CHARACTER FORMULA FOR THE SUPERCUSPIDAL REPRESENTATIONS OF
\( \text{GL}_l, \ l \ \text{A PRIME} \neq p \)

TETSUYA TAKAHASHI

Abstract. The article gives a character formula for the irreducible supercuspidal representation of \( \text{GL}_l(F) \) for \( F \) a local field of the residual characteristic \( p \neq l \).

Introduction

Let \( l \) be a prime, \( A \) a central simple algebra of dimension \( l^2 \) over a non-archimedean local field \( F \) and \( E/F \) an extension of degree \( l \) in \( A \). Recall that any compact (mod center) Cartan subgroup of \( A^\times \) is isomorphic to \( E^\times \) for some extension \( E/F \) of degree \( l \). As is well-known ([15], [5]), any irreducible supercuspidal representation is obtained from a quasi-character of \( E^\times \). The aim of this paper is to get a character formula for the irreducible supercuspidal representations of \( A^\times \) on the set of elliptic regular elements in \( A^\times \). To calculate the character on the split torus is another problem. See Murnaghan’s papers ([16], [17] and [18]) for this topic. We only remark that the character value on the elliptic regular conjugacy classes determines uniquely the supercuspidal representation.

When \( E/F \) is unramified, a character formula was obtained in [21]. Therefore in this paper we treat the case \( E/F \) is ramified. When the residual characteristic \( p \) of \( F \) equals to \( l \), the ramification is wild. This case is very hard to treat (see e.g. [22]). In this paper, we assume \( p \neq l \). Since the case \( l = 2 \) was solved in [13], we assume \( l \) is odd. We note that \( A \) is isomorphic to a division algebra \( D = D_l \) of dimension \( l^2 \) over \( F \) or the algebra \( \text{M}_l(F) \) of \( l \times l \) matrices over \( F \).

Let \( D_n \) be a division algebra of dimension \( n^2 \) over \( F \). Deligne-Kazhdan-Vignéras [8] and Rogawski [20] proved an abstract matching theorem: there is a bijection between the set of equivalence classes of irreducible representations of \( D_n^\times \) and that of essentially square-integrable representations of \( \text{GL}_n(F) \) which preserves the characters up to \((-1)^{n-1}\). In the tame case, i.e., when \( n \) is prime to the residual characteristic of \( F \), Moy [15] has proved that there is a bijection between the same sets as above using the concrete construction of the representations given by Howe [12]. Henniart [10] has shown that two correspondences coincide when \( n = l \neq p \). Thus we only treat the \( \text{GL} \) case.

In the earlier paper [21], the author gave a character formula of the representation of \( \text{GL}_l(F) \) and \( D_l^\times \) which is obtained from a quasi-character of \( E^\times \) where \( E \) is an unramified extension of \( F \) with degree \( l \). (When \( l = 2 \), the character formula is given in [13]). Then we essentially use the fact that \( E/F \) is a Galois extension. In our case, we need to treat the case where \( E/F \) is non-Galois. In order to treat the non-Galois case, we use the
result of Bushnell-Henniart [3] on the base change lift of simple characters. Since the base
change lift is available only for GL case, we do not treat the division algebra case directly.
In addition, the formula of the character near the conductor becomes simpler than that of
the division algebra case (see Lemma 2.5 and Theorem 4.2 (d) in [6]). Our main result is
Theorem 3.12. As in the unramified case, the analogue of Weyl’s character formula holds
for our formula. This does not hold when l = p (cf. [23]).

Section 1 is devoted to the review of the construction of an irreducible supercuspidal
representation πθ (resp. π′ θ) of GLd(F) (resp. Dl) from a generic quasi-character θ of E×
and the known results about the representation. We note that πθ is not always monomial,
and we note that πθ is not always monomial, i.e., induced from a one-dimensional representation, but it can be written as a Q-linear combination of monomial representations. In fact πθ is written as a Q-linear combination of the forms indH GL(F) ρ θ where H is a compact mod center subgroup of GLd(F) and ρ θ is a quasi-characters of H.

In section 2, we compute the character of πθ up to some root numbers. Let G = GLd(F),
B the normalizer of an Iwahori subgroup of G containing H and ηθ = indB H ρ θ. Since we
treat only elliptic regular conjugacy classes, we consider the character χηθ on L × where
L/F are extensions of fields of degree l. Moreover the case L = E is essential. By the
Frobenius formula and the result of Kutzko ([14]), we have only to calculate the sum
\[ \chi_{\eta\theta}(x) = \sum_{a \in H \backslash B} \rho \theta(axa^{-1}) \]
for x ∈ E in order to get the character formula of πθ. Therefore it is essential to know
when axa−1 ∈ H, which is determined in Lemma 2.1. From this, we get the character
formula of ηθ except near the conductor (Proposition 2.2). But this formula contains the
Gauss sum part G(y, j), which is calculated later. The exceptional part can be calculated
directly by taking the explicit matrix form of E× (Lemma 2.4). Except this lemma, there is
no new result in this section. But the proof becomes short and simple. Moreover since
we use the property “intertwining implies conjugacy ” of E/F-minimal (very cuspidal in the
terminology of Carayol [4]) element as the key tool, the result may be extended to
GLn, at least when n is prime to p. Section 3 is devoted to the calculation of the Gauss
sum part G(y, j). It appears in the character formula on E×. For this purpose, it is the
point that we have only to treat the character of πθ on U1* = F×(1 + PE) − F×(1 + P2 E).
For this calculation, we use the E×-module structure of various objects. We first assume
E/F is a Galois extension since E×-module structure can be described easily for this
case. This part is analogous to section 1 of [21], but everything becomes easier since we
have only to treat U1*. When E/F is non-Galois, we use the base change lift. Let ζ be a
primitive l-th root of unity and L = F(ζ). Then L is an unramified extension of F
and EL/L is Galois. Therefore we can use the tools of the Galois case for GLd(L). Let
Gal(L/F) = ⟨τ⟩. By the result of Bushnell-Henniart [3], there is a base change lift ηL of
ηθ to H1 such that the twisted trace of ηL by τ gives the trace of ηθ. (See Proposition 3.7
and Lemma 3.8). We remark that we need not assume the characteristic of F is 0 since we
do not use the Arthur-Clozel base change lift [1]. The method of calculating the twisted
trace of ηL is similar to that of Galois case. The complete character formula is stated as
Theorem 3.12.

At the end of this introduction, we compare our formula with the known results. The
same type of character formula for the division algebra case was given by Corwin, Moy
and Sally, Jr. [6] and for GLd case by Debacker [7]. Their formulas agree with the result
given in section 2. It contains some root number associated with a quadratic form. They have shown that this root number is a root of unity when $p \neq 2$. In this paper, we have determined it completely including the case $p = 2$ in section 3. Moreover we find the Kloosterman sum appears in the character formula. These are new results of this paper. In [23], the author gave the character formula of $\pi_\theta$ for GL$_3$ by using the decomposition of $\pi_\theta$ as $E^\times$-module. But this needs the explicit matrix form of an inverse matrix which is hard to treat for large $l$. We can simplify the proof of the main theorem, although we treat a general prime $l$.

**Notation**

Let $F$ be a non-archimedean local field. We denote by $O_F$, $P_F$, $\varpi_F$, $k_F$ and $v_F$ the maximal order of $F$, the maximal ideal of $O_F$, a prime element of $P_F$, the residue field of $F$ and the valuation of $F$ normalized by $v_F(\varpi_F) = 1$. We set $q$ to be the number of elements in $k_F$. Hereafter we fix an additive character $\psi$ of $F$ whose conductor is $P_F$, i.e., $\psi$ is trivial on $P_F$ and not trivial on $O_F$. For an extension $E$ over $F$, we denote by $\text{tr}_E$, $n_E$ the trace and norm to $F$ respectively. We set $\psi_E = \psi \circ \text{tr}_E$. The trace of matrix is denoted by $\text{Tr}$. For an irreducible admissible representation $\pi$ of GL$_l(F)$, the conductoral exponent of $\pi$ is defined to be the integer $f(\pi)$ such that the local constant $\varepsilon(s, \pi, \psi)$ of Godement-Jacquet [9] is the form $aq^{-k(f(\pi)-l)}$.

