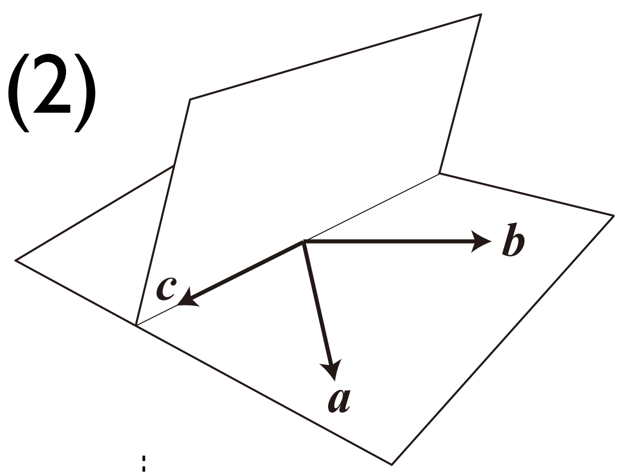
P2 $$\left\{ \begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3 \,\middle|\, x+y+2z=0 \right\}$$

	correct	computational error	conceptual error	no answer
P1	56.07%	2.80%	26.17%	14.95%
P2	74.77%	11.21%	2.80%	11.22%

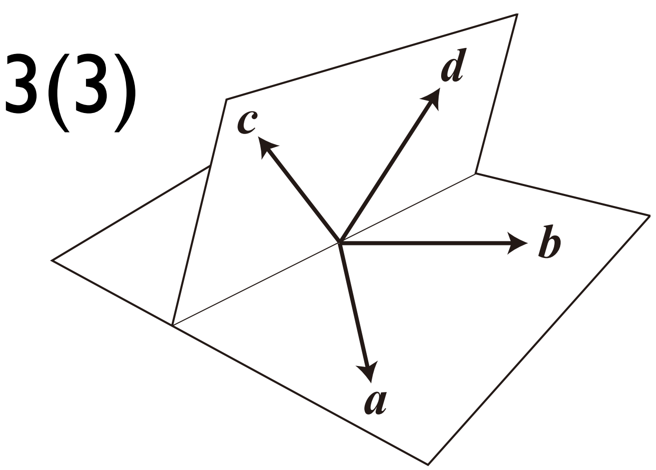
P3(1)



P3(2)



P3(3)



	correct	incorrect	no answer
P3(1)	85.05%	5.61%	9.35%
P3(2)	87.85%	3.74%	8.41%
P3(3)	62.62%	23.37%	14.02%

Results for determining linearly independence

Results

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41.1%

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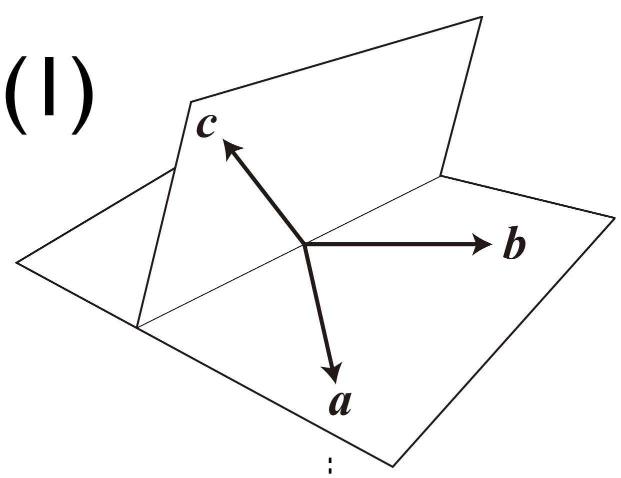
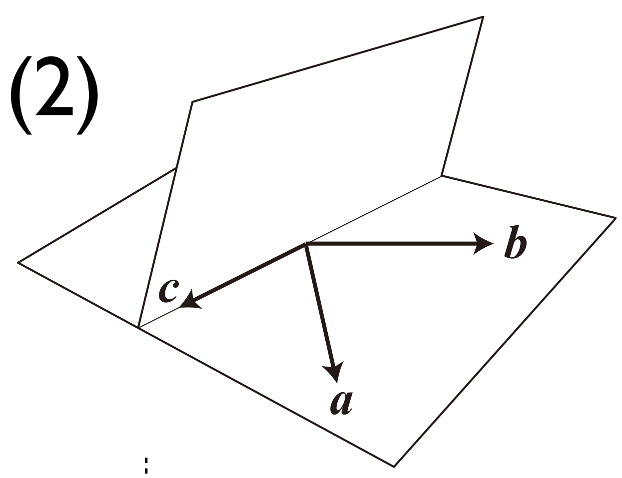


