ま え が き

本講究録は、数理解析研究所研究集会「保型形式と局所体上の代数群の表現」（平成１５年１月２０日ー２４日）の報告集です。

保型形式に関する最近の結果に加えて、保型形式と関連する局所体上の表現についての講演を中心にプログラムを準備しました。局所体上の代数群の表現と保型形式とは密接な関係を持っておりますが、この方面の日本における研究は活発とは言い難い状況にあったと思います。これまで局所体上の表現に関心を持っておられなかった方々にも興味を持って頂くきっかけとなれば幸いです。

また今回は、国外から若手の研究者を招待し、日本の研究者との交流をはかることを一つの目標としました。そのため、プログラムの編成について例年と異なり、制限が加わったと感じられた方々もおられることと存じます。このことについては御寛恕をお願いする次第です。幸いに８名の国外の研究者の参加を得、講演の合間を縫って熱心な討論が行われました。これが今後の交流、共同研究の端緒となればと思っております。

これらの国外の研究者を招待するにつきましては、上野健爾氏、伊吹山知義氏、行者明彦氏の科学研究費からの援助を頂きました。ここに御礼申し上げます。

最後に、研究集会参加者、講演者の方々、厳寒の京都の集会に参加頂きました、国外からの参加者の方々に御礼申し上げます。

２００３年7月

研究代表者　齋藤　裕

副代表者　　高橋哲也

Preface

This is the proceeding of the conference "automorphic forms and representations of algebraic groups over local fields" held at Research Institute of Mathematical Science, 20 -24 January 2003.

During the conference, there were 19 talks concerning with automorphic forms and related topics. This volume contains the texts of all lectures.

The organizers wish to express their gratitude to all participants, speakers, especially foreign mathematicians, who visited Kyoto in the coldest season and gave wonderful talks.

Kyoto, July, 2003

Hiroshi Saito

Tetusya Takahashi

保型形式と局所体上の代数群の表現

研　究　集　会

京都大学数理解析研究所の共同事業の一つとして、下記のように研究集会を催しますのでご案内申しあげます。

研究代表者　　齋藤 裕
（京都大学人間環境学研究科）
高橋　哲也
（大阪府立大学総合科学部）

記

日時：２００３年　1月２０日（月）１３：３０～
1月２４日（金）１１：４５

場所：京都大学数理解析研究所４階４２０号室
京都市左京区追分町
市バス　京大農学部前　または　北白川　下車

プログラム

1月２０日（月）

13:30 - 14:30　Tsuneo Arakawa (Rikkyo Univ.)
Shimura correspondence for Maass wave forms and Selberg zeta functions

14:45 - 15:45　Shuichi Hayashida（Osaka Univ.)
Skew holomorphic Jacobi forms of general degree

16:00 - 17:00　Shin-ichiro Mizumoto (Tokyo I. T.)
Certain series attached to an even number of elliptic modular forms

1月21日（火）

9:30 - 10:30　Taku Ishii (Univ. Tokyo)
Principal series Whittaker functions on symplectic groups

10:45 - 12:15　Kyo Nishiyama (Kyoto Univ.)
Theta correspondence and representation theory

14:00 - 15:00　Takayuki Oda(Univ. Tokyo)
Whittaker functions of nonspherical principal series on $SL(3, \mathbf{R})$

15:15 - 16:15　Richard Hill (Univ. College London)
Fractional weights and non-congruence subgroups

18:00 -　party（Kyodai-kaikan）

１月２２日（水）

9:30 - 10:30	Noritomo Kojima (Tokyo I.T.) Standard L-functions attached to vector valued Siegel modular forms
10:45 - 11:45	Yumiko Hironaka（Waseda Univ.） Spherical functions on certain spherical homogeneous spaces over p-adic fields
13:15 - 14:15	Brooks Roberts (Univ Idaho) Canonical vectors for representations of GSp(4):results and conjectures
14:30 - 15:30	Jeffrey Hakim (American Univ) Supercuspidal representations attached to symmetric spaces
15:45 - 16:45	Kaoru Hiraga (Kyoto Univ.) On functoriality of Zelevensky involution

１月２３日（木）

9:30 - 10:30	Takuya Konno(Kyushu Univ.) TBA
10:45 - 11:45	Jean Francois Dat (Univ Strasbourg) Parabolic induction and paraholic induction
13:15 - 14:15	Ralf Schmidt (Univ Saarlandes) On Siegel modular forms with square-free level
14:30 - 15:30	Bao Chau Ngo (Univ Paris 13) Shtukas with multiples modifications and base change identities
15:45 - 15:45	Atsushi Ichino (Osaka City Univ.) Restrictions of hermitian Maass lifts and the Gross-Prasad conjecture(joint work with Ikeda)

1月24日（金）

9:30 - 10:30	Wee Teck Gan (Princeton Univ) Multiplicities of cusp forms
10:45 - 11:45	Tamotsu Ikeda(Kyoto Univ.) On the lifting for hermitian modular forms

ホームページ
http://wwwmi.cias.osakafu-u.ac.jp/ takahasi/workshop/january/index.html
に当研究集会関連の情報を随時掲載しますので、こちらもご覧ください。

保型形式と局所体上の代数群の表現
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研究代表者　齋藤　裕 (Hiroshi Saito)

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Shimura correspondence for Maass wave forms and Selberg zeta functions

Tsuneo ARAKAWA (Rikkyo University)

荒川 恒男 (立教大学 理学部)

0 Introduction

Shimura in [Shm] established a significant correspondence from holomorphic modular forms of even integral weight $2k-2$ to modular forms of half integral weight $k-1/2$ which is consistent with the actions of Hecke operators. The converse correspondence was given by Shintani [Shn] in terms of period integrals. After these results, Kohnen ([Koh]) showed that this correspondence yields a bijection from the space S_{2k-2} of holomorphic modular forms of weight $2k-2$ on $SL_2(\mathbb{Z})$ to the plus space $S^+_{k-1/2}$ of modular cusp forms of weight $k-1/2$ on $\Gamma_0(4)$. On the other hand the plus space corresponds bijectively to the space $J^{cusp}_{k,1}$ of holomorphic Jacobi cusp forms (resp. the space $J^{sk,\,cusp}_{k,1}$ of skew holomorphic Jacobi cusp forms ([Sk1], [Sk2])) of weight k and index 1 on $SL_2(\mathbb{Z})$ if k is even (resp. odd). We exhibit here the isomorphisms in the case of $k>1$ being odd:

$$S_{2k-2} \cong S^+_{k-1/2} \cong J^{sk,\,cusp}_{k,1}. \tag{0.1}$$

As for the Maass wave forms Katok-Sarnak in [KS] formed the Shimura correspondence from the space of even Maass wave forms to a certain plus space consisting of automorphic forms of weight 1/2. This work is understood to give an analogue of Shintani's converse correspondence to the case of Maass wave forms.

A purpose of this article is to explain an analogue of the right correspondence in the above (0.1) in the case of Maass wave forms. Another purpose is to interpret this Shimura correspondence for Maass wave forms from viewpoints of Selberg zeta functions and resolvent Selberg trace formulas. Finally we discuss some arithmetic aspects of Selberg zeta functions and also some applications.

We explain a little more in details. Let $\Gamma = SL_2(\mathbb{Z})$ and $\mathcal{H}^{even}_0$ denote the space of even functions $f \in \mathcal{H}_0 = L^2(\Gamma\backslash\mathfrak{H})$ satisfying $f(-\overline{z}) = f(z)$. It is known by Katok-Sarnak [KS] that to each Hecke eigen Maass wave form $f \in \mathcal{H}^{even}_0$ there corresponds an automorphic form g in the plus space of weight 1/2 having reasonable properties. The whole plus space corresponds to the space $\mathcal{H}_{-1/4,\chi}$ of automorphic forms attached to the theta multiplier system χ defined by (1.2). This space plays an alternative role of the space of skew holomorphic Jacobi cusp forms in (0.1). We have computed the resolvent trace formula for $\mathcal{H}_0^{even}$ and that of $\mathcal{H}_{-1/4,\chi}$. There attached to the space $\mathcal{H}_0^{even}$ the Selberg zeta function $Z_{even}(s)$ is introduced, while associated to the multiplier system χ we have the Selberg zeta function $Z_\chi(s)$ (see (2.1), (2.3)). By comparing the both resolvent trace formulas for $\mathcal{H}_0^{even}$ and $\mathcal{H}_{-1/4,\chi}$ the conjectural bijectivity of the Katok-Sarnak correspondence will be reduced to some simple relationship of the two Selberg zeta functions concerned, which will be presented as a new conjecture (Conjecture 4). Towards the solution of our conjecture we discuss an explicit arithmetic expression of the Selberg zeta function $Z_\chi(s)$. The explicit espression of $Z_{even}(s)$ can easily be obtained similarly from that of $Z(s)$, the original Selberg zeta function for $SL_2(\mathbb{Z})$.

Finally as an application of the trace formula for $\mathcal{H}_0^{even}$ the prime geodesic theorem ((4.4), Theorem 6) for $GL_2(\mathbb{Z})$ will be given. This will be a refinement of the original result for the group $SL_2(\mathbb{Z})$ due to Sarnak [Sa].

1 Shimura correspondence for Maass wave forms

We use the symbol $e(w)$ for $\exp(2\pi i w)$. Throughout this article Γ denotes the modular group $SL_2(\mathbb{Z})$. Let $\mathfrak{H}$ denote the upper half plane. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathfrak{H}$, $J(A,z) := cz + d$ denotes the usual factor of automorphy for $SL_2(\mathbb{R})$. For a non-zero complex number w, $\arg w$ is chosen so that $-\pi < \arg w \leqq \pi$ and the branch of a holomorphic function $w^s = \exp(s \log w)$ $(w \neq 0)$ is fixed once and for all. For $A,\ B \in SL_2(\mathbb{R})$, the cocycle $\sigma_{2k}(A,B)$ is given by

$$\sigma_{2k}(A,B) = \exp\bigl(2ik\{\arg J(A,Bz) + \arg J(B,z) - \arg J(AB,z)\}\bigr)$$

(note here that the right hand side is independent of z).

Following [Fi], we give a definition of a multiplier system of Γ. Let V be a finite dimensional $\mathbb{C}$-vector space equipped with a positive definite hermitian scalar product $\langle v, w\rangle$ $(v, w \in V)$ and let $\mathcal{U}(V)$ denote the group of unitary tansformations of V with respect to the scalar product. A map $\chi : \Gamma \longrightarrow \mathcal{U}(V)$ is called a *multipier system* of Γ of weight $2k$ $(k \in \mathbb{R})$, if it satisfies

(i) $\chi(-1_2) = e^{-2\pi i k} id_V$, id_V being the identity map of V.

(ii) $\chi(AB) = \sigma_{2k}(A,B)\chi(A)\chi(B)$ for all $A,\ B \in \Gamma$.

We set, for $A \in SL_2(\mathbb{R})$ and a function f on $\mathfrak{H}$,

$$f|[M,k](z) := j_M(z)^{-1} f(Mz)$$

with $j_M(z) = \exp(2ik \arg J(M,z))$. Let $\mathcal{H}_{k,\chi}$ denote the space of V-valued measurable functions on $\mathfrak{H}$ with the properties

(i) $f|[M,k] = \chi(M)f$ for all $M \in \Gamma$,

(ii) $(f,f) := \int_{\Gamma\backslash\mathfrak{H}} \langle f(z), f(z)\rangle d\omega(z) < +\infty.$

Then $\mathcal{H}_{k,\chi}$ forms a Hilbert space with respect to the scalar product

$$(f,g) = \int_{\Gamma\backslash\mathfrak{H}} \langle f(z), g(z)\rangle d\omega(z), \qquad (f,\ g \in \mathcal{H}_{k,\chi}).$$

The differential operator Δ_k which is consistent with the action $f|[A,k]$ is given by

$$\Delta_k := y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) - 2iky\frac{\partial}{\partial x}.$$

A fundamental subspace $\mathcal{D}$ of $\mathcal{H}_{k,\chi}$ consists of C^2-class functions f satisfying $(\Delta_k f, \Delta_k f) < \infty$. Since $-\Delta_k$ is symmetric on $\mathcal{D}$, it is known by [Ro],I, Satz3.2 that there exists the unique self-adjoint extension $-\widetilde{\Delta}_k : \widetilde{\mathcal{D}} \longrightarrow \mathcal{H}_{k,\chi}$, where $\widetilde{\mathcal{D}}$ denotes the domain of definition of $-\widetilde{\Delta}_k$. By the self-adjointness of $-\widetilde{\Delta}_k$, eigen values of $-\widetilde{\Delta}_k$ are all real numbers. So we let

$$\lambda_n = \frac{1}{4} + r_n^2 \quad (\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots)$$

denote all distinct eigen values of $-\widetilde{\Delta}_k$. We may choose r_n so that $r_n \in i(0,\infty) \cup [0,\infty)$. Denote by $\mathcal{H}_{k,\chi}(s)$ the space of C^2-class functions $f \in \mathcal{H}_{k,\chi}$ satisfying $-\Delta_k f = s(1-s)f$. It is known that $\mathcal{H}_{k,\chi}(s)$ is a finite dimensional $\mathbb{C}$-vector space. Moreover

$$d_n := \dim \mathcal{H}_{k,\chi}\Big(\frac{1}{2} + ir_n\Big)$$

gives the multiplicity of $\lambda_n = \frac{1}{4} + r_n^2$ of $-\widetilde{\Delta}_k$. Let $s,\ a \in \mathbb{C}$. The spectral series attached to the multiplier system (Γ, χ) is defined by

$$S_{\Gamma,\chi}(s,a) := \sum_{n=0}^{\infty} \left(\frac{d_n}{\big(s - 1/2\big)^2 + r_n^2} - \frac{d_n}{\big(a - 1/2\big)^2 + r_n^2} \right). \tag{1.1}$$

It is known that the infinite series is absolutely convergent for s, a with $s \neq \frac{1}{2} \pm ir_n$, $a \neq \frac{1}{2} \pm ir_n$. Then $S_{\Gamma,\chi}(s,a)$ indicates a meromorphic function of s whose poles are located at $s = \frac{1}{2} \pm ir_n$. They are simple poles except for $s = 1/2$ $(r_n = 0)$.

In this note we exclusively concider the following two cases. First let $k = 0$, $V = \mathbb{C}$ and χ be the trivial character of Γ. Then

$$\mathcal{H}_0 := \mathcal{H}_{0,\chi} = L^2(\Gamma\backslash\mathfrak{H}).$$

A function f of $\mathcal{H}_0$ is called an even function if it satisfies $f(-\overline{z}) = f(z)$. Let $\mathcal{H}_0^{even}$ (resp. $\mathcal{H}_0^{even}(s)$ $(s \in \mathbb{C})$) be the subspace of $\mathcal{H}_0$ consisting of even functions (resp. even C^2-class functions with $-\Delta_k f = s(1-s)f$). We denote by $S_\Gamma^{even}(s,a)$ the spectral series attached to the space $\mathcal{H}_0^{even}$ and the differential operator $\Delta_0 = y^2\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\big)$ which is similarly defined as in (1.1).

Another one is the multiplier system obtained from the theta transformation formula. Let $\theta_i(\tau,z)$ $(i = 0,\ 1)$ be the usual theta series defined by

$$\theta_i(\tau,z) = \sum_{n\in\mathbb{Z}} e\big((n+i/2)^2\tau + (2n+i)z\big).$$

The theta transformation law for these theta series is described as follows:

$$\begin{pmatrix} \theta_0(M(\tau,z)) \\ \theta_1(M(\tau,z)) \end{pmatrix} = e\Big(\frac{cz^2}{J(M,\tau)}\Big) J(M,z)^{1/2} U(M) \begin{pmatrix} \theta_0(\tau,z) \\ \theta_1(\tau,z) \end{pmatrix} \quad \Big(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\Big),$$

where $U(M)$ is a unitary matrix of size two. For the convenience we consider the complex conjugate χ of U:

$$\chi(M) = \overline{U(M)} \quad (M \in \Gamma). \tag{1.2}$$

Since we have $\chi(-1_2) = e^{\pi i/2}1_2$, χ forms a multiplier system of Γ with weight $-1/2$. Let $\mathcal{H}_{-1/4,\chi}$ and $\mathcal{H}_{-1/4,\chi}(s)$ be the spaces defined as above for this multiplier χ and Γ.

We explain here the Maass wave form version of the correspondences in (0.1). Denote by $j(M,\tau)$ $(M \in \Gamma_0(4))$ Shimura's factor of automorphy on $\Gamma_0(4)$ given by

$$j(M,\tau) = \theta(M\tau)/\theta(\tau),$$

$\theta(\tau)$ being the theta series $\theta_0(\tau,0) = \sum_{n\in\mathbb{Z}} e(n^2\tau)$. Katok-Sarnak defined a certain plus space consisting of Maass wave forms of weight 1/2. For $s \in \mathbb{C}$ let T_s^+ denote the space consisting of C^2-class functions $g : \mathfrak{H} \longrightarrow \mathbb{C}$ satisfying the following two conditions:

(i) $g(Mz) = g(z)j(M,z)|cz+d|^{-1/2}$ for all $M \in \Gamma_0(4)$ and $\displaystyle\int_{\Gamma_0(4)\backslash\mathfrak{H}} |g(z)|^2\, d\omega(z) < +\infty$,

(ii) g has a Fourier expansion of the form:

$$g(z) = \sum_{n\in\mathbb{Z}} B(n,y)e(nx),$$

where we impose the condition that if $n \equiv 2,\, 3 \bmod 4$, then necessarily $B(n,y) = 0$. Moreover we assume the Fourier coefficients $B(n,y)$ for $n \neq 0$ are given of the form

$$B(n,y) = b(n) W_{\mathrm{sign} n/4,\, s-1/2}(4\pi y|n|), \tag{1.3}$$

where $W_{\alpha,\beta}$ denotes the usual Whittaker function.

Then a modified version of the second isomorphism in (0.1) generalized to Maass wave forms is given by

Proposition 1 *There exists the following anti $\mathbb{C}$-linear isomorphism*

$$\mathcal{H}_{-1/4,\chi}(s) \cong T^{+}_{\bar{s}} \tag{1.4}$$

given by $\mathcal{H}_{-1/4,\chi}(s) \ni g = \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} \mapsto G(\tau) = \overline{g_0(4\tau)} + \overline{g_1(4\tau)} \in T^{+}_{\bar{s}}$.

Remark. We note that, if s is real or of the form $s = \frac{1}{2} + ir$ with r real, then $T^{+}_{s} = T^{+}_{\bar{s}}$, and moreover that $T^{+}_{s} = \{0\}$, otherwise. In particular if $s = 1/4$, then the space $T^{+}_{1/4} = T^{+}_{3/4}$ is nothing but $M^{+}_{1/2}(\Gamma_0(4))$.

For the proof of the proposition we refer to [Ar4].

An analogue of the correspondences in (0.1) to Maass wave forms is described as follows:

$$\mathcal{H}^{even}_{0}(2s - \frac{1}{2}) \sim T^{+}_{s} \cong \mathcal{H}_{-1/4,\chi}(s).$$

Here the symbol "$\sim$" means that there exists a certain correspondence from $\mathcal{H}^{even}_{0}(2s - \frac{1}{2})$ to T^{+}_{s} described as in the following theorem due to Katok-Sarnak [KS].

Theorem 2 ([KS]) *Let $s \in \mathbb{C}$ and let f be an even Hecke eigen Maass wave form of $\mathcal{H}^{even}_{0}(2s - 1/2)$. Then there exists $g = \sum_{n\in\mathbb{Z}} B(n,y)e(nx) \in T^{+}_{s}$ whose Fourier coefficients satisfy the relation*

$$b(-n) = n^{-3/4} \sum_{T,\, \det 2T = n} f(z_T) |Aut T|^{-1} \qquad (n \in \mathbb{Z}_{>0}),$$

where T runs through all the $SL_2(\mathbb{Z})$-equivalence classes of positive definite half-integral symmetric matrices T with $\det 2T = n$ and $|Aut T|$ denotes the order of the unit group of T. Moreover z_T is the point in $\mathfrak{H}$ corresponding to T; namely if we write $T = {}^t g^{-1} g^{-1}$ with $g \in GL^{+}_{2}(\mathbb{R})$, then $z_T = g(i)$.