We call $\pi$ **minimal** if

$$f(\pi) = \min_{\eta} f(\pi \otimes (\eta \circ \text{Nr})),$$

where $\eta$ runs through the quasi-characters of $F^\times$. Let $G$ be a totally disconnected, locally compact group. We denote by $\hat{G}$ the set of (equivalence classes of) irreducible admissible representations of $G$. For a closed subgroup $H$ of $G$ and a representation $\rho$ of $H$, we denote by $\text{Ind}^G_H \rho$ (resp. $\text{ind}^G_H \rho$) the induced representation (resp. compactly induced representation) of $\rho$ to $G$. For a representation $\pi$ of $G$, we denote by $\pi|_H$ the restriction of $\pi$ to $H$.

1. **Construction of the representation**

Let $l \neq p$ be an odd prime and $E$ a ramified extension of $F$ of degree $l$. Then $E$ can be embedded into $M_l(F)$ and, up to conjugacy, the embedding is unique. Let $G = \text{GL}_l(F)$. In this section, we review the construction of supercuspidal representations of $G$ which are parameterized by the quasi-characters of $E^\times$. Of course, this construction is well-known ([4], [15]).

**Definition 1.1.** Let $\theta$ be a quasi-character of $E^\times$ and $f(\theta)$ the exponent of the conductor of $\theta$ i.e. the minimum integer such that $\text{Ker} \theta \subset 1 + P^l_E$. Then $\theta$ is called generic if $f(\theta) \neq 1$ mod $l$. For a generic character $\theta$ of $E^\times$, $\gamma_\theta \in P^{l-f(\theta)}_E - P^{l-2f(\theta)}_E$ is defined by

\begin{equation}
(1.1) \quad \theta(1 + x) = \psi_E(\gamma_\theta x) \quad \text{for} \quad x \in P^{f(\theta)+1)/2}_E.
\end{equation}

Then $F(\gamma_\theta) = E$. We denote by $\hat{E}_\text{gen}^\times$ the set of generic quasi-characters of $E^\times$.

We construct an irreducible supercuspidal representation of $G = \text{GL}_l(F)$ from $\theta \in \hat{E}_\text{gen}^\times$. For simplicity, we set $\gamma = \gamma_\theta$. Since $E/F$ is tamely ramified, there exists a prime element $\varpi_E$ of $O_E$ satisfying $\varpi_E^l \in F$. Put $\varpi_F = \varpi_E^l$. We identifies $M_l(F)$ with $\text{End}_F E$ and $G$ with $\text{Aut}_F E$ by the $F$-basis $\{\varpi_E^{-1}, \varpi_E^{-2}, \ldots, \varpi_E, 1\}$ of $E$, which is also an $O_F$-basis of $O_E$. 
By the lattice flag $\{P_E^i\}_{i \in \mathbb{Z}}$, we construct a maximal compact modulo center subgroup. The construction of the representation is well-known. For details, see [15].

**Definition 1.2.** For $i \in \mathbb{Z}$, set

$$A^i = \{ f \in M_i(F) \mid f(P_E^j) \subset P_E^{j+i} \text{ for all } j \in \mathbb{Z} \}.$$ 

Put $K = (A^0)^\times, B = E^\times K$ and $K^i = 1 + A^i$ for $i \geq 1$.

Then $K$ is an Iwahori subgroup of $G$ and $B$ is a normalizer of $K$. At first we construct an irreducible representation of $B$ from a generic quasi-character of $E^\times$.

Let $\theta$ be a generic quasi-character of $E^\times$, i.e., $f(\theta) = n \neq 1 \mod l$. There exists an element $\gamma \in P_E^{1-n}$ such that $\theta(1 + x) = \psi_E(\gamma x)$ for $x \in P_E^m$ where $m = [(n + 1)/2]$. Define $\psi_\gamma$ on $K^m$ by $\psi_\gamma(1 + x) = \psi(\text{Tr}(\gamma x))$ for $x \in A^m$. Then $\psi_\gamma$ is a quasi-character of $K^m$. Put $H = E^\times K^m$ and define a quasi-character $\rho_\theta$ of $H$ by

$$\rho_\theta(h \cdot g) = \theta(h)\psi_\gamma(g) \quad \text{for } h \in E^\times, \ g \in K^m.$$ 

Let $J$ be the normalizer of $\psi_\gamma$ in $B$, i.e.,

$$J = \{ a \in B \mid \psi_\gamma^a = \psi_\gamma \},$$

where $\psi_\gamma^a(x) = \psi_\gamma(a^{-1}xa)$ for $x \in K^m$. Then $J = E^\times K^m'$ where $m' = [n/2]$. Put $\eta_\theta = \text{Ind}_H^B \rho_\theta$.

When $n$ is even, i.e., $n = 2m$, then $J = H = E^\times K^m$. By the Clifford theory, $\eta_\theta$ is an irreducible representation of $B$. We put

$$\kappa_\theta = \eta_\theta.$$ 

When $n$ is odd, i.e., $n = 2m - 1$, then $J = E^\times K^{m-1}$. Thus $\eta_\theta$ is not irreducible. In this case, we put

$$\kappa_\theta = \frac{1 - \left(\frac{q}{l}\right)q^{(l-1)/2}}{lq^{(l-1)/2}} \sum_{\chi \in (E^\times/F^\times(1+P_E))^\times} \eta_\theta \otimes \chi + \left(\frac{q}{l}\right) \eta_\theta,$$

where $\left(\frac{q}{l}\right)$ is the Legendre symbol. The following result is well-known ([15],[19]).

**Theorem 1.3.** Let the notation be as above. Then $\kappa_\theta$ is an irreducible representation of $B$. Put $\pi_\theta = \text{ind}_B^G \kappa_\theta$. Then $\pi_\theta$ is an irreducible supercuspidal representation of $G$ such that

1. the $L$-function of $\pi_\theta$ is 1;
2. $\varepsilon(\pi_\theta, \psi) = \varepsilon(\theta, \psi_E)$; in particular $f(\pi_\theta) = f(\theta) + l$;
3. $\bigcup_{\pi \in \hat{E}_{\text{gen}}} = \{ \pi \in \text{A}_0(G) \mid f_{\text{min}}(\pi) \not\equiv 0 \mod l \}$, where $E$ runs through isomorphism classes of ramified extensions of degree $l$ over $F$ and $\text{A}_0(G)$ be the set of equivalent classes of the supercuspidal representations of $G$.

**Remark.** If $\pi \in \text{A}_0(G)$ and $f_{\text{min}}(\pi) \equiv 0 \mod l$, $\pi$ can be constructed from a regular quasi-characters of $L^\times$, where $L$ is an unramified extension of $F$ of degree $l$. The character formula for such a representation was given in [21].
Next we construct an irreducible representation of $D^x = D^x_l$ from $\theta \in E^\times_{gen}$. Let $f(\theta) = n$. We recall $n \not\equiv 1 \mod l$. We define a function $\psi_\gamma$ on $1 + P_D^m$ by $\psi_\gamma(1 + x) = \psi(\text{Tr}(\gamma x))$ for $x \in P_D^m$. Then $\psi_\gamma$ is a quasi-character of $1 + P_D^m$. $H' = E^x(1 + P_D^m) \subset D^x$ and define a quasi-character $\rho'_0$ of $H'$ by
\[ \rho'_0(h \cdot g) = \theta(h)\psi_\gamma(g) \quad \text{for} \quad h \in E^x, \quad g \in 1 + P_D^m. \]

When $n$ is even, i.e., $n = 2m$, we set
\[ \pi'_0 = \text{Ind}_{H'}^{D^x} \rho'_0. \]
When $n$ is odd, i.e., $n = 2m - 1$, we set
\[ \pi'_0 = \frac{1 - \left(\frac{q}{l}\right)q^{l(l-1)/2}}{lq^{l(l-1)/2}} \sum_{\chi \in (E^x/F^x(1+P))} \text{Ind}_{H'}^{D^x} \rho'_0 \otimes \chi + \left(\frac{q}{l}\right) \text{Ind}_{H'}^{D^x} \rho'_0, \]
where $\left(\frac{q}{l}\right)$ is the Legendre symbol. The following result is essentially well-known. (See [2], [15]).

