CHARACTER FORMULA FOR THE SUPERCUSPIDAL REPRESENTATIONS OF GL_l, l A PRIME $\neq p$

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Abstract. The article gives a character formula for the irreducible supercuspidal representation of $\mathrm{GL}_l(F)$ for F a local field of the residual characteristic $p \neq l$.

Introduction

Let l be a prime, A a central simple algebra of dimension l^2 over a non-archimedean local field F and E/F an extension of degree l in A. Recall that any compact (mod center) Cartan subgroup of $A^\times$ is isomorphic to $E^\times$ for some extension E/F of degree l. As is well-known ([15], [5]), any irreducible supercuspidal representation is obtained from a quasi-character of $E^\times$. The aim of this paper is to get a character formula for the irreducible supercuspidal representations of $A^\times$ on the set of elliptic regular elements in $A^\times$. To calculate the character on the split torus is another problem. See Murnaghan's papers ([16], [17] and [18]) for this topic. We only remark that the character value on the elliptic regular conjugacy classes determines uniquely the supercuspidal representation.

When E/F is unramified, a character formula was obtained in [21]. Therefore in this paper we treat the case E/F is ramified. When the residual characteristic p of F equals to l, the ramification is wild. This case is very hard to treat (see e.g. [22]). In this paper, we assume $p \neq l$. Since the case $l = 2$ was solved in [13], we assume l is odd. We note that A is isomorphic to a division algebra $D = D_l$ of dimension l^2 over F or the algebra $\mathrm{M}_l(F)$ of $l \times l$ matrices over F.

Let D_n be a division algebra of dimension n^2 over F. Deligne-Kazhdan-Vignéras [8] and Rogawski [20] proved an abstract matching theorem: there is a bijection between the set of equivalence classes of irreducible representations of $D_n^\times$ and that of essentially square-integrable representations of $\mathrm{GL}_n(F)$ which preserves the characters up to $(-1)^{n-1}$. In the tame case, i.e., when n is prime to the residual characteristic of F, Moy [15] has proved that there is a bijection between the same sets as above using the concrete construction of the representations given by Howe [12]. Henniart [10] has shown that two correspondences coincide when $n = l \neq p$. Thus we only treat the GL case.

In the earlier paper [21], the author gave a character formula of the representation of $\mathrm{GL}_l(F)$ and $D_l^\times$ which is obtained from a quasi-character of $E^\times$ where E is an unramified extension of F with degree l. (When $l = 2$, the character formula is given in [13]). Then we essentially use the fact that E/F is a Galois extension. In our case, we need to treat the case where E/F is non-Galois. In order to treat the non-Galois case, we use the

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result of Bushnell-Henniart [3] on the base change lift of simple characters. Since the base change lift is available only for GL case, we do not treat the division algebra case directly. In addition, the formula of the character near the conductor becomes simpler than that of the division algebra case (see Lemma 2.5 and Theorem 4.2 (d) in [6]). Our main result is Theorem 3.12. As in the unramified case, the analogue of Weyl's character formula holds for our formula. This does not hold when $l = p$ (cf. [23]).

Section 1 is devoted to the review of the construction of an irreducible supercuspidal representation π_θ (resp. π'_θ) of $\mathrm{GL}_l(F)$ (resp. $D_l^\times$) from a generic quasi-character θ of $E^\times$ and the known results about the representation. We note that π_θ is not always monomial, i.e., induced from a one-dimensional representation, but it can be written as a $\mathbb{Q}$-linear combination of monomial representations. In fact π_θ is written as a $\mathbb{Q}$-linear combination of the forms $\mathrm{ind}_H^{\mathrm{GL}_l(F)} \rho_\theta$ where H is a compact mod center subgroup of $\mathrm{GL}_l(F)$ and ρ_θ is a quasi-characters of H.

In section 2, we compute the character of π_θ up to some root numbers. Let $G = \mathrm{GL}_l(F)$, B the normalizer of an Iwahori subgroup of G containing H and $\eta_\theta = \mathrm{ind}_H^B \rho_\theta$. Since we treat only elliptic regular conjugacy classes, we consider the character χ_{π_θ} on $L^\times$ where L/F are extensions of fields of degree l. Moreover the case $L = E$ is essential. By the Frobenius formula and the result of Kutzko ([14]), we have only to calculate the sum

$$\chi_{\eta_\theta}(x) = \sum_{a \in H\backslash B} \rho_\theta(axa^{-1})$$

for $x \in E$ in order to get the character formula of π_θ. Therefore it is essential to know when $axa^{-1} \in H$, which is determined in Lemma 2.1. From this, we get the character formula of η_θ except near the conductor (Proposition 2.2). But this formula contains the Gauss sum part $G(y, j)$, which is calculated later. The exceptional part can be calculated directly by taking the explicit matrix form of $E^\times$ (Lemma 2.4). Except this lemma, there is no new result in this section. But the proof becomes short and simple. Moreover since we use the property "intertwining implies conjugacy "of E/F-minimal (very cuspidal in the terminology of Carayol [4]) element as the key tool, the result may be extended to GL_n, at least when n is prime to p. Section 3 is devoted to the calculation of the Gauss sum part $G(y, j)$. It appears in the character formula on $E^\times$. For this purpose, it is the point that we have only to treat the character of π_θ on $U_1^* = F^\times(1 + P_E) - F^\times(1 + P_E^2)$. For this calculation, we use the $E^\times$-module structure of various objects. We first assume E/F is a Galois extension since $E^\times$-module structure can be described easily for this case. This part is analogous to section 1 of [21], but everything becomes easier since we have only to treat U_1^*. When E/F is non-Galois, we use the base change lift. Let ζ be a primitive l-th root of unity and $L = F(\zeta)$. Then L is an unramified extension of F and EL/L is Galois. Therefore we can use the tools of the Galois case for $\mathrm{GL}_l(L)$. Let $\mathrm{Gal}(L/F) = \langle\tau\rangle$. By the result of Bushnell-Henniart [3], there is a base change lift η_L of η_θ to H_L^1 such that the twisted trace of η_L by τ gives the trace of η_θ. (See Proposition 3.7 and Lemma 3.8). We remark that we need not assume the characteristic of F is 0 since we do not use the Arthur-Clozel base change lift [1]. The method of calculating the twisted trace of η_L is similar to that of Galois case. The complete character formula is stated as Theorem 3.12.

At the end of this introduction, we compare our formula with the known results. The same type of character formula for the division algebra case was given by Corwin, Moy and Sally, Jr. [6] and for GL_l case by Debacker [7]. Their formulas agree with the result given in section 2. It contains some root number associated with a quadratic form. They have shown that this root number is a root of unity when $p \neq 2$. In this paper, we have determined it completely including the case $p = 2$ in section 3. Moreover we find the Kloosterman sum appears in the character formula. These are new results of this paper. In [23], the author gave the character formula of π_θ for GL_3 by using the decomposition of π_θ as $E^\times$-module. But this needs the explicit matrix form of an inverse matrix which is hard to treat for large l. We can simplify the proof of the main theorem, although we treat a general prime l.

Notation

Let F be a non-archimedean local field. We denote by $\mathcal{O}_F$, P_F, ϖ_F, k_F and v_F the maximal order of F, the maximal ideal of $\mathcal{O}_F$, a prime element of P_F, the residue field of F and the valuation of F normalized by $v_F(\varpi_F) = 1$. We set q to be the number of elements in k_F. Hereafter we fix an additive character ψ of F whose conductor is P_F, i.e., ψ is trivial on P_F and not trivial on $\mathcal{O}_F$. For an extension E over F, we denote by tr_E, n_E the trace and norm to F respectively. We set $\psi_E = \psi \circ \mathrm{tr}_E$. The trace of matrix is denoted by Tr. For an irreducible admissible representation π of $\mathrm{GL}_l(F)$, the conductoral exponent of π is defined to be the integer $f(\pi)$ such that the local constant $\varepsilon(s, \pi, \psi)$ of Godement-Jacquet [9] is the form $aq^{-s(f(\pi)-l)}$.

We call π *minimal* if

$$f(\pi) = \min_\eta f(\pi \otimes (\eta \circ \mathrm{Nr})),$$

where η runs through the quasi-characters of $F^\times$. Let G be a totally disconnected, locally compact group. We denote by $\widehat{G}$ the set of (equivalence classes of) irreducible admissible representations of G. For a closed subgroup H of G and a representation ρ of H, we denote by $\mathrm{Ind}_H^G \rho$ (resp. $\mathrm{ind}_H^G \rho$) the induced representation (resp. compactly induced representation) of ρ to G. For a representation π of G, we denote by $\pi|_H$ the restriction of π to H.