Remark. It is expected that for each Hecke eigen Maass wave form f there exists at least one non-zero g corresponding to f. Under this expectation

$$\dim \mathcal{H}^{even}_{0}(2s - 1/2) \leqq \dim T^{+}_{s} \qquad (?). \tag{1.5}$$

2 Selberg zeta functions concerned

The Selberg zeta functions $Z_{even}(s)$ has been introduced in [Ar3] to describe the trace formula for $\mathcal{H}_0^{even}$. Let $Prm^+(\Gamma)$ be the set of primitive hyperbolic elements P of Γ with $\mathrm{tr}P > 2$ and $Prm^+(\Gamma)^I$ the set consisting of $P \in Prm^+(\Gamma)$ that are primitive even in $GL_2(\mathbb{Z})$. Set $\widetilde{\Gamma} = GL_2(\mathbb{Z}) - SL_2(\mathbb{Z})$. An elemnet of $\widetilde{\Gamma}$ is called primitive hyperbolic, if $\mathrm{tr}P \neq 0$ and P cannot be represented as any power of any element of $\widetilde{\Gamma}$. Let $Prm^+(\widetilde{\Gamma})$ be the set of primitive hyperbolic elements P of $\widetilde{\Gamma}$ with $\mathrm{tr}P > 0$. For any element $P \in Prm^+(\Gamma)$ (or $P \in Prm^+(\widetilde{\Gamma})$) let $N(P)$ denote the square of the eigen value (> 1) of P. For any subset S of $GL_2(\mathbb{Z})$ which is stable under the $SL_2(\mathbb{Z})$-conjugation we denote by S/Γ the set of Γ-conjugacy classes in S. We define $Z_{even}(s)$ by

$$(2.1) \quad Z_{even}(s) = \prod_{\{P_0\}_\Gamma} \prod_{m=0}^{\infty} \Big(1-(-1)^m N(P_0)^{-s-m}\Big)^2 \times \prod_{\{P\}_\Gamma}{}^{I} \prod_{m=0}^{\infty} \Big(1 - N(P)^{-s-m}\Big),$$

where $\{P_0\}_\Gamma$ is taken over $Prm^+(\widetilde{\Gamma})/\Gamma$ and the product $\prod_{\{P\}_\Gamma}^{I}$ indicates that $\{P\}_\Gamma$ runs through $Prm^+(\Gamma)^I/\Gamma$. The zeta function $Z_{even}(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$. Moreover it is immediate to see from (2.1) that the logarithmic derivative of $Z_{even}(s)$ is given by

$$(2.2) \quad \frac{Z'_{even}}{Z_{even}}(s) = \sum_{\{P\}_\Gamma} \sum_{m=1}^{\infty} \frac{\log N(P)}{1 - N(P)^{-m}} N(P)^{-ms} + \sum_{\{P_0\}_\Gamma} \sum_{\substack{n>0 \\ odd}} \frac{\log N(P_0)^2}{1 + N(P_0)^{-n}} N(P_0)^{-ns}.$$

In [Ar1] we obtained the resolvent trace formula for the space $\mathcal{H}_{-1/4,\chi}$ involving the zeta function $Z_\chi(s)$ given by

$$(2.3) \qquad Z_\chi(s) = \prod_{\{P\}_\Gamma \in Prm^+(\Gamma)/\Gamma} \prod_{m=0}^{\infty} \det\Big(1_2 - \chi(P) N(P)^{-s-m}\Big).$$

On the other hand in [Ar3], [Ar4] we computed the resolvent trace formula for the space $\mathcal{H}_0^{even}$ and compared the both trace formulas for $\mathcal{H}_{-1/4,\chi}$ and $\mathcal{H}_0^{even}$ in an explicit manner. As an important consequence of this comparison we have the following fundamental theorem which connects the spectral series with the Selberg zeta functions concerned.

Theorem 3 *Let* $s' = 2s - 1/2$ *and* $a' = 2a - 1/2$ *with* $\mathrm{Re}(s) > 1$, $\mathrm{Re}(a) > 1$. *Then*

$$(2.4) \qquad S_{\Gamma,\chi}(s,a) - \Big(\frac{1}{2s-1}\frac{Z'_\chi}{Z_\chi}(s) - \frac{1}{2a-1}\frac{Z'_\chi}{Z_\chi}(a)\Big)$$
$$= 4\left(S_\Gamma^{even}(s',a') - \Big(\frac{1}{2(2s'-1)}\frac{Z'_{even}}{Z_{even}}(s') - \frac{1}{2(2a'-1)}\frac{Z'_{even}}{Z_{even}}(a')\Big)\right).$$

For the proof we refer to [Ar4].

In (2.4) we expect that the hyperbolic contributions of the both hand sides should coincide. Therefore we may present the following conjecture.

Conjecture 4 *We have*

$$\frac{Z'_\chi}{Z_\chi}(s) = \frac{Z'_{even}}{Z_{even}}(2s-1/2) \quad \textit{or eqivalently,} \quad Z_\chi(s)^2 = Z_{even}(2s-1/2).$$

Towards the solution of the conjecture it will be necessary to obtain explicit arithmetic expressions of the zeta functions $Z_\chi(s)$ and $Z_{even}(2s-1/2)$; in particular that of $Z_\chi(s)$.

3 Arithmetic forms

For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ $(M \neq \pm 1_2)$, we write

$$\widetilde{M} = \begin{pmatrix} b & (d-a)/2 \\ (d-a)/2 & -c \end{pmatrix} \qquad \text{and} \quad n(M) = \frac{1}{\beta}\widetilde{M},$$

where $\beta = \gcd(b, d-a, c)$ (β is often denoted by $\beta(M)$). By a straightforward computation it is not dificult to see that, for $P \in GL_2(\mathbb{Z})$,

$$n(PMP^{-1}) = (\det P)^{-1} P n(M) {}^tP.$$

Let $t := a+d$ be the trace of M. The trace of $U(M)$ for $M \in \Gamma$, $t > 2$ is given by

$$\text{(3.1)} \qquad \mathrm{tr}U(M) = \frac{1}{(t-2)^{3/2}} \sum_{\lambda,\mu \in \mathbb{Z}/(t-2)\mathbb{Z}} e\Big(\frac{1}{t-2}(\lambda,\mu)\widetilde{M}\begin{pmatrix} \lambda \\ \mu \end{pmatrix}\Big).$$

The matrix entries of $U(M)$ have been computed by Skoruppa-Zagier [SZ] in terms of Gaussian sums. The formula above is easily derived from their results. We note that this trace depends only on Γ-conjugacy class of M:

$$\mathrm{tr}U(P^{-1}MP) = \mathrm{tr}U(M) \qquad (P \in \Gamma).$$

Let D range over all positive discriminants and $C_{pr}(D)$ denote the set of all primitive half integral symmetric matrices $N = \begin{pmatrix} n_1 & n_2/2 \\ n_2/2 & n_3 \end{pmatrix}$ with $n_2^2 - 4n_1n_3 = D$. Denote by C_{pr} the collection of such $N \in C_{pr}(D)$ with D varying in all positive discriminants D. The modular group Γ acts on C_{pr} (also on $C_{pr}(D)$) in a usual manner by $N \longmapsto PN{}^tP$

$(N \in C_{pr},\ P \in \Gamma)$. Denote by $C_{pr}(D)//\Gamma$ (resp. $C_{pr}//\Gamma$) the set of the Γ-equivalence classes in $C_{pr}(D)$ (resp. C_{pr}) and by $h(D)$ the cardinality of this finite set $C_{pr}(D)//\Gamma$; namely $h(D)$ is the class number of primitive binary integral quadratic forms with discriminant D. Let $\epsilon_D = \frac{t+\beta\sqrt{D}}{2}$ denote the minimal solution of the Pell equation $t^2 - \beta^2 D = 4$ with $t,\ \beta \in \mathbb{Z}_{>0}$. Moreover we denote by $\epsilon_D^0 = \frac{t_0+\beta_0\sqrt{D}}{2}$ the minimal solution of the Pell equation $t_0^2 - \beta_0^2 D = -4$ with $t_0,\ \beta_0 \in \mathbb{Z}_{>0}$ if it exists (in this case $\epsilon_D = (\epsilon_D^0)^2$).

It is known that there exists a bijection from $Prm^+(\Gamma)$ to C_{pr}:

$$Prm^+(\Gamma) \ni P \longmapsto n(P) \in C_{pr}. \tag{3.2}$$

and that it induces a bijective map from the set $Prm^+(\Gamma)/\Gamma$ of all the Γ-conjugacy classes in $Prm^+(\Gamma)$ onto $C_{pr}//\Gamma$. For each $N = \begin{pmatrix} n_1 & n_2/2 \\ n_2/2 & n_3 \end{pmatrix} \in C_{pr}(D)$ the opposite map is given by

$$N \longmapsto P = \begin{pmatrix} (t-\beta n_2)/2 & \beta n_1 \\ -\beta n_3 & (t+\beta n_2)/2 \end{pmatrix} \in Prm^+(\Gamma).$$

We define, for each positive discriminant D and a positive integer m,

$$C_{\chi,m}(D) := \sum_{N \in C_{pr}(D)//\Gamma} \mathrm{tr}(\chi(P)^m),$$

where P corresponds to N by the above bijective map, namely, $n(P) = N$. Then we have another expression of $(Z'_\chi/Z_\chi)(s)$:

$$\frac{Z'_\chi}{Z_\chi}(s) = \sum_{D>0}\sum_{m=1}^{\infty} C_{\chi,m}(D)\frac{\log(\epsilon_D^2)}{1-\epsilon_D^{-2m}}\epsilon_D^{-2ms},$$

where $\epsilon_D = \dfrac{t+\beta\sqrt{D}}{2}$ with (t,β) denoting the minimal solution of the Pell equation $t^2 - D\beta^2 = 4,\ t,\ \beta \in \mathbb{Z}_{>0}$. To obtain this expression we note that $\mathrm{tr}(\chi(P^m)) = \mathrm{tr}(\chi(P)^m)$. Since $\chi(P)^m$ are unitary matrices of size two, the values which $\mathrm{tr}(\chi(P)^m)$ can take are rather limited. We have tried to compute $C_{\chi,m}(D)$, but at present we have got only partial results.

Proposition 5 *Let D be a positive discriminant with $D \equiv 1 \bmod 4$. Assume that there exists a fundamental unit $\epsilon_D^0 = \dfrac{t_0+\beta_0\sqrt{D}}{2}$ $(t_0,\ \beta_0 \in \mathbb{Z}_{>0})$ with (t_0,β_0) giving the minimal solution of the Pell equation $t_0^2 - D\beta_0^2 = -4$ (namely, $N(\epsilon_D^0) = -1$) and* *moreover assume that t_0 is odd. For each $N \in C_{pr}(D)$, chosse $P \in Prm^+(\Gamma)$ which corresponds to N by $n(P) = N$. Then we have*

$$\mathrm{tr}(\chi(P)^m) = 2\cos\frac{m\pi}{3} \quad (m \in \mathbb{Z}_{>0}).$$

Accordingly,

$$C_{\chi,m}(D) = 2h(D)\cos\frac{m\pi}{3}.$$

Proof. Let ϵ_D, ϵ_D^0, P and N be the same as above. We note that $\epsilon_D = (\epsilon_D^0)^2$, $\widetilde{P} = \beta N$, from which we have $t - 2 = t_0^2$ and $\beta = t_0\beta_0$. The expression (3.1) implies that

$$\mathrm{tr}U(P) = \frac{1}{(t-2)^{3/2}} \prod_{p\,|\,t-2} J_p \tag{3.3}$$

where for each prime p dividing $t - 2$ we set

$$J_p := \sum_{\lambda,\mu\in\mathbb{Z}/p^e\mathbb{Z}} e\Big(\frac{1}{t-2}(\lambda,\mu)\widetilde{P}\begin{pmatrix}\lambda\\ \mu\end{pmatrix}\Big) = \sum_{\lambda,\mu\in\mathbb{Z}/p^e\mathbb{Z}} e\Big(\frac{\beta_0}{t_0}(\lambda,\mu)N\begin{pmatrix}\lambda\\ \mu\end{pmatrix}\Big)$$

with $p^e \| t-2$ (this means that p^e divides $t-2$ and p^{e+1} does not). For each prime p the function $e(x)$ restricted to $\mathbb{Q}$ extends to a continuous function $e_p(x)$ on $\mathbb{Q}_p$ in such a manner that $e_p(x) = e(x)$ for $x \in \mathbb{Q}$. Let a prime p divide $t-2$. By the assumption on t_0, p is an odd prime. We may assume that N is $SL_2(\mathbb{Z}_p)$-equivalent to $\begin{pmatrix}u & 0\\ 0 & -u^{-1}D\end{pmatrix}$ with $u \in \mathbb{Z}_p^\times$. Then,

$$J_p = \Big(\sum_{\lambda \bmod p^e} e_p\Big(\frac{\beta_0}{t_0}u\lambda^2\Big)\Big)\Big(\sum_{\mu \bmod p^e} e_p\Big(-\frac{\beta_0}{t_0}u^{-1}D\mu^2\Big)\Big).$$

If we write $t_0 = p^f t_0'$ with $(t_0', p) = 1$, then $e = 2f$ and

$$J_p = p^e G_{p^f}\Big(\frac{\beta_0 u}{t_0'}\Big) G_{p^f}\Big(-\frac{\beta_0 u^{-1}D}{t_0'}\Big),$$

where we put, for $a \in \mathbb{Z}_p^\times$,

$$G_{p^f}(a) = \sum_{\lambda \bmod p^f} e_p\Big(\frac{a\lambda^2}{p^f}\Big).$$

It is well-known and easy to see that

$$G_{p^f}(a) = \begin{cases} p^{f/2} & \text{if } f \text{ is even,}\\ p^{(f-1)/2}\psi_p(a)G(\psi_p) & \text{if } f \text{ is odd,}\end{cases}$$

where ψ_p is the non-trivial quadratic character modulo p (ψ_p is extended to $\mathbb{Z}_p^\times$) and $G(\psi_p)$ is the usual Gaussian sum associated to ψ_p:

$$G(\psi_p) = \sum_{x \bmod p} \psi_p(x) e_p(x).$$

Using the identity $G(\psi_p)^2 = \psi_p(-1)p$, one can compute J_p in an explicit manner:

$$J_p = \begin{cases} p^{3e/2} & \text{if } f \text{ is even,} \\ p^{3e/2}\psi_p(D) & \text{if } f \text{ is odd.} \end{cases}$$

Since $t_0^2 - \beta_0^2 D = -4$, we have $\psi_p(D) = 1$. Therefore by (3.3) we conclude that

$$\mathrm{tr} U(P) = 1, \qquad \text{namely,} \quad \mathrm{tr}\chi(P) = 1.$$

Set, for any $M \in \Gamma$,

$$\omega(M) = \det U(M).$$

Then ω forms a character of Γ. We now borrow some notations and results from [Ar2]. We may assume $N = \begin{pmatrix} n_1 & n_2/2 \\ n_2/2 & n_3 \end{pmatrix} \in C_{pr}(D)$ to be reduced; namely, n_1, $n_3 > 0$ and $n_2 > n_1 + n_3$. Set

$$\alpha = \frac{n_2 + \sqrt{D}}{2n_1}.$$

Then N is reduced, if and only if α satisfies the condition

$$\alpha > 1 \quad \text{and} \quad 0 < \alpha' < 1, \tag{3.4}$$

which amounts to saying that α has a purely periodic continued fraction expansion:

$$\alpha = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots\, b_r - \cfrac{1}{b_1 - \ddots}}} \qquad (b_j \in \mathbb{Z},\ b_1, \ldots, b_r \geqq 2).$$

This expansion is denoted by

$$\alpha = [[\, \overline{b_1, b_2, \ldots, b_r}\,]] \tag{3.5}$$

(for this type of continued fraction expansion and the relationship with quadratic forms we refer to Zagier [Za]). Here r is called the period of α. Let B denote the Γ-equivalence class in $C_{pr}(D)$ represented by N. Then the period r depends only on the class B and is denoted by $r(B)$. Let B^* be the class of $C_{pr}(D)$ represented by $N^* = -{}^tQNQ$ with $Q = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$. Then we know in the proof of Proposition 5.1 of [Ar2] that

$$\omega(P) = i^{r(B)-r(B^*)}.$$

Moreover it is known that if there exists ϵ_D^0 with norm -1, then $r(B) = r(B^*)$. Therefore $\det U(P) = \omega(P) = 1$. This means that $U(P)$ is $GL_2(\mathbb{C})$-conjugate to some $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ with $\theta \in \mathbb{R}$. Then $\mathrm{tr}U(P) = 2\cos\theta = 1$, which implies $\theta = \pm\pi/3 + 2n\pi$ $(n \in \mathbb{Z})$. Thus,

$$\mathrm{tr}U(P^m) = \mathrm{tr}(U(P)^m) = 2\cos m\theta = 2\cos\frac{m\pi}{3}.$$

We have completed the proof of Proposition 5. ∎

Let $Z(s)$ denote the ordinary Selberg zeta function for Γ:

$$Z(s) = \prod_{\{P\}_\Gamma \in Prm^+(\Gamma)/\Gamma} \prod_{m=0}^{\infty} \Big(1 - N(P)^{-s-m}\Big).$$

It is well-known ([Sa], [He]) and easy to see from the bijection (3.2) that $Z(s)$ has the following arithmetic expression:

$$\begin{aligned} Z(s) &= \prod_{D>0} \prod_{m=0}^{\infty} \Big(1 - \epsilon_D^{-2(s+m)}\Big)^{h(D)}, \\ \frac{Z'}{Z}(s) &= \sum_{D>0} \sum_{m=1}^{\infty} h(D) \frac{\log(\epsilon_D^2)}{1 - \epsilon_D^{-2m}} \epsilon_D^{-2ms}. \end{aligned}$$

For each positive discriminant D let $C_{pr}^-(D)$ be the subset of $C_{pr}(D)$ consisting of N for which there exists a $P \in \widetilde{\Gamma}$ with $PN\,{}^tP = -N$. Denote by C_{pr}^- the union of all $C_{pr}^-(D)$ with D varying in all positive discriminants. We see easily that for each D only the case of either $C_{pr}^-(D) = \phi$ or $C_{pr}^-(D) = C_{pr}(D)$ occurs and moreover that $C_{pr}^-(D) = C_{pr}(D)$ if and only if ϵ_D^0 with norm -1 exists.

Therefore one can consider the set $C_{pr}^-(D)//\Gamma$ (or $C_{pr}^-//\Gamma$) of Γ-equivalence classeis in $C_{pr}^-(D)$ (in C_{pr}^-). Then it is easy to show in a similar manner that there exists a bijection from $Prm^+(\widetilde{\Gamma})$ onto C_{pr}^- via the map $Prm^+(\Gamma) \ni P \longmapsto n(P) \in C_{pr}^-$ and that it induces a bijective map from the set $Prm^+(\widetilde{\Gamma})/\Gamma$ onto $C_{pr}^-//\Gamma$.

Consequently by (2.1), (2.2), we have the expression for $Z_{even}(s)$:

$$Z_{even}(s) = \prod_{D>0}^{\sharp} \prod_{m=0}^{\infty} \Big(1-(-1)^m \epsilon_D^{-(s+m)}\Big)^{2h(D)} \times \prod_{D>0}^{I} \prod_{m=0}^{\infty} \Big(1-\epsilon_D^{-2(s+m)}\Big)^{h(D)},$$

$$\frac{Z'_{even}}{Z_{even}}(s) = \sum_{D>0}\sum_{m=1}^{\infty} \frac{2h(D)\log\epsilon_D}{1-(\epsilon_D)^{-2m}}(\epsilon_D)^{-2ms} + \sum_{D>0}^{\sharp}\sum_{\substack{n>0\\ odd}} \frac{2h(D)\log\epsilon_D}{1+(\epsilon_D)^{-n}}(\epsilon_D)^{-ns},$$

where $^{\sharp}$ (resp. ^{I}) indicates that D runs over all positive discriminants for which ϵ_D^0 with norm -1 exist (resp. for which ϵ_D^0 do not exist).