**Theorem 1.4.** Let the notation be as above. Then $\pi'_0$ is an irreducible minimal representation of $D^x$ such that
\begin{enumerate}
\item the degree of $\pi'_0$ is $q^{(n-2)(l-1)/2(l-1)} \cdot q^{l(l-1)/2} q^{-1}$;
\item $\varepsilon(\pi'_0, \psi) = \varepsilon(\theta, \psi_F)$; in particular $f(\pi'_0) = f(\theta) + l$;
\item $\bigcup_{E \in \widetilde{E}^x_{reg}} \{ \pi'_0 \mid \theta \in \widetilde{E}^x_{reg} \} = \{ \pi' \mid \theta \in \widetilde{D}^x_{\min} | f_{\text{min}}(\pi') \not\equiv 0 \mod l \}$, where $E$ runs through the isomorphism classes of ramified extensions of degree $l$ over $F$.
\item The correspondence $\pi'_0 \leftrightarrow \pi_0$ by way of generic quasi-characters of $E^x$ is a bijection and preserves $\varepsilon$-factors and conductoral exponents. (This correspondence is a special case of Howe’s bijection (see [15]).)
\end{enumerate}

On the other hand, there exists an abstract matching theorem, which is called the Deligne-Kazhdan correspondence ([8], [20]).

**Theorem 1.5.** There is a bijection between the set of irreducible representations of $D^x$ and the set of essentially square-integrable representations of $G$ which preserves the characters on elliptic regular elements. In particular, it preserves $\varepsilon$-factors and conductoral exponents.

By the result of Henniart ([10] Theorem 8.1), these two correspondences coincide.

**Theorem 1.6.** If $l \neq p$ is a prime, Howe’s bijection (1.4) coincides with Deligne-Kazhdan correspondence (1.5) between the set of essentially square-integrable representations of $GL_l$ and the set of irreducible representations of $D^x_l$.

At the end of this section, we quote the result of Kutzko [14] in the form that the character formula of $\pi_0$ on elliptic regular elements is essentially given by the one of $\kappa_0$.

**Theorem 1.7.** Let $x$ be an elliptic regular element of $G$.
\begin{enumerate}
\item If $F(x)/F$ is ramified and $x \not\in F^x(1 + P^m_{F(x)})$, \[ \chi_{\pi_0}(x) = \chi_{\kappa_0}(x). \]
(2) If $F(x)/F$ is unramified and $x \not\in F^\times (1 + P_{F(x)}^{[n/l]} + 1)$, 
$$
\chi_{\pi}(x) = 0.
$$

Proof. These are obtained by applying Proposition 5.5 in [14] to our case. \hfill \Box

Remark. Since

$$
\chi_{\eta}(x) = \begin{cases}
\left(\frac{q}{i}\right) \chi_{\eta}(x) & x \in E^\times - F^\times (1 + P_E), \\
q^{(l-1)/2} \chi_{\eta}(x) & x \in F^\times (1 + P_E),
\end{cases}
$$

we have only to calculate $\chi_{\eta}$.

2. Calculation of the character

Now we begin to calculate the characters of the representations constructed in the previous section. In this section, we shall get a character formula up to some root numbers. These root numbers are calculated explicitly in the next section.

Hereafter we fix a generic character $\theta$ and put $\rho = \rho_0$, $\eta = \eta_0$ and so on. Since $E/F$ is a totally tamely ramified extension, there exists a prime element $\wp$ of $O_E$ such that $\wp^l \in P_F - P_F^2$. Put $\omega_E = \omega_F$. As in the previous section, we identify $M_l(E)$ with $\text{End}_F(E)$ by the $F$-basis $\{\omega_E^{-1}, \omega_E^{-2}, \ldots, \omega_E, 1\}$, which is an $O_F$-basis of $O_E$. Thus we get the explicit matrix forms of various objects:

$$
\omega_E = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
\omega_F & 0 & \cdots & \cdots & 0
\end{pmatrix},
$$

$$
K = \begin{cases}
\begin{array}{c}
\begin{pmatrix}
a_{11} & \cdots & a_{1l} \\
\cdots & \cdots & \cdots \\
a_{l1} & \cdots & a_{ll}
\end{pmatrix}
\end{array} & \begin{array}{c}
a_{ij} \in O_F \text{ if } i < j \\
a_{ij} \in O_F^c \text{ if } i > j
\end{array} \\
\begin{array}{c}
a_{ij} \in P_F \text{ if } i < j \\
a_{ij} \in P_F \text{ if } i > j
\end{array}
\end{cases},
$$

$$
A_0 = \begin{cases}
\begin{array}{c}
\begin{pmatrix}
a_{11} & \cdots & a_{1l} \\
\cdots & \cdots & \cdots \\
a_{l1} & \cdots & a_{ll}
\end{pmatrix}
\end{array} & \begin{array}{c}
a_{ij} \in O_F \text{ if } i \leq j \\
a_{ij} \in P_F \text{ if } i > j
\end{array} \\
\begin{array}{c}
a_{ij} \in O_F \text{ if } i \leq j \\
a_{ij} \in P_F \text{ if } i > j
\end{array}
\end{cases},
$$

$$
A_1 = \begin{cases}
\begin{array}{c}
\begin{pmatrix}
a_{11} & \cdots & a_{1l} \\
\cdots & \cdots & \cdots \\
a_{l1} & \cdots & a_{ll}
\end{pmatrix}
\end{array} & \begin{array}{c}
a_{ij} \in O_F \text{ if } i < j \\
a_{ij} \in P_F \text{ if } i \geq j
\end{array} \\
\begin{array}{c}
a_{ij} \in O_F \text{ if } i < j \\
a_{ij} \in P_F \text{ if } i \geq j
\end{array}
\end{cases}.
$$

If $q \equiv 1 \mod l$, $F$ has a primitive $l$-th root of unity $\zeta$ and $E/F$ is a Galois extension. Let $\sigma$ be a generator of $\text{Gal}(E/F)$ determined by $\sigma \omega_E = \omega_E \zeta$. We denote the diagonal matrix $\text{diag}(1, \zeta^{l-1}, \zeta^{l-2}, \ldots, \zeta)$ by $\xi$. Then $\xi$ satisfies $\xi^l = 1$ and $\xi x \xi^{-1} = \sigma x$ for $x \in E$.

Define a natural ring morphism $R$ from $A_0$ to $k_F^l$ by the identification of $A_0/A_1$ with $k_F^l$. We note that if $R(\alpha) = (\alpha_0, \alpha_1, \cdots, \alpha_{l-1})$, $R(\omega_E \omega_E^{-1}) = (\alpha_1, \alpha_2, \cdots, \alpha_0)$. For convenience, we extend the suffix to $\mathbb{Z}$ by putting $\alpha_i = \alpha_{i \mod l}$. The next lemma is the key tool for the character calculation.
Lemma 2.1. Let \( x \in P_E^* - (F + P_E^{+1}) \), \( g \in B \) and \( j \) a positive integer. If \( gxg^{-1} \in E^\times (1 + A^l) \), then
\[
g \in \begin{cases} E^\times (1 + A^l) & \text{if } q \not\equiv 1 \mod l, \\ \bigcup_{k=0}^{l-1} E^\times (1 + A^l)\xi^k & \text{if } q \equiv 1 \mod l. \end{cases}
\]

Proof. We may assume \( g \in A_0 \) by replacing \( g \) by \( \varpi_E^{-k}g \) if \( g \in A_k \). Let \( x = \varpi_E x_0 \) for \( x_0 \in O_E^\times \) and \( R(g) = (\alpha_0, \alpha_1, \ldots, \alpha_{l-1}) \). Then
\[
R(gxg^{-1}x^{-1}) = (\alpha_0\alpha_i^{-1}, \alpha_1\alpha_{i+1}^{-1}, \ldots, \alpha_{l-1}\alpha_{i+1}^{-1}),
\]
where \( \alpha_s = \alpha_{s \mod l} \) for \( s \in \mathbb{Z} \). Since \( g \not\in F + P_E^{+1} \), \( i \not\equiv 0 \mod l \). Therefore \( gxg^{-1}x^{-1} \in E^\times \) implies
\[
g \in \begin{cases} E^\times (1 + A^l) & \text{if } q \equiv 1 \mod l, \\ \bigcup_{k=0}^{l-1} E^\times (1 + A^l)\xi^k & \text{otherwise}. \end{cases}
\]
Thus we may assume \( g - 1 \in A^k - (P_E^{+1} + A^{k+1}) \) for \( k \geq 1 \). Put \( g - 1 = \varpi_E^k g_0 \) and \( R(g_0) = (\beta_0, \beta_1, \ldots, \beta_{l-1}) \). Since
\[
gxg^{-1}x^{-1} \equiv 1 + (g - 1) - x(g - 1)x^{-1} \mod A^{k+1}
\]
\[
\equiv 1 + \varpi_E^k (g_0 - xg_0x^{-1}) \mod A^{k+1},
\]
\[
R(g_0 - xg_0x^{-1}) = (\beta_0 - \beta_k, \beta_1 - \beta_{1+k}, \ldots, \beta_{l-1} - \beta_{l-1+k}).
\]
Therefore \( gxg^{-1}x^{-1} \in E^\times K^{k+1} \) contradicts \( g - 1 \in A^k - (P_E^{+1} + A^{k+1}) \). It implies that if \( gxg^{-1}x^{-1} \in E^\times K^j \),
\[
g \in \begin{cases} E^\times (1 + A^l) & \text{if } q \not\equiv 1 \mod l, \\ \bigcup_{k=0}^{l-1} E^\times (1 + A^l)\xi^k & \text{if } q \equiv 1 \mod l. \end{cases}
\]
Hence our lemma. \( \square \)

Put \( U_{-1} = E^\times, U_0 = F^\times O_E^\times, U_i = F^\times (1 + P_E^i) \) for \( i \geq 1 \) and \( U_i = U_i - U_{i+1} \) for \( j \geq -1 \). The previous lemma gives the character of \( \eta_\theta \) on \( E^\times - U_{-1} \). We remark \( \text{Aut}_F E = \{1\} \) if \( q \not\equiv 1 \mod l \).