P3(2)



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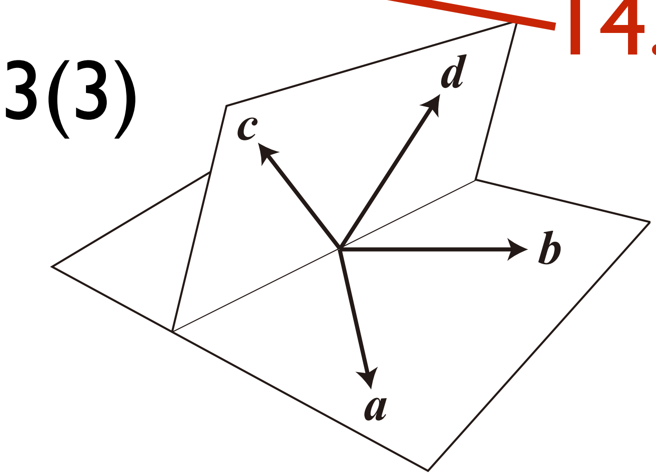

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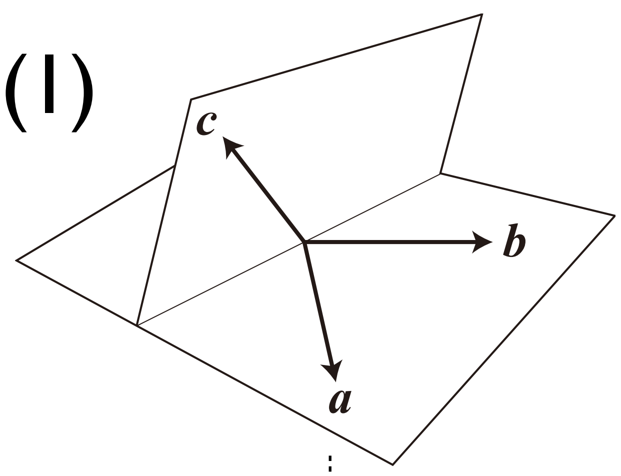
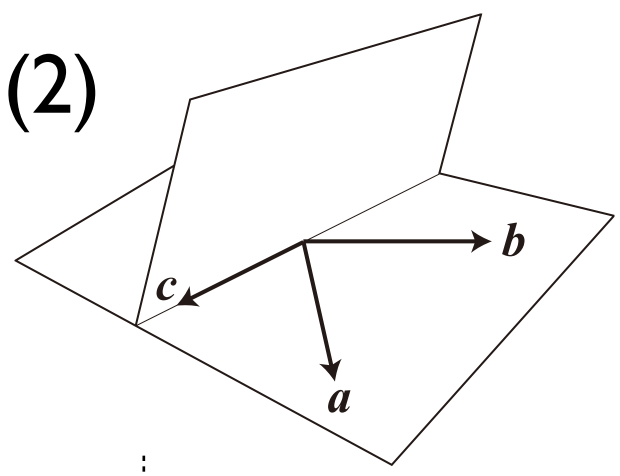
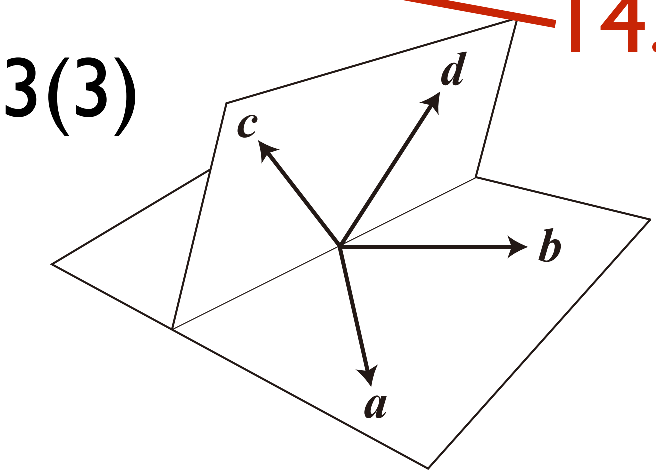