4 Prime geodesic theorem

It is known originally by Sarnak [Sa] that

(4.1) $$\sum_{\substack{\{P\}_\Gamma\\ N(P)\leqq X}} \log N(P) = X + O(X^{\frac{3}{4}+\varepsilon}).$$

and hence that

(4.2) $$\sum_{\substack{D>0\\ \varepsilon_D\leqq X}} h(D)\log((\epsilon_D)^2) = X^2 + O(X^{\frac{3}{2}+\epsilon})$$

(note that (4.2) is easily derived from (4.1) with the help of the bijection from $Prm^+(\Gamma)/\Gamma$ onto $C_{pr}//\Gamma$). The best possible error term in the right hand side of (4.2) is $O(X^{\frac{7}{5}+\epsilon})$ which is given by Luo-Sarnak [LS].

Similarly by using the Selberg trace formula for the space $\mathcal{H}_0^{even}$ and by a general procedure (cf. [Iw], [He]) the following estimate follows:

(4.3) $$\frac{1}{2}\Bigg(\sum_{\substack{\{P\}_\Gamma\\ N(P)\leqq X}} \log N(P) + \sum_{\substack{\{P_0\}_\Gamma\\ N(P_0)\leqq X}} \log N(P_0)^2\Bigg) = X + O(X^{\frac{3}{4}+\varepsilon}) \qquad (\varepsilon>0),$$

where the summations indicate that $\{P\}_\Gamma$ and $\{P_0\}_\Gamma$ run through $Prm^+(\Gamma)/\Gamma$ and $Prm^+(\widetilde{\Gamma})/\Gamma$ with the conditions $N(P)\leqq X$ and $N(P_0)\leqq X$, respectively. Then by comparing (4.1) and (4.3) we have

(4.4) $$\sum_{\substack{\{P_0\}_\Gamma\\ N(P_0)\leqq X}} \log N(P_0)^2 = X + O(X^{\frac{3}{4}+\varepsilon}).$$

Therefore in the arithmetic terminology we have

Theorem 6 *Assume $\varepsilon > 0$. We have*

$$\sum_{\substack{D>0 \\ \epsilon_D^0 \leqq X}}{}^{\sharp} h(D) \log((\epsilon_D^0)^2) = \frac{X^2}{2} + O(X^{\frac{3}{2}+\epsilon}) \tag{4.5}$$

and

$$\sum_{\substack{D>0 \\ \varepsilon_D \leqq X}}{}^{I} h(D) \log((\epsilon_D)^2) = X^2 + O(X^{\frac{3}{2}+\epsilon}),$$

where the second summation indicates that D runs through all positive discriminants for which fundamental units with norm -1 do not exist.

Proof. The former identity is a direct consequence of (4.4) and the bijectivity of the map from $Prm^+(\widetilde{\Gamma})/\Gamma$ onto $C_{pr}^-//\Gamma$, while the latter one is derived from (4.2) and (4.5).

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Skew holomorphic Jacobi forms of general degree

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Introduction

In the study of modular forms of half integral weight, it is well known that Kohnen's plus space (a certain subspace of elliptic modular forms of half integral weight) of weight "even integer minus 1/2" is isomorphic to the space of Jacobi forms of index 1 (cf. Eichler-Zagier[3] Theorem 5.4). Moreover, Skoruppa[14] introduced the notion of skew holomorphic Jacobi forms which satisfy a certain transformation formula like Jacobi forms but not holomorphic functions, and he constructed a linear isomorphism between skew holomorphic Jacobi forms of index 1 and Kohnen's plus space of weight "odd integer minus 1/2" in the case of degree 1. This notion of skew holomorphic Jacobi forms was generalised for higher degree by Arakawa[1]. There are several works about the Jacobi form of general degree (cf. [1],[2],[8],[10],[11],[15],[18] etc), but there are few results about skew holomorphic Jacobi forms of general degree except Arakawa[1].

The purpose of this article is to investigate skew holomorphic Jacobi forms of general degree. This article is a summarisation of three papers of Hayashida[4],[5],[6]. In Section 1 we describe the definition of skew holomorphic Jacobi forms following Arakawa[1]. Skew holomorphic Jacobi forms are not holomorphic functions but vanish under a certain differential operator $\Delta_{\mathcal{M}}$ which will be defined in Section 1. In Section 2 we give an isomorphism between plus space of general degree and the space of skew holomorphic Jacobi forms of index 1 of general degree. In Section 3 we construct Klingen type Eisenstein series of skew holomorphic Jacobi forms. In order to obtain this construction, we used a certain differential operator $\Delta_{\mathcal{M}}$. In Section 4 we give an analogue of the Zharkovskaya's theorem for Siegel modular forms of half integral weight.

1 Skew holomorphic Jacobi forms

We denote $Sp_n(\mathbb{R})$ the real symplectic group of size $2n$. Let $\mathfrak{H}_n$ denote Siegel upper half space of degree n, and let $\mathfrak{D}_{n,l} = \mathfrak{H}_n \times M_{n,l}(\mathbb{C})$.

Skew holomorphic Jacobi forms was first introduced by Skoruppa[14] as function on $\mathfrak{D}_{1,1}$, and he showed the isomorphism between Kohnen's plus space and the space of skew holomorphic Jacobi forms of index 1 in the case of degree 1. This notion of skew holomorphic Jacobi forms was generalised for higher degree by Arakawa[1]. In this section, we would like to describe the definition of skew holomorphic Jacobi forms following Arakawa[1]. We prepare some notations.

Let $G^J_{n,l}$ be the Jacobi group, $G^J_{n,l}$ is a subgroup of $Sp_{n+l}(\mathbb{R})$ as follows,

$$G^J_{n,l} := \left\{ \begin{pmatrix} * & 0 & * & * \\ * & 1_l & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & 1_l \end{pmatrix} \in Sp_{n+l}(\mathbb{R}) \right\}$$

We put $\Gamma^J_{n,l} = G^J_{n,l} \cap Sp_{n+l}(\mathbb{Z})$.

We denote the action of $Sp_n(\mathbb{R})$ on $\mathfrak{H}_n$ by

$$M \cdot Z \ := \ (AZ+B)(CZ+D)^{-1}$$

where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp_n(\mathbb{R})$, and $Z \in \mathfrak{H}_n$.

Let $M > 0$ be a symmetric half integral matrix of size l. Now we describe the definition of the skew holomorphic Jacobi forms.

Definition 1 (skew holomorphic Jacobi forms cf. [1]). *Let $F(\tau, z)$ be a function on $\mathfrak{D}_{n,l}$, holomorphic on $\mathfrak{H}_n$ and real analytic on $M_{n,l}(\mathbb{C})$. We say F is a skew holomorphic Jacobi form of weight k of index $\mathcal{M}$ belongs to $\Gamma^J_{n,l}$, if F satisfies the following two conditions :*

(1) *We put $F_{\mathcal{M}}(Z) := F(\tau, z)\, e(tr(\mathcal{M}\tau'))$ for $Z = \begin{pmatrix} \tau & z \\ {}^t z & \tau' \end{pmatrix} \in \mathfrak{H}_{n+l}$, then $F_{\mathcal{M}}$ satisfies*

$$F_{\mathcal{M}}(\gamma \cdot Z) \ = \ \overline{\det(CZ+D)}^{k-l} |\det(CZ+D)|^l F_{\mathcal{M}}(Z) \ ,$$

for every $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^J_{n,l}$.

(2) *F has the Fourier expansion as follows :*

$$F(\tau, z) = \sum_{N \in Sym_n, R \in M_{n,l}(\mathbb{Z})} C(N,R)\, e(N\tau - \frac{i}{2}(4N - R\mathcal{M}^{-1t}R) Im\, \tau +^t Rz) \,,$$

where we denote by Sym_n the set of all half integral symmetric matrices of size n, and $C(N,R)=0$ unless $4N - R\mathcal{M}^{-1t}R \le 0$.

Moreover, if Fourier coefficients satisfy a condition that $C(N,R)=0$ unless $4N - R\mathcal{M}^{-1t}R < 0$, we say F is a skew holomorphic Jacobi cusp form.

We set differential operators $\frac{\delta}{\delta\tau} := \left(\frac{1+\delta_{s,t}}{2}\frac{\delta}{\delta\tau_{s,t}}\right)$, $\frac{\delta}{\delta z} := \left(\frac{\delta}{\delta z_{i,j}}\right)$ for $(\tau,z)\in \mathfrak{D}_{n,l}$, where $\delta_{s,t}$ is the Kronecker's delta symbol, and $\frac{\delta}{\delta\tau_{s,t}} := \frac{1}{2}\left(\frac{\delta}{\delta x_{s,t}} - i\frac{\delta}{\delta y_{s,t}}\right)$, where $x_{s,t}$ (resp. $y_{s,t}$) is the real part (resp. the imaginary part) of $\tau_{s,t}$. We define a differential operator

$$\Delta_{\mathcal{M}} := 8\pi i\frac{\delta}{\delta\tau} - \frac{\delta}{\delta z}\mathcal{M}^{-1t}\frac{\delta}{\delta z}.$$

We note the following equivalence. If a function F on $\mathfrak{D}_{n,l}$ satisfies the condition (1) of the definition of skew holomorphic Jacobi forms, and if $n>1$, then the condition (2) is equivalent to the following condition

(2') $\Delta_{\mathcal{M}} F = 0_n$.

We denote the vector space of skew holomorphic Jacobi forms (resp. skew holomorphic Jacobi cusp forms) of weight k of index $\mathcal{M}$ by $J^{sk}_{k,\mathcal{M}}(\Gamma^J_n)$ (resp. $J^{sk,cusp}_{k,\mathcal{M}}(\Gamma^J_n)$).

2 Isomorphisms between skew holomorphic Jacobi forms of index 1 and plus spaces

First, we shall describe the definition of Siegel modular forms of half integral weight.

For positive integer q, we put

$$\Gamma_0^{(n)}(q) := \{M = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in Sp_n(\mathbb{Z}) \mid C \equiv 0 \,(mod\, q)\}$$

is the congruence subgroup of the symplectic group $Sp_n(\mathbb{Z})$.

We define a character ψ on $\Gamma_0^{(n)}(4)$, ψ is given by $\psi(M) := \left(\frac{-4}{\det D}\right)$ for $M = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \Gamma_0^{(n)}(4)$.

We put the standard theta series $\theta^n(Z)$ and put a function $j(M,Z)$ as follows:

$$\begin{aligned} \theta^n(Z) &:= \sum_{m\in\mathbb{Z}^n} e({}^tmZm), \qquad (Z\in\mathfrak{H}_n) \\ j(M,Z) &:= \frac{\theta^n(M\cdot Z)}{\theta^n(Z)}, \qquad (M\in\Gamma_0^{(n)}(4), Z\in\mathfrak{H}_n), \end{aligned}$$

then this $j(M,Z)$ satisfies

$$j(M,Z)^2 \;=\; \psi(M)\,\det(CZ+D) \;\text{ for any } M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^{(n)}(4)\ .$$

Let k be an integer, χ be a Dirichlet character modulo q, and $4|q$. A holomorphic function $F(Z)$ on $\mathfrak{H}_n$ is said to be a *Siegel modular form* of weight $k-1/2$ with character χ belongs to $\Gamma_0^{(n)}(q)$ if F satisfies

$$F(M\cdot Z) = \chi(\det D) j(M,Z)^{2k-1} F(Z)\,,\ \text{for any } \ M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^{(n)}(q),$$

and in the case of $n=1$ we need that the function $F(Z)$ is holomorphic at all cusps of $\Gamma_0^{(1)}(q)$. We denote the set of such functions by $M_{k-1/2}(\Gamma_0^{(n)}(q),\chi)$. If $n=0$ then we denote $M_{k-1/2}(\Gamma_0^{(0)}(q),\chi) = \mathbb{C}$ for $k>0$. Also, we denote the set of cusp forms in $M_{k-1/2}(\Gamma_0^{(n)}(q),\chi)$ by $S_{k-1/2}(\Gamma_0^{(n)}(q),\chi)$.

Next, we define a subspace $M^+_{k-1/2}(\Gamma_0^{(n)}(4),\psi^u)$ of $M_{k-1/2}(\Gamma_0^{(n)}(4),\psi^u)$ ($u=0$ or 1) by

$$\begin{aligned} & M^+_{k-1/2}(\Gamma_0^{(n)}(4),\psi^u) \\ := \ & \Big\{ h(\tau) \in M_{k-1/2}(\Gamma_0^{(n)}(4),\psi^u) \mid \text{the coefficients} \quad c(T) = 0\ , \\ & \ \text{unless} \ \ T \equiv -(-1)^{k+u} \mu\,{}^t\mu \mod 4\,Sym_n \ \text{ for some } \ \mu \in \mathbb{Z}^n \Big\}\,. \end{aligned}$$

We also define $S^+_{k-1/2}(\Gamma_0^{(n)}(4),\psi^u)$ by

$$S^+_{k-1/2}(\Gamma_0^{(n)}(4),\psi^u) \;:=\; M^+_{k-1/2}(\Gamma_0^{(n)}(4),\psi^u) \cap S_{k-1/2}(\Gamma_0^{(n)}(4),\psi^u)\ .$$

We say that $M^+_{k-1/2}(\Gamma_0^{(n)}(4),\psi^u)$ is the plus space.

Let $\mathcal{M}>0$ be a half integral symmetric matrix of size l and let $R \in M_{n,l}(\mathbb{Z})$, we put the theta series

$$\vartheta_{R,\mathcal{M}}(\tau,z) \;=\; \sum_{\lambda \in M_{n,l}(\mathbb{Z})} e(tr\left(\mathcal{M}\left(\tau[(\lambda + R(2\mathcal{M})^{-1})] + 2\,{}^t z(\lambda + R(2\mathcal{M})^{-1})\right)\right)),$$

where $\tau[(\lambda + R(2\mathcal{M})^{-1})] = {}^t(\lambda + R(2\mathcal{M})^{-1})\tau(\lambda + R(2\mathcal{M})^{-1})$.

Let $F(\tau,z) \in J^{sk}_{k,\mathcal{M}}(\Gamma^J_n)$, then F satisfies the condition (1) of the definition of skew holomorphic Jacobi forms, we can see

$$F(\tau, z+\tau\lambda+\mu) \;=\; e(-\mathrm{tr}(\mathcal{M}({}^t\lambda\tau\lambda + 2\,{}^t\lambda z)))\,F(\tau,z)$$

for every $\lambda,\mu \in M_{n,l}(\mathbb{Z})$. Hence, we have the following equation,

$$F(\tau,z) \;=\; \sum_{R \in M_{n,l}(\mathbb{Z})/(M_{n,l}(\mathbb{Z})(2\mathcal{M}))} F_R(\tau)\,\vartheta_{R,\mathcal{M}}(\tau,z)\ ,$$

where $F_R(\tau)$ are uniquely determined and $F_R(-\overline{\tau})$ are holomorphic functions on $\mathfrak{H}_n$. If we set $F(\tau,z) = \sum_{N,R'} C(N,R') e(N\tau - (4N - R'\mathcal{M}^{-1t}R')Im\,\tau + {}^tR'z)$, then we can write F_R by

$$F_R(\tau) \;=\; \sum_{\substack{N\in sym_n \\ 4N-R\mathcal{M}^{-1t}R\le 0}} C(N,R)\, e(\frac{1}{4}\mathrm{tr}(4N - R\mathcal{M}^{-1t}R)\overline{\tau}) \;.$$

In this section, from here, we consider only the case $l=1$, $\mathcal{M}=1$, and we put $\vartheta_r := \vartheta_{r,1}$.

Let $F(\tau,z) = \sum_{r\in(\mathbb{Z}/2\mathbb{Z})^n} F_r(\tau)\vartheta_r(\tau,z) \in J^{sk}_{k,1}(\Gamma^J_n)$. We define a holomorphic function $\sigma(F)(\tau) = \sum_{r\in(\mathbb{Z}/2\mathbb{Z})^n} F_r(-4\,\overline{\tau})$, then we have the following theorem.

Theorem 1. *$\sigma(F)$ is an element of $M^+_{k-1/2}(\Gamma^{(n)}_0(4),\psi^{k-1})$. Moreover, the map $\sigma : J^{sk}_{k,1}(\Gamma^J_n) \to M^+_{k-1/2}(\Gamma^{(n)}_0(4),\psi^{k-1})$ induces the linear isomorphism over $\mathbb{C}$. The space of skew holomorphic Jacobi cusp forms corresponds with the space of cusp forms of plus space by this isomorphism. This isomorphism map σ commutes with Hecke operators of both spaces.*

We note that if degree n is odd and integer k is even, then it is easy to see that $M_{k-1/2}(\Gamma^{(n)}_0(4),\psi) = J^{sk}_{k,1}(\Gamma^J_n) = \{0\}$.

We denote the space of holomorphic Jacobi forms of weight k of index 1 of degree n by $J_{k,1}(\Gamma^J_n)$ (cf. Ibukiyama [8]), then the table of linear isomorphisms between the plus space and the holomorphic (or skew holomorphic) Jacobi forms of index 1 is given as follows.

$M^+_{k-1/2}(\Gamma^{(n)}_0(4),\psi^u) \;\cong$

u \ k	even	odd
0	$J_{k,1}(\Gamma^J_n)$	$J^{sk}_{k,1}(\Gamma^J_n)$
1	$J^{sk}_{k,1}(\Gamma^J_n)$	$J_{k,1}(\Gamma^J_n)$

3 Klingen type Eisenstein series

We shall construct Klingen type Eisenstein series of skew holomorphic Jacobi forms. Let r be an integer ($0\le r\le n$). We prepare the following subgroups,

$$\Gamma_{[n,r]} \;:=\; \left\{ g = \begin{pmatrix} A_1 & 0 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ C_1 & 0 & D_1 & D_2 \\ 0 & 0 & 0 & D_4 \end{pmatrix} \in Sp_n(\mathbb{Z}) \mid A_1,B_1,C_1,D_1 \in M_r(\mathbb{Z}) \right\},$$

$$\Gamma^J_{[n,r],l} \;:=\; \left\{ \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1_l & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1_l \end{pmatrix} \begin{pmatrix} 1_n & 0 & 0 & \mu \\ {}^t\lambda & 1_l & {}^t\mu & \kappa \\ 0 & 0 & 1_n & -\lambda \\ 0 & 0 & 0 & 1_l \end{pmatrix} \in \Gamma^J_{n,l} \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{[n,r]}\;, \right.$$
$$\left. \lambda = \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} \in M_{n,l}(\mathbb{Z})\;,\; \lambda_1 \in M_{r,l}(\mathbb{Z}) \right\}.$$

Let $F(\tau_1, z_1) \in J^{sk,cusp}_{k,\mathcal{M}}(\Gamma^J_r)$ and let k be an integer satisfies $k \equiv l \mod 2$ (l is the size of $\mathcal{M}$). We define a function F^* on $\mathfrak{D}_{n,l}$ as

$$F^*(\tau, z) := F(\tau_1, z_1) \ , \tag{3.1}$$

where $\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ {}^t\tau_2 & \tau_3 \end{pmatrix}$, $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ and $(\tau_1, z_1) \in \mathfrak{D}_{r,l}$.

We consider the following function

$$E^{sk}_{n,r}(F; (\tau, z)) \ = \sum_{\gamma \in \Gamma^J_{[n,r],l} \backslash \Gamma^J_{n,l}} (F^*|_{k,\mathcal{M}}\gamma)(\tau, z) \ , \qquad (\tau, z) \in \mathfrak{D}_{n.l}. \tag{3.2}$$

The above sum does not depend on the choice of the representative elements. Because F is a cusp form, we can show the fact that there exists a constant C which satisfies

$$|F(\tau_1, z_1)| \det(Y_1)^{\frac{k}{2}} e(-tr(\mathcal{M}{}^t\beta_1(iY_1)^{-1}\beta_1)) \ < \ C,$$

for every $(\tau_1, z_1) \in \mathfrak{D}_{r,l}$, where β_1 and Y_1 are the imaginary part of z_1 and τ_1 respectively. Hence, by the same calculation as Ziegler[18] Theorem2.5, we can show the fact that if $k > n + l + r + 1$ then $E^{sk}_{n,r}$ is uniformly absolutely convergent in the wider sense on $\mathfrak{D}_{n,l}$. It is clear that $E^{sk}_{n,r}(F; (\tau, z))$ satisfies the condition (1) of the definition of skew holomorphic Jacobi forms of weight k of index $\mathcal{M}$ belongs to Γ^J_n

We can show the following equation :

$$\Delta_{\mathcal{M}} \left(E^{sk}_{n,r}(F; (\tau, z)) \right) \ = \ 0_n. \tag{3.3}$$

Because this equation induces the shape of the Fourier expansion of $E^{sk}_{n,r}(F; (\tau, z))$, and by using Shimura correspondence and Köcher principle, we can show the fact that $E^{sk}_{n,r}(F; (\tau, z))$ satisfies the condition (2) of the definition of skew holomorphic Jacobi forms. Hence, we have the following theorem.