Proposition 2.2. Let \( x \in U_i^* \) for \(-1 \leq i < n - 1\). If \( i \not\equiv 0 \mod l \), \( x \) is written in the form \( x = c(1 + y) \) for \( c \in F \) and \( y = \varpi_E y_0 \in \varpi_E O_E^\times \). For \( u \in k_E^\times \) and \( j \in (\mathbb{Z}/l\mathbb{Z})^\times \), we define the Gauss sum part \( G(u, j) \) by
\[
G(u, j) = \sum_{(\alpha_0, \ldots, \alpha_{l-1})} \psi \left( \sum_{k=0}^{l-1} u(\alpha_{k+1} - \alpha_k)\alpha_{j+k} \right),
\]
(2.5)
Thus it follows from Lemma 2.1 that

\[ \sum_{k} \text{map} \]

where \( \Delta = \{ (\alpha, \ldots, \alpha) | \alpha \in k_F \} \) is the image of the diagonal embedding of \( k_F \) into \( k_F^l \).

Then \( \chi_{\eta_{\theta}} \) on \( U_1^* \) is given as follows:

\[
\chi_{\eta_{\theta}}(x) = \begin{cases} 
\sum_{\sigma \in \text{Aut}_{\theta} E} \theta(\sigma x) & i = -1, \\
 q^{l(i+1)/2}(l-1) \sum_{\sigma \in \text{Aut}_{\theta} E} \theta(\sigma x) & i > 0 \text{ and } n - i \text{ even}, \\
 q^{l(1/2)(l-1)} \sum_{\sigma \in \text{Aut}_{\theta} E} \theta(\sigma x) & G(\gamma \sigma = y(\sigma \omega_E / \omega_E), c) \quad i > 0 \text{ and } n - i \text{ odd},
\end{cases}
\]

where \( c = i^{-1}(n + i - 1)/2 \in (\mathbb{Z}/\mathbb{Z})^\times \).

**Proof.** Put \( a_x = axa^{-1} \) for \( a, x \in G \). At first we treat the case \( x \in U_1 = E^\times - F^\times \mathcal{O}_E^\times \).

Since

\[
\chi_{\eta_{\theta}}(x) = \sum_{a \in H \setminus B} \rho_{\theta}(a x),
\]

we have only to show that if \( a x \in H \) for \( a \in B \), then

\[
a \in \begin{cases} 
H & \text{if } q \not\equiv 1 \text{ mod } l, \\
\bigcup_{k=0}^{l-1} H \xi_k & \text{if } q \equiv 1 \text{ mod } l.
\end{cases}
\]

This follows immediately from Lemma 2.1.

Now we treat the case \( x = c(1 + y) \) for \( c \in F \) and \( y \in P_{E}^i - (F + P_{E}^{i+1}) \). We may assume \( c = 1 \) since \( F^\times \) is the center of \( B \). For \( 1 + k \in K^{[(n-i+1)/2]} \) and \( a \in B \), we have

\[
\chi_{\eta_{\theta}}(1 + y) = \sum_{a \in H \setminus B} \rho_{\theta}(q(1 + y)) = C \sum_{1 + k \in K^{1-n-i} \setminus K^{[(n-i+1)/2]}_1} \sum_{a \in H \setminus B} \rho_{\theta}(a_{1+k}(1 + y)),
\]

where \( C = \frac{1}{K^{n-i} \setminus K^{[(n-i+1)/2]}} \).

In the above expression,

\[
\rho_{\theta}(a_{1+k}(1 + y)) = \rho_{\theta}(1 + a y + a(ky - yk))
\]

\[
= \rho_{\theta}(1 + a y)\rho_{\theta}(1 + a(1 + y)^{-1}(ky - yk))
\]

\[
= \rho_{\theta}(1 + a y)\psi(\text{Tr} \gamma \psi((1 + y)^{-1}(ky - yk)))
\]

\[
= \rho_{\theta}(1 + a y)\psi(\text{Tr} a^{-1} \gamma(1 + y)^{-1}(ky - yk))
\]

\[
= \rho_{\theta}(1 + a y)\psi(\text{Tr}(y^{a^{-1} \gamma - a^{-1} \gamma} y)(1 + y)^{-1} k)
\]

since \( y k^2 \in A^n \) and \( a(1 + y)^{-1}(ky - yk)a^{-1} \in A^m \). If \( y^{a^{-1} \gamma - a^{-1} \gamma} y \not\in A^{1-[(n-i+1)/2]} \), the map \( k \mapsto \psi(\text{Tr}(y^{a^{-1} \gamma - a^{-1} \gamma} y)(1 + y)^{-1} k) \) is a non-trivial character of \( A^{n-i-1} \setminus A^{(n-i+1)/2} \); thus

\[
\sum_{k \in A^{n-i-1} \setminus A^{(n-i+1)/2}} \psi(\text{Tr}(y^{a^{-1} \gamma - a^{-1} \gamma} y)(1 + y)^{-1} k) = 0.
\]

By Lemma 3.3 in [4], \( y^{a^{-1} \gamma - a^{-1} \gamma} y \in A^{1-[(n-i+1)/2]} \) is equivalent to \( a^{-1} \gamma \in E^\times K^{n-i-1} \setminus [(n-i+1)/2] \).

Thus it follows from Lemma 2.1 that

\[
\chi_{\eta_{\theta}}(1 + y) = \sum_{\sigma \in \text{Aut}_{\theta} E 1 + a \in H \setminus E^\times \setminus K^{[(n-i)/2]}} \rho_{\theta}(1 + (1 + a)^{\eta_{\theta}} y(1 + a)^{-1}).
\]
By virtue of $(1 + y)^{-1}(1 + (1 + a)y) \in K^m$ and $(1 + y)^{-1}(1 + (1 + a)y) \equiv 1 + (1 + y)^{-1}(ay - ya) + (ya - ay)a \mod K^n$,

$$\rho_y(1 + (1 + a)y) = \theta(1 + y)\psi_\gamma((1 + y)^{-1}(ay - ya))\psi_\gamma((1 + y)^{-1}(ya - ay)a).$$

Since

$$\psi_\gamma((1 + y)^{-1}(ay - ya)) = \psi(\text{Tr}(y\gamma(1 + y)^{-1} - \gamma(1 + y)^{-1}y)a) = 1,$$

$$\psi_\gamma((1 + y)^{-1}(ya - ay)a) = \psi_\gamma((ya - ay)a)$$

and $|E^* K^j/E^* K^m| = q^{(l-1)(m-j)}$, we obtain

$$\chi_{\eta}(1 + y) = \left\{ \begin{array}{ll}
q^{m-(n-i)/2} \sum_{\sigma \in \text{Aut}_F E} \theta(1 + \sigma y) & n - i \text{ even}, \\
q^{m-(n-i+1)/2} \sum_{\sigma \in \text{Aut}_F E} \theta(1 + \sigma y)S(n - i, \sigma) & n - i \text{ odd},
\end{array} \right.$$  

where

$$S(n - i, \sigma) = \sum_{a \in A^{(n-i+1)/2} \cap E \cap A^{(n-i-1)/2}} \psi_\gamma((\sigma y a - a^* y a)).$$

Now we may assume $n - i$ odd and $\sigma = 1$. Put $y = \varpi_E^ty_0$, $a = \varpi_E^{(n-i)/2}a_0$ and $S = S(n - i, 1)$. Since

$$(ya - ay)a = \varpi_E^{n-1}(y_0 \varpi_E^{-(n-i)/2}a_0 \varpi_E^{(n-i)/2} - \varpi_E^{-(n+i)/2}a_0 \varpi_E^{(n-i)/2}y_0)a_0,$$

we have by way of the map $R : A_0/A_1 \to k_F^l$ that

$$S = \sum_{(a_j) \in k_F^l/\Delta} \psi\left(\sum_{j=0}^{l-1} \varpi_E^{-n+1}y_0(\alpha_{j-(n-i-1)/2} - \alpha_{j-(n+i-1)/2})a_j\right).$$

(The suffix is extended to $\mathbb{Z}$ by $\alpha_j = \alpha_j \mod l$.) At first replacing the suffix $j$ by $j + (n + i - 1)/2$ and then replacing $\alpha_{ij}$ by $\alpha_j$, we get our lemma. \[ \square \]

**Remark.** It is proved that the Gauss sum $q^{-(l-1)/2}G(u, j)$ is a fourth root of unity when $p \neq 2$ in [6] and [7].