P3(2)



P3(3)



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P3(3)	62.62%	23.37%	14.02%	37.4%

Results for determining linearly independence

Errors: Giving Coefficients

P1 $$\left\{ x\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + y\begin{pmatrix}0\\0\\0\\1\end{pmatrix} + z\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

$(x, y, z) = (1, 1, 1), (1, 1, 2). (1, 1, 3)\ (1, 1, 4)$

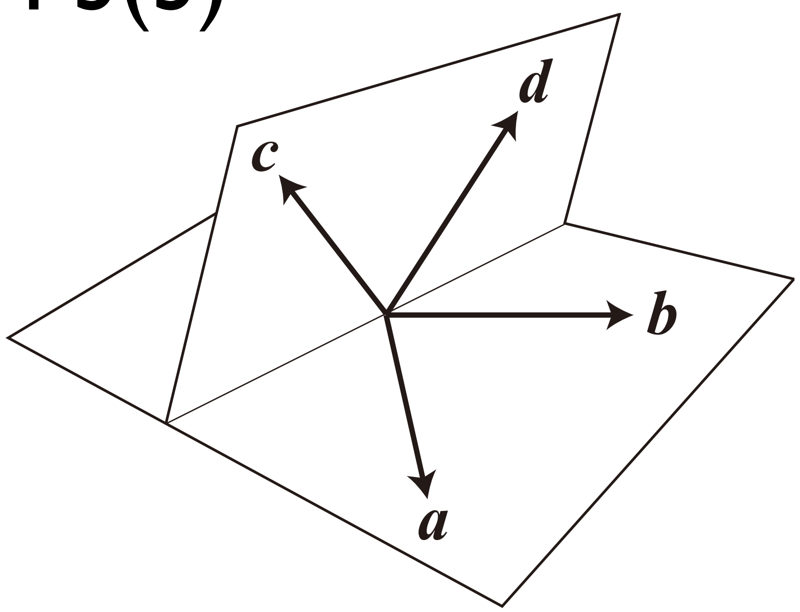
$x + z = 0 \qquad x = y$

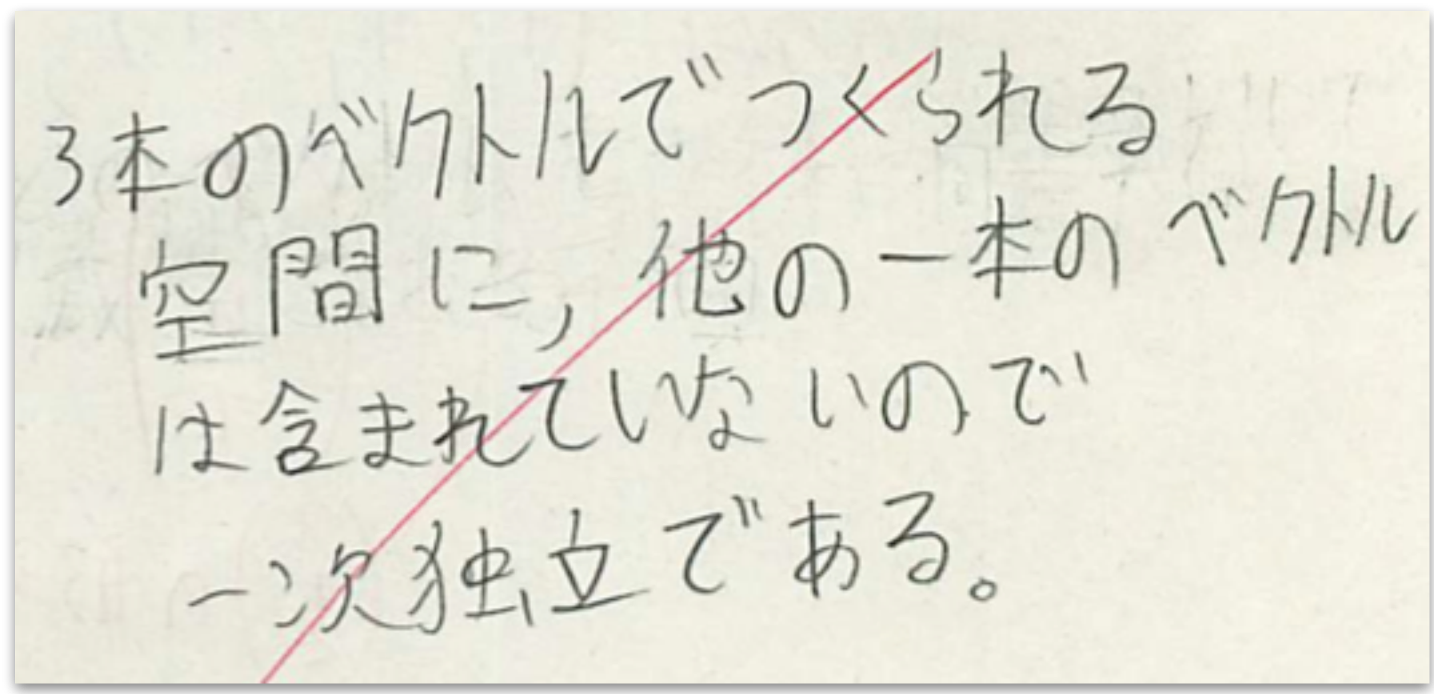
$y + z = 0 \qquad z = -x$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-3\\-3\\3\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}2\\2\\-2\end{pmatrix}, \begin{pmatrix}-4\\-4\\4\end{pmatrix}$$

Errors: Incorrect Image

P3(3)