Theorem 2. *Let $\mathcal{M} > 0$ and $F \in J^{sk}_{k,\mathcal{M}}(\Gamma^J_r)$. If $k > n + l + r + 1$ satisfies $k \equiv l \mod 2$, then $E^{sk}_{n,r}(F; (\tau, z))$ is an element of $J^{sk}_{k,\mathcal{M}}(\Gamma^J_n)$.*

We note that we can obtain the above theorem under the assumption on $\mathcal{M} \geq 0$ (cf.[4]).

We shall show that the Siegel operator of skew holomorphic Jacobi forms has same properties as holomorphic Jacobi forms case (cf. Ziegler[18]).

For a function $F(\tau, z)$ on $\mathfrak{D}_{n,l}$, we define a function

$$\Phi^n_r(F)(\tau_1, z_1) \ := \ \lim_{t \to +\infty} F\left(\left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & it1_{n-r} \end{smallmatrix} \right), \left(\begin{smallmatrix} z_1 \\ 0 \end{smallmatrix} \right) \right), \quad (\tau_1, z_1) \in \mathfrak{D}_{n,r}.$$

Then $\Phi^n_r(F)$ is a function on $\mathfrak{D}_{r,l}$. This Φ^n_r is called the Siegel operator.

By direct calculation, we have the following proposition.

Proposition 3. *Let $F(\tau, z) \in J^{sk}_{k,\mathcal{M}}(\Gamma^J_n)$ be a skew holomorphic Jacobi form, then $\Phi^n_r(F)$ is also a skew holomorphic Jacobi form in $J^{sk}_{k,\mathcal{M}}(\Gamma^J_r)$.*

Moreover, we have the following theorem.

Theorem 4. *If integer k satisfies $k > n + l + r + 1$ and $k \equiv l \mod 2$, then we have $\Phi^n_r(E^{sk}_{n,r}(F; (\tau, z))) = F(\tau_1, z_1)$ for every $F(\tau_1, z_1) \in J^{sk,cusp}_{k,\mathcal{M}}(\Gamma^J_r)$. Hence, the Siegel operator Φ^n_r induces a surjective map from $J^{sk}_{k,\mathcal{M}}(\Gamma^J_n)$ to $J^{sk,cusp}_{k,\mathcal{M}}(\Gamma^J_r)$.*

Now, we imitate some Arakawa's work[2]. We assume the following condition on $\mathcal{M} > 0$.

(4.1) If $\mathcal{M}[x] \in \mathbb{Z}$ for $x \in (2\mathcal{M})^{-1}M_{l,1}(\mathbb{Z})$, then necessarily, $x \in M_{l,1}(\mathbb{Z})$.

By the same argument with Arakawa [2] (Proposition 4.1, Theorem 4.2 of [2]), we deduce the following Proposition 5 and Theorem 6.

Proposition 5. *Let $F \in J^{sk}_{k,\mathcal{M}}(\Gamma^J_n)$. Under the condition* (4.1) *on $\mathcal{M}$, we have $F \in J^{sk,cusp}_{k,\mathcal{M}}(\Gamma^J_n)$ if and only if $\Phi^n_{n-1}(F) = 0$.*

Theorem 6. *Assume that $\mathcal{M}$ satisfies the condition* (4.1). *Let k be a positive integer which satisfies $k > 2n + l + 1$ and $k \equiv l \mod 2$. Then we have the direct sum decomposition $J^{sk}_{k,\mathcal{M}}(\Gamma^J_n) = \bigoplus_{r=0}^n J^{sk,(r)}_{k,\mathcal{M}}(\Gamma^J_n)$, where $J^{sk,(r)}_{k,\mathcal{M}} = \{E^{sk}_{n,r}(F; (\tau, z)) | F \in J^{sk,cusp}_{k,\mathcal{M}}(\Gamma^J_r)\}$.*

In section 2 theorem 1, we obtained the isomorphism between the plus space and the space of skew holomorphic Jacobi forms of index 1. Hence, by using theorem 6, if k is an odd integer which satisfies $k > 2n+2$, we can also obtain a similar decomposition for the plus space of degree n of weight $k - \frac{1}{2}$ with trivial character. Namely, under these conditions, we can deduce the fact that the plus space of weight $k - \frac{1}{2}$ is spanned by Klingen-Cohen type Eisenstein series (which corresponds to the Klingen type Eisenstein series of skew holomorphic Jacobi form of index 1) and cusp forms.

4 Zharkovskaya's theorem

In this section, we give an analogue of the Zharkovskaya's theorem for Siegel modular forms of half integral weight, and quote a conjecture.

Let $q > 0$ be an integer divisible by 4. Let $F \in M_{k-1/2}(\Gamma^{(n)}_0(q), \chi)$ be an eigenfunction for the action of a certain Hecke ring. This F has a Fourier expansion

$$F(Z) = \sum_{N \in Sym^*_n} f(N) e(NZ),$$

where we denote by Sym_n^* the set of all semi positive definite half integral symmetric matrices of size n. From the definition of $M_{k-1/2}(\Gamma_0^{(n)}(q),\chi)$, it follows that $f({}^tUNU)=f(N)$ for every $U\in SL_n(\mathbb{Z})$.

Here, we describe a result of Zhuravlev[17]. Let λ be a completely multiplicative function which grows no faster than some power of argument, and let $N>0$ be a matrix in Sym_n^*.

Theorem 7 (Zhuravlev). *When the real part of s is sufficiently large, The following series has Euler expansion,*

$$\sum_{\substack{M\in SL_n(\mathbb{Z})\backslash M_n^+(\mathbb{Z})\\(\det M,q)=1}}\frac{\lambda(\det M)f(MN{}^tM)}{(\det M)^{s+k-3/2}} \;=\; \prod_{p:prime}\frac{P_{F,p}(N,\lambda,p^{-s})}{Q_{F,p}(\lambda,p^{-s})}, \tag{4.1}$$

where we denote by $M_n^+(\mathbb{Z})$ all positive determinant matrices in $M_{n,n}(\mathbb{Z})$, and $P_{F,p}(N,\lambda,z)$ is the polynomial of z which degree is at most $2n$, $Q_{F,p}(\lambda,z)$ is the polynomial of z which degree is $2n$. Especially $Q_{F,p}(\lambda,z)$ is not depend on the choice of N. The polynomial $Q_{F,p}(\lambda,z)$ was defined as follows,

$$Q_{F,p}(\lambda,z) \;=\; \prod_{i=0}^{n}(1-\alpha_{i,p}\chi(p)\lambda(p)z)(1-\alpha_{i,p}^{-1}\chi(p)\lambda(p)z), \tag{4.2}$$

where $\alpha_{i,p}^{\pm 1}$ are the p-parameters of F.

We denote the Siegel operator by Φ. Oh-Koo-Kim [12] showed the existence of a commuting relation between Hecke operators and the Siegel operator acting on the Siegel modular forms of half integral weight, and they showed also the fact that if $F\in M_{k-1/2}(\Gamma_0^{(n)}(q),\chi)$ is a Hecke eigen form then $\Phi(F)\in M_{k-1/2}(\Gamma_0^{(n-1)}(q),\chi)$ is also a Hecke eigen form.

We put $L(s,\lambda,F)=\prod_{(p,q)=1}Q_{F,p}(\lambda,p^{-s+k-3/2})^{-1}$ (see eq(4.1), eq(4.2)), then we obtain the following theorem, this is an analogue of the theorem of Zharkovskaya [16].

Theorem 8. *We assume $\Phi(F)\neq 0$, then we have*

$$L(s,\lambda,F) \;=\; L_1(s-n+1,\lambda,E_{2k-2n,\chi^2})L(s,\lambda,\Phi(F)),$$

where we put

$$L_1(s,\lambda,E_{2k-2n,\chi^2}) := \prod_{p,(p,q)=1}(1-\lambda(p)p^{-s})^{-1}(1-\lambda(p)\chi(p)^2p^{2k-2n-1-s})^{-1}.$$

If $k>n+1$ then $L_1(s,\lambda,E_{2k-2n,\chi^2})$ is the L-function of Eisenstein series of degree 1 of weight $2k-2n$ with character χ^2 twisted by λ.

Above theorem was first observed by Hayashida-Ibukiyama [7] in the case of $n = 2$, $\lambda \equiv 1$, and $\chi \equiv 1$. Here, we have the case of higher degree with character.

Let $F \in M_{k-1/2}(\Gamma_0^{(2)}(4))$ be a Hecke eigen form, and we assume $\Phi(F) \neq 0$, then

$$L(s, F) \;=\; L(s, \Phi(F))\, L(s, E_{2k-4}),$$

up to Euler 2-factor. Let $f \in M_{2k-2}(SL(2,\mathbb{Z}))$ be a Hecke eigen form which corresponds to $\Phi(F) \in M_{k-1/2}(\Gamma_0^{(1)}(4))$ by Shimura correspondence, then we have

$$L(s, F) \;=\; L(s, f)\, L(s, E_{2k-4})\;.$$

Similar figure seems valid for the case of cusp forms. We quote a following conjecture from Hayashida-Ibukiyama [7].

Conjecture 1 (cf. [7]). *Let k be an integer, and let $f \in S_{2k-2}(SL(2,\mathbb{Z}))$, $g \in S_{2k-4}(SL(2,\mathbb{Z}))$. We assume both f and g are normalised Hecke eigen forms. Then there exits $F \in S^+_{k-1/2}(\Gamma_0^{(2)}(4))$, such that F is a Hecke eigen form and satisfy*

$$L(s, F) \;=\; L(s, f)\, L(s-1, g)$$

up to Euler 2-factor, and where $L(s, f)$ and $L(s, g)$ are usual L-functions of f and g respectively.

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Certain series attached to an even number of elliptic modular forms

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1 Results

Let $n \in \mathbf{Z}_{>0}$, $k := (k_1, \dots, k_n) \in (\mathbf{Z}_{>0})^n$, $m = (m_1, \dots, m_n) \in (\mathbf{Z}_{>0})^n$ and $s \in \mathbf{C}$. We put

$$Q_k^{(n)}(m,s) := \int_0^\infty t^{s+|k|-n-1} dt \cdot \prod_{j=1}^n \int_0^\infty u_j^{k_j-2} e^{-4\pi m_j u_j t} (\sqrt{u_j}\theta(iu_j) - 1) du_j \; ; \tag{1}$$

here $|k| := \sum_{j=1}^n k_j$ and

$$\theta(z) := \sum_{l=-\infty}^{\infty} e^{\pi i l^2 z}$$

is the Jacobi theta function. The right-hand side of (1) converges absolutely and locally uniformly for $\mathrm{Re}(s) > \frac{n}{2}$. It is easy to see

$$Q_k^{(n)}(m,\sigma) > 0 \quad \text{for} \quad \frac{n}{2} < \sigma \in \mathbf{R}.$$

For $w \in \mathbf{Z}$ let M_w be the space of holomorphic modular forms of weight w for $SL_2(\mathbf{Z})$ and S_w be the space of cusp forms in M_w. Let f_j and g_j be elements of M_{k_j} such that $f_j(z)g_j(z)$ is a cusp form for each $j = 1, \dots, n$. Let

$$f_j(z) = \sum_{l=0}^\infty a_j(l) e^{2\pi i l z} \quad \text{and} \quad g_j(z) = \sum_{l=0}^\infty b_j(l) e^{2\pi i l z} \tag{2}$$

be the Fourier expansions. The series we treat here is the following:

$$\begin{aligned} &\mathcal{D}(s; f_1, \ldots, f_n; g_1, \ldots, g_n) \\ := &\sum_{m=(m_1,\ldots,m_n)\in(\mathbf{Z}_{>0})^n} \left(\prod_{j=1}^{n} a_j(m_j)\overline{b_j(m_j)} \right) Q_k^{(n)}(m, s). \end{aligned} \tag{3}$$

The right-hand side of (3) converges absolutely and locally uniformly for

$$\mathrm{Re}(s) > \frac{n}{2}(\max_{1\le j\le n}(k_j) + 1).$$

Theorem 1.
(i) *The series* (3) *has a meromorphic continuation to the whole* s*-plane.*
(ii) *Let* (,) *be the Petersson inner product. Then the function*

$$\sum_{\nu=1}^{n} \sum_{1\le i_1<\ldots<i_\nu\le n} \left(\prod_{\substack{j\ne i_1,\ldots,i_\nu \\ 1\le j\le n}} (f_j, g_j) \right) \cdot \mathcal{D}(s; f_{i_1}, \ldots, f_{i_\nu}; g_{i_1}, \ldots, g_{i_\nu})$$

is invariant under the substitution $s \mapsto n - s$ *; it has possible simple poles at* $s = 0$ *and* $s = n$ *with residues* $-\prod_{j=1}^{n}(f_j, g_j)$ *and* $\prod_{j=1}^{n}(f_j, g_j)$ *respectively, and is holomorphic elsewhere.*

In case where every g_j is the Eisenstein series we have

Corollary. *Suppose* $f_j \in S_{k_j}$ $(j = 1, \ldots, n)$ *with Fourier expansions as in* (2)*. For* $l \in \mathbf{Z}_{>0}$ *put*

$$\sigma_\nu(l) := \sum_{d|l} d^\nu \quad \textit{for} \quad \nu \in \mathbf{C}.$$

Then the series

$$\mathcal{S}(s; f_1, \ldots, f_n) := \sum_{m=(m_1,\ldots,m_n)\in(\mathbf{Z}_{>0})^n} \left(\prod_{j=1}^{n} a_j(m_j)\sigma_{k_j-1}(m_j) \right) Q_k^{(n)}(m, s)$$

has a holomorphic continuation to the whole s*-plane and satisfies the functional equation*

$$\mathcal{S}(s; f_1, \ldots, f_n) = \mathcal{S}(n - s; f_1, \ldots, f_n).$$

2 A key to the proof: an integral of Rankin-Selberg type

We use the following type of Eisenstein series for the Siegel modular group $\Gamma_n := Sp_{2n}(\mathbf{Z})$ whose properties were studied by Kohnen-Skoruppa [2], Yamazaki [5], and Deitmar-Krieg [1]:

$$E_s^{(n)}(Z) := \sum_{M \in \Delta_{n,n-1} \backslash \Gamma_n} \left(\frac{\det(\mathrm{Im}(M\langle Z \rangle))}{\det(\mathrm{Im}(M\langle Z \rangle^*))} \right)^s . \tag{4}$$

Here $s \in \mathbf{C}$, Z is a variable on H_n, the Siegel upper half space of degree n,

$$\Delta_{n,n-1} := \left\{ \begin{pmatrix} * & * \\ 0^{(1,2n-1)} & * \end{pmatrix} \in \Gamma_n \right\},$$

M runs over a complete set of representatives of $\Delta_{n,n-1} \backslash \Gamma_n$; for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with A, B, C, D being $n \times n$ blocks ,

$$M\langle Z \rangle := (AZ + D)(CZ + D)^{-1}$$

and $M\langle Z \rangle^*$ is the upper left $(n-1) \times (n-1)$ block of $M\langle Z \rangle$. We understand that

$$\det(\mathrm{Im}(M\langle Z \rangle^*)) = 1$$

if $n = 1$. The right-hand side of (4) converges absolutely and locally uniformly for $\mathrm{Re}(s) > n$. Put

$$\xi(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

By [1][5], the Eisenstein series (4) has meromorphic continuation in s to the whole s-plane; the function $\xi(2s)E_s^{(n)}(Z)$ is invariant under the substitution $s \mapsto n - s$ and is holomorphic except for the simple poles at $s = 0$ and $s = n$ with residues $-1/2$ and $1/2$, respectively.

Theorem 1 follows from the following integral representation:

Theorem 2. *For*

$$F_j(z) := \overline{f_j(z)} g_j(z) \mathrm{Im}(z)^{k_j}$$

we have

$$\left(\left(\cdots \left(E_s^{(n)} \begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{pmatrix}, F_1(z_1) \right), \cdots \right), F_n(z_n) \right)$$

$$= \frac{1}{2\xi(2s)} \sum_{\nu=1}^{n} \sum_{1\le i_1<\ldots<i_\nu\le n} \left(\prod_{\substack{j\neq i_1,\ldots,i_\nu \\ 1\le j\le n}} (f_j, g_j) \right)$$
$$\cdot \mathcal{D}(s; f_{i_1}, \ldots, f_{i_\nu}; g_{i_1}, \ldots, g_{i_\nu}).$$

Remark. Define a symmetric positive definite matrix

$$P_Z := \begin{pmatrix} 1_n & {}^tX \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ X & 1_n \end{pmatrix}.$$

Then

$$E_s^{(n)}(Z) = \frac{1}{2\zeta(2s)} \sum_{h\in \mathbf{Z}^{(2n,1)}-\{0\}} ({}^thP_Zh)^{-s} \quad \text{for} \quad \mathrm{Re}(s) > n.$$

3 Supplementary remarks

(i) Let

$$\varphi_j(z) = \sum_{l=1}^{\infty} c_j(l) e^{2\pi i l z}$$

be holomorphic primitive cusp forms of weight 1 for $\Gamma_0(N_j)$ with odd characters χ_j where $N_j \in \mathbf{Z}_{>0}$ and $j = 1, \ldots, n$. Suppose $n \ge 3$. Then by Kurokawa [3, Theorem 5], the Dirichlet series

$$\sum_{l=1}^{\infty} c_1(l) \cdots c_n(l) l^{-s}$$

has meromorphic continuation in the region $\mathrm{Re}(s) > 0$ but has the line $\mathrm{Re}(s) = 0$ as a natural boundary. (Cf. also [4, Theorem 8].) Thus it is a nontrivial problem to find a series associated with more than two elliptic modular forms which has analytic continuation to the whole s-plane.
(ii) In case $n = 1$ we have

$$\mathcal{D}(s; f_1; g_1) = 2\xi(2s)(4\pi)^{1-k_1-s}\Gamma(s + k_1 - 1)D(s + k_1 - 1, f_1, g_1)$$

for $\mathrm{Re}(s) > (k_1 + 1)/2$, where

$$D(s, f_1, g_1) := \sum_{m=1}^{\infty} a_1(m)\overline{b_1(m)}m^{-s}.$$

Thus in this case Theorem 1 states nothing but the well-known properties of the Rankin series $D(s, f_1, g_1)$.

(iii) In case $n = 2$ we have

$$\begin{aligned}
&\mathcal{D}(s; f_1, f_2; g_1, g_2)\\
&= 2^{6-2|k|}\pi^{2-|k|}(2\pi)^{-2s}\frac{\Gamma(s)\Gamma(s+|k|-2)\Gamma(s+k_1-1)\Gamma(s+k_2-1)}{\Gamma(2s+|k|-2)}\\
&\cdot \sum_{m_1,m_2\in\mathbf{Z}_{>0}} a_1(m_1)a_2(m_2)\overline{b_1(m_1)b_2(m_2)}m_1^{1-k_1-s}m_2^{1-k_2}\\
&\cdot \sum_{\lambda_1,\lambda_2\in\mathbf{Z}_{>0}} \lambda_1^{-2s}F\left(s, s+k_1-1; 2s+|k|-2; 1-\frac{m_2\lambda_2^2}{m_1\lambda_1^2}\right)
\end{aligned}$$

for $\mathrm{Re}(s) > \max(k_1, k_2) + 1$, where $F = {}_2F_1$ is the hypergeometric function.
(iv) The function $Q_k^{(n)}(m, s)$ has another representation:

$$\begin{aligned}
Q_k^{(n)}(m,s) \quad &= 2^{3n-|k|+1}\pi^{\frac{n-|k|}{2}-s}\left(\prod_{j=1}^{n} m_j^{\frac{1-k_j}{2}}\right)\cdot \sum_{\lambda_1,\dots,\lambda_n\in\mathbf{Z}_{>0}}\left(\prod_{j=1}^{n}\lambda_j^{k_j-1}\right)\\
&\cdot\int_0^\infty t^{2s-1+|k|-n}\prod_{j=1}^{n}K_{k_j-1}(4\sqrt{\pi m_j}\lambda_j t)dt
\end{aligned}$$

for $\mathrm{Re}(s) > n/2$, where K_ν is the modified Bessel function of order ν.