Next we calculate the character on $K^{n-1} - K^n$. We state the character formula including the case $x \notin E$. On $K^{n-1} - K^n$, the Kloosterman sum appears in the formula.

**Definition 2.3.** For $a \in k_F^\times$, we define the Kloosterman sum $\text{Kl}(a)$ by

$$\text{Kl}(a) = \sum_{(y_0, \ldots, y_{l-1}) \in k_F^l} \psi\left(y_0 + \cdots + y_{l-1}\right).$$

**Lemma 2.4.** Let $x = 1 + \varpi_E^{n-1}x_0$ for $x_0 = \text{diag}(k_0, \ldots, k_{l-1}), (k_i \in \mathcal{O}_F^\times)$. Then

$$\chi_{\eta}(x) = q^{(l-1)(m-1)} \text{Kl}\left(\gamma \varpi_E^{n-1}\prod_{j=0}^{l} k_j\right).$$

(Since $\gamma \varpi_E^{n-1} \in \mathcal{O}_E$ and $k_F = k_F$, we regard $\gamma \varpi_E^{n-1} \mod P_E$ as an element of $k_F$.)
Proof. By the definition of $\eta_0$, we have
\[
\chi_{\eta_0}(1 + \omega^n_E \text{diag}(k_0, \ldots, k_{l-1}))
= q^{(l-1)(m-1)} \sum_{a \in E^x K^1 \setminus B} \psi(\text{Tr} \gamma a \omega^n_E \text{diag}(k_0, \ldots, k_{l-1}) a^{-1}).
\]
It follows from (2.2) and (2.4) that the set \{\text{diag}(1, y_1, \ldots, y_{l-1}) \mid y_i \in k_F^x\} makes a complete system of representatives of $E^x K^1 \setminus B$. For convenience, put $y_0 = 1$. Since
\[
\omega_E \text{diag}(1, y_1, \ldots, y_{l-1}) \omega^{-1} = \text{diag}(y_1, \ldots, y_{l-1}, 1),
\]
we have
\[
\text{Tr} \gamma \text{diag}(1, y_1, \ldots, y_{l-1}) \omega^n_E^{-1} \text{diag}(k_0, \ldots, k_{l-1}) \text{diag}(1, y_1, \ldots, y_{l-1})^{-1}
= \gamma \omega^n_E^{-1} \sum_{i=0}^{l-1} k_i y_{i-n+1}/y_i \text{ mod } P_F,
\]
where $y_i = y_i \text{ mod } l$. By replacing $y_i$ by $k_i y_{i-n+1}/y_i$, we get our lemma. \hfill \Box

On $K^n$, the character of $\pi = \pi_0$ becomes a constant function on elliptic regular conjugacy classes.

**Lemma 2.5.** Let $x$ be an elliptic regular element in $K^n$. Then
\[
\chi_\pi(x) = q^{(n-2)(l-1)/2} \left( q^l - 1 \right) / q - 1.
\]

**Proof.** We use the Deligne-Kazhdan correspondence (Theorem 1.5). Since the correspondence preserves the conductorial exponents, there exists a generic character $\theta'$ such that $f(\theta') = n$ and $\chi_\pi' = \chi_\pi$. Since $\pi_0'$ is trivial on $1 + P_B^n$ and its degree is $q^{(n-2)(l-1)/2} \left( q^l - 1 \right) / q - 1$, $\chi_\pi_0'(x) = q^{(n-2)(l-1)/2} \left( q^l - 1 \right)'$ for $x \in 1 + P_B^n$. Consequently we have $\chi_\pi_0(x) = q^{n-1}(q^2 + q + 1)$ if $x \in K^n$ is elliptic regular. \hfill \Box

The character formula on elliptic regular conjugacy classes outside $E^x$ can be obtained easily.

**Lemma 2.6.** Let $x$ be an elliptic regular element of $B$. If $x$ satisfies the condition that $F(x) \not\in E$ and $x$ is not conjugate to an element of $F^x K^n$, then $\chi_\pi(x) = 0$.

**Proof.** See Lemma 3.3 in [14]. \hfill \Box

## 3. Calculation of Gauss sums

In this section, we determine the Gauss sum part $G(y, n-i)$ explicitly. Since $G(y, n-i)$ depends only on $n - i \text{ mod } l$ and $y \text{ mod } P_E$, we have only to treat the character of $\eta_0$ on $U_1^*$ by replacing $n$ big enough.

**Lemma 3.1.** Assume $n = 2m$. Then for $x \in U_1^*$,
\[
(3.1) \quad \chi_{\eta_0}(x) = \sum_{\sigma \in \text{Aut}_E} \sum_{a \in H \setminus E^x K^{m-1}} \rho_0(a^\sigma x a^{-1}).
\]

**Proof.** It follows from Lemma 2.1 that $axa^{-1} \in H$ implies $a \in E^x K^{m-1}$. Hence our lemma. \hfill \Box
For the calculation of the sum in the above lemma, we use the $E^\times$-module structure of various objects. When $E/F$ is a Galois extension, it is easy to treat. Thus we first assume $E/F$ is Galois, i.e., $q \equiv 1 \mod l$. We recall $\xi$ is the diagonal matrix $\text{diag}(1, \zeta^{l-1}, \zeta^{l-2}, \ldots, \zeta)$ where $\zeta$ is an $l$-th root of unity in $F$ and $\xi$ satisfies $\xi^l = 1$ and $\xi x \xi^{-1} = \gamma x$ for $x \in E$ where $\gamma$ is the generator of $\text{Gal}(E/F)$ determined by $\gamma \omega_E = \omega_E \zeta$. By the explicit matrix form of $E$ and $A_i$, we obtain:

$$\begin{align*}
M_i(F) &= E \oplus E\xi \oplus \cdots \oplus E\xi^{l-1} \\
A^0 &= \mathcal{O}_E \oplus \mathcal{O}_E\xi \oplus \cdots \oplus \mathcal{O}_E\xi^{l-1} \\
A^1 &= P_E \oplus P_E\xi \oplus \cdots \oplus P_E\xi^{l-1} \\
\cdots \\
A^{l-1} &= P_E^{l-1} \oplus P_E^{l-1}\xi \oplus \cdots \oplus P_E^{l-1}\xi^{l-1}.
\end{align*}$$

(3.2)

Lemma 3.2. A complete system of representatives of $H \backslash E^\times K^{m-1}$ is given by

$$\{1 + \omega_E^{-m+1}\alpha_1\xi + \cdots + \omega_E^{-m+1}\alpha_{l-1}\xi^{l-1} \mid \alpha_i \in k_F\}.$$

Proof. It is obvious from (3.2). \hfill \Box

For $a = 1 + \alpha_1\xi + \cdots + \alpha_{l-1}\xi^{l-1} \in A^{m-1}$, $\rho(axa^{-1})$ for $x \in U_1^\times$ can be expressed explicitly in terms of $\alpha_1, \ldots, \alpha_{l-1}$. At first, we determine the coefficients of $a^{-1}$ with respect to the $F$-basis $\{1, \xi, \ldots, \xi^{l-1}\}$.