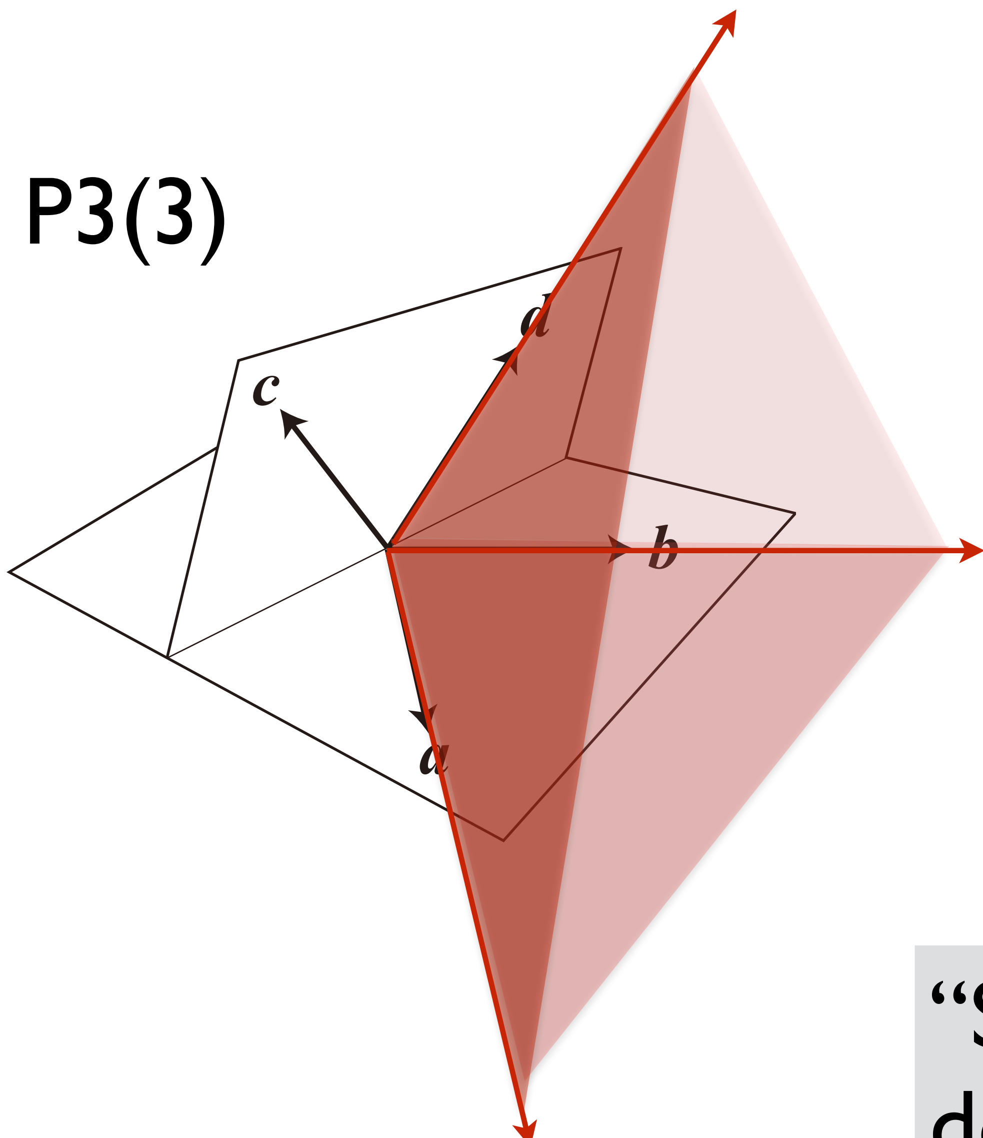


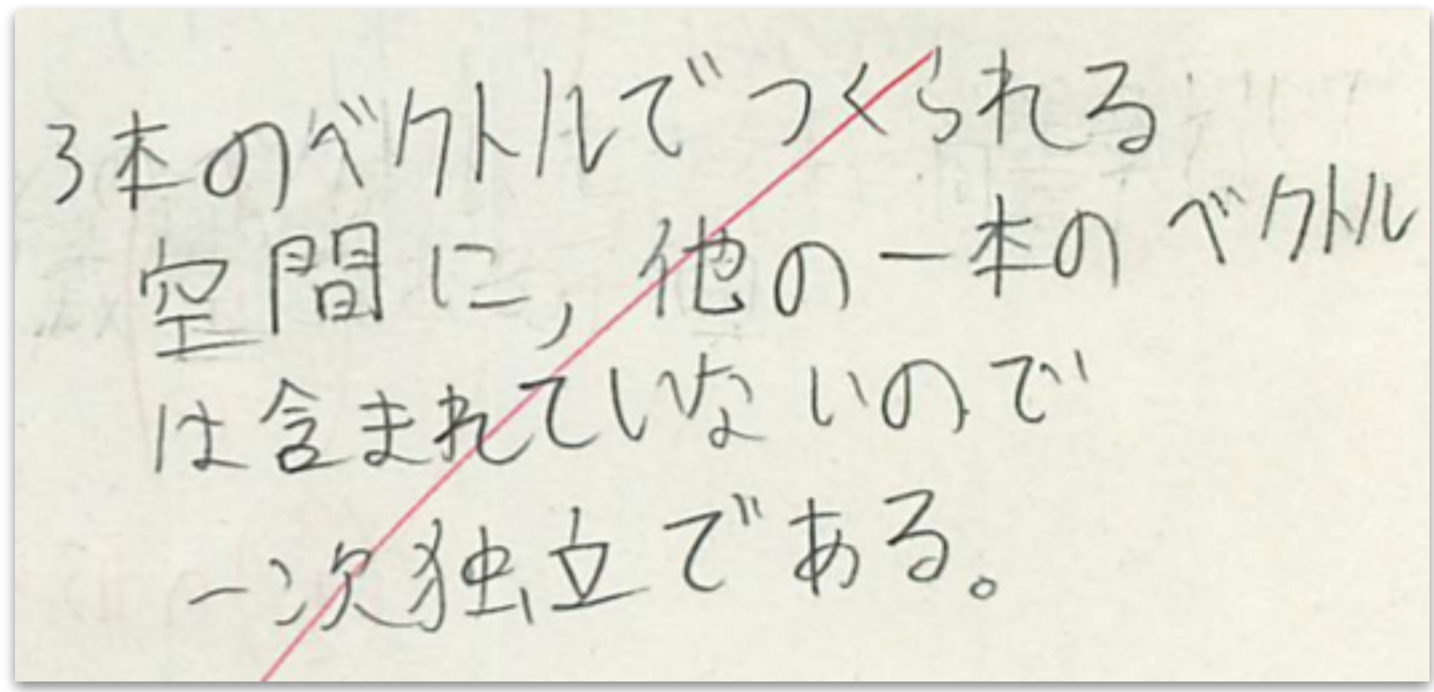

3本のベクトルでつくられる
空間に、他の一本のベクトル
は含まれていないので
一次独立である。

"Since the space generated by 3 vectors does not contain