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PRINCIPAL SERIES WHITTAKER FUNCTIONS ON SYMPLECTIC GROUPS

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§1. Class one Whittaker functions

(1.1) Definitions and notation Let G be a semisimple Lie group with finite center and $\mathfrak{g}$ its Lie algebra. Fix a maximal compact subgroup K of G and put $\mathfrak{k} = \mathrm{Lie}(K)$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ and θ the corresponding Cartan involution. For a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ and $\alpha \in \mathfrak{a}^*$, put $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X$ for all $H \in \mathfrak{a}\}$ and $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ the restricted root system. Denoted by Δ^+ the positive system in Δ and Π the set of simple roots. Then we have an Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ with $\mathfrak{n} = \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$. Let $G = NAK$ be the Iwasawa decomposition corresponding to that of $\mathfrak{g}$. We denote by W the Weyl group of the root system Δ.

Let $P_0 = MAN$ be the minimal parabolic subgroup of G with $M = Z_K(A)$. For a linear form $\nu \in \mathfrak{a}^*_{\mathbf{C}} = \mathfrak{a}^* \otimes_{\mathbf{R}} \mathbf{C}$, define a character e^ν on A by $e^\nu(a) = \exp(\nu(\log a))$ $(a \in A)$. We call the induced representation

$$\pi_\nu = L^2\text{-}\mathrm{Ind}_{P_0}^G(1_M \otimes e^{\nu+\rho} \otimes 1_N)$$

the *class one principal series representation* of G. Here $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} m_\alpha \alpha$ $(m_\alpha = \dim \mathfrak{g}_\alpha)$.

Let $U(\mathfrak{g}_{\mathbf{C}})$ and $U(\mathfrak{a}_{\mathbf{C}})$ be the universal enveloping algebras of $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{a}_{\mathbf{C}}$, the complexifications of $\mathfrak{g}$ and $\mathfrak{a}$ respectively. Set

$$U(\mathfrak{g}_{\mathbf{C}})^K = \{X \in U(\mathfrak{g}_{\mathbf{C}}) \mid \mathrm{Ad}(k)X = X \text{ for all } k \in K\}.$$

Let p be the projection $U(\mathfrak{g}_{\mathbf{C}}) \to U(\mathfrak{a}_{\mathbf{C}})$ along the decomposition $U(\mathfrak{g}_{\mathbf{C}}) = U(\mathfrak{a}_{\mathbf{C}}) \oplus (\mathfrak{n}U(\mathfrak{g}_{\mathbf{C}}) + U(\mathfrak{g}_{\mathbf{C}})\mathfrak{k})$. Define the automorphism γ of $U(\mathfrak{a}_{\mathbf{C}})$ by $\gamma(H) = H + \rho(H)$ for $H \in \mathfrak{a}_{\mathbf{C}}$. For $\nu \in \mathfrak{a}^*_{\mathbf{C}}$, define the algebra homomorphism $\chi_\nu : U(\mathfrak{g}_{\mathbf{C}})^K \to \mathbf{C}$ by

$$\chi_\nu(z) = \nu(\gamma \circ p(z))$$

for $z \in U(\mathfrak{g}_{\mathbf{C}})^K$. Note that χ_ν is trivial on $U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k}$ and the restriction of χ_ν to the center $Z(\mathfrak{g}_{\mathbf{C}})$ of $U(\mathfrak{g}_{\mathbf{C}})$ coincides with the infinitesimal character of the class one principal series representation π_ν. Let η be a unitary character of N. Since $\mathfrak{n} = [\mathfrak{n}, \mathfrak{n}] \oplus \sum_{\alpha\in\Pi} \mathfrak{g}_\alpha$, η is determined by the restriction $\eta_\alpha := \eta|_{\mathfrak{g}_\alpha}$ $(\alpha \in \Pi)$. The length $|\eta_\alpha|$ of η_α is defined as $|\eta_\alpha|^2 = \sum_{1\le i\le m_\alpha} \eta(X_{\alpha,i})$, where the root vector $X_{\alpha,j}$ is chosen as $B(X_{\alpha,i}, \theta X_{\alpha,j}) = -\delta_{i,j}$ $(1 \le i, j \le m_\alpha)$. Here $B(\ ,\ )$ is the Killing form on $\mathfrak{g}$. In this article we assume that η is nondegenerate, that is, $\eta_\alpha \ne 0$ for all $\alpha \in \Pi$.

Definition 1.1 Under the above notation, a smooth function $w = w_{\nu,\eta}$ on G is called *class one Whittaker function* if

(i) $w(ngk) = \eta(n)w(g)$, for all $n \in N$, $g \in G$ and $k \in K$,

(ii) $Zw = \chi_\nu(Z)w$, for all $Z \in U(\mathfrak{g}_{\mathbf{C}})^K$.

We denote by $\mathrm{Wh}(\nu, \eta)$ the space of class one Whittaker functions and $\mathrm{Wh}(\nu, \eta)^{\mathrm{mod}}$ the subspace consisting of moderate growth functions.

Remark. Because of the Iwasawa decomposition, $w \in \mathrm{Wh}(\nu, \eta)$ is determined by its restriction $w|_A$ to A. We call $w|_A$ the *radial part* of w.

(1.2) M and W-Whittaker functions

Theorem 1.2 *The dimension of the space* $\mathrm{Wh}(\nu, \eta)$ *is the order of the Weyl group* W *and the the dimension of* $\mathrm{Wh}(\nu, \eta)^{\mathrm{mod}}$ *is at most one. Moreover the unique (up to constant) element in* $\mathrm{Wh}(\nu, \eta)^{\mathrm{mod}}$ *is given by Jacquet integral*:

$$W(\nu, \eta; g) = \int_N a(s_0^{-1} ng)^{\nu+\rho} \eta(n)^{-1} dn.$$

Here s_0 is the longest element in W and $g = n(g)a(g)k(g)$ the Iwasawa decomposition of $g \in G$.

Hashizume ([3]) gave a basis of $\mathrm{Wh}(\nu, \eta)$ and express the Jacquet integral as a linear combination of the basis functions. Let $\langle\ ,\ \rangle$ be the inner product on $\mathfrak{a}_{\mathbf{C}}^*$ induced by the Killing form $B(\ ,\ )$. We denote by L the set of linear functions on $\mathfrak{a}_{\mathbf{C}}$ of the form $\sum_{\alpha \in \Pi} n_\alpha \alpha$ with $n_\alpha \in \mathbf{Z}_{\geq 0}$.

For each $\lambda \in L$, we can define the rational function a_λ on $\mathfrak{a}_{\mathbf{C}}^*$ as follows. Put $a_0(\nu) = 1$ and determine a_λ for $\lambda \in L \backslash \{0\}$ by

$$(\langle \lambda, \lambda \rangle + 2\langle \lambda, \nu \rangle) a_\lambda(\nu) = 2 \textstyle\sum_{\alpha \in \Pi} |\eta_\alpha|^2 a_{\lambda - 2\alpha}(\nu), \tag{1.1}$$

inductively. Here we assumed that $\langle \lambda, \lambda \rangle + 2\langle \lambda, \nu \rangle \neq 0$ for all $\lambda \in L \backslash \{0\}$.

Definition 1.3 For $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ and unitary character η of N, define a series $M(\nu, \eta; a)$ on A by

$$M(\nu, \eta; a) = a^{\nu+\rho} \sum_{\lambda \in L} a_\lambda(\nu) a^\lambda \quad (a \in A)$$

and extend it to the function on G by

$$M(\nu, \eta; g) = \eta(n(g)) M(\nu, \eta; a(g)).$$

Definition 1.4 We denote by $'\mathfrak{a}_{\mathbf{C}}^*$ the set of elements $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ satisfying the following:

(i) $\langle \lambda, \lambda \rangle + 2\langle \lambda, s\nu \rangle \neq 0$ *for all* $\lambda \in L \backslash \{0\}$ *and* $s \in W$,

(ii) $s\nu - t\nu \notin \{\sum_{\alpha \in \Pi} m_\alpha \alpha \mid m_\alpha \in \mathbf{Z}\}$ *for all* $s \neq t \in W$.

Theorem 1.5 ([3, Theorem 5.4]) *Let* $\nu \in {'\mathfrak{a}_{\mathbf{C}}^*}$. *Then the set* $\{M(s\nu, \eta; g) \mid s \in W\}$ *forms a basis of* $\mathrm{Wh}(\nu, \eta)$.

We call $W(\nu, \eta; g)$ (resp. $M(\nu, \eta; g)$) W-Whittaker function (resp. M-Whittaker function). Let us recall the linear relation between W and M-Whittaker functions.

Proposition 1.6 ([4, cf. Ch IV]) *Let $c(\nu)$ be the Harish Chandra c-function. Then*

$$c(\nu) := \int_N a(s_0^{-1}n)^{\nu+\rho}dn$$
$$= \prod_{\alpha\in\Delta_0^+} 2^{\frac{m\alpha - m_{2\alpha}}{2}} \Big(\frac{\pi}{\langle\alpha,\alpha\rangle}\Big)^{\frac{m\alpha+m_{2\alpha}}{2}} \frac{\Gamma(\nu_\alpha)\Gamma(\frac{1}{2}(\nu_\alpha+\frac{m_\alpha}{2}))}{\Gamma(\nu_\alpha+\frac{m_\alpha}{2})\Gamma(\frac{1}{2}(\nu_\alpha+\frac{m_\alpha}{2}+m_{2\alpha}))}.$$

Here $\Delta_0^+ = \{\alpha\in\Delta^+ \mid \frac{1}{2}\alpha \notin \Delta\}$.

Definition 1.7 For $\eta \in \widehat{N}$, $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ and $s \in W$, we define $\gamma(s;\nu,\eta)$ as follows. If $s = s_\alpha$ ($\alpha \in \Pi$), the simple reflection,

$$\gamma(s;\nu,\eta) = \Big(\frac{|\eta_\alpha|}{2\sqrt{2\langle\alpha,\alpha\rangle}}\Big)^{2\nu_\alpha} \frac{\Gamma(\frac{1}{2}(-\nu_\alpha+\frac{m_\alpha}{2}+1))\Gamma(\frac{1}{2}(-\nu_\alpha+\frac{m_\alpha}{2}+m_{2\alpha}))}{\Gamma(\frac{1}{2}(\nu_\alpha+\frac{m_\alpha}{2}+1))\Gamma(\frac{1}{2}(-\nu_\alpha+\frac{m_\alpha}{2}+m_{2\alpha}))}.$$

For $s \in W$ and $\alpha \in \Pi$ such that $l(s_\alpha s) = l(s)+1$, then

$$\gamma(s_\alpha s;\nu,\eta) = \gamma(s;\nu,\eta)\gamma(s_\alpha; s\nu,\eta).$$

Here $l(s)$ means the length of s.

Theorem 1.8 ([3, Theorem 7.8]) *If $\nu \in {}'\mathfrak{a}_C^*$,*

$$W(\nu,\eta;g) = \sum_{s\in W} \gamma(s_0 s;\nu,\eta)c(s_0 s\nu)M(s\nu,\eta;g).$$

Problem : Find explicit formulas of $W(\nu,\eta;g)$ and $M(\nu,\eta;g)$.

Known results (G is real semisimple) :

(1) G is real rank 1 : W (resp. M)-Whittaker functions can be written by modified K (resp. I)-Bessel functions.

(2) $G = SL(n,\mathbf{R})$: In case of $n = 3$, Tahtajan-Vinogradov ([14]) and Bump ([1]) obtained explicit formulas of W and M-Whittaker functions. For general n, Stade ([11]) found a recursive integral formula of W-Whittaker function and I ([7]) proved a similar recursive formula of M-Whittaker function conjectured in [13]. When $n = 4$, Stade ([12]) also gave a explicit formula of $a_\lambda(\nu)$ by solving the recurrence relation (1.1) and his formula included (terminating) generalized hypergeometric series ${}_4F_3(1)$ (cf. [7]).

(3) $G = Sp(2,\mathbf{R}), SO_o(2,q) (q \geq 3)$: As for W-Whittaker function on $Sp(2,\mathbf{R})$, Niwa ([9]) obtained the formula (3.5) in section (3.1). In the similar way to Proskurin's evaluation of Jacquet integral for $G = Sp(2,\mathbf{C})$ ([10]), I ([5]) found the integral expression (3.7). The explicit formula (3.4) of M-Whittaker function is also obtained in [5]. These results can be extended to $SO_o(2,q)$ in [6] ($\mathfrak{so}(2,3) \cong \mathfrak{sp}(2,\mathbf{R})$, $\mathfrak{so}(2,4) \cong \mathfrak{su}(2,2)$).

Extending the work of Niwa, we consider the problem in case of $G = Sp_n(\mathbf{R})$ and $SO_{n,n}$ in this article.

(1.3) Structure theory for $Sp_n(\mathbf{R})$ and $SO_{n,n}$ We give precise description of the notation in the above subsections. Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be algebraic groups over $\mathbf{Q}$ defined as

$$\mathbf{G}_1 = \mathbf{SO}_{n,n} = \left\{ g \in \mathbf{SL}_{2n} \,\middle|\, {}^t g \begin{pmatrix} & J_n \\ J_n & \end{pmatrix} g = \begin{pmatrix} & J_n \\ J_n & \end{pmatrix} \right\},$$

and

$$\mathbf{G}_2 = \mathbf{Sp}_n = \left\{ g \in \mathbf{SL}_{2n} \,\middle|\, {}^t g \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix} g = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix} \right\}.$$

Here $J_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$ ($n \times n$ matrix). Hereafter we use the notation in sections (1.1) and (1.2) with subscript $_1$ for $G_1 := \mathbf{G}_1(\mathbf{R}) = SO_{n,n}$ and $_2$ for $G_2 := \mathbf{G}_2(\mathbf{R}) = Sp_n(\mathbf{R})$.

< Iwasawa decompositions >

$$\begin{aligned}
\mathfrak{a}_1 &= \{\mathrm{diag}(a_1, \ldots, a_n, -a_n, \ldots, -a_1) \mid a_i \in \mathbf{R}\}, \\
\mathfrak{a}_2 &= \{\mathrm{diag}(t_1, \ldots, t_n, -t_n, \ldots, -t_1) \mid t_i \in \mathbf{R}\}, \\
A_1 &= \{\mathrm{diag}(a_1, \ldots, a_n, a_n^{-1}, \ldots, a_1^{-1}) \mid a_i > 0\}, \\
A_2 &= \{\mathrm{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \mid t_i > 0\}, \\
N_i &= \left\{ \begin{pmatrix} n_0 & * \\ 0 & J_n {}^t n_0^{-1} J_n \end{pmatrix} \in G_i \,\middle|\, n_0 = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}.
\end{aligned}$$

< principal series >

$$\begin{aligned}
&\nu = (\nu_1, \ldots, \nu_n) \in \mathfrak{a}_{i,\mathbf{C}}^* \quad (i = 1, 2), \\
&\rho_1 = \rho_1^{(n)} = (n-1, n-2, \ldots, 1, 0), \quad \rho_2 = \rho_2^{(n)} = (n, n-1, \ldots, 2, 1).
\end{aligned}$$

< Weyl groups > $W_1 = \mathfrak{S}_n \ltimes (\mathbf{Z}/2\mathbf{Z})^{n-1}, \quad W_2 = \mathfrak{S}_n \ltimes (\mathbf{Z}/2\mathbf{Z})^n.$

< unitary characters >

$$\begin{aligned}
\eta_1(u) &= \exp\big(2\pi\sqrt{-1}(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n} + u_{n-1,n+1})\big), \\
\eta_2(u) &= \exp\big(2\pi\sqrt{-1}(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n} + u_{n,n+1})\big),
\end{aligned}$$

for $u = (u_{k,l}) \in N_i$.

< $c_i(\nu)$ and $\gamma_i(s; \nu, \eta)$ >

$$\begin{aligned}
c_1(\nu) &= \pi^{\frac{n(n-1)}{2}} \prod_{1 \le i < j \le n} \frac{\Gamma(\frac{\nu_i - \nu_j}{2})\Gamma(\frac{\nu_i + \nu_j}{2})}{\Gamma(\frac{\nu_i - \nu_j + 1}{2})\Gamma(\frac{\nu_i + \nu_j + 1}{2})}, \\
c_2(\nu) &= \frac{\pi^{\frac{n^2}{2}}}{2^{\frac{n}{2}}} \prod_{1 \le i \le n} \frac{\Gamma(\frac{\nu_i}{2})}{\Gamma(\frac{\nu_i + 1}{2})} \prod_{1 \le i < j \le n} \frac{\Gamma(\frac{\nu_i - \nu_j}{2})\Gamma(\frac{\nu_i + \nu_j}{2})}{\Gamma(\frac{\nu_i - \nu_j + 1}{2})\Gamma(\frac{\nu_i + \nu_j + 1}{2})},
\end{aligned}$$

$$c_1(s_0s\nu)\gamma_1(s_0s;\nu,\eta_1) = \pi^{\frac{n(n-1)}{2}+\langle\nu,\rho_1\rangle} \frac{s\big[\pi^{\langle\nu,\rho_1\rangle}\prod_{1\le i<j\le n}\Gamma(\frac{-\nu_i+\nu_j}{2})\Gamma(\frac{-\nu_i-\nu_j}{2})\big]}{\prod_{1\le i<j\le n}\Gamma(\frac{\nu_i-\nu_j+1}{2})\Gamma(\frac{\nu_i+\nu_j+1}{2})},$$

$$c_2(s_0s\nu)\gamma_2(s_0s;\nu,\eta_2) = 2^{-\frac{n}{2}}\pi^{\frac{n^2}{2}+\langle\nu,\rho_2\rangle-\frac{1}{2}\sum_{i=1}^n\nu_i}$$
$$\cdot\frac{s\big[\pi^{\langle\nu,\rho_2\rangle-\frac{1}{2}\sum_{i=1}^n\nu_i}\prod_{1\le i\le n}\Gamma(-\frac{\nu_i}{2})\prod_{1\le i<j\le n}\Gamma(\frac{-\nu_i+\nu_j}{2})\Gamma(\frac{-\nu_i-\nu_j}{2})\big]}{\prod_{1\le i\le n}\Gamma(\frac{\nu_i+1}{2})\prod_{1\le i<j\le n}\Gamma(\frac{\nu_i-\nu_j+1}{2})\Gamma(\frac{\nu_i+\nu_j+1}{2})}.$$

§2. Symplectic orthogonal theta lifts and main theorem

(2.1) Weil representation and theta lift Let k be a local field and ψ a nontrivial character of k. For a finite dimensional k-vector space Z equipped with symplectic form $\langle\ ,\ \rangle$, put

$$Sp(Z,k) = \{g \in GL(Z,k) \mid \langle z_1g, z_2g\rangle = \langle z_1,z_2\rangle,\quad \forall z_1,z_2 \in Z\}.$$

Let $Z = Z^+ + Z^-$ be a polarization, that is, $Z^\pm$ are maximal isotropic subspace of Z. Let ω_ψ be the Weil representation of $\widetilde{Sp}(Z,k)$ on $\mathscr{S}(Z^+)$, the space of Schwartz-Bruhat functions on Z^+. When k is a global field and ψ a nontrivial character on $k\backslash\mathbf{A}$, we can also define Weil representation ω_ψ of $\widetilde{Sp}(Z,\mathbf{A})$ on $\mathscr{S}(Z^+_{\mathbf{A}})$.

Let k be a global field and X a $2n$-dimensional k-vector space of column vectors with symmetric form $(\ ,\ )$ given by $(x,y) = {}^tx\begin{pmatrix} & J_n\\ J_n & \end{pmatrix}y$. Then $\mathbf{G}_1(k) = SO_{n,n}(k)$ acts on X from the left and preserves $(\ ,\ )$. Also let Y be a $2n$-dimensional k-vector space of row vectors with symplectic form $\langle\ ,\ \rangle$ given by $\langle x,y\rangle = x\begin{pmatrix} & J_n\\ -J_n & \end{pmatrix}{}^ty$. Then $\mathbf{G}_2(k) = Sp_n(k)$ acts on Y from the right and preserves $\langle\ ,\ \rangle$. The space $Z := X\otimes Y$ has a symplectic form $(\ ,\ )\otimes\langle\ ,\ \rangle$ and we have a homomorphism $SO_{n,n}(\mathbf{A})\times Sp_n(\mathbf{A}) \to Sp(Z,\mathbf{A})$. Let $\{e_1,\dots,e_n,e_{-n},\dots,e_{-1}\}$ be the standard basis of X. Then $X^+ = \mathrm{Span}\{e_1,\dots,e_n\}$ and $X^- = \mathrm{Span}\{e_{-n},\dots,e_{-1}\}$ give a polarization of X. Also take the standard basis of Y by $\{\varepsilon_1,\dots,\varepsilon_n,\varepsilon_{-n},\dots,\varepsilon_{-1}\}$ and put $Y^+ = \mathrm{Span}\{\varepsilon_1,\dots,\varepsilon_n\}$, $Y^- = \mathrm{Span}\{\varepsilon_{-n},\dots,\varepsilon_{-1}\}$. We choose a polarization of Z by $Z^\pm = X\otimes Y^\pm$ and denote $\sum_{i=1}^n x_i\otimes\varepsilon_i \in Z^+$ by $(x_1,\dots,x_n)$.