Lemma 3.3. For $a = \sum_{j=0}^{l-1} \alpha_j\xi^j$ ($\alpha_j \in E$), put

$$\Lambda(a) = (\sigma^j\alpha_{1-j \mod l})_{0 \leq i,j \leq l-1}$$

$$= \begin{pmatrix}
\alpha_0 & \sigma\alpha_{l-1} & \cdots & \sigma^{l-1}\alpha_1 \\
\alpha_1 & \sigma\alpha_0 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\alpha_{l-1} & \cdots & \sigma^{l-2}\alpha_1 & \sigma^{l-1}\alpha_{0-1} \\
\end{pmatrix} \in M_l(E)$$

and let $\Lambda_k(a)$ be the $(1, k+1)$-cofactor of $\Lambda(a)$. Then

$$a^{-1} = \sum_{j=0}^{l-1} \frac{\Lambda_j(a)}{|\Lambda(a)|} \xi^j,$$

where $|\Lambda(a)|$ is the determinant of $\Lambda(a)$.

Proof. By the map $\Lambda : M_l(F) \rightarrow M_l(E)$, we can embed $M_l(F)$ into $M_l(E)$. Then our lemma follows from Cramer’s formula. \hfill \Box

Lemma 3.4. Assume $n = 2m$ and $3(m-1) \geq 2m$. Let $c \in F^\times$, $y \in P_E^{m-1}$ and $a = 1 + \sum_{j=1}^{l-1} \alpha_j\xi^j \in K^{m-1}$. Then

$$\rho_\theta(ac(1 + y)a^{-1}) = \theta(c(1 + y))\psi_E \left(\sum_{j=1}^{l-1} (\gamma \alpha_j \sigma^j \alpha_{1-j} - \sigma^{-j} \gamma \alpha_{1-j} \sigma^{-j} \alpha_j) y\right).$$
Proof. It is obvious that we may assume $c = 1$. Since
\[
g^{-1}a^{-1}a = 1 + (g^{-1}(a - 1)g)(a^{-1}) = 1 + \left(\sum_{j=1}^{l-1}(\gamma gg^{-1} - 1)\alpha_j \xi^j\right)a^{-1},
\]
\[
\sum_{j=1}^{l-1}(\gamma gg^{-1} - 1)\alpha_j \xi^j \in A^n\text{ and } \text{Tr}(\gamma x \xi^j) = 0 \text{ for all } x \in E, \text{ we have:}
\]
\[
\rho_\theta(g^{-1}a^{-1}) = \psi_\gamma \left(\sum_{j=1}^{l-1}(\gamma gg^{-1} - 1)\alpha_j \xi^j\right)
\]
\[
= \psi_\gamma \left(\sum_{j=1}^{l-1}(\gamma gg^{-1} - 1)\alpha_j \xi^j(f_{l-j}(a))\right),
\]
where $f_j(a) \in E$ is defined by $a^{-1} = \sum_{j=0}^{l-1}f_j(a)\xi^j$. Put $g = 1 + y$. In the last equation, $\gamma \in P_E^{1-n}$, $f_{l-j} \in P_E^{m-1}$ and $\gamma gg^{-1} - 1 \equiv \gamma^j y - y \mod P_E^{2m-2}$. Thus we get
\[
\rho_\theta(g^{-1}a^{-1}) = \psi_E \left(\sum_{j=1}^{l-1}(\gamma^j f_{l-j}(a)\xi^j - \gamma^j f_{l-j}(a)\alpha_j)\right)
\]
by virtue of $\text{tr}_E u^\sigma v = \text{tr}_E \sigma^j uv$ for any $u, v \in E$. It follows from Lemma 3.3 that
\[
f_{l-j}(a) \equiv \frac{\Lambda_{l-j}(a)}{|\Lambda(a)|} \equiv \alpha_{l-j} \mod P_E^{2m-2}.
\]
By the assumption $3m - 3 \geq 2m$, we obtain the desired formula. \qed

**Proposition 3.5.** Assume $q \equiv 1 \mod l$, $n = 2m$ and $m \geq 3$.

1. For $x \in U_1^+$,
\[
\chi_{\eta_0}(x) = q^{(l-1)/2} \sum_{j=0}^{l-1} \theta(\sigma^j x).
\]

2. For an even integer $n$ and $y \in O_E$, $G(y, n-1) = q^{(l-1)/2}$. In particular, $G(y, n-1)$ depends neither on $n$ nor on $y$.

Proof. By Lemmas 3.1, 3.2 and 3.4, we have for $c \in F^\times$ and $y \in 1 + P_E$
\[
\chi_{\eta_0}(c(1 + y)) = \sum_{i=0}^{l-1} \theta(c(1 + \sigma_i y)) \sum_{(\alpha_1, \ldots, \alpha_{l-1}) \in (P_E^{m-1}/P_E^n)^{l-1}} f(\alpha_1, \ldots, \alpha_{l-1}, \sigma_i),
\]
where
\[
f(\alpha_1, \ldots, \alpha_{l-1}; y) = \psi_E \left(\sum_{j=1}^{l-1}(\gamma \alpha_j \sigma^j - \gamma \alpha_{l-j} \sigma^{-j})\right).
\]
Put $S_j = \{(\alpha_1, \ldots, \alpha_{l-1}) \in (P_E^{m-1}/P_E^n)^{l-1} | \alpha_k = 0 \text{ for } k < j, \alpha_j \neq 0\}$ and $I_j(y) = \sum_{(\alpha_1, \ldots, \alpha_{l-1}) \in S_j} f(\alpha_1, \ldots, \alpha_{l-1}; y)$. Then
\[
\chi_{\eta_0}(c(1 + y)) = \sum_{i=0}^{l-1} \theta(c(1 + \sigma_i y)) \sum_{j=1}^{l-1} I_j(\sigma^i y).
\]
If $\alpha_1 = \cdots = \alpha_{(l-1)/2} = 0$, $f(\alpha_1, \ldots, \alpha_{l-1}; y) = 0$. Thus we have

$$ \sum_{j=(l+1)/2}^{l-1} I_j(y) = q^{(l-1)/2}. $$

For $1 \leq j \leq (l-1)/2$, $I_j(y)$ is proportional to

$$ \psi_E((\gamma \alpha_j)^{-1} \gamma \alpha_{l-j} - \sigma^{-1} \gamma \alpha_{l-j} \sigma^{-1} \alpha_j)y). $$

Since $\alpha_j \neq 0$, the map

$$ \alpha_{l-j} \mapsto \gamma \alpha_j^{-1} \gamma \alpha_{l-j} - \sigma^{-1} \gamma \alpha_{l-j} \sigma^{-1} \alpha_j $$

is a bijection from $P_{E_l}^{m-1}/P_{E_l}^m$ to $k_F$. Therefore $I_j(y) = 0$. Consequently we get the first part of our lemma. $G(y, n - 1) = q^{(l-1)/2}$ follows from Proposition 2.2 and the first part.

Next we assume $q - 1 \neq 0 \mod l$. In this situation, it is rather difficult to describe $E^\times$-module structure of various objects since $F$ has no $l$-th primitive root of unity and $E/F$ is not Galois. In order to apply the result of Galois case, we use the base change lift of simple characters by Bushnell-Henniart [3]. Let $\zeta$ be a primitive $l$-th root of unity and $L = F(\zeta)$. Then $L/F$ is an unramified extension of degree $d$ where $d$ is the smallest integer satisfying $q^d \equiv 1 \mod l$. The generator $\tau$ of $\Gal(L/F)$ is determined by $\zeta = \zeta^k$ where $k = r(l-1)/d$ and $r$ is a generator of $(\mathbb{Z}/l\mathbb{Z})^\times$. We add the subscript $L$ to the base changed objects. Then $M_i(L) = M_i(F) \otimes_L L$ and $E_L = E \otimes_L L \simeq EL$. $E_L$ is a ramified Galois extension over $L$ of degree $l$, an unramified extension over $E$ of degree $d$ with $\Gal(E_L/E) = \Gal(L/F) = \langle \tau \rangle$ and a non-Abelian Galois extension over $F$ of degree $ld$. (We embed $E$ into $E_L$ by the map: $x \mapsto x \otimes 1$).