the other vector, these vectors are linearly independent."

Errors: Incorrect Image

P3(3)

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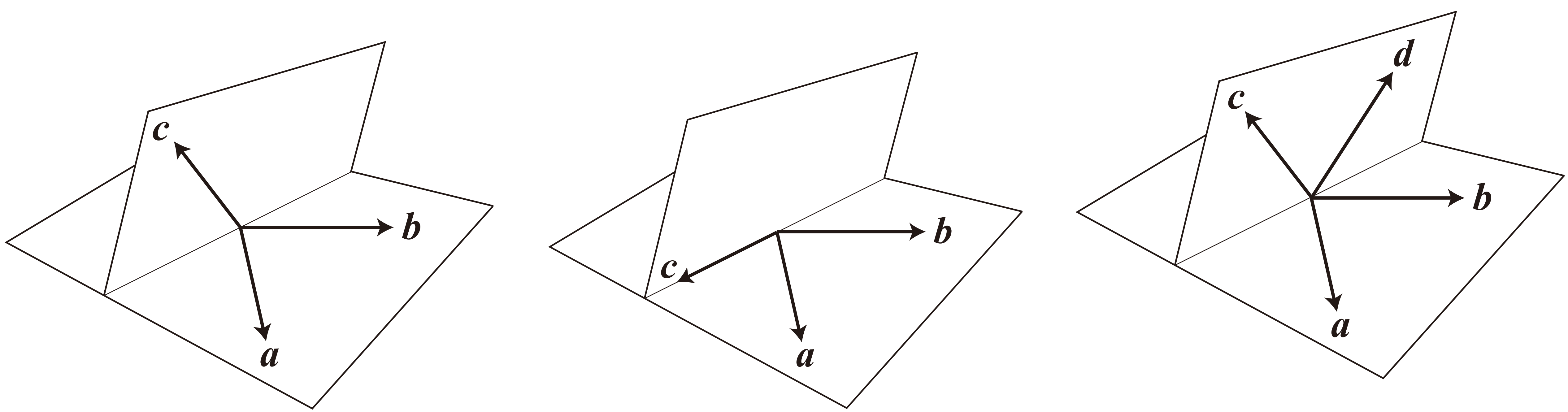