1. Construction of the representation

Let $l \neq p$ be an odd prime and E a ramified extension of F of degree l. Then E can be embedded into $\mathrm{M}_l(F)$ and , up to conjugacy, the embedding is unique. Let $G = \mathrm{GL}_l(F)$. In this section, we review the construction of supercuspidal representations of G which are parameterized by the quasi-characters of $E^\times$. Of course, this construction is well-known ([4], [15]).

Definition 1.1. Let θ be a quasi-character of $E^\times$ and $f(\theta)$ the exponent of the conductor of θ i.e. the minimum integer such that $\mathrm{Ker}\,\theta \subset 1 + P_E^n$. Then θ is called generic if $f(\theta) \not\equiv 1 \bmod l$. For a generic character θ of $E^\times$, $\gamma_\theta \in P_E^{1-f(\theta)} - P_E^{2-f(\theta)}$ is defined by

$$\theta(1 + x) = \psi_E(\gamma_\theta x) \quad \text{for} \quad x \in P_E^{[(f(\theta)+1)/2]}. \tag{1.1}$$

Then $F(\gamma_\theta) = E$. We denote by $\widehat{E}_{gen}^\times$ the set of generic quasi-characters of $E^\times$.

We construct an irreducible supercuspidal representation of $G = \mathrm{GL}_l(F)$ from $\theta \in \widehat{E}_{gen}^\times$. For simplicity, we set $\gamma = \gamma_\theta$. Since E/F is tamely ramified, there exists a prime element ϖ_E of $\mathcal{O}_E$ satisfying $\varpi_E^l \in F$. Put $\varpi_F = \varpi_E^l$. We identifies $\mathrm{M}_l(F)$ with $\mathrm{End}_F\, E$ and G with $\mathrm{Aut}_F\, E$ by the F-basis $\{\varpi_E^{l-1}, \varpi_E^{l-2}, \ldots, \varpi_E, 1\}$ of E, which is also an $\mathcal{O}_F$-basis of $\mathcal{O}_E$.

By the lattice flag $\{P_E^i\}_{i\in\mathbb{Z}}$, we construct a maximal compact modulo center subgroup. The construction of the representation is well-known. For details, see [15].

Definition 1.2. For $i \in \mathbb{Z}$, set

$$A^i = \{f \in \mathrm{M}_l(F) | f(P_E^j) \subset P_E^{j+i} \quad \text{for all} \ \ j \in \mathbb{Z}\}.$$

Put $K = (A^0)^\times, B = E^\times K$ and $K^i = 1 + A^i$ for $i \geq 1$.

Then K is an Iwahori subgroup of G and B is a normalizer of K. At first we construct an irreducible representation of B from a generic quasi-character of $E^\times$.

Let θ be a generic quasi-character of $E^\times$, i.e., $f(\theta) = n \not\equiv 1 \mod l$. There exists an element $\gamma \in P_E^{1-n}$ such that $\theta(1+x) = \psi_E(\gamma x)$ for $x \in P_E^m$ where $m = [(n+1)/2]$. Define ψ_γ on K^m by $\psi_\gamma(1+x) = \psi(\mathrm{Tr}(\gamma x))$ for $x \in A^m$. Then ψ_γ is a quasi-character of K^m. Put $H = E^\times K^m$ and define a quasi-character ρ_θ of H by

$$\rho_\theta(h \cdot g) = \theta(h)\psi_\gamma(g) \qquad \text{for} \quad h \in E^\times, \quad g \in K^m. \tag{1.2}$$

Let J be the normalizer of ψ_γ in B, i.e.,

$$J = \{a \in B \mid \psi_\gamma^a = \psi_\gamma\},$$

where $\psi_\gamma^a(x) = \psi_\gamma(a^{-1}xa)$ for $x \in K^m$. Then $J = E^\times K^{m'}$ where $m' = [n/2]$. Put $\eta_\theta = \mathrm{Ind}_H^B \rho_\theta$.

When n is even, i.e., $n = 2m$, then $J = H = E^\times K^m$. By the Clifford theory, η_θ is an irreducible representation of B. We put

$$\kappa_\theta = \eta_\theta. \tag{1.3}$$

When n is odd, i.e., $n = 2m - 1$, then $J = E^\times K^{m-1}$. Thus η_θ is not irreducible. In this case, we put

$$\kappa_\theta = \frac{1 - \left(\dfrac{q}{l}\right) q^{(l-1)/2}}{lq^{(l-1)/2}} \sum_{\chi \in (E^\times/F^\times(1+P_E))^\wedge} \eta_{\theta\otimes\chi} + \left(\frac{q}{l}\right)\eta_\theta, \tag{1.4}$$

where $\left(\dfrac{q}{l}\right)$ is the Legendre symbol. The following result is well-known ([15],[19]).

Theorem 1.3. *Let the notation be as above. Then κ_θ is an irreducible representation of B. Put $\pi_\theta = \mathrm{ind}_B^G \kappa_\theta$. Then π_θ is an irreducible supercuspidal representation of G such that*

(1) *the L-function of π_θ is 1;*
(2) *$\varepsilon(\pi_\theta, \psi) = \varepsilon(\theta, \psi_E)$; in particular $f(\pi_\theta) = f(\theta) + l$;*
(3) *$\bigcup_E \{\pi_\theta | \theta \in \widehat{E}_{gen}^\times\} = \{\pi \in A_0(G) | f_{\min}(\pi) \not\equiv 0 \mod l\}$, where E runs through isomorphism classes of ramified extensions of degree l over F and $A_0(G)$ be the set of equivalent classes of the supercuspidal representations of G.*

Remark. If $\pi \in A_0(G)$ and $f_{\min}(\pi) \equiv 0 \mod l$, π can be constructed from a regular quasi-characters of $L^\times$, where L is an unramified extension of F of degree l. The character formula for such a representation was given in [21].

Next we construct an irreducible representation of $D^\times = D_l^\times$ from $\theta \in E_{gen}^\times$. Let $f(\theta) = n$. We recall $n \not\equiv 1 \bmod l$. We define a function ψ_γ on $1+P_D^m$ by $\psi_\gamma(1+x) = \psi(\mathrm{Tr}(\gamma x))$ for $x \in P_D^m$. Then ψ_γ is a quasi-character of $1+P_D^m$. $H' = E^\times(1+P_D^m) \subset D^\times$ and define a quasi-character ρ'_θ of H' by

$$\rho'_\theta(h \cdot g) = \theta(h)\psi_\gamma(g) \qquad \text{for} \quad h \in E^\times, \quad g \in 1+P_D^m. \tag{1.5}$$

When n is even, i.e., $n = 2m$, we set

$$\pi'_\theta = \mathrm{Ind}_{H'}^{D^\times} \rho'_\theta. \tag{1.6}$$

When n is odd, i.e., $n = 2m-1$, we set

$$\pi'_\theta = \frac{1 - \left(\frac{q}{l}\right) q^{(l-1)/2}}{lq^{(l-1)/2}} \sum_{\chi \in (E^\times/F^\times(1+P_E))^\wedge} \mathrm{Ind}_{H'}^{D^\times} \rho'_{\theta\otimes\chi} + \left(\frac{q}{l}\right) \mathrm{Ind}_{H'}^{D^\times} \rho'_\theta, \tag{1.7}$$

where $\left(\frac{q}{l}\right)$ is the Legendre symbol. The following result is essentially well-known. (See [2], [15]).

Theorem 1.4. *Let the notation be as above. Then π'_θ is an irreducible minimal representation of $D^\times$ such that*

1. *the degree of π'_θ is $q^{(n-2)(l-1)/2}\frac{(q^l-1)}{q-1}$;*
2. *$\varepsilon(\pi'_\theta, \psi) = \varepsilon(\theta, \psi_E)$; in particular $f(\pi'_\theta) = f(\theta) + l$;*
3. *$\bigcup_E \{\pi'_\theta | \theta \in \widehat{E}_{reg}^\times\} = \{\pi' \in \widehat{D}^\times | f_{\min}(\pi') \not\equiv 0 \mod l\}$, where E runs through the isomorphism classes of ramified extensions of degree l over F.*
4. *The correspondence $\pi'_\theta \leftrightarrow \pi_\theta$ by way of generic quasi-characters of $E^\times$ is a bijection and preserves ε-factors and conductoral exponents. (This correspondence is a special case of Howe's bijection (see* [15]*).)*

On the other hand, there exists an abstract matching theorem, which is called the Deligne-Kazhdan correspondence ([8], [20]).

Theorem 1.5. *There is a bijection between the set of irreducible representations of $D^\times$ and the set of essentially square-integrable representations of G which preserves the characters on elliptic regular elements. In particular, it preserves ε-factors and conductoral exponents.*

By the result of Henniart ([10] Theorem 8.1), these two correspondences coincide.