As in the previous section, we identifies $M_i(L)$ with $\End_L E_L$ and $G_L = GL_i(L)$ with $\Aut_L E_L$ by the $L$-basis $\{x_{E_L}^{-1}, \cdots, x_{E_L}, 1\}$ of $E_L$, which is also an $\mathcal{O}_L$-basis of $\mathcal{O}_{E_L}$. By the lattice flag $\{P_{E_L}^i\}_{i \in \mathbb{Z}}$, we define

$$ A_L^i = \{f \in M_i(L) | f(P_{E_L}^j) \subset P_{E_L}^{j+i} \text{ for all } j \in \mathbb{Z} \}. $$

Put $K_L = (A_L^i)^\times$, $B_L = E^\times K_L$ and $K_L^i = 1 + A_L^i$ for $i \geq 1$. For a subgroup $M_L \subset B_L$ (resp. $M \subset B$), we set $M_L^i = M \cap L^\times K_L$ (resp. $M^i = M \cap F^\times K$). By the result of Kutzko (Theorem 1.7), it suffices to calculate the character of $\kappa = \kappa_\theta$ instead of $\pi_\theta$. In fact, we have only to get the character of $\eta|_{B^i}$. Therefore we have only to treat the base change of $\eta|_{B^i}$ to $B_L^i$ where $B_L^i = L^\times K_L^i$.

**Definition 3.6.** Let $\theta$ be a generic character of $E^\times$ with $f(\theta) = n$ and $\theta(1 + x) = \psi(\mathrm{tr}_E(\gamma x))$ for $x \in P_{E_L}^m$. We define a base change lift $\theta_L$ of $\theta$ to $L^\times$ by $\theta_L = \theta \circ n_{E_L/E}$. Then $\theta_L(1 + x) = \psi_L(\mathrm{tr}_{E_L/L}(\gamma x))$ for $x \in P_{E_L}^m$. (Recall $m = [(n + 1)/2]$.) The base change lift $\rho_L$ of $\rho|_{H^i}$ to $H_L^i = L^\times (1 + P_{E_L}) K_L^m$ is defined by

$$ \rho_L(h \cdot g) = \theta_L(h) \psi_L(\mathrm{Tr} \gamma(g - 1)) \quad \text{for} \quad h \in L^\times (1 + P_{E_L}), \ g \in K_L^m. $$

We define the base change $\eta_L$ of $\eta|_{B^i}$ to $B_L^i$ by

$$ \eta_L = \text{Ind}_{H_L^i}^{B_L^i} \rho_L. $$
By virtue of \( \theta_L \circ \tau = \theta_L \), we have \( \rho_L \circ \tau = \rho_L \). Thus we can define an extension \( \tilde{\rho}_L \) of \( \rho_L \) to \( H^1_L \times \langle \tau \rangle \) by

\[
\tilde{\rho}_L(x \times \tau) = \rho_L(x) \quad \text{for } x \in H^1_L.
\]

Now we apply the result of Bushnell-Henniart [3] to our case and get the character relation between \( \eta_L \) and \( \tilde{\eta}_L \). Put \( U_{E_L,i} = L^\times (1 + P_{E_L}^i) \) for \( i > 0 \) and \( U^*_{E_L,i} = U_{E_L,i} - U_{E_L,i+1} \). By (12.19) Corollary in [3] and the fact \( \langle \tau \rangle \)-fixed space \( (L^\times K_L^i) \) is equal to \( F^\times K^i \), the following result follows.

**Proposition 3.7.** Let \( x \in U_{E_L,1} \). Between the set

\[
\{ g \in H^1 \setminus (E^\times K^{m-1})^1 \mid gn_{E_L/E}(x)g^{-1} \in H^1 \}
\]

and the set

\[
\{ h \in H^1 \setminus (E^\times K^{m-1})^1 \mid hx^\tau h^{-1} \in H^1 \},
\]

there is a bijection \( \psi \) with the property

\[
\rho_L(\psi(g)x^\tau(\psi(g))^{-1}) = \rho(gn_{E_L/E}(x)g^{-1}).
\]

Combining this with Lemma 3.1, we have:

**Lemma 3.8.**

\[
\chi_{\eta_L}(n_{E_L/E}(x)) = \sum_{a \in H^1 \setminus (E^\times K^{m-1})^1} \rho_L(ax^\tau a^{-1}).
\]

Since \( n_{E_L/E}(L^\times (1 + P_{E_L}^i)) = F^\times (1 + P_E^i) \), it suffices to calculate the right hand side of (3.3) for \( x \in U_{E_L,1}^\times \).

As in the Galois case, set \( \xi = \text{diag}(1, \zeta^{l-1}, \zeta^{l-2}, \ldots, \zeta) \in M_l(L) \). Then \( \xi \) satisfies \( \xi^l = 1 \), \( \zeta = \xi^k \) and

\[
\xi x^\tau \xi^{-1} = \sigma \quad \text{for any } x \in E_L,
\]

where \( \sigma \) is the generator of \( \text{Gal}(E_L/L) \) determined by \( \sigma \psi_E = \psi_E \). Moreover we have \( \tau \sigma \tau^{-1} = \sigma^k \) and

\[
\begin{align*}
M_l(L) &= E_L \oplus E_L \xi \oplus \cdots \oplus E_L \xi^{l-1} \\
A_l^0 &= E_L \oplus E_L \xi \oplus \cdots \oplus E_L \xi^{l-1} \\
A_l^1 &= P_{E_L} \oplus P_{E_L} \xi \oplus \cdots \oplus P_{E_L} \xi^{l-1} \\
&\vdots \\
A_{l-1}^1 &= P_{E_L}^{l-1} \oplus P_{E_L}^{l-1} \xi \oplus \cdots \oplus P_{E_L}^{l-1} \xi^{l-1}
\end{align*}
\]

(3.4)

We note that any element of \( K^i_L \) can be written in the form \((1+\alpha_1 \xi+\alpha_2 \xi^2+\cdots+\alpha_{l-1} \xi^{l-1})\) for \( \alpha_i \in P_{E_L} \).

**Lemma 3.9.** Let \( i < m \) and \( a = 1 + \alpha_1 \xi + \alpha_2 \xi^2 + \cdots + \alpha_{l-1} \xi^{l-1} \) for \( \alpha_j \in \mathcal{O}_E \) and \( x \in U_{E_L,i}^\times \).

Then \( ax^{-1} \in H_L \) is equivalent to \( \alpha_j \in P_{E_L}^{m-i} \) and \( \alpha_{kj} = \tau^j \alpha_h \) for \( j = 0, 1, \ldots, d - 1 \) and \( h = 1, r, \ldots, r^{(l-1)/d-1} \). (The suffix of \( \alpha_j \) is extended to \( \mathbb{Z} \) by \( \alpha_j = \alpha_{j \mod l} \).)

**Proof.** It follows from Lemma 3.2 that if \( a^{-1}x^\tau a \in H_L \), there exist \( \gamma_0 \in \mathcal{O}_E^\times \) and \( \gamma_j \in P_{E_L}^m \) for \( 1 \leq j \leq l - 1 \) such that

\[
(1 + \alpha_1 \xi + \cdots + \alpha_{l-1} \xi^{l-1})x = \gamma_0 (1 + \gamma_1 \xi + \gamma_2 \xi^2 + \cdots + \gamma_{l-1} \xi^{l-1})
\]

\[
(1 + \tau \alpha_1 \xi^k + \tau \alpha_2 \xi^{2k} + \cdots + \tau \alpha_{l-1} \xi^{(l-1)k}).
\]
It implies
\[ x = \gamma_0(1 + \gamma_l \alpha_1 + \gamma_l^2 \alpha_2 + \cdots + \gamma_l^k \alpha_{l-1}) \]
\[ \alpha_k \sigma^k x = \gamma_0(\gamma_k + \gamma_l \alpha_1 + \gamma_l^2 \alpha_2 + \cdots + \gamma_l^k \alpha_{l-1}) \]
\[ \cdots \]
\[ \alpha_{l-k} \sigma^{l-k} x = \gamma_0(\gamma_l + \gamma_l^2 \alpha_1 + \cdots + \gamma_{l-1}) \].
Thus we have
\[ \alpha_h \sigma^k x = x^* \alpha_h \mod \mathbb{P}_{E_L}^m, \quad (h \in (\mathbb{Z}/l\mathbb{Z})^\times). \]
By eliminating \( \alpha_h, \alpha_h, \alpha_h, \cdots, \alpha_h, \) we get
\[ \alpha_h = n_{E_L/E}(x) \sigma^k n_{E_L/E}(x)^{-1} \alpha_h \mod \mathbb{P}_{E_L}^m. \]
Since \( n_{E_L}(x) \sigma^k n_{E_L/E}(x)^{-1} \in 1 + P_{E_L}^i \), we obtain
\[ \alpha_h \in P_{E_L}^{m-i}, \quad \alpha_h \sigma^k \in P_{E_L}^{m-i}, \quad \text{for } j = 0, 1, \ldots, d - 1. \]

**Lemma 3.10.** Assume \( n = 2m \) and \( m \geq 3 \). Let \( x \in 1 + P_{E_L} - P^2_{E_L} \) and \( a = 1 + \sum_{i=1}^{(l-1)/d} \sum_{j=1}^d \tau_i \sigma^r \). Then
\[ (3.5) \quad \rho_L(ax \sigma^a x^{-1}) = \psi_E \left( \sum_{i=1}^{(l-1)/d} \tau_i \sigma^r \right) \]

where \( u_i = \gamma \alpha_i \sigma^r \alpha_{-r} \).