For ω_ψ and $\phi\in\mathscr{S}(Z^+_{\mathbf{A}})$, define the theta series θ^ϕ_ψ on $\mathbf{G}_1(\mathbf{A})\times\mathbf{G}_2(\mathbf{A})$ by

$$\theta^\phi_\psi(g_1,g_2) = \sum_{z\in Z^+_k}\omega_\psi(g_1,g_2)\phi(z).$$

Let σ be an irreducible cuspidal automorphic representation of $\mathbf{G}_1(\mathbf{A})$. For a cusp form $f\in\sigma$, put

$$F^\phi_f(g_2) = \int_{\mathbf{G}_1(k)\backslash\mathbf{G}_1(\mathbf{A})}\theta^\phi_\psi(g_1,g_2)f(g_1)dg_1.$$

It is known that F^ϕ_f defines a cusp form on $\mathbf{G}_2(\mathbf{A})$ and the space $\Theta_\psi(\sigma) = \langle F^\phi_f \mid f\in\sigma, \phi\in\mathscr{S}(Z^+_{\mathbf{A}})\rangle$ is called the *theta lift* of σ with respect to ψ.

(2.2) Whittaker coefficients To describe Whittaker coefficient, we fix unitary characters ψ_1 and ψ_2 of $\mathbf{N}_1(\mathbf{A})$ and $\mathbf{N}_2(\mathbf{A})$ as follows (cf. section (1.3)).

$$\psi_1(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n} + u_{n-1,n+1}),$$
$$\psi_2(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n} + u_{n,n+1})$$

for $u = (u_{k,l}) \in \mathbf{N}_i(\mathbf{A})$. We say an irreducible cuspidal representation σ_i on $\mathbf{G}_i(\mathbf{A})$ *has a nontrivial ψ_i^{-1}-Whittaker coefficient*, if the integral

$$W_f(g_i) = \int_{\mathbf{N}_i(k)\backslash\mathbf{N}_i(\mathbf{A})} f(ng_i)\psi_i^{-1}(n)dn$$

does not vanish for some $f \in \sigma_i$. Ginzburg, Rallis and Soudry ([2]) proved the following:

Proposition 2.1 ([2, Proposition 3.5]) *We assume that the irreducible cuspidal representation σ of $\mathbf{G}_1(\boldsymbol{A})$ has a nontrivial ψ_1^{-1}-Whittaker coefficient. Then the theta lift $\Theta_\psi(\sigma)$ to $\mathbf{G}_2(\boldsymbol{A})$ is nontrivial and has a ψ_2^{-1}-Whittaker coefficient. Moreover, the ψ_2^{-1}-Whittaker coefficient of $F_f^\phi \in \Theta_\psi(\sigma)$ is*

$$W_{F_f^\phi}(g_2) = \int_{E(\mathbf{A})\backslash\mathbf{G}_1(\mathbf{A})} \omega_\psi(g_1, g_2)\phi(u_0)W_f(g_1)dg_1. \tag{2.1}$$

Here E is the stabilizer of $u_0 = (e_1, \ldots, e_{n-1}, e_n + e_{-n}) \in Z^+$.

If we decompose the right hand side of (2.1) to the local factors, the integral

$$\int_{E(\mathbf{R})\backslash\mathbf{G}_1(\mathbf{R})} \omega_\psi(g_1, g_2)\phi(u_0)W(g_1)dg_1.$$

is expected to represent the Whittaker function on $Sp_n(\mathbf{R})$. Here W is the Whittaker function on $SO_{n,n}$. Then, if we take

$$\phi(X) = \exp[-\pi(\mathrm{tr}({}^tXX))],$$

and compute the integral by using the formulas of Weil representation, we can propose the following:

Theorem 2.2 *For $a \in A_1$ and $t \in A_2$, put*

$$\theta(a,t) = \exp\Big[-\pi\Big\{\Big(\frac{t_1^2}{a_1^2} + \frac{a_1^2}{t_2^2}\Big) + \cdots + \Big(\frac{t_{n-1}^2}{a_{n-1}^2} + \frac{a_{n-1}^2}{t_n^2}\Big) + \Big(\frac{t_n^2}{a_n^2} + t_n^2a_n^2\Big)\Big\}\Big].$$

Then, for $\nu \in {}'\mathfrak{a}_{1,\mathbf{C}}^* \cap {}'\mathfrak{a}_{2,\mathbf{C}}^*$,

$$\frac{\pi^{-\frac{1}{2}\sum_{i=1}^n \nu_i}}{(2\pi)^{\frac{n}{2}}} \prod_{i=1}^n \Gamma\Big(\frac{\nu_i+1}{2}\Big) \cdot t^{-\rho_2}W_2(\nu;t) = \int_{(\mathbf{R}_{\geq 0})^n} \theta(a,t)\cdot a^{-\rho_1}W_1(\nu;a)\prod_{i=1}^n \frac{da_i}{a_i}. \tag{2.2}$$

The right hand side of (2.2) represent a Whittaker function, however, to see that it is just the Whittaker function we want to seek, it seems to need further argument. For example, if we use the similar result of [2] from $\mathbf{Sp}_n$ to $\mathbf{SO}_{n+1,n+1}$, we obtain Whittaker function on $SO_{n+1,n+1}$ from one on $Sp_n(\mathbf{R})$ (see (3.11)). Though in this formula, the parameter of principal series is not general ($\nu_{n+1} = 0$). Then in case of $n = 2$, Niwa proved this theorem by checking the right hand side (=(3.5)) satisfy the system of partial differential equation for $Sp_2(\mathbf{R})$-Whittaker function by using computer. But in case of general n, the explicit form of differential equation is not known. So we first prove the lifting of M-Whittaker functions (which also seems to be interesting result) and by using Theorem 1.7 we establish the lifting of W-Whittaker functions.

(2.3) Lifting of M-Whittaker functions We first write down the recurrence relation (1.1) explicitly.

Proposition 2.3 *Let*

$$M_1(\nu; a) = a^{\nu+\rho_1} \sum_{\mathbf{m}=(m_1,\ldots,m_n)\in(\mathbf{Z}_{\geq 0})^n} c_{1,\mathbf{m}}(\nu)\Big(2\pi\frac{a_1}{a_2}\Big)^{2m_1}\cdots\Big(2\pi\frac{a_{n-1}}{a_n}\Big)^{2m_{n-1}}(2\pi a_{n-1}a_n)^{2m_n}$$

be the radial part of M-Whittaker function on $SO_{n,n}$. If $\nu \in {}'\mathfrak{a}^*_{1,\mathbf{C}}$, the coefficients $c_{1,\mathbf{m}}(\nu)$ are determined by the following recurrence relation:

$$\begin{aligned}(2.3)\qquad &\Big[4\Big(\sum_{i=1}^{n} m_i^2 - \sum_{i=1}^{n-2} m_i m_{i+1} - m_{n-2}m_n\Big)\\ &+2\Big(\sum_{i=1}^{n-1} m_i(\nu_i-\nu_{i+1}) + m_n(\nu_{n-1}+\nu_n)\Big)\Big]c_{1,\mathbf{m}}(\nu) = \sum_{i=1}^{n} c_{1,\mathbf{m}-\mathbf{e}_i}(\nu),\end{aligned}$$

with $\mathbf{e}_i = (0,\ldots,1,\ldots,0)$.

Proposition 2.4 *Let*

$$M_2(\nu; t) = t^{\nu+\rho_2} \sum_{\mathbf{k}=(k_1,\ldots,k_n)\in(\mathbf{Z}_{\geq 0})^n} c_{2,\mathbf{k}}(\nu)\Big(2\pi\frac{t_1}{t_2}\Big)^{2k_1}\cdots\Big(2\pi\frac{t_{n-1}}{t_n}\Big)^{2k_{n-1}}(2\pi t_n^2)^{2k_n}$$

be the radial part of M-Whittaker function on $Sp_n(\boldsymbol{R})$. If $\nu \in {}'\mathfrak{a}^*_{2,\mathbf{C}}$, the coefficients $c_{2,\mathbf{k}}(\nu)$ are determined by the following recurrence relation:

$$\begin{aligned}(2.4)\qquad &\Big[4\Big(\sum_{i=1}^{n-1} k_i^2 + 2k_n^2 - \sum_{i=1}^{n-2} k_i k_{i+1} - 2k_{n-1}k_n\Big)\\ &+2\Big(\sum_{i=1}^{n-1} k_i(\nu_i-\nu_{i+1}) + 2k_n\nu_n\Big)\Big]c_{2,\mathbf{k}}(\nu) = \sum_{i=1}^{n-1} c_{2,\mathbf{k}-\mathbf{e}_i}(\nu) + 2c_{2,\mathbf{k}-\mathbf{e}_n}(\nu).\end{aligned}$$

From the above propositions we can prove the following:

Theorem 2.5 *If* $\nu \in {}'\mathfrak{a}^*_{1,\mathbf{C}} \cap {}'\mathfrak{a}^*_{2,\mathbf{C}}$,

$$c_{2,\mathbf{k}}(\nu) = \frac{1}{\prod_{i=1}^{n}(\frac{\nu_i}{2}+1)_{k_i}}$$

$$\cdot \sum_{\mathbf{m}\in S(\mathbf{k})} \frac{(-1)^{m_1+\cdots+m_{n-1}} 4^{\sum_{i=1}^{n}(m_i-k_i)} \prod_{i=1}^{n-1}(-k_{i+1}-\frac{\nu_{i+1}}{2})_{m_i}\cdot c_{1,\mathbf{m}}(\nu)}{(k_1-m_1)!\dots(k_{n-2}-m_{n-2})!(k_{n-1}-m_{n-1}-m_n)!(k_n-m_n)!}.$$

Here we use the notation

$$S(\mathbf{k})=\left\{\mathbf{m}\in\mathbf{Z}^n_{\geq 0}\ \middle|\ \begin{array}{l} 0\le m_1\le k_1,\dots,0\le m_{n-2}\le k_{n-2}, \\ 0\le m_{n-1},\ m_{n-1}+m_n\le k_{n-1},\ 0\le m_n\le k_n \end{array}\right\}$$

and $(a)_n=\Gamma(a+n)/\Gamma(a)$.

By using this Theorems 2.5 and 1.7, we compute the right hand side of (2.2), then we can reach the Theorem 2.2 after somewhat complicated but elementary calculus.

§3. Examples of explicit formulas

From now on we adopt the notation $W_1^{(n)}(\nu;a)$ (resp. $W_2^{(n)}(\nu;t)$) for the radial part of W-Whittaker function on $SO_{n,n}$ (resp. $Sp_n(\mathbf{R})$), etc.

(3.1) From $SO_{2,2}$ to $Sp_2(\mathbf{R})$

Proposition 3.1

$$M_1^{(2)}(\nu;a)=a_1^{\nu_1+1}a_2^{\nu_2}\sum_{m_1,m_2\geq 0}\frac{(\pi a_1/a_2)^{2m_1}(\pi a_1a_2)^{2m_2}}{m_1!m_2!(\frac{\nu_1-\nu_2}{2}+1)_{m_1}(\frac{\nu_1+\nu_2}{2}+1)_{m_2}}. \tag{3.1}$$

Proposition 3.2 $W_1^{(2)}(\nu;a)$ *has the following expressions.*

$$c_1^{(2)}a_1K_{\frac{\nu_1-\nu_2}{2}}\Big(2\pi\frac{a_1}{a_2}\Big)K_{\frac{\nu_1+\nu_2}{2}}(2\pi a_1a_2), \tag{3.2}$$

$$c_1^{(2)}a_1\int_{(\mathbf{R}_{\geq 0})^2}\exp\Big[-\pi\Big\{\frac{a_1^2}{t^2}+\Big(\frac{t^2}{a_2^2}+a_2^2t^2\Big)+\Big(\frac{t^2}{b^2}+t^2b^2\Big)\Big\}\Big]\cdot b^{\nu_1}\Big(\frac{a_1a_2b}{t^2(1+a_2^2b^2)}\Big)^{\nu_2}\frac{dt}{t}\frac{db}{b}, \tag{3.3}$$

with some constant $c_1^{(2)}$.

From the above two propositions, we have the followings:

Proposition 3.3

$$M_2^{(2)}(\nu;t)=t_1^{\nu_1+2}t_2^{\nu_2+1}\sum_{m_1,m_2\geq 0} {}_3F_2\left(\begin{array}{c}-m_2,\ -m_1-\frac{\nu_1}{2},\ m_1+\frac{\nu_1}{2}+1\\ \frac{\nu_1}{2}+1,\ \frac{\nu_2}{2}+1\end{array}\middle|\ 1\right)\cdot\frac{(\pi t_1/t_2)^{2m_1}(\pi t_2^2)^{2m_2}}{m_1!\,m_2!\,(\frac{\nu_1-\nu_2}{2}+1)_{m_1}(\frac{\nu_1+\nu_2}{2}+1)_{m_2}}. \tag{3.4}$$

Proposition 3.4 $W_2^{(2)}(\nu;t)$ *has following integral expressions.*

$$c_2^{(2)}t_1^2t_2\int_{(\mathbf{R}_{\geq 0})^2}\exp\Big[-\pi\Big\{\Big(\frac{t_1^2}{a_1^2}+\frac{a_1^2}{t_2^2}\Big)+\Big(\frac{t_2^2}{a_2^2}+t_2^2a_2^2\Big)\Big\}\Big]\cdot K_{\frac{\nu_1-\nu_2}{2}}\Big(2\pi\frac{a_1}{a_2}\Big)K_{\frac{\nu_1+\nu_2}{2}}(2\pi a_1a_2)\frac{da_1da_2}{a_1a_2}, \tag{3.5}$$

$$c_2^{(2)} t_1^2 t_2 \int_{(\mathbf{R}_{\geq 0})^4} \exp\Big[-\pi\Big\{\Big(\frac{t_1^2}{a_1^2}+\frac{a_1^2}{t_2^2}\Big)+\Big(\frac{t_2^2}{a_2^2}+t_2^2a_2^2\Big)+\frac{a_1^2}{u^2}+\Big(\frac{u^2}{a_2^2}+a_2^2u^2\Big) + \Big(\frac{u^2}{b^2}+u^2b^2\Big)\Big\}\Big]\cdot b^{\nu_1}\Big(\frac{a_1a_2b}{u^2(1+a_2^2b^2)}\Big)^{\nu_2}\frac{da_1da_2}{a_1a_2}\frac{du}{u}\frac{db}{b}, \tag{3.6}$$

$$\frac{1}{4}c_2^{(2)} t_1^{2+\frac{\nu_1}{2}} t_2^{1-\frac{3\nu_2}{2}} \int_{(\mathbf{R}_{\geq 0})^2} K_{\frac{\nu_1}{2}}\Big(2\pi\frac{t_1}{t_2}\sqrt{1+x+y}\Big)K_{\frac{\nu_2}{2}}\Big(2\pi t_2^2\sqrt{(1+1/x)(1+1/y)}\Big) \cdot\Big(\frac{x^2y^2}{1+x+y}\Big)^{\frac{\nu_1}{4}}\Big(\frac{x(1+x)}{y(1+y)}\Big)^{\frac{\nu_2}{4}}\frac{dxdy}{xy}, \tag{3.7}$$

with some constant $c_2^{(2)}$.

Remark. As mentioned before, (3.5) is the result of [9] and (3.7) is of [5]. The equivalence of these two expressions can be checked by way of (3.6) and slight change of variables.

(3.2) From $SO_{3,3}$ to $Sp_3(\mathbf{R})$ By virtue of $\mathfrak{so}_{3,3} \cong \mathfrak{sl}_4(\mathbf{R})$, we can find the integral expressions of $W_1^{(3)}(\nu;a)$ by the result of Stade ([11]) for W-Whittaker functions on $SL(n,\mathbf{R})$.

Proposition 3.5 $W_1^{(3)}(\nu;a)$ can be written as follows.

$$c_1^{(3)} a_1^2 a_2 \int_{(\mathbf{R}_{\geq 0})^2} K_{\frac{\nu_1+\nu_2}{2}}\Big(2\pi a_2a_3\sqrt{1+u_1^{-2}}\Big)K_{\frac{\nu_1+\nu_2}{2}}\Big(2\pi\frac{a_2}{a_3}\sqrt{1+u_2^2}\Big) \cdot K_{\frac{\nu_1+\nu_2}{2}}\Big(2\pi\frac{a_1}{a_2}\sqrt{(1+u_1^2)(1+u_2^{-2})}\Big)K_{\frac{\nu_1-\nu_2}{2}}\Big(2\pi\frac{a_1}{a_2}\frac{u_1}{u_2}\Big) \cdot\Big(\frac{a_3}{u_1u_2}\Big)^{\nu_3}\frac{du_1du_2}{u_1u_2}, \tag{3.8}$$

$$c_1^{(3)} a_1^2 a_2 \int_{(\mathbf{R}_{\geq 0})^6} \exp\Big[-\pi\Big\{\frac{a_1^2}{t_1^2}+\Big(\frac{t_1^2}{a_2^2}+\frac{a_2^2}{t_2^2}\Big)+\Big(\frac{t_2^2}{a_3^2}+a_3^2t_2^2\Big)+\Big(\frac{t_1^2}{b_1^2}+\frac{b_1^2}{t_2^2}\Big) + \Big(\frac{t_2^2}{b_2^2}+t_2^2b_2^2\Big)+\frac{b_1^2}{s^2}+\Big(\frac{s^2}{b_2^2}+b_2^2s^2\Big)+\Big(\frac{s^2}{c^2}+s^2c^2\Big)\Big\}\Big] \cdot c^{\nu_1}\Big(\frac{b_1b_2c}{s^2(1+b_2^2c^2)}\Big)^{\nu_2}\Big(\frac{a_1a_2a_3b_1b_2}{t_1^2t_2^2(1+a_3^2b_2^2)}\Big)^{\nu_3}\frac{dt_1dt_2}{t_1t_2}\frac{db_1db_2}{b_1b_2}\frac{ds}{s}\frac{dc}{c}, \tag{3.9}$$

with some constant $c_1^{(3)}$.

Proposition 3.6 $W_2^{(3)}(\nu;t)$ *is of the form*

$$c_2^{(3)} t_1^3 t_2^2 t_3 \int_{(\mathbf{R}_{\geq 0})^5} K_{\frac{\nu_1+\nu_2}{2}}\Big(2\pi a_2 a_3 \sqrt{1+u_1^{-2}}\Big) K_{\frac{\nu_1+\nu_2}{2}}\Big(2\pi \frac{a_2}{a_3}\sqrt{1+u_2^2}\Big)$$
$$\cdot K_{\frac{\nu_1+\nu_2}{2}}\Big(2\pi \frac{a_1}{a_2}\sqrt{(1+u_1^2)(1+u_2^{-2})}\Big) K_{\frac{\nu_1-\nu_2}{2}}\Big(2\pi \frac{a_1}{a_2}\frac{u_1}{u_2}\Big) \tag{3.10}$$
$$\cdot \exp\Big[-\pi\Big\{\Big(\frac{t_1^2}{a_1^2}+\frac{a_1^2}{t_2^2}\Big)+\Big(\frac{t_2^2}{a_2^2}+\frac{a_2^2}{t_3^2}\Big)+\Big(\frac{t_3^2}{a_3^2}+t_3^2a_3^2\Big)\Big\}\Big]$$
$$\cdot\Big(\frac{a_3}{u_1u_2}\Big)^{\nu_3}\frac{du_1du_2}{u_1u_2}\frac{da_1da_2da_3}{a_1a_2a_3},$$

with some constant $c_2^{(3)}$.