Theorem 1.6. *If $l \neq p$ is a prime, Howe's bijection (1.4) coincides with Deligne-Kazhdan correspondence (1.5) between the set of essentially square-integrable representations of GL_l and the set of irreducible representations of $D_l^\times$.*

At the end of this section, we quote the result of Kutzko [14] in the form that the character formula of π_θ on elliptic regular elements is essentially given by the one of κ_θ.

Theorem 1.7. *Let x be an elliptic regular element of G.*

(1) *If $F(x)/F$ is ramified and $x \notin F^\times(1+P_{F(x)}^n)$,*

$$\chi_{\pi_\theta}(x) = \chi_{\kappa_\theta}(x).$$

(2) *If $F(x)/F$ is unramified and $x \notin F^\times(1+P_{F(x)}^{[n/l]+1})$,*

$$\chi_{\pi_\theta}(x) = 0.$$

Proof. These are obtained by applying Proposition 5.5 in [14] to our case. □

Remark. Since

$$\chi_{\kappa_\theta}(x) = \begin{cases} \left(\dfrac{q}{l}\right) \chi_{\eta_\theta}(x) & x \in E^\times - F^\times(1+P_E), \\ \dfrac{1}{q^{(l-1)/2}} \chi_{\eta_\theta}(x) & x \in F^\times(1+P_E), \end{cases} \tag{1.8}$$

we have only to calculate χ_{η_θ}.

2. Calculation of the character

Now we begin to calculate the characters of the representations constructed in the previous section. In this section, we shall get a character formula up to some root numbers. These root numbers are calculated explicitly in the next section.

Hereafter we fix a generic character θ and put $\rho = \rho_\theta$, $\eta = \eta_\theta$ and so on. Since E/F is a totally tamely ramified extension, there exists a prime element ϖ_E of $\mathcal{O}_E$ such that $\varpi_E^l \in P_F - P_F^2$. Put $\varpi_E^l = \varpi_F$. As in the previous section, we identify $\mathrm{M}_l(F)$ with $\mathrm{End}_F(E)$ by the F-basis $\{\varpi_E^{l-1}, \varpi_E^{l-2}, \dots, \varpi_E, 1\}$, which is an $\mathcal{O}_F$-basis of $\mathcal{O}_E$. Thus we get the explicit matrix forms of various objects:

$$\varpi_E = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ \varpi_F & 0 & \cdots & \cdots & 0 \end{pmatrix}, \tag{2.1}$$

$$K = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \cdots & \cdots & \cdots \\ a_{l1} & \cdots & a_{ll} \end{pmatrix} \middle| \begin{array}{ll} a_{ij} \in \mathcal{O}_F & \text{if } i < j \\ a_{ii} \in \mathcal{O}_F^\times & \\ a_{ij} \in P_F & \text{if } i > j \end{array} \right\}, \tag{2.2}$$

$$A^0 = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \cdots & \cdots & \cdots \\ a_{l1} & \cdots & a_{ll} \end{pmatrix} \middle| \begin{array}{ll} a_{ij} \in \mathcal{O}_F & \text{if } i \le j \\ a_{ij} \in P_F & \text{if } i > j \end{array} \right\}, \tag{2.3}$$

$$A^1 = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \cdots & \cdots & \cdots \\ a_{l1} & \cdots & a_{ll} \end{pmatrix} \middle| \begin{array}{ll} a_{ij} \in \mathcal{O}_F & \text{if } i < j \\ a_{ij} \in P_F & \text{if } i \ge j \end{array} \right\}. \tag{2.4}$$

If $q \equiv 1 \bmod l$, F has a primitive l-th root of unity ζ and E/F is a Galois extension. Let σ be a generator of $\mathrm{Gal}(E/F)$ determined by ${}^\sigma\varpi_E = \varpi_E\zeta$. We denote the diagonal matrix $\mathrm{diag}(1, \zeta^{l-1}, \zeta^{l-2}, \dots, \zeta)$ by ξ. Then ξ satisfies $\xi^l = 1$ and $\xi x \xi^{-1} = {}^\sigma x$ for $x \in E$.

Define a natural ring morphism R from A^0 to k_F^l by the identification of A^0/A^1 with k_F^l. We note that if $R(a) = (\alpha_0, \alpha_1, \cdots, \alpha_{l-1})$, $R(\varpi_E a \varpi_E^{-1}) = (\alpha_1, \alpha_2, \cdots, \alpha_0)$. For convenience, we extend the suffix to $\mathbb{Z}$ by putting $\alpha_i = \alpha_{i \bmod l}$. The next lemma is the key tool for the character calculation.

Lemma 2.1. *Let $x \in P_E^i - (F + P_E^{i+1})$, $g \in B$ and j a positive integer. If $gxg^{-1} \in E^\times(1 + A^j)$, then*

$$g \in \begin{cases} E^\times(1+A^j) & \text{if} \quad q \not\equiv 1 \mod l, \\ \bigcup_{k=0}^{l-1} E^\times(1+A^j)\xi^k & \text{if} \quad q \equiv 1 \mod l. \end{cases}$$

Proof. We may assume $g \in A_0$ by replacing g by $\varpi_E^{-k} g$ if $g \in A_k$. Let $x = \varpi_E^i x_0$ for $x_0 \in \mathcal{O}_E^\times$ and $R(g) = (\alpha_0, \alpha_1, \dots, \alpha_{l-1})$. Then

$$R(gxg^{-1}x^{-1}) = (\alpha_0\alpha_i^{-1}, \alpha_1\alpha_{i+1}^{-1}, \dots, \alpha_{l-1}\alpha_{l-1+i}^{-1}),$$

where $\alpha_s = \alpha_{s \bmod l}$ for $s \in \mathbb{Z}$. Since $x \notin F + P_E^{i+1}$, $i \not\equiv 0 \bmod l$. Therefore $gxg^{-1}x^{-1} \in E^\times$ implies

$$\begin{cases} \alpha_0 = \alpha_1 = \cdots = \alpha_{l-1} & \text{if} \quad q \not\equiv 1 \bmod l, \\ \alpha_k = \zeta^j \alpha_0 \ (0 \le k \le l-1) & \text{otherwise,} \end{cases}$$

for some integer j. Since $\xi\varpi_E\xi^{-1} = \zeta\varpi_E$, we get:

$$g \in \begin{cases} E^\times(1+A^1) & \text{if} \quad q \equiv 1 \bmod l, \\ \bigcup_{k=0}^{l-1} E^\times(1+A^1)\xi^k & \text{otherwise.} \end{cases}$$

Thus we may assume $g - 1 \in A^k - (P_E^{k+1} + A^{k+1})$ for $k \ge 1$. Put $g - 1 = \varpi_E^k g_0$ and $R(g_0) = (\beta_0, \beta_1, \dots, \beta_{l-1})$. Since

$$\begin{aligned} gxg^{-1}x^{-1} &\equiv 1 + (g-1) - x(g-1)x^{-1} \mod A^{k+1} \\ &\equiv 1 + \varpi_E^k(g_0 - xg_0x^{-1}) \mod A^{k+1}, \end{aligned}$$

$R(g_0 - xg_0x^{-1}) = (\beta_0 - \beta_k, \beta_1 - \beta_{1+k}, \dots, \beta_{l-1} - \beta_{l-1+k})$. Therefore $gxg^{-1}x^{-1} \in E^\times K^{k+1}$ contradicts $g - 1 \in A^k - (P_E^{k+1} + A^{k+1})$. It implies that if $gxg^{-1}x^{-1} \in E^\times K^j$,

$$g \in \begin{cases} E^\times(1+A^j) & \text{if} \quad q \not\equiv 1 \bmod l, \\ \bigcup_{k=0}^{l-1} E^\times(1+A^j)\xi^k & \text{if} \quad q \equiv 1 \bmod l. \end{cases}$$

Hence our lemma. □

Put $U_{-1} = E^\times, U_0 = F^\times\mathcal{O}_E^\times, U_i = F^\times(1 + P_E^i)$ for $i \ge 1$ and $U_i^* = U_i - U_{i+1}$ for $j \ge -1$. The previous lemma gives the character of η_θ on $E^\times - U_{n-1}$. We remark $\mathrm{Aut}_F E = \{1\}$ if $q \not\equiv 1 \bmod l$.