**Proof.** By Lemma 3.9, \( \gamma a a^{-1} \in H_L \). Since \( \rho_L(\gamma a a^{-1}) = 1 \), it implies \( \rho_L(ax \sigma^a x^{-1}) = \rho_L(axa^{-1}g^{-1}) \). By the same way as Lemma 3.4, we have:
\[ \rho_L(ax \sigma^a x^{-1}) = \psi_E \left( \sum_{i=1}^{(l-1)/d} \sum_{j=1}^d (v_{i,j} - \sigma^r \alpha_{-r}) \right) \]

where \( v_{i,j} = \gamma \alpha_i \sigma^r \alpha_{-r} \). Since \( \sigma^r \tau_i \tau_j = \tau_i \sigma^r \) and \( \tau_i = \gamma \), we have
\[ \sum_{j=1}^d (v_{i,j} - \sigma^r \alpha_{-r}) = \sum_{j=1}^d (\gamma \alpha_i \sigma^r \alpha_{-r} - \sigma^r \alpha_{-r} \gamma \alpha_i \alpha_{-r}) \]
\[ = \sum_{j=1}^d \tau_i (\gamma \alpha_i \sigma^r \alpha_{-r} - \sigma^r \alpha_{-r} \gamma \alpha_i \alpha_{-r}) \]
This implies the equation (3.5). \( \square \)

It is time to get the character value of \( \chi_\eta \) on \( U_1^* \).

**Proposition 3.11.** Let \( x \in 1 + P_{E_L} - P^2_{E_L} \) and \( n = 2m > 6 \). Then
\[ \chi_\eta(n_{E_L/E}(x)) = \left( \frac{q}{l} \right)^{(l-1)/2} \theta(n_{E_L/E}(x)) \]
and
\[ G(y, j) = \left( \frac{q}{l} \right)^{(l-1)/2} \]
for all \( y \in k_F \) and \( j \) odd.

**Proof.** By Proposition 3.7, Lemmas 3.8, 3.9, 3.10 and 3.11, we have:

\[
\chi_n(n_{E_L/E}(x)) = \theta_L(x) \sum_{(\alpha, r)} \psi_E \left( \sum_{i=1}^{(l-1)/d} \text{tr}_{E_L/E}(u_i - \sigma^{r_i} u_i) \text{tr}_{E_L/E}(x - 1) \right)
\]

where \( u_i = \gamma \alpha r_i \sigma^{r_i} \alpha - r_i \) and \((\alpha, r)_1 \leq i \leq (l-1)/d \in (\mathbb{P}^{m-1}_{E_L}/P^m_{E_L})/(l-1)/d\). First we assume \((l-1)/d\) is odd. Then \((q^\frac{d}{2}) = -1\), \( d \) is even and \( \sigma^{r^2/2} \alpha_{r^2} = \alpha_{-r^2} \). Let \( E_i \) be the \( \langle \sigma^r \tau^{d/2} \rangle \)-fixed field. Then \( E_L/E_i \) is a quadratic unramified extension, \( \alpha_{r^2} \sigma^{r^2} \tau^{d/2} \alpha_{r^2} = n_{E_L/E_i}(\alpha_{r^2}) \), \( n_{E_L/E_i} \)
induces a surjection from \( \mathbb{w}_E^{n-1}O_{E_L}/1 + P_{E_L} \) to \( \mathbb{w}_{E_i}^{m-2}O_{E_i}/1 + P_{E_i} \) and each fiber of the induced map has \( q^{d/2} + 1 \) elements. Moreover the map \( x \mapsto \text{tr}_{E_L/E_i}(x - \sigma^{r^2} x) \) induces a surjective \( k_F \)-linear map from \( P_{E_i}^{2m-2}/P_{E_i}^{2m-1} \) to \( P_E^{-1}/O_E \). Thus we have:

\[
\sum_{\alpha, r \in P_{E_i}^{m-1}/P_{E_i}^{m}} \psi_E \left( \sum_{i=1}^{(l-1)/d} \text{tr}_{E_L/E} (u_i - \sigma^{r_i} u_i) \text{tr}_{E_L/E} (x - 1) \right) = (1 - (q^{d/2} + 1)).
\]

Putting this into 3.6, we get:

\[
\chi_n(n_{E_L/E}(x)) = \theta_L(x) (1 - (q^{d/2} + 1))^{(l-1)/d} = -q^{(l-1)/2} \theta_L(x)
\]

and it follows from Proposition 2.2 that \( G(y, j) = -q^{(l-1)/2} \) for all \( y \in k_F \) and \( j \) odd. Now we assume \((l-1)/d\) is even. Then \((q^\frac{d}{2}) = 1\) and it follows from the same argument as in the proof of Proposition 3.5 that

\[
\chi_n(n_{E_L/E}(x)) = \theta_L(x)|k_{E_L}|^{(l-1)/2d} = q^{(l-1)/2} \theta_L(x).
\]

By Proposition 2.2, we have \( G(y, j) = q^{(l-1)/2} \) for all \( y \in k_F \) and \( j \) odd in this case. \( \square \)

Putting all these together, we can state the character formula.

**Theorem 3.12.** Let \( E \) be a ramified extension of \( F \) with degree \( l \), \( \theta \) a generic quasi-character of \( E^\times \) with \( f(\theta) = n \) and \( \pi = \pi_\theta \) the irreducible supercuspidal representation of \( \text{GL}_n(F) \) defined in section 1. Put \( U_0 = F^\times O_E^\times \), \( U_j = F^\times (1 + P_E) \) and \( U_j^* = U_j - U_{j+1} \) for \( j \geq 1 \). Let \( x \) be an elliptic regular element of \( \text{GL}_n(F) \) and \( \text{Aut}_F E \) the group of automorphism of \( E \) over \( F \).

(1) If \( F(x)/F \) is unramified, then

\[
\chi_\pi(x) = \begin{cases} 
0 & x \notin F^\times (1 + P^m_{F(x)}), \\
q^{(n-2)(l-1)/2(l-1)/2} \theta(c) & x = c(1 + y) \\
\text{for } c \in F^\times, y \in P^m_{F(x)}.
\end{cases}
\]

(2) If \( F(x)/F \) is ramified, then

\[
\chi_\pi(x) = \begin{cases} 
0 & x \notin F^\times (1 + P^m_{F(x)}), \\
q^{2n(l-1)/2(l-1)/2} \theta(c) & x = c(1 + y) \\
\text{for } c \in F^\times, y \in P^m_{F(x)}.
\end{cases}
\]
Remark. By Theorem 1.6, the character formula of the representation

\[ \chi_\pi(x) = \begin{cases} 0 & \text{if } x \not\in F^\times(1 + P_{F(x)}^{n-1}), \\ q^{(n-2)(l-1)/2} \theta(c) \text{ Kl}( (\gamma \varpi_E^{-1})^l \prod_{j=0}^{l-1} k_j ) & \text{if } x = c(1 + \varpi_E^{-1} \text{ diag}(k_0, \ldots, k_{l-1}) + z) \\ \text{for } c \in F^\times, k_i \in k_F^\times, z \in P_{F(x)}^n, \\ q^{(n-2)(l-1)/2 (\frac{q-1}{q-1}) \theta(c)} & \text{if } x = c(1 + y) \text{ for } c \in F^\times, y \in P_{F(x)}^n. \end{cases} \]

(See (2.6) for the definition of the Kloosterman sum Kl(\(a\)).)

Proof. It follows from (1.8), Theorem 1.7, Lemmas 2.4, 2.5, 2.6, Propositions 2.2, 3.5 and 3.11.

Remark. By Theorem 1.6, the character formula of the representation \(\pi_\theta\) of \(D^\times\) is given by the same formula for \(\pi_\theta\).

References


Department of Mathematics and Information Science, College of Integrated Arts and Sciences, Osaka Prefecture University

E-mail address: takahasi@mi.cias.osakafu-u.ac.jp