Remark. We also have a formula for $M_2^{(3)}(\nu;t)$ by using the formula in [12], however, our result is not satisfactory form now.

(3.3) Conjecture for general n [2, Proposition 2.7] also computed Whittaker coefficient of theta lift from $\mathbf{Sp}_n$ to $\mathbf{SO}_{n+1,n+1}$. In view of the result, it seems to hold

$$a^{-\rho_1^{(n+1)}} W_1^{(n+1)}((\nu_1,\dots,\nu_n,0);a)$$
$$= c\int_{\mathbf{R}^n_{\geq 0}} \widetilde{\theta}(a,t)\cdot t^{-\rho_2^{(n)}} W_2^{(n)}((\nu_1,\dots,\nu_n);t)\prod_{i=1}^n \frac{dt_i}{t_i}, \tag{3.11}$$

where

$$\widetilde{\theta}(a,t)=\exp\Big[-\pi\Big\{\frac{a_1^2}{t_1^2}+\Big(\frac{t_1^2}{a_1^2}+\frac{a_2^2}{t_2^2}\Big)+\cdots+\Big(\frac{t_{n-1}^2}{a_{n-1}^2}+\frac{a_n^2}{t_n^2}\Big)+\Big(\frac{t_n^2}{a_{n+1}^2}+a_{n+1}^2t_n^2\Big)\Big\}\Big].$$

It may be impossible to extend $(\nu_1,\dots,\nu_n,0)\to(\nu_1,\dots,\nu_{n+1})$ by adding some terms containing ν_{n+1} to the integrand, however, we can propose the following conjecture from the results for $n=2,3$ ((3.3), (3.9)).

Conjecture 3.7 *Let* $b=\mathrm{diag}(b_1,\dots,b_{n+1},b_{n+1}^{-1},\dots,b_1^{-1})$. *Then* $W_1^{(n+1)}((\nu_1,\dots,\nu_{n+1});b)$ *has the following expressions.*

$$c\, b^{\rho_1^{(n+1)}}\int_{(\mathbf{R}_{\geq 0})^{2n}} \widetilde{\theta}(b,t)\theta(t,a)\cdot a^{-\rho_1^{(n)}} W_1^{(n)}((\nu_1,\dots,\nu_n);a)$$
$$\cdot\Big(\frac{b_1\cdots b_{n+1}a_1\cdots a_n}{(t_1\cdots t_n)^2(1+b_{n+1}^2a_n^2)}\Big)^{\nu_{n+1}}\prod_{i=1}^n\frac{dt_i}{t_i}\frac{da_i}{a_i}, \tag{3.12}$$

$$c\, b^{\rho_1^{(n+1)}}\int_{(\mathbf{R}_{\geq 0})^n}\prod_{i=1}^{n-1}K_{\nu_{n+1}}\Big(2\pi\frac{b_i}{b_{i+1}}\sqrt{\Big(1+\frac{a_{i-1}^2}{b_i^2}\Big)\Big(1+\frac{b_{i+1}^2}{a_i^2}\Big)}\Big)$$
$$\cdot K_{\nu_{n+1}}\Big(2\pi b_nb_{n+1}\sqrt{\Big(1+\frac{a_{n-1}^2}{b_n^2}\Big)\Big(1+\frac{a_n^2}{b_{n+1}^2}\Big)\Big(1+\frac{1}{a_n^2b_{n+1}^2}\Big)}\Big) \tag{3.13}$$
$$\cdot a^{-\rho_1^{(n)}}W_1^{(n)}((\nu_1,\dots,\nu_n);a)\Big(\frac{a_n^2+b_{n+1}^2}{1+a_n^2b_{n+1}^2}\Big)^{\frac{\nu_{n+1}}{2}}\prod_{i=1}^n\frac{da_i}{a_i},$$

with some constant c.

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THETA CORRESPONDENCE AND REPRESENTATION THEORY

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Abstract. After reviewing the relation between the theta integral (= theta lifting) and Howe correspondence, we give an example of the preservation of the associated cycles by the theta lifting (joint work with C.-B. Zhu).

Namely, let (G, G') be a type I dual pair strictly in the stable range (we assume that G' is the smaller member), and π' a unitary highest weight module of $\widetilde{G'}$. Then the associated cycle of the theta lift $\pi = \theta(\pi')$ of π' can be given as $\mathcal{AC}(\theta(\pi')) = \theta(\mathcal{AC}(\pi'))$, where the theta lifting of associated cycle is naturally defined using the lifting of nilpotent orbits. We also give a naive introduction to the basic property of associated cycles and the lifting of nilpotent orbits.

1. Theta integral

The content of this section is mainly quoted from [10, 11] and [4].

Let F be a number field and $\mathbb{A}$ a ring of adeles of F. For simplicity, we consider *one of type I dual pairs* defined over F in the following. It is constructed as follows. Take a vector space

$$V\ /F \text{ with non-degenerate symmetric bilinear form } (\,,\,) = (\,,\,)_V$$
$$V'\ /F \text{ with non-degenerate skew-symmetric bilinear form } (\,,\,)' = (\,,\,)_{V'}$$

Then $W = V \otimes_F V'$ inherits a skew-symmetric form defined by $(\,,\,)_W = (\,,\,)_V \otimes_F (\,,\,)_{V'}$. We put

$$\begin{cases} G = O(V) & \text{orthogonal group} \\ G' = Sp(V') & \text{symplectic group} \end{cases}$$

They are naturally subgroups of $Sp(W)$ commuting with each other, which form a type I dual pair (G, G') in $Sp(W)$. We denote by $G(\mathbb{A}), G'(\mathbb{A})$ or $Sp(W)_{\mathbb{A}}$, the global adelic groups. For each place v of F, let F_v be the completion of F at v, and G_v or G'_v denotes the corresponding groups over the local field F_v.

$Sp(W)_{\mathbb{A}}$ has a non-trivial double cover $Mp(W)_{\mathbb{A}}$ called the metaplectic group. This group has a distinguished representation called the Weil representation. We do not give an exact construction of the representation but use an explicit realization. For this, we refer the readers to [20], [4], [3], [17], et al.

Let $W = X \oplus Y$ be a complete polarization, and take a character χ of $\mathbb{A}$ which is trivial on F. Then the Weil representation $\omega = \omega_\chi$ of $Mp(W)_{\mathbb{A}}$ is realized on the Hilbert space $L^2(X(\mathbb{A}))$. It is unitary, and the space of smooth vectors coincides with the space of Schwartz-Bruhat functions $\mathcal{S} = \mathcal{S}(X(\mathbb{A}))$ on $X(\mathbb{A})$.

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Let θ be a tempered distribution on $\mathcal{S}$ defined by

$$\theta(\varphi) = \sum_{\xi \in X(F)} \varphi(\xi) \qquad (\varphi \in \mathcal{S}), \tag{1.1}$$

which converges absolutely. Then θ is $Sp(W)_F$-invariant distribution, i.e.,

$$\theta(\omega(\gamma)\varphi) = \theta(\varphi) \qquad (\gamma \in Sp(W)_F,\ \varphi \in \mathcal{S}).$$

Note that $Sp(W)_F$ is embedded into $Mp(W)_{\mathbb{A}}$ as a discrete subgroup. This property characterizes θ up to constant multiple ([4]). Let $\widetilde{G}(\mathbb{A})$ denote the inverse image of $G(\mathbb{A})$ of the covering map $Mp(W)_{\mathbb{A}} \to Sp(W)_{\mathbb{A}}$. The same notation applies to arbitrary subgroup of $Sp(W)_{\mathbb{A}}$. For $(g, h) \in \widetilde{G}(\mathbb{A}) \times \widetilde{G'}(\mathbb{A})$, we put

$$\theta_\varphi(g, h) = \theta(\omega(g \cdot h)\varphi) \qquad (\varphi \in \mathcal{S}). \tag{1.2}$$

Then, appropriate choice of φ and G, G' will give various types of classical theta functions.

Assume that π' is an automorphic representation realized on a Hilbert space

$$\mathcal{H}_{\pi'} \subset L^2(G'(F)\backslash\widetilde{G'}(\mathbb{A})).$$

For $f \in \mathcal{H}_{\pi'}^\infty =$ (smooth vectors), we define the *theta integral* by

$$\theta_\varphi^f(g) = \int_{G'(F)\backslash\widetilde{G'}(\mathbb{A})} \theta_\varphi(g, h) f(h) dh. \tag{1.3}$$

If π' is a cuspidal representation, then the integral converges and defines a slowly increasing function on $G(F)\backslash\widetilde{G}(\mathbb{A})$. In the following (in this section), *we assume π' to be cuspidal.*

Formally θ_φ^f gives an automorphic form on $G(F)\backslash\widetilde{G}(\mathbb{A})$ and one may expect that

$$\{\theta_\varphi^f \mid f \in \mathcal{H}_{\pi'}^\infty, \varphi \in \mathcal{S}(X(\mathbb{A}))\}$$

gives an automorphic representation π of $\widetilde{G}(\mathbb{A})$ after some *completion.* Thus we want to see when the integral

$$\langle \theta_{\varphi_1}^{f_1}, \theta_{\varphi_2}^{f_2} \rangle = \int_{G(F)\backslash\widetilde{G}(\mathbb{A})} \theta_{\varphi_1}^{f_1}(g)\, \overline{\theta_{\varphi_2}^{f_2}(g)}\, dg \tag{1.4}$$

converges (under some assumption), and gives a non-zero value for some choice of $\{f_i\}$ and $\{\varphi_i\}$.

Theorem 1.1 (Rallis' inner product formula)**.** *Assume* $\dim V > 2\dim V' + 2$. *Then the above inner product* (1.4) *converges absolutely. Moreover, we have*

$$\begin{aligned}\langle \theta_{\varphi_1}^{f_1}, \theta_{\varphi_2}^{f_2} \rangle &= \int_{\widetilde{G'}(\mathbb{A})} \langle \omega(h)\varphi_1, \varphi_2 \rangle \langle \pi'(h) f_1, f_2 \rangle dh \\ &= \prod_{v \in P(F)} \int_{\widetilde{G'}(F_v)} \langle \omega_v(h)\varphi_{1\,v}, \varphi_{2\,v} \rangle \langle \pi'_v(h) f_{1\,v}, f_{2\,v} \rangle dh\end{aligned}$$

Here, $P(F)$ denotes the set of all places of F.

Proof. For proof, see [10, Theorem 2.1]. Essentially, the following two ingredients prove the theorem; (i) Howe's technique of doubling variables; (ii) Siegel-Weil formula, which claims that θ-integral coincides with an Eisenstein series. □

Thus we should consider the integral

$$\int_{\widetilde{G'}(F_v)} \langle \omega_v(h)\varphi_{1\,v}, \varphi_{2\,v}\rangle \langle \pi'_v(h) f_{1\,v}, f_{2\,v}\rangle dh$$

at various places $v \in P(F)$. For finite places at which F is not ramified, it is given by special values of L-functions. In the following sections, we will concentrate on real places.

2. Theta correspondence over reals

From now on, we assume the ground field is $\mathbb{R}$, thus V, V' are now considered as vector spaces over $\mathbb{R}$ with symmetric (respectively skew-symmetric) non-degenerate bilinear form. We also write $G = O(V)$ and $G' = Sp(V')$, which are *real Lie groups*. The Weil representation $\omega = \omega_\chi$ is realized on the space of L^2-functions $L^2(X)$ on a maximal totally isotropic space X of W.

Let us consider the integral

$$\int_{\widetilde{G'}} \langle \omega(h)\varphi_1, \varphi_2\rangle \langle \pi'(h) f_1, f_2\rangle dh \qquad (\varphi_i \in \mathcal{S}(X),\ f_i \in \mathcal{H}_{\pi'}^\infty) \tag{2.1}$$

for a genuine irreducible unitary representation π' of $\widetilde{G'}$ on a Hilbert space $\mathcal{H}_{\pi'}$. A representation of $\widetilde{G'}$ is called *genuine* if it is not factor through to the representation of G', i.e., if it is non-trivial on the kernel of the covering map. Note that π' is not necessarily automorphic nor cuspidal now, and everything is considered over $\mathbb{R}$. If we put $\Phi_i = \varphi_i \otimes f_i \in \mathcal{S} \otimes \mathcal{H}_{\pi'}^\infty$, the above formula becomes

$$(\Phi_1, \Phi_2)_{\pi'} = \int_{G'} \langle (\omega \otimes \pi')(h)\Phi_1, \Phi_2\rangle dh. \tag{2.2}$$

Since ω and π' are both genuine, $\omega \otimes \pi'$ factors through to a representation of G', and we do not need a cover anymore.

It may be useful to consider this integral for a compact group (or a compact dual pair) as a toy model. Thus, only in this short paragraph, let us pretend as if G' was a compact group and ω was a representation of G'. Then ω decomposes discretely as

$$\omega \simeq \sum\nolimits^{\oplus}_{\xi \in \widehat{G'}} \mathrm{Hom}_{G'}(\xi, \omega) \otimes \xi, \qquad \text{hence}$$
$$\omega \otimes \pi' \simeq \sum\nolimits^{\oplus}_{\xi \in \widehat{G'}} \mathrm{Hom}_{G'}(\xi, \omega) \otimes (\xi \otimes \pi').$$

Then an integral $\int_{G'} \langle (\omega \otimes \pi')(h) v_1, v_2\rangle dh$ survives only if $\xi \simeq (\pi')^*$ for some $\xi \in \widehat{G'}$ and the collection of $(\Phi_1, \Phi_2)_{\pi'}$ will give an inner product on the space of multiplicities $\mathrm{Hom}_{G'}((\pi')^*, \omega)$. In some sense, this is carried over to our present situation.

Now let us return to our original settings in this section. For the convergence of the integral, the following theorem holds.

Theorem 2.1 (Li [8], [11, Theorem 2.1])**.** *Suppose we are in one of the following two situations.*

(1) *The pair* (G, G') *is in the* stable range*, i.e.,*

$$\dim(\textit{maximal totally isotropic space in } V) \geq \dim V'. \tag{2.3}$$

(2) π' *is in the discrete series and* $\dim V \geq \dim V'$*.*

Then the above integral (2.1) *converges absolutely for any choice of* $\{\varphi_i\} \subset \mathcal{S}$ *and* $\{f_i\} \subset \mathcal{H}_{\pi'}^{\infty}$.

Now assume the above (1) or (2) from now on.

Put $R = $ (kernel of $(\ ,\ )_{\pi'}$), and make a completion of $(\mathcal{S} \otimes \mathcal{H}_{\pi'}^{\infty})/R$ by the inner product $(\ ,\ )_{\pi'}$.

$$\mathfrak{H} = (\text{completion of } (\mathcal{S} \otimes \mathcal{H}_{\pi'}^{\infty})/R) \tag{2.4}$$

Since $\widetilde{G}$ acts on $\mathcal{S} \otimes \mathcal{H}_{\pi'}^{\infty}$ which leaves R stable, the resulting Hilbert space $\mathfrak{H}$ carries a unitary representation π of $\widetilde{G}$ (but still it may be zero). The following theorem is proved by Li for general type I dual pairs ([9, Proposition 2.4]), and independently by Moeglin for the pair $(O(2p, 2q), Sp(2n, \mathbb{R}))$.

Theorem 2.2 (Moeglin, Li)**.** (1) *If* $\mathfrak{H}$ *is not zero, then it carries a genuine unitary representation* $(\pi, \mathfrak{H})$ *of* $\widetilde{G}$*, which is the theta lift of* $(\pi')^*$ *in the sense of Howe* ([6]*; see* §3) *;* $\pi = \theta((\pi')^*)$.

(2) *If* (G, G') *is in the stable range,* $\mathfrak{H}$ *is non-zero for any unitary irreducible representation* π' *of* $\widetilde{G}$*, which is genuine.*

(3) *If* π' *is in the discrete series which is "sufficiently regular", then* $\mathfrak{H}$ *is non-zero and* $\pi = A_{\mathfrak{q}}(\lambda)$ *; a representation with non-zero cohomology defined by Vogan and Zuckerman* ([19])*.*

This theorem tells us that the representation $(\pi, \mathfrak{H})$ of $\widetilde{G}$ so-obtained is in correspondence with the dual of $(\pi', \mathcal{H})$ in the sense of Howe (one may call it Howe correspondence). In this sense, the notion of theta lifting and Howe correspondence are almost the same.

In the next section, we briefly review the definition and basic properties of Howe correspondence.

3. Howe correspondence

Let ω be the Weil representation of $Mp(W)$, and we choose a complete polarization $W = X \oplus Y$. Let $\Omega = \mathfrak{H}_{\boldsymbol{K}}$ be the space of $\boldsymbol{K}$-finite vectors of ω, where $\boldsymbol{K}$ is a maximal compact subgroup of $Mp(W)$. Then Ω is a $(\mathfrak{G}, \boldsymbol{K})$-module, where $\mathfrak{G}$ is the complexified Lie algebra of $Mp(W)$, and Ω is called the Harish-Chandra module of ω.

In this section, we only consider the Harish-Chandra module Ω, and by abuse of notation, we often denote the action of $(\mathfrak{G}, \boldsymbol{K})$ by the same letter ω, or simply write it by module notation. It is well known that Ω can be identified with the space of polynomials on $X_{\mathbb{C}} = X \otimes_{\mathbb{R}} \mathbb{C}$. In this realization, $\boldsymbol{K}$ is identified with the determinantal double cover of the unitary group $U(X_{\mathbb{C}})$, and the action of $\boldsymbol{K}$ is given by the left translation of polynomials times its determinant, i.e.,

$$\omega(k)f(x) = \sqrt{\det(k)}\, f(k^{-1}x) \qquad (k \in \boldsymbol{K}, f(x) \in \mathbb{C}[X_{\mathbb{C}}], x \in X_{\mathbb{C}}) \tag{3.1}$$

Let $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}$ be the complexified Cartan decomposition along the Lie algebra of the maximal compact subgroup. Then, the action of $\mathfrak{K}$ is given by the differential of $\omega(\boldsymbol{K})$, and the action of $\mathfrak{P}$ is given by the multiplication of polynomials of degree two (if it is a root vector of a positive root), or the differentiation by a constant coefficient differential operator of degree two (if it is a root vector of a negative root). For more detailed realization, we refer to [5] (or [12], for example).

Take an irreducible Harish-Chandra module π' of $\widetilde{G'}$, i.e., π' is a $(\mathfrak{g}', \widetilde{K'})$-module, where $\mathfrak{g}'$ denotes the complexified Lie algebra of $\widetilde{G'}$ and K' is a maximal compact subgroup of G' (similarly we will denote by $\mathfrak{g}$ the complexified Lie algebra of $\widetilde{G}$ and by K a maximal compact subgroup of G). Put

$$\mathbb{H} = \mathrm{Hom}_{(\mathfrak{g}', \widetilde{K'})}(\Omega, \pi') \qquad \text{(morphisms of Harish-Chandra } (\mathfrak{g}', \widetilde{K'})\text{-modules)}, \tag{3.2}$$

and consider

$$\Omega/N; \quad N = \bigcap_{\varphi \in \mathbb{H}} \operatorname{Ker} \varphi.$$

Then there exists a quasi-simple $(\mathfrak{g}, \widetilde{K})$-module $\Omega(\pi')$ of finite length such that

$$\Omega/N \simeq \Omega(\pi') \otimes \pi'$$

as $(\mathfrak{g} \oplus \mathfrak{g}', \widetilde{K} \times \widetilde{K'})$-modules. $\Omega(\pi')$ is called *Howe's maximal quotient* for π'.

Theorem 3.1 (Howe)**.** *If $\Omega(\pi')$ is not zero, it has a unique irreducible quotient, which is denoted by $\theta(\pi')$ and called the* theta lift *of π'.*

In fact, the correspondence $\pi' \leftrightarrow \pi = \theta(\pi')$ is bijective between the genuine irreducible representations which appear in ω as quotients. Note that π and π' are in correspondence if and only if there exists a non-trivial $(\mathfrak{g} \oplus \mathfrak{g}', \widetilde{K} \times \widetilde{K'})$-module morphism $\Omega \to \pi \otimes \pi'$. As a formal convention, we put $\theta(\pi') = 0$ if $\Omega(\pi') = 0$, i.e., π' does not appear as a quotient of ω.