Proposition 2.2. *Let $x \in U_i^*$ for $-1 \le i < n-1$. If $i \not\equiv 0 \bmod l$, x is written in the form $x = c(1+y)$ for $c \in F$ and $y = \varpi_E^i y_0 \in \varpi_E^i \mathcal{O}_E^\times$. For $u \in k_F^\times$ and $j \in (\mathbb{Z}/l\mathbb{Z})^\times$, we define the Gauss sum part $G(u, j)$ by*

$$G(u,j) = \sum_{(\alpha_0,\dots,\alpha_{l-1}) \in k_F^l/\Delta} \psi\left(\sum_{k=0}^{l-1} u(\alpha_{k+1} - \alpha_k)\alpha_{j+k}\right), \tag{2.5}$$

where $\Delta = \{(\alpha, \dots, \alpha) | \alpha \in k_F\}$ *is the image of the diagonal embedding of* k_F *into* k_F^l. *Then* χ_{η_θ} *on* U_i^* *is given as follows:*

$$\chi_{\eta_\theta}(x) = \begin{cases} \sum_{\sigma \in \operatorname{Aut}_F E} \theta({}^\sigma x) & i = -1, \\ q^{[(i+1)/2](l-1)} \sum_{\sigma \in \operatorname{Aut}_F E} \theta({}^\sigma x) & i > 0 \text{ and } n-i \text{ even}, \\ q^{[i/2](l-1)} \sum_{\sigma \in \operatorname{Aut}_F E} \theta({}^\sigma x) & \\ \quad G(\gamma \varpi_E^{n-1} y_0 ({}^\sigma \varpi_E / \varpi_E)^i, c) & i > 0 \text{ and } n-i \text{ odd}, \end{cases}$$

where $c = i^{-1}(n+i-1)/2 \in (\mathbb{Z}/l\mathbb{Z})^\times$.

Proof. Put ${}^a x = axa^{-1}$ for $a, x \in G$. At first we treat the case $x \in U_{-1}^* = E^\times - F^\times \mathcal{O}_E^\times$. Since

$$\chi_{\eta_\theta}(x) = \sum_{a \in H \backslash B} \rho_\theta({}^a x),$$

we have only to show that if ${}^a x \in H$ for $a \in B$, then

$$a \in \begin{cases} H & \text{if } \; q \not\equiv 1 \bmod l, \\ \bigcup_{k=0}^{l-1} H\xi^k & \text{if } \; q \equiv 1 \bmod l. \end{cases}$$

This follows immediately from Lemma 2.1.

Now we treat the case $x = c(1+y)$ for $c \in F$ and $y \in P_E^i - (F + P_E^{i+1})$. We may assume $c = 1$ since $F^\times$ is the center of B. For $1 + k \in K^{[(n-i+1)/2]}$ and $a \in B$, we have

$$\begin{aligned} \chi_{\eta_\theta}(1+y) &= \sum_{a \in H \backslash B} \rho_\theta({}^a(1+y)) \\ &= C \sum_{1+k \in K^{n-i} \backslash K^{[(n-i+1)/2]}} \sum_{a \in H \backslash B} \rho_\theta({}^{a(1+k)}(1+y)), \end{aligned}$$

where $C = \frac{1}{K^{n-i} \backslash K^{[(n-i+1)/2]}}$. In the above expression,

$$\begin{aligned} \rho_\theta({}^{a(1+k)}(1+y)) &= \rho_\theta(1 + {}^a y + {}^a(ky - yk)) \\ &= \rho_\theta(1 + {}^a y) \rho_\theta(1 + {}^a((1+y)^{-1}(ky - yk))) \\ &= \rho_\theta(1 + {}^a y) \psi(\operatorname{Tr} \gamma {}^a((1+y)^{-1}(ky - yk))) \\ &= \rho_\theta(1 + {}^a y) \psi(\operatorname{Tr} {}^{a^{-1}}\gamma (1+y)^{-1}(ky - yk)) \\ &= \rho_\theta(1 + {}^a y) \psi(\operatorname{Tr}(y {}^{a^{-1}}\gamma - {}^{a^{-1}}\gamma y)(1+y)^{-1} k) \end{aligned}$$

since $yk^2 \in A^n$ and $a(1+y)^{-1}(ky - yk)a^{-1} \in A^m$. If $y {}^{a^{-1}}\gamma - {}^{a^{-1}}\gamma y \notin A^{1-[(n-i+1)/2]}$, the map $k \mapsto \psi(\operatorname{Tr}(y {}^{a^{-1}}\gamma - {}^{a^{-1}}\gamma y)(1+y)^{-1} k))$ is a non-trivial character of $A^{n-i} \backslash A^{[(n-i+1)/2]}$; thus

$$\sum_{k \in A^{n-i} \backslash A^{[(n-i+1)/2]}} \psi(\operatorname{Tr}(y {}^{a^{-1}}\gamma - {}^{a^{-1}}\gamma y)(1+y)^{-1} k) = 0.$$

By Lemma 3.3 in [4], $y {}^{a^{-1}}\gamma - {}^{a^{-1}}\gamma y \in A^{1-[(n-i+1)/2]}$ is equivalent to ${}^{a^{-1}}\gamma \in E^\times K^{n-i-[(n-i+1)/2]}$. Thus it follows from Lemma 2.1 that

$$\chi_{\eta_\theta}(1+y) = \sum_{\sigma \in \operatorname{Aut}_F E} \sum_{1+a \in H \backslash E^\times K^{[(n-i)/2]}} \rho_\theta(1 + (1+a)^\sigma y (1+a)^{-1}).$$

By virtue of $(1+y)^{-1}(1+{}^{(1+a)}y) \in K^m$ and $(1+y)^{-1}(1+{}^{(1+a)}y) \equiv 1+(1+y)^{-1}((ay-ya)+(ya-ay)a) \mod K^n$,

$$\rho_\theta(1+{}^{(1+a)}y) = \theta(1+y)\psi_\gamma((1+y)^{-1}(ay-ya))\psi_\gamma((1+y)^{-1}(ya-ay)a).$$

Since

$$\psi_\gamma((1+y)^{-1}(ay-ya)) = \psi(\mathrm{Tr}(y\gamma(1+y)^{-1}-\gamma(1+y)^{-1}y)a) = 1,$$

$\psi_\gamma((1+y)^{-1}(ya-ay)a) = \psi_\gamma((ya-ay)a)$ and $|E^\times K^j/E^\times K^m| = q^{(l-1)(m-j)}$, we obtain

$$\chi_{\eta_\theta}(1+y) = \begin{cases} q^{m-(n-i)/2} \displaystyle\sum_{\sigma\in\mathrm{Aut}_F E} \theta(1+{}^\sigma y) & n-i \text{ even}, \\ q^{m-(n-i+1)/2} \displaystyle\sum_{\sigma\in\mathrm{Aut}_F E} \theta(1+{}^\sigma y)S(n-i,\sigma) & n-i \text{ odd}, \end{cases}$$

where

$$S(n-i,\sigma) = \sum_{a\in A^{(n-i+1)/2}+E\cap A^{(n-i-1)/2}\setminus A^{(n-i-1)/2}} \psi_\gamma(({}^\sigma ya-a{}^\sigma y)a).$$

Now we may assume $n-i$ odd and $\sigma=1$. Put $y=\varpi_E^i y_0$, $a=\varpi_E^{(n-i-1)/2}a_0$ and $S=S(n-i,1)$. Since

$$\begin{aligned}(ya-ay)a = \varpi_E^{n-1}(&y_0\varpi_E^{-(n-i-1)/2}a_0\varpi_E^{(n-i-1)/2} \\ &- \varpi_E^{-(n+i-1)/2}a_0\varpi_E^{(n+i-1)/2}y_0)a_0,\end{aligned}$$

we have by way of the map $R: A_0/A_1 \to k_F^l$ that

$$S = \sum_{(\alpha_j)\in k_F^l/\Delta} \psi\left(\sum_{j=0}^{l-1} \gamma\varpi_E^{n-1}y_0(\alpha_{j-(n-i-1)/2}-\alpha_{j-(n+i-1)/2})\alpha_j\right).$$

(The suffix is extended to $\mathbb{Z}$ by $\alpha_j = \alpha_{j \bmod l}$.) At first replacing the suffix j by $j+(n+i-1)/2$ and then replacing α_{ij} by α_j, we get our lemma. ☐

Remark. It is proved that the Gauss sum $q^{-(l-1)/2}G(u,j)$ is a fourth root of unity when $p\neq 2$ in [6] and [7],

Next we calculate the character on $K^{n-1}-K^n$. We state the character formula including the case $x\notin E$. On $K^{n-1}-K^n$, the Kloosterman sum appears in the formula.

Definition 2.3. For $a\in k_F^\times$, we define the Kloosterman sum $\mathrm{Kl}(a)$ by

$$\mathrm{Kl}(a) = \sum_{\substack{(y_0,\dots,y_{l-1})\in k_F^l \\ y_0\cdots y_{l-1}=a}} \psi(y_0+\cdots+y_{l-1}). \tag{2.6}$$

Lemma 2.4. *Let $x = 1+\varpi_E^{n-1}x_0$ for $x_0 = \mathrm{diag}(k_0,\dots,k_{l-1}), (k_i\in\mathcal{O}_F^\times)$. Then*

$$\chi_{\eta_\theta}(x) = q^{(l-1)(m-1)}\,\mathrm{Kl}\left((\gamma\varpi_E^{n-1})^l\prod_{j=0}^{l} k_j\right).$$

(Since $\gamma\varpi_E^{n-1}\in\mathcal{O}_E$ and $k_E=k_F$, we regard $\gamma\varpi_E^{n-1} \bmod P_E$ as an element of k_F.)