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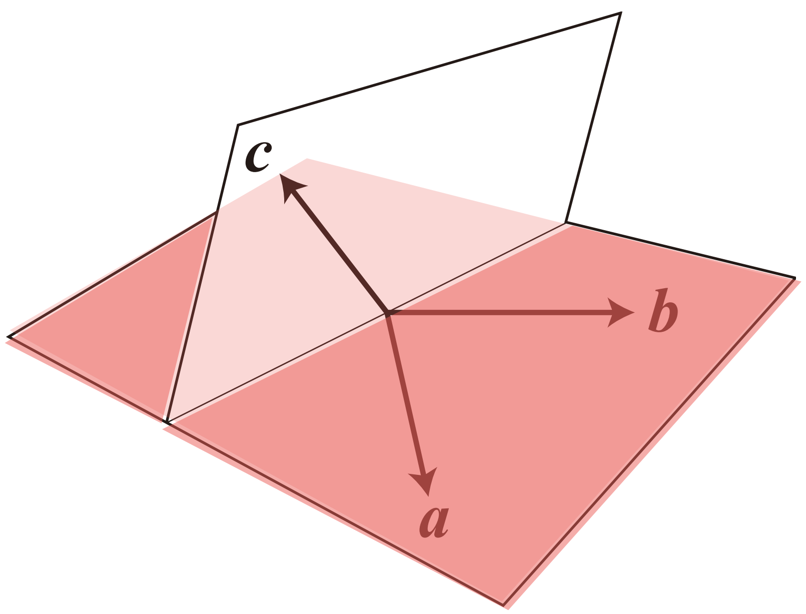
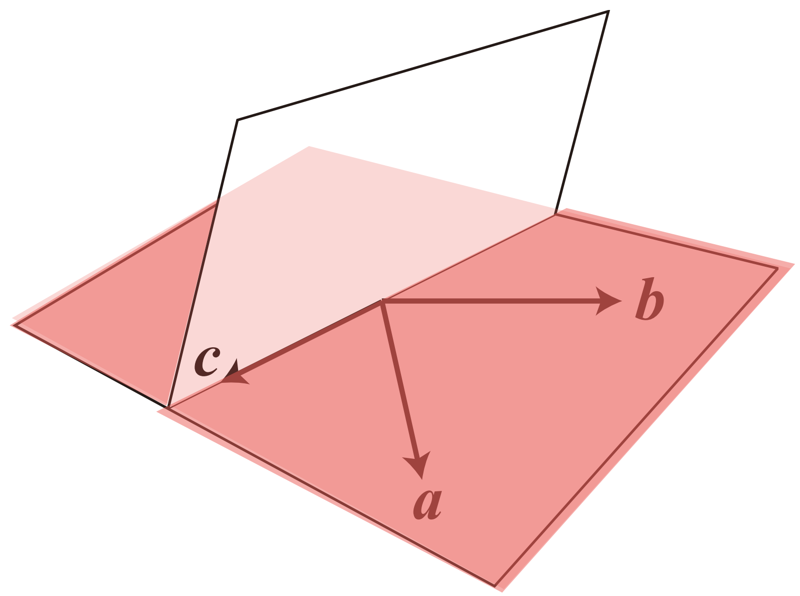
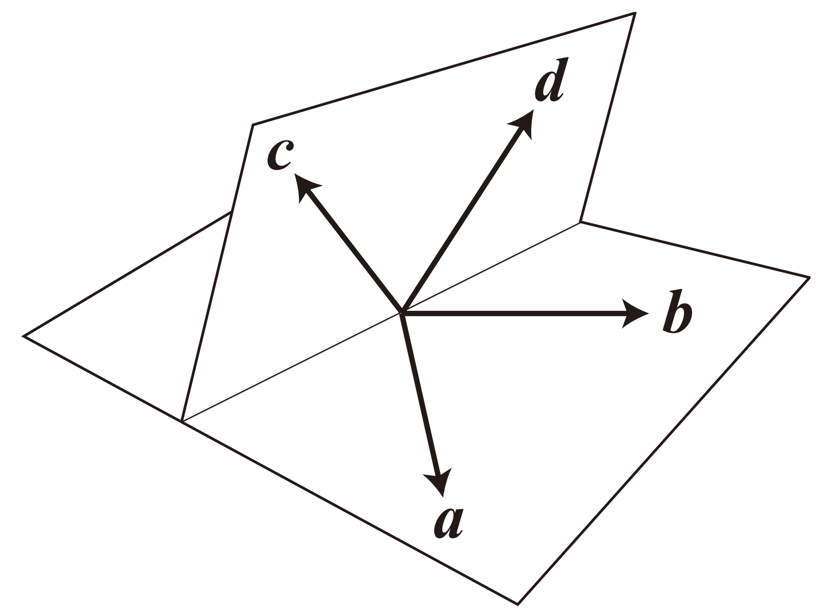




Two vectors generate a plane.



Three vectors generate ???

Discussion & Conclusion

- The results indicate that some students do not have rich mental representations of spanned spaces.
    - They do not have a correct image for the spanned space in terms of geometric vectors and in terms of sets.
    - They do not have an image that 3 geometric vectors not on the same plane span the whole space.
- Future research will address whether and how these mental representations influence the learning of linear algebra.