Proof. By the definition of η_θ, we have

$$\begin{aligned}&\chi_{\eta_\theta}(1+\varpi_E^n \operatorname{diag}(k_0,\dots,k_{l-1}))\\ =&q^{(l-1)(m-1)}\sum_{a\in E^\times K^1\backslash B}\psi(\operatorname{Tr}\gamma a\varpi_E^n \operatorname{diag}(k_0,\dots,k_{l-1})a^{-1}).\end{aligned}$$

It follows from (2.2) and (2.4) that the set $\{\operatorname{diag}(1,y_1,\dots,y_{l-1}) \mid y_i \in k_F^\times\}$ makes a complete system of representatives of $E^\times K^1\backslash B$. For convenience, put $y_0 = 1$. Since

$$\varpi_E \operatorname{diag}(1,y_1,\dots,y_{l-1})\varpi_E^{-1} = \operatorname{diag}(y_1,\dots,y_{l-1},1),$$

we have

$$\begin{aligned}\operatorname{Tr}\gamma \operatorname{diag}(1,y_1,\dots,y_{l-1})\varpi_E^{n-1}\operatorname{diag}(k_0,\dots,k_{l-1})\operatorname{diag}(1,y_1,\dots,y_{l-1})^{-1}&\\ \equiv \gamma\varpi_E^{n-1}\sum_{i=0}^{l-1}k_i y_{i-n+1}/y_i \bmod P_F,&\end{aligned}$$

where $y_i = y_{i \bmod l}$. By replacing y_i by $k_i y_{i-n+1}/y_i$, we get our lemma. □

On K^n, the character of $\pi = \pi_\theta$ becomes a constant function on elliptic regular conjugacy classes.

Lemma 2.5. *Let x be an elliptic regular element in K^n. Then*

$$\chi_\pi(x) = q^{(n-2)(l-1)/2}\frac{(q^l-1)}{q-1}.$$

Proof. We use the Deligne-Kazhdan correspondence (Theorem 1.5). Since the correspondence preserves the conductoral exponents, there exists a generic character θ' such that $f(\theta') = n$ and $\chi_{\pi'_{\theta'}} = \chi_{\pi_\theta}$. Since $\pi'_{\theta'}$ is trivial on $1+P_D^n$ and its degree is $q^{(n-2)(l-1)/2}\frac{(q^l-1)}{q-1}$, $\chi_{\pi'_{\theta'}}(x) = q^{(n-2)(l-1)/2}\frac{(q^l-1)}{q-1}$ for $x \in 1+P_D^n$. Consequently we have $\chi_{\pi_\theta}(x) = q^{n-1}(q^2+q+1)$ if $x \in K^n$ is elliptic regular. □

The character formula on elliptic regular conjugacy classes outside $E^\times$ can be obtained easily.

Lemma 2.6. *Let x be an elliptic regular element of B. If x satisfies the condition that $F(x) \not\simeq E$ and x is not conjugate to an element of $F^\times K^n$, then $\chi_\pi(x) = 0$.*

Proof. See Lemma 3.3 in [14]. □

3. Calculation of Gauss sums

In this section, we determine the Gauss sum part $G(y, n-i)$ explicitly. Since $G(y, n-i)$ depends only on $n-i \bmod l$ and $y \bmod P_E$, we have only to treat the character of η_θ on U_1^* by replacing n big enough.

Lemma 3.1. *Assume $n = 2m$. Then for $x \in U_1^*$,*

$$\chi_{\eta_\theta}(x) = \sum_{\sigma\in \operatorname{Aut}_F E}\ \sum_{a\in H\backslash E^\times K^{m-1}}\rho_\theta(a^\sigma x a^{-1}). \tag{3.1}$$

Proof. It follows from Lemma 2.1 that $axa^{-1} \in H$ implies $a \in E^\times K^{m-1}$. Hence our lemma. □

For the calculation of the sum in the above lemma, we use the $E^\times$-module structure of various objects. When E/F is a Galois extension, it is easy to treat. Thus we first assume E/F is Galois, i.e., $q \equiv 1 \bmod l$. We recall ξ is the diagonal matrix $\mathrm{diag}(1, \zeta^{l-1}, \zeta^{l-2}, \dots, \zeta)$ where ζ is an l-th root of unity in F and ξ satisfies $\xi^l = 1$ and $\xi x \xi^{-1} = {}^\sigma x$ for $x \in E$ where σ is the generator of $\mathrm{Gal}(E/F)$ determined by ${}^\sigma\varpi_E = \varpi_E \zeta$. By the explicit matrix form of E and A_i, we obtain:

$$\begin{array}{cccccccc} \mathrm{M}_l(F) & = E & \oplus & E\xi & \oplus & \cdots & E\xi^{l-1} \\ A^0 & = \mathcal{O}_E & \oplus & \mathcal{O}_E\xi & \oplus & \cdots & \mathcal{O}_E\xi^{l-1} \\ A^1 & = P_E & \oplus & P_E\xi & \oplus & \cdots & P_E\xi^{l-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A^{l-1} & = P_E^{l-1} & \oplus & P_E^{l-1}\xi & \oplus & \cdots & P_E^{l-1}\xi^{l-1}. \end{array} \tag{3.2}$$

Lemma 3.2. *A complete system of representatives of $H\backslash E^\times K^{m-1}$ is given by*

$$\{1 + \varpi_E^{m-1}\alpha_1\xi + \cdots + \varpi_E^{m-1}\alpha_{l-1}\xi^{l-1} \mid \alpha_i \in k_F\}.$$

Proof. It is obvious from (3.2). □

For $a = 1+\alpha_1\xi+\cdots+\alpha_{l-1}\xi^{l-1} \in A^{m-1}$, $\rho(axa^{-1})$ for $x \in U_1^*$ can be expressed explicitly in terms of $\alpha_1, \dots, \alpha_{l-1}$. At first, we determine the coefficients of a^{-1} with respect to the F-basis $\{1, \xi, \dots, \xi^{l-1}\}$.

Lemma 3.3. *For $a = \sum_{j=0}^{l-1} \alpha_j \xi^j$ $(\alpha_j \in E)$, put*

$$\Lambda(a) = ({}^{\sigma^j}\alpha_{i-j \bmod l})_{0\le i,j\le l-1} = \begin{pmatrix} \alpha_0 & {}^\sigma\alpha_{l-1} & \cdots & {}^{\sigma^{l-1}}\alpha_1 \\ \alpha_1 & {}^\sigma\alpha_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & {}^{\sigma^{l-1}}\alpha_{l-1} \\ \alpha_{l-1} & \cdots & {}^{\sigma^{l-2}}\alpha_1 & {}^{\sigma^{l-1}}\alpha_0 \end{pmatrix} \in \mathrm{M}_l(E)$$

and let $\Lambda_k(a)$ be the $(1, k+1)$-cofactor of $\Lambda(a)$. Then

$$a^{-1} = \sum_{j=0}^{l-1} \frac{\Lambda_j(a)}{|\Lambda(a)|}\xi^j,$$

where $|\Lambda(a)|$ is the determinant of $\Lambda(a)$.

Proof. By the map $\Lambda : \mathrm{M}_l(F) \to \mathrm{M}_l(E)$, we can embed $\mathrm{M}_l(F)$ into $\mathrm{M}_l(E)$. Then our lemma follows from Cramer's formula. □

Lemma 3.4. *Assume $n = 2m$ and $3(m-1) \ge 2m$. Let $c \in F^\times, y \in P_E^{m-1}$ and $a = 1 + \sum_{j=1}^{l-1} \alpha_j\xi^j \in K^{m-1}$. Then*

$$\rho_\theta(ac(1+y)a^{-1}) = \theta(c(1+y))\psi_E\left(\sum_{j=1}^{l-1}(\gamma\alpha_j{}^{\sigma^j}\alpha_{l-j} - {}^{\sigma^{-j}}\gamma\alpha_{l-j}{}^{\sigma^{-j}}\alpha_j)y\right).$$

Proof. It is obvious that we may assume $c = 1$. Since

$$\begin{aligned} g^{-1}aga^{-1} &= 1 + (g^{-1}(a-1)g - (a-1))a^{-1} \\ &= 1 + \left(\sum_{j=1}^{l-1} ({}^{\sigma^j}gg^{-1} - 1)\alpha_j \xi^j\right) a^{-1}, \end{aligned}$$

$\sum_{j=1}^{l-1}({}^{\sigma^j}gg^{-1} - 1)\alpha_j\xi^j \in A^m$ and $\mathrm{Tr}(\gamma x \xi^j) = 0$ for all $x \in E$, we have:

$$\begin{aligned} \rho_\theta(g^{-1}aga^{-1}) &= \psi_\gamma\left((\sum_{j=1}^{l-1} ({}^{\sigma^j}gg^{-1} - 1)\alpha_j\xi^j)a^{-1}\right) \\ &= \psi_\gamma\left(\sum_{j=1}^{l-1} ({}^{\sigma^j}gg^{-1} - 1)\alpha_j{}^{\sigma^j}(f_{l-j}(a))\right), \end{aligned}$$

where $f_j(a) \in E$ is defined by $a^{-1} = \sum_{j=0}^{l-1} f_j(a)\xi^j$. Put $g = 1 + y$. In the last equation, $\gamma \in P_E^{1-n}$, $f_{l-j} \in P_E^{m-1}$ and ${}^{\sigma^j}gg^{-1} - 1 \equiv {}^{\sigma^j}y - y \bmod P_E^{2m-2}$. Thus we get

$$\rho_\theta(g^{-1}aga^{-1}) = \psi_E\left(\sum_{j=1}^{l-1} ({}^{\sigma^{-j}}\gamma f_{l-j}(a)^{\sigma^{-j}}\alpha_j - \gamma^{\sigma^j}(f_{l-j}(a))\alpha_j)y\right)$$

by virtue of $\mathrm{tr}_E\, u^{\sigma^j}v = \mathrm{tr}_E\, {}^{\sigma^{-j}}uv$ for any $u, v \in E$. It follows from Lemma 3.3 that

$$f_{l-j}(a) = \frac{\Lambda_{l-j}(a)}{|\Lambda(a)|} \equiv \alpha_{l-j} \bmod P_E^{2m-2}.$$

By the assumption $3m - 3 \geq 2m$, we obtain the desired formula. □

Proposition 3.5. *Assume $q \equiv 1 \bmod l$, $n = 2m$ and $m \geq 3$.*

(1) *For $x \in U_1^*$,*

$$\chi_{\eta\theta}(x) = q^{(l-1)/2}\sum_{j=0}^{l-1}\theta({}^{\sigma^j}x).$$

(2) *For an even integer n and $y \in \mathcal{O}_F$, $G(y, n-1) = q^{(l-1)/2}$. In particular, $G(y, n-1)$ depends neither on n nor on y.*

Proof. By Lemmas 3.1, 3.2 and 3.4, we have for $c \in F^\times$ and $y \in 1 + P_E$

$$\chi_{\eta\theta}(c(1+y)) = \sum_{i=0}^{l-1}\theta(c(1 + {}^{\sigma^i}y)) \sum_{(\alpha_1,\ldots,\alpha_{l-1})\in(P_E^{m-1}/P_E^m)^{l-1}} f(\alpha_1, \ldots, \alpha_{l-1}; {}^{\sigma^i}y),$$

where

$$f(\alpha_1, \ldots, \alpha_{l-1}; y) = \psi_E\left(\sum_{j=1}^{l-1}(\gamma\alpha_j{}^{\sigma^j}\alpha_{l-j} - {}^{\sigma^{-j}}\gamma\alpha_{l-j}{}^{\sigma^{-j}}\alpha_j)y\right).$$

Put $S_j = \{(\alpha_1, \ldots, \alpha_{l-1}) \in (P_E^{m-1}/P_E^m)^{l-1} \mid \alpha_k = 0 \text{ for } k < j, \alpha_j \neq 0\}$ and $I_j(y) = \sum_{(\alpha_1,\ldots,\alpha_{l-1})\in S_j} f(\alpha_1, \ldots, \alpha_{l-1}; y)$. Then

$$\chi_{\eta\theta}(c(1+y)) = \sum_{i=0}^{l-1}\theta(c(1 + {}^{\sigma^i}y))\sum_{j=1}^{l-1} I_j({}^{\sigma^i}y).$$

If $\alpha_1 = \cdots = \alpha_{(l-1)/2} = 0$, $f(\alpha_1, \ldots, \alpha_{l-1}; y) = 0$. Thus we have

$$\sum_{j=(l+1)/2}^{l-1} I_j(y) = q^{(l-1)/2}.$$

For $1 \leq j \leq (l-1)/2$, $I_j(y)$ is proportional to

$$\sum_{\alpha_{l-j} \in P_E^m / P_E^{m+1}} \psi_E((\gamma \alpha_j {}^{\sigma^j}\alpha_{l-j} - {}^{\sigma^{-j}}\gamma \alpha_{l-j} {}^{\sigma^{-j}}\alpha_j) y).$$

Since $\alpha_j \neq 0$, the map

$$\alpha_{l-j} \mapsto \gamma \alpha_j {}^{\sigma^j}\alpha_{l-j} - {}^{\sigma^{-j}}\gamma \alpha_{l-j} {}^{\sigma^{-j}}\alpha_j$$

is a bijection from P_E^{m-1}/P_E^m to k_F. Therefore $I_j(y) = 0$. Consequently we get the first part of our lemma. $G(y, n-1) = q^{(l-1)/2}$ follows from Proposition 2.2 and the first part. $\square$

Next we assume $q - 1 \not\equiv 0 \bmod l$. In this situation, it is rather difficult to describe $E^\times$-module structure of various objects since F has no l-th primitive root of unity and E/F is not Galois. In order to apply the result of Galois case, we use the base change lift of simple characters by Bushnell-Henniart [3]. Let ζ be a primitive l-th root of unity and $L = F(\zeta)$. Then L/F is an unramified extension of degree d where d is the smallest integer satisfying $q^d \equiv 1 \bmod l$. The generator τ of $\mathrm{Gal}(L/F)$ is determined by ${}^\tau\zeta = \zeta^k$ where $k = r^{(l-1)/d}$and r is a generator of $(\mathbb{Z}/l\mathbb{Z})^\times$. We add the subscript L to the base changed objects. Then $\mathrm{M}_l(L) = \mathrm{M}_l(F) \otimes_F L$ and $E_L = E \otimes_F L \simeq EL$. E_L is a ramified Galois extension over L of degree l, an unramified extension over E of degree d with $\mathrm{Gal}(E_L/E) = \mathrm{Gal}(L/F) = \langle \tau \rangle$ and a non-Abelian Galois extension over F of degree ld. (We embed E into E_L by the map: $x \mapsto x \otimes 1$).

As in the previous section, we identifies $\mathrm{M}_l(L)$ with $\mathrm{End}_L E_L$ and $G_L = \mathrm{GL}_l(L)$ with $\mathrm{Aut}_L E_L$ by the L-basis $\{\varpi_E^{l-1}, \cdots, \varpi_E, 1\}$ of E_L, which is also an $\mathcal{O}_L$-basis of $\mathcal{O}_{E_L}$. By the lattice flag $\{P_{E_L}^i\}_{i \in \mathbb{Z}}$, we define

$$A_L^i = \{f \in \mathrm{M}_l(L) | f(P_{E_L}^j) \subset P_{E_L}^{j+i} \quad \text{for all } \ j \in \mathbb{Z}\}.$$

Put $K_L = (A_L^0)^\times, B_L = E_L^\times K_L$ and $K_L^i = 1 + A_L^i$ for $i \geq 1$. For a subgroup $M_L \subset B_L$ (resp. $M \subset B$), we set $M_L^1 = M \cap L^\times K_L$ (resp. $M^1 = M \cap F^\times K$). By the result of Kutzko (Theorem 1.7), it suffices to calculate the character of $\kappa = \kappa_\theta$ instead of π_θ. In fact, we have only to get the character of $\eta_\theta|_{B^1}$. Therefore we have only to treat the base change of $\eta_\theta|_{B^1}$ to B_L^1 where $B_L^1 = L^\times K_L$.

Definition 3.6. Let θ be a generic character of $E^\times$ with $f(\theta) = n$ and $\theta(1+x) = \psi(\mathrm{tr}_E(\gamma x))$ for $x \in P_E^m$. We define a base change lift θ_L of θ to $L^\times$ by $\theta_L = \theta \circ n_{E_L/E}$. Then $\theta_L(1+x) = \psi_L(\mathrm{tr}_{E_L/L} \gamma x)$ for $x \in P_{E_L}^m$. (Recall $m = [(n+1)/2]$.) The base change lift ρ_L of $\rho|_{H^1}$ to $H_L^1 = L^\times(1 + P_{E_L})K_L^m$ is defined by

$$\rho_L(h \cdot g) = \theta_L(h) \psi_L(\mathrm{Tr}\, \gamma(g-1)) \qquad \text{for} \quad h \in L^\times(1 + P_{E_L}), \quad g \in K_L^m.$$

We define the base change η_L of $\eta|_{B^1}$ to B_L^1 by

$$\eta_L = \mathrm{Ind}_{H_L^1}^{B_L^1} \rho_L.$$

By virtue of $\theta_L \circ \tau = \theta_L$, we have $\rho_L \circ \tau = \rho_L$. Thus we can define an extension $\tilde{\rho}_L$ of ρ_L to $H_L^1 \rtimes \langle\tau\rangle$ by

$$\tilde{\rho}_L(x \rtimes \tau) = \rho_L(x) \quad \text{for } x \in H_L^1.$$

Now we apply the result of Bushnell-Henniart [3] to our case and get the character relation between η_L and $\tilde{\eta}_L$. Put $U_{E_L,i} = L^\times(1+P_{E_L}^i)$ for $i > 0$ and $U_{E_L,i}^* = U_{E_L,i} - U_{E_L,i+1}$. By (12.19) Corollary in [3] and the fact $\langle\tau\rangle$-fixed space ${}^{\langle\tau\rangle}(L^\times K_L^i)$ is equal to $F^\times K^i$, the following result follows.

Proposition 3.7. *Let $x \in U_{E_L,1}$. Between the set*

$$\{g \in H^1 \backslash (E^\times K^{m-1})^1 \mid g n_{E_L/E}(x) g^{-1} \in H^1\}$$

and the set

$$\{h \in H_L^1 \backslash (E_L^\times K_L^{m-1})^1 \mid h x^\tau h^{-1} \in H_L^1\},$$

there is a bijection ψ with the property

$$\rho_L(\psi(g) x^\tau (\psi(g))^{-1}) = \rho(g n_{E_L/E}(x) g^{-1}).$$

Combining this with Lemma 3.1, we have:

Lemma 3.8.

$$\chi_{\eta_\theta}(n_{E_L/E}(x)) = \sum_{\substack{a \in H_L^1 \backslash (E_L^\times K_L^{m-1})^1 \\ a x^\tau a^{-1} \in H_L}} \rho_L(a x^\tau a^{-1}). \tag{3.3}$$

Since $n_{E_L/E}(L^\times(1+P_{E_L}^i)) = F^\times(1+P_E^i)$, it suffices to calculate the right hand side of (3.3) for $x \in U_{E_L,1}^*$.

As in the Galois case, set $\xi = \mathrm{diag}(1, \zeta^{l-1}, \zeta^{l-2}, \ldots, \zeta) \in \mathrm{M}_l(L)$. Then ξ satisfies $\xi^l = 1$, ${}^\tau\xi = \xi^k$ and

$$\xi x \xi^{-1} = {}^\sigma x \quad \text{for any} \quad x \in E_L,$$

where σ is the generator of $\mathrm{Gal}(E_L/L)$ determined by ${}^\sigma\varpi_E = \varpi_E\zeta$. Moreover we have $\tau\sigma\tau^{-1} = \sigma^k$ and

$$\begin{array}{lllllll} \mathrm{M}_l(L) & = E_L & \oplus & E_L\xi & \oplus & \cdots & E_L\xi^{l-1} \\ A_L^0 & = \mathcal{O}_{E_L} & \oplus & \mathcal{O}_{E_L}\xi & \oplus & \cdots & \mathcal{O}_{E_L}\xi^{l-1} \\ A_L^1 & = P_{E_L} & \oplus & P_{E_L}\xi & \oplus & \cdots & P_{E_L}\xi^{l-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_L^{l-1} & = P_{E_L}^{l-1} & \oplus & P_{E_L}^{l-1}\xi & \oplus & \cdots & P_{E_L}^{l-1}\xi^{l-1}. \end{array} \tag{3.4}$$

We note that any element of K_L^1 can be written in the form $(1+\alpha_1\xi+\alpha_2\xi^2+\cdots+\alpha_{l-1}\xi^{l-1})$ for $\alpha_i \in P_{E_L}$.

Lemma 3.9. *Let $i < m$ and $a = 1+\alpha_1\xi+\alpha_2\xi^2+\cdots+\alpha_{l-1}\xi^{l-1}$ for $\alpha_j \in \mathcal{O}_E$ and $x \in U_{E_L,i}^*$. Then $a x^\tau a^{-1} \in H_L$ is equivalent to $\alpha_j \in P_{E_L}^{m-i}$ and $\alpha_{hk^j} = {}^{\tau^j}\alpha_h$ for $j = 0, 1, \ldots, d-1$ and $h = 1, r, \ldots, r^{(l-1)/d-1}$. (The suffix of α_j is extended to $\mathbb{Z}$ by $\alpha_j = \alpha_{j \bmod l}$.)*

Proof. It follows from Lemma 3.2 that if $a^{-1} x^\tau a \in H_L$, there exist $\gamma_0 \in \mathcal{O}_E^\times$ and $\gamma_j \in P_{E_L}^m$ for $1 \le j \le l-1$ such that

$$(1+\alpha_1\xi+\cdots+\alpha_{l-1}\xi^{l-1})x = \gamma_0(1+\gamma_1\xi+\gamma_2\xi^2+\cdots+\gamma_{l-1}\xi^{l-1})$$
$$(1+{}^\tau\alpha_1\xi^k+{}^\tau\alpha_2\xi^{2k}+\cdots+{}^\tau\alpha_{l-1}\xi^{(l-1)k}).$$

It implies

$$\begin{aligned}
x &= \gamma_0(1+{\gamma_{l-k}}^{\sigma^{l-k}\tau}\alpha_1+{\gamma_{l-2k}}^{\sigma^{l-2k}\tau}\alpha_2+\cdots+{\gamma_k}^{\sigma^k\tau}\alpha_{l-1})\\
{\alpha_k}^{\sigma^k}x &= \gamma_0(\gamma_k+{}^{\tau}\alpha_1+{\gamma_{l-k}}^{\sigma^{l-k}\tau}\alpha_2+\cdots+{\gamma_{2k}}^{\sigma^{2k}\tau}\alpha_{l-1})\\
&\cdots\cdots\\
{\alpha_{l-k}}^{\sigma^{l-k}}x &= \gamma_0(\gamma_{l-k}+{\gamma_{l-2k}}^{\sigma^{l-2k}\tau}\alpha_1+\cdots+{}^{\tau}\alpha_{l-1}).
\end{aligned}$$

Thus we have

$${\alpha_{hk}}^{\sigma^{hk}}x = x^{\tau}\alpha_h \bmod P_{E_L}^m \quad (h\in(\mathbb{Z}/l\mathbb{Z})^{\times}).$$

By eliminating $\alpha_{hk},\alpha_{hk^2},\dots,\alpha_{hk^{d-1}}$, we get

$$\alpha_h = n_{E_L/E}(x)^{\sigma^k}n_{E_L/E}(x)^{-1}\alpha_h \bmod P_{E_L}^m.$$

Since $n_{E_L}(x)^{\sigma^k}n_{E_L/E}(x)^{-1}\in 1+P_E^i-P_E^{i+1}$, $\alpha_k\in P_{E_L/E}^{m-i}$. By $x^{\sigma^k}x^{-1}\in 1+P_{E_L}^i$, we obtain $\alpha_h\in P_{E_L}^{m-i}$ and $\alpha_{hk^j}={}^{\tau^j}\alpha_h \bmod P_{E_L}^m$ for $j=0,1,\dots,d-1$ and $h=1,r,\dots,r^{(l-1)/d-1}$. □

Lemma 3.10. *Assume $n=2m$ and $m\geq 3$. Let $x\in 1+P_{E_L}-P_{E_L}^2$ and $a=1+\sum_{i=1}^{(l-1)/d}\sum_{j=1}^{d}{}^{\tau^j}\alpha_{r^i}\xi^{r^ik^j}$ for $\alpha_{r^i}\in P_{E_L}^{m-1}$. Then*

$$\rho_L(ax^{\tau}a^{-1}x^{-1})=\psi_E\left(\sum_{i=1}^{(l-1)/d}\operatorname{tr}_{E_L/E}(u_i-{}^{\sigma^{-r^i}}u_i)\operatorname{tr}_{E_L/E}(x-1)\right) \tag{3.5}$$

where $u_i=\gamma{\alpha_{r^i}}^{\sigma^{r^i}}\alpha_{-r^i}$.

Proof. By Lemma 3.9, ${}^{\tau}aa^{-1}\in H_L$. Since $\rho_L({}^{\tau}aa^{-1})=1$, it implies $\rho_L(ax^{\tau}a^{-1}g^{-1})=\rho_L(axa^{-1}g^{-1})$. By the same way as Lemma 3.4, we have:

$$\rho_L(ax^{\tau}a^{-1}x^{-1})=\psi_{E_L}\left(\sum_{i=1}^{(l-1)/d}\sum_{j=1}^{d}(v_{i,j}-{}^{\sigma^{-r^ik^j}}v_{i,j})(x-1)\right)$$

where $v_{i,j}=\gamma^{\tau^j}{\alpha_{r^i}}^{\tau^j\sigma^{r^i}}\alpha_{-r^i}$. Since $\sigma^{r^ik^j}\tau^j=\tau^j\sigma^{r^i}$ and ${}^{\tau}\gamma=\gamma$, we have

$$\begin{aligned}
\sum_{j=1}^{d}(v_{i,j}-{}^{\sigma^{-r^ik^j}}v_{i,j}) &= \sum_{j=1}^{d}(\gamma^{\tau^j}{\alpha_{r^i}}^{\tau^j\sigma^{r^i}}\alpha_{-r^i}-{}^{\tau^j\sigma^{-r^i}}\gamma^{\tau^j}{\alpha_{-r^i}}^{\tau^j\sigma^{-r^i}}\sigma_{-r^i})\\
&= \sum_{j=1}^{d}{}^{\tau^j}(\gamma{\alpha_{r^i}}^{\sigma^{r^i}}\alpha_{-r^i}-{}^{\sigma^{-r^i}}\gamma^{\sigma^{-r^i}}\alpha_{r^i}\alpha_{-r^i})\\
&= \operatorname{tr}_{E_L/E}(\gamma{\alpha_{r^i}}^{\sigma^{r^i}}\alpha_{-r^i}-{}^{\sigma^{-r^i}}\gamma^{\sigma^{-r^i}}\alpha_{r^i}\alpha_{-r^i}).
\end{aligned}$$

This implies the equation (3.5). □

It is time to get the character value of χ_η on U_1^*.

Proposition 3.11. *Let $x\in 1+P_{E_L}-P_{E_L}^2$ and $n=2m>6$. Then*

$$\chi_\eta(n_{E_L/E}(x))=\left(\frac{q}{l}\right)q^{(l-1)/2}\theta(n_{E_L/E}(x))$$

and

$$G(y,j)=\left(\frac{q}{l}\right)q^{(l-1)/2}$$

for all $y \in k_F$ and j odd.

Proof. By Proposition 3.7, Lemmas 3.8, 3.9, 3.10 and 3.11, we have:

$$\chi_\eta(n_{E_L/E}(x)) = \theta_L(x) \sum_{(\alpha_{r^i})} \psi_E \left(\sum_{i=1}^{(l-1)/d} \operatorname{tr}_{E_L/E}(u_i - {}^{\sigma^{-r^i}}u_i) \operatorname{tr}_{E_L/E}(x-1) \right) \tag{3.6}$$

where $u_i = \gamma \alpha_{r^i} {}^{\sigma^{r^i}}\alpha_{-r^i}$ and $(\alpha_{r^i})_{1 \le i \le (l-1)/d} \in (P_{E_L}^{m-1}/P_{E_L}^m)^{(l-1)/d}$. First we assume $(l-1)/d$ is odd. Then $\left(\frac{q}{l}\right) = -1$, d is even and ${}^{\tau^{d/2}}\alpha_{r^i} = \alpha_{-r^i}$. Let E_i be the $\langle \sigma^{r^i}\tau^{d/2}\rangle$-fixed field. Then E_L/E_i is a quadratic unramified extension, $\alpha_{r^i}{}^{\sigma^{r^i}\tau^{d/2}}\alpha_{r^i} = n_{E_L/E_i}(\alpha_{r^i})$, n_{E_L/E_i} induces a surjection from $\varpi_E^{m-1}\mathcal{O}_{E_L}/1+P_{E_L}$ to $\varpi_E^{2m-2}\mathcal{O}_{E_i}/1+P_{E_i}$ and each fiber of the induced map has $q^{d/2}+1$ elements. Moreover the map $x \mapsto \operatorname{tr}_{E_L/E_i}(x - {}^{\sigma^{-r^i}}x$ induces a surjective k_F-linear map from $P_{E_i}^{2m-2}/P_{E_i}^{2m-1}$ to $P_E^{-1}/\mathcal{O}_E$. Thus we have:

$$\begin{aligned}\sum_{\alpha_{r^i} \in P_{E_L}^{m-1}/P_{E_L}^m} \psi_E \left(\sum_{i=1}^{(l-1)/d} \operatorname{tr}_{E_L/E}(u_i - {}^{\sigma^{-r^i}}u_i) \operatorname{tr}_{E_L/E}(x-1) \right) \\ = (1 - (q^{d/2}+1)).\end{aligned}$$

Putting this into 3.6, we get:

$$\begin{aligned}\chi_\eta(n_{E_L/E}(x)) &= \theta_L(x)(1-(q^{d/2}+1))^{(l-1)/d} \\ &= -q^{(l-1)/2}\theta_L(x)\end{aligned}$$

and it follows from Proposition 2.2 that $G(y,j) = -q^{(l-1)/2}$ for all $y \in k_F$ and j odd. Now we assume $(l-1)/d$ is even. Then $\left(\frac{q}{l}\right) = 1$ and it follows from the same argument as in the proof of Proposition 3.5 that

$$\begin{aligned}\chi_\eta(n_{E_L/E}(x)) &= \theta_L(x)|k_{E_L}|^{(l-1)/2d} \\ &= q^{(l-1)/2}\theta_L(x).\end{aligned}$$

By Proposition 2.2, we have $G(y,j) = q^{(l-1)/2}$ for all $y \in k_F$ and j odd in this case. □

Putting all these together, we can state the character formula.

Theorem 3.12. *Let E be a ramified extension of F with degree l, θ a generic quasi-character of $E^\times$ with $f(\theta) = n$ and $\pi = \pi_\theta$ the irreducible supercuspidal representation of $\mathrm{GL}_l(F)$ defined in section 1. Put $U_0 = F^\times\mathcal{O}_E^\times$, $U_j = F^\times(1+P_E^j)$ and $U_j^* = U_j - U_{j+1}$ for $j \ge 1$. Let x be an elliptic regular element of $\mathrm{GL}_l(F)$ and $\operatorname{Aut}_F E$ the group of automorphism of E over F.*

(1) *If $F(x)/F$ is unramified, then*

$$\chi_\pi(x) = \begin{cases} 0 & x \notin F^\times(1+P_{F(x)}^n), \\ q^{(n-2)(l-1)/2}\frac{(q^l-1)}{q-1}\theta(c) & x = c(1+y) \\ & \quad \text{for } c \in F^\times, y \in P_{F(x)}^n. \end{cases}$$

(2) *If $F(x)/F$ is ramified and $F(x) \not\simeq E$, then*

$$\chi_\pi(x) = \begin{cases} 0 \quad \text{if} \quad x \notin F^\times(1 + P_{F(x)}^{n-1}), \\ q^{(n-2)(l-1)/2}\theta(c)\,\mathrm{Kl}((\gamma\varpi_E^{n-1})^l \prod_{j=0}^{l-1} k_j) \\ \qquad \text{if} \quad x = c(1 + \varpi_E^{n-1}\,\mathrm{diag}(k_0, \ldots, k_{l-1}) + z) \\ \qquad\quad \text{for} \quad c \in F^\times, k_i \in k_F^\times, z \in P_{F(x)}^n, \\ q^{(n-2)(l-1)/2}\frac{(q^l-1)}{q-1}\theta(c) \\ \qquad \text{if} \quad x = c(1+y) \quad \text{for} \quad c \in F^\times, y \in P_{F(x)}^n. \end{cases}$$

(3) *When $x \in E$, then*

$$\chi_\pi(x) = \begin{cases} \left(\frac{q}{l}\right)^{n-j} q^{j(l-1)/2} \sum_{\sigma \in \mathrm{Aut}_F E} \theta({}^\sigma x) \\ \qquad \text{if} \quad x \in U_j^* \quad \text{for} \quad 0 \le j < n-1, \\ q^{(n-2)(l-1)/2}\theta(c)\,\mathrm{Kl}((\gamma\varpi_E^{n-1}x_0)^l) \\ \qquad \text{if} \quad x = c(1 + \varpi_E^{n-1}x_0) \quad \text{for} \quad c \in F^\times, x_0 \in \mathcal{O}_E^\times, \\ q^{(n-2)(l-1)/2}\frac{(q^l-1)}{q-1}\theta(c) \\ \qquad \text{if} \quad x = c(1+y) \quad \text{for} \quad c \in F^\times, y \in P_E^n. \end{cases}$$

(See (2.6) *for the definition of the Kloosterman sum* $\mathrm{Kl}(a)$*.)*

Proof. It follows from (1.8), Theorem 1.7, Lemmas 2.4, 2.5, 2.6, Propositions 2.2, 3.5 and 3.11 □

Remark. By Theorem 1.6, the character formula of the representation π'_θ of $D^\times$ is given by the same formula for π_θ.

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