Lemma 3.2. *Let us denote by $\Omega(\pi')^*$ the $\widetilde{K}$-finite dual of $\Omega(\pi')$. Then we have*

$$\Omega(\pi')^* \simeq \mathrm{Hom}_{(\mathfrak{g}', \widetilde{K'})}(\Omega, \pi')_{\widetilde{K}\text{-finite}} = \mathbb{H}_{\widetilde{K}}.$$

Proof. Take $v^* \in \Omega(\pi')^*$. Then

$$\Omega \xrightarrow{\text{proj.}} \Omega/N \simeq \Omega(\pi') \otimes \pi' \xrightarrow{v^* \otimes 1} \pi'$$

gives an element of $\mathbb{H}_{\widetilde{K}}$.

Conversely, any $f \in \mathbb{H}_{\widetilde{K}}$ factors through Ω/N by the definition of N. Thus we get $\Omega(\pi') \otimes \pi' \simeq \Omega/N \xrightarrow{\overline{f}} \pi'$, which is $(\mathfrak{g}', \widetilde{K'})$-equivariant. Since π' is irreducible, $\overline{f} : v \otimes \pi' \xrightarrow{\sim} \pi'$ gives a scalar $v_f^*(v)$. This gives the inverse map. □

Theorem 3.3 (N-Zhu [15])**.** *Assume that (G, G') is an irreducible type I dual pair strictly in the stable range. If π' is a unitary highest weight module for $\widetilde{G'}$ (so that G'/K' must be a Hermitian symmetric space), then $\Omega(\pi')$ is irreducible, hence $\Omega(\pi') = \theta(\pi')$ gives the theta lift.*

Remark 3.4. We say that the pair (G, G') is *strictly in the stable range* if (G, G') is in the following list.

This condition is a little bit stronger than the stable range condition (due to J.-S. Li) given above. Note that it is ambiguously called "stable range" in [15]. Though the theorem itself is valid for all the above three pairs, we are only treating Case $\mathbb{R}$ in this note.

Table 1. The dual pairs strictly in the stable range

	the pair (G, G')	strictly stable range condition
Case $\mathbb{R}$:	$(O(p,q), Sp(2n,\mathbb{R}))$	$2n < \min(p,q)$
Case $\mathbb{C}$:	$(U(p,q), U(m,n))$	$m+n \le \min(p,q)$
Case $\mathbb{H}$:	$(Sp(p,q), O^*(2n))$	$n \le \min(p,q)$

To give an idea of the proof of this theorem, let us briefly indicate how to compute K-types of $\Omega(\pi')$ (which is proved to be $\theta(\pi')$ afterwards).

Let us remind that V is an indefinite quadratic space over $\mathbb{R}$, and $G = O(V)$. Let $V = V^+ \oplus V^-$ be a decomposition for which

$$\begin{cases} V^+ \text{ is positive definite} & p = \dim V^+, \\ V^- \text{ is negative definite} & q = \dim V^-. \end{cases} \tag{3.3}$$

We denote $K^\pm = O(V^\pm)$, so that $K = K^+ \times K^-$ gives a maximal compact subgroup of G. Recall the complete polarization $V' = X' \oplus Y'$ of the symplectic space V'. Then according to the decomposition, we can take a maximal totally isotropic space X as

$$X = V \otimes Y' = (V^+ \otimes Y') \oplus (V^- \otimes Y'). \tag{3.4}$$

Therefore, the Weil representation ω is realized on the L^2-space

$$L^2(X) = L^2(V^+ \otimes Y') \otimes L^2(V^- \otimes Y').$$

We note that $L^2(V^\pm \otimes Y')$ carries the Weil representation for compact dual pairs $(K^\pm, G')$ $= (O(V^\pm), Sp(V'))$, whose decomposition is well known by the work of Kashiwara and Vergne [7]. Up to twisting by a genuine character of the double cover of $O(V^\pm)$, we have

$$\begin{cases} L^2(V^+ \otimes Y') \simeq \sum_{\sigma_1 \in O(V^+)^\wedge} \sigma_1^{\chi_1} \otimes L^+(\sigma_1), \\ L^2(V^- \otimes Y') \simeq \sum_{\sigma_2 \in O(V^-)^\wedge} \sigma_2^{\chi_2} \otimes L^-(\sigma_2), \end{cases} \tag{3.5}$$

where $L^+(\sigma_1)$ (respectively $L^-(\sigma_2)$) is a unitary highest (respectively lowest) weight module of $\widetilde{G'} = \widetilde{Sp}(V')$, which is genuine; and $\sigma_i^{\chi_i} = \chi_i \otimes \sigma_i$ is a genuine irreducible finite dimensional representation of the double cover $\widetilde{O}(V^\pm)$ obtained from the irreducible representation $\sigma_i \in O(V^\pm)^\wedge$ twisted by a certain genuine character χ_i. Note, however, the double covers $\widetilde{G'}$ differ according to $V^\pm$ if the parities of p and q are different. The reason is that the cover is taken in the different metaplectic groups $Mp(V^\pm \otimes V')$. Similarly, $\widetilde{G'} \subset Mp(W)$ may be different from $\widetilde{G'} \subset Mp(V^\pm \otimes V')$. But it is too subtle to denote the dependence, so we will omit it.

Under the condition that the pair is strictly in the stable range, $L^+(\sigma_1)$ is a holomorphic discrete series, and $L^-(\sigma_2)$ is an anti-holomorphic one. This will make our arguments particularly simple.

Using (3.5), we get

$$\begin{aligned} \operatorname{Hom}_{\widetilde{G'}}(\omega, \pi')_{\widetilde{K}} &= \operatorname{Hom}_{\widetilde{G'}}(L^2(V^+ \otimes Y') \otimes L^2(V^- \otimes Y'), \pi')_{\widetilde{K}} \\ &= \sum_{\sigma_1, \sigma_2} \operatorname{Hom}_{\widetilde{G'}}(L^+(\sigma_1) \otimes L^-(\sigma_2), \pi') \otimes (\sigma_1^{\chi_1} \otimes \sigma_2^{\chi_2})^*. \end{aligned}$$

Since π' is a unitary highest weight module, the multiplicity

$$\mathrm{Hom}_{\widetilde{G'}}(L^+(\sigma_1) \otimes L^-(\sigma_2), \pi') \simeq \mathrm{Hom}_{\widetilde{G'}}(L^+(\sigma_1), L^-(\sigma_2)^* \otimes \pi')$$

is of finite dimension. Moreover, it can be described in terms of finite dimensional representations. Namely, if

$$\begin{cases} \tau_1 \text{ is the minimal } \widetilde{K'}\text{-type of } L^+(\sigma_1), \text{ and} \\ \tau_2 \text{ is the minimal } \widetilde{K'}\text{-type of } L^-(\sigma_2)^*, \end{cases}$$

then the above multiplicity equals to

$$\mathrm{Hom}_{\widetilde{K'}}(\tau_1, \tau_2 \otimes (\pi'\big|_{\widetilde{K'}})).$$

Thus, by Lemma 3.2, finally we obtain

$$\Omega(\pi')\big|_{\widetilde{K}} \simeq \sum_{\sigma_1 \in O(V^+)^\wedge, \sigma_2 \in O(V^-)^\wedge} \mathrm{Hom}_{\widetilde{K'}}(\tau_1, \tau_2 \otimes (\pi'\big|_{\widetilde{K'}}))^* \otimes (\sigma_1^{\chi_1} \otimes \sigma_2^{\chi_2}). \tag{3.6}$$

Since $\theta(\pi')$ is the unique irreducible quotient of $\Omega(\pi')$, the multiplicity of $\widetilde{K}$-types in $\theta(\pi')$ cannot exceed $\dim \mathrm{Hom}_{\widetilde{K'}}(\tau_1, \tau_2 \otimes (\pi'\big|_{\widetilde{K'}}))^*$. However, if we choose appropriate $\widetilde{K}$-types in $\mathcal{S} \otimes \mathcal{H}^\infty_{(\pi')^*}$ and its $\widetilde{K'}$-finite vectors, the theta integral (2.2) converges for such vectors and gives a non-degenerate inner product. This means that the above multiplicity should survive after taking the quotient by $R = \mathrm{Ker}(\ ,\ )_{(\pi')^*}$ (cf. (2.4)). This means that the multiplicity of $\widetilde{K}$-types in $\theta(\pi')$ and $\Omega(\pi')$ is the same, which proves $\theta(\pi') = \Omega(\pi')$.

We summarize the above arguments into

Theorem 3.5 (N-Zhu). *Let π' be a genuine unitary highest weight module of $\widetilde{G'}$ and $\pi = \theta(\pi')$ its theta lift. Then the $\widetilde{K}$-type decomposition of π is given by*

$$\pi\big|_{\widetilde{K}} \simeq \sum_{\sigma_1 \in O(V^+)^\wedge, \sigma_2 \in O(V^-)^\wedge} \mathrm{Hom}_{\widetilde{K'}}(\tau_1, \tau_2 \otimes (\pi'\big|_{\widetilde{K'}}))^* \otimes (\sigma_1^{\chi_1} \otimes \sigma_2^{\chi_2}).$$

The multiplicities in the above decomposition formula is efficiently computable. See [12] for example.

4. Associated Cycle

Let $(\pi, \mathfrak{X})$ be a Harish-Chandra $(\mathfrak{g}, K)$ module, where $\mathfrak{g}$ is the complexified Lie algebra of G and K is a maximal compact subgroup. For simplicity, we assume that $\mathfrak{X}$ is quasi-simple, i.e., the center $\mathfrak{Z}(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ acts on $\mathfrak{X}$ as scalars.

Take a finite dimensional generating space $\mathfrak{X}_0 \subset \mathfrak{X}$, which is K-stable. Let $\{U_n(\mathfrak{g})\}_{n=0}^\infty$ be the standard filtration of the enveloping algebra $U(\mathfrak{g})$. We define a filtration of $\mathfrak{X}$ as $\mathfrak{X}_n = U_n(\mathfrak{g})\mathfrak{X}_0$, which is K-stable. Moreover, it satisfies $U_m(\mathfrak{g})\mathfrak{X}_n = \mathfrak{X}_{n+m}$. If a filtration satisfies this condition for sufficiently large n and arbitrary $m \geq 0$, it is called *good*. Thus $\{\mathfrak{X}_n\}_{n=0}^\infty$ is a K-stable good filtration of $\mathfrak{X}$. Let

$$\mathrm{gr}\, \mathfrak{X} = \sum_{n \geq 0}^{\oplus} \mathfrak{X}_n / \mathfrak{X}_{n-1} \qquad (\mathfrak{X}_{-1} = 0) \tag{4.1}$$

be the associated graded module of $\mathrm{gr}\, U(\mathfrak{g}) = S(\mathfrak{g})$ (the symmetric algebra of $\mathfrak{g}$).

In general, let M be a finitely generated module over a Noetherian ring A. (In our present case, we take $A = S(\mathfrak{g})$ and $M = \operatorname{gr} \mathfrak{X}$.) Let $\{P_i\}_{i=1}^{l}$ be the set of all the minimal prime ideals containing the annihilator ideal $\operatorname{Ann} M = \{a \in A \mid aM = 0\}$. Clearly,

$$\operatorname{Supp} M := \operatorname{Spec}(\operatorname{Ann} M) = \bigcup\nolimits_{i=1}^{l} \operatorname{Spec}(A/P_i) \tag{4.2}$$

gives an irreducible decomposition of $\operatorname{Supp} M$. In addition to this, we associate a multiplicity $m_i = m(M, P_i)$ with each irreducible component $\operatorname{Spec}(A/P_i)$, where m_i is defined as the length of A_{P_i}-module M_{P_i}. Here A_{P_i} or M_{P_i} denotes the localization at P_i. Note that M_{P_i} is an Artinian A_{P_i}-module, so that the multiplicity is a positive integer. The *associated cycle* $\mathcal{AC}(M)$ of M is defined to be a formal sum

$$\mathcal{AC}(M) = \sum\nolimits_{i=1}^{l} m_i \cdot \operatorname{Spec}(A/P_i). \tag{4.3}$$

If $\mathcal{M}$ is a coherent sheaf over $\operatorname{Spec} A$ corresponding to M, then the support of $\mathcal{M}$ is $\operatorname{Supp} M$ above, and the usual notion of characteristic cycle $\operatorname{Ch}(\mathcal{M})$ coincides with $\mathcal{AC}(M)$.

Let us return to the $S(\mathfrak{g})$-module $\operatorname{gr} \mathfrak{X}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition, and we identify $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}]$ via Killing form, so that m-Spec $S(\mathfrak{g}) \simeq \mathfrak{g}$, where m-Spec A denotes the set of maximum spectrum of A.

Theorem 4.1 (Vogan [18])**.** *The associated cycle $\mathcal{AC}(\operatorname{gr} \mathfrak{X})$ does not depend on the choice of the generating space $\mathfrak{X}_0$. We denote it by $\mathcal{AC}(\mathfrak{X})$. Then $\mathcal{AC}(\mathfrak{X})$ is a finite union of the closure of $K_{\mathbb{C}}$-nilpotent orbits in $\mathfrak{p}$ with multiplicity.*

$$\mathcal{AC}(\mathfrak{X}) = \sum_{i=1}^{l} m_i \cdot [\overline{O_i}] \qquad (O_i:\ K_{\mathbb{C}}\textit{-nilpotent orbit in } \mathfrak{p}) \tag{4.4}$$

We call $\operatorname{Supp}(\operatorname{gr} \mathfrak{X}) = \cup_i \overline{O_i}$ *the* associated variety *of $\mathfrak{X}$, and denote it by $\mathcal{AV}(\mathfrak{X})$.*

Proof. We skip the proof of independency of $\mathcal{AC}(\mathfrak{X})$ from the choice of the $K_{\mathbb{C}}$-stable generating space.

Since the filtration $\{\mathfrak{X}_n\}_{n=0}^{\infty}$ is K-stable, the action of $\mathfrak{k} = \operatorname{Lie}(K)_{\mathbb{C}}$ kills $\operatorname{gr} \mathfrak{X}$. Thus, in fact, $\operatorname{gr} \mathfrak{X}$ is an $S(\mathfrak{g}/\mathfrak{k})$-module. This means the support of $\operatorname{gr} \mathfrak{X}$ is contained in $\mathfrak{p} \simeq (\mathfrak{g}/\mathfrak{k})^*$. Moreover, there is an action of $K_{\mathbb{C}}$ on $\operatorname{Supp}(\operatorname{gr} \mathfrak{X})$ induced by $K_{\mathbb{C}}$-module structure of $\operatorname{gr} \mathfrak{X}$, hence $\mathcal{AV}(\mathfrak{X})$ is a union of $K_{\mathbb{C}}$-orbits.

Since $\mathfrak{X}$ is assumed to be quasi-simple, $\operatorname{gr} \mathfrak{Z}(\mathfrak{g}) = S(\mathfrak{g})^G$ acts on $\operatorname{gr} \mathfrak{X}$ trivially. Thus the invariants of positive degree $S(\mathfrak{g})_+^G$ kills $\operatorname{gr} \mathfrak{X}$. By the result of Kostant, it is known that $S(\mathfrak{g})_+^G$ generates a prime ideal, which is an annihilator ideal of the nilpotent variety. Thus $\operatorname{Supp}(\operatorname{gr} \mathfrak{X}) = \mathcal{AV}(\mathfrak{X})$ is contained in the nilpotent variety. □

We give some examples of associated cycles here.

Example 4.2. If τ is a finite dimensional representation of G, its associated cycle is supported on the point $\{0\}$. The multiplicity is given by the dimension $\dim \tau$.

Example 4.3 (Yamashita, N-Ochiai-Taniguchi)**.** We will give the associated cycles of unitary highest/lowest weight modules. For details, we refer the readers to [13].

First we describe certain nilpotent orbits. Let $(G, G') = (O(p, q), Sp(2n, \mathbb{R}))$ be our type I dual pair. A choice of a maximal compact subgroup $K' \subset G'$ determines a complexified Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$. Since $G' = Sp(V')$ is a Hermitian symmetric type, there is a $K'_{\mathbb{C}}$-stable decomposition $\mathfrak{p}' = \mathfrak{p}'_+ \oplus \mathfrak{p}'_-$. One can identify $\mathfrak{p}'_+ = \operatorname{Sym}_n(\mathbb{C})$ (the space of symmetric matrices of order n), where $n = \frac{1}{2}\dim V'$ is the real rank of G'. The action of $K'_{\mathbb{C}} \simeq GL_n(\mathbb{C})$ is the usual one; $gX\,{}^tg$ ($g \in GL_n(\mathbb{C}), X \in \mathrm{Sym}_n(\mathbb{C})$). Then $\mathfrak{p}'_\pm$ is contained in the nilpotent variety, and the nilpotent $K'_{\mathbb{C}}$-orbits in $\mathfrak{p}'_\pm$ is classified by the rank of symmetric matrices. We denote the nilpotent orbit in $\mathfrak{p}'_+$ of rank k by $\mathcal{O}_k$. In particular, $\mathcal{O}_0 = \{0\}$ is the trivial orbit, and $\mathcal{O}_n = \{A \in \mathrm{Sym}_n(\mathbb{C}) \mid \det A \neq 0\}$ is dense open in $\mathfrak{p}'_+$.

Let us recall the decomposition (3.5). Thus unitary highest weight modules $L^+(\sigma)$ are parametrized by irreducible finite dimensional representations $\sigma \in O(V^+)^\wedge$ for various positive definite quadratic space V^+. Here (only in this example), we do *not* assume any condition between the dimensions of $V^\pm$ and V'. Therefore, $L^+(\sigma)$ need not be in holomorphic discrete series, but it can be an arbitrary unitary highest weight representation including singular unitary highest weight modules.

The associated cycle of $L^+(\sigma)$ ($\sigma \in O(V^+)^\wedge$) is given by

$$\mathcal{AC}(L^+(\sigma)) = \begin{cases} \dim\sigma \cdot [\overline{\mathcal{O}_p}] & \text{if } p = \dim V^+ \leq n \\ \dim\sigma^{O(p-n)} \cdot [\overline{\mathcal{O}_n}] & \text{if } p = \dim V^+ > n \end{cases} \tag{4.5}$$

Here $O(p-n)$ is embedded into $O(p)$ diagonally, and $\sigma^{O(p-n)}$ denotes $O(p-n)$-invariants in σ. Note that $\overline{\mathcal{O}_n} = \mathfrak{p}'_+$.

By a result of Yamashita ([21]), the multiplicity of $\mathcal{AC}(L^+(\sigma))$ is also interpreted as the dimension of the space of generalized Whittaker vectors. This is one of the motivation to calculate associated cycles.

5. Theta lift of associated cycles

First we recall the notion of the lifting of nilpotent orbits for symmetric pairs [14]:

$$\begin{gathered} G' = Sp(2n,\mathbb{R}) \longrightarrow G = O(p,q), \\ \mathcal{N}(\mathfrak{p}') \supset \mathcal{O}' \longrightarrow \mathcal{O} \subset \mathcal{N}(\mathfrak{p}), \end{gathered}$$

where for a subset $\mathfrak{s} \subset \mathfrak{g}$ we denote the set of nilpotent elements in $\mathfrak{s}$ by $\mathcal{N}(\mathfrak{s})$. We always assume that the pair (G, G') is strictly in the stable range, which amounts to assume that $2n < \min(p,q)$.

Now $W = \mathbb{R}^{p,q} \otimes \mathbb{R}^{2n}$ has a complex structure such that the imaginary part of the standard Hermitian form gives our symplectic form. With this complex structure, we consider W as a complex vector space:

$$W = M_{p+q,n}(\mathbb{C}) = \left\{ \begin{pmatrix} A \\ B \end{pmatrix} \mid A \in M_{p,n}(\mathbb{C}), B \in M_{q,n}(\mathbb{C}) \right\} = M_{p,n}(\mathbb{C}) \oplus M_{q,n}(\mathbb{C}).$$

Then the action of $K_{\mathbb{C}} = O(p,\mathbb{C}) \times O(q,\mathbb{C})$ and $K'_{\mathbb{C}} = GL_n(\mathbb{C})$ on W can be given as

$$\begin{pmatrix} kA\,{}^tg \\ hBg^{-1} \end{pmatrix}; \qquad \begin{pmatrix} A \\ B \end{pmatrix} \in W,\ (k,h) \in O(p,\mathbb{C}) \times O(q,\mathbb{C}),\ g \in GL_n(\mathbb{C}).$$

We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (resp. $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$) as

$$\mathfrak{g} = \mathfrak{o}(p+q, \mathbb{C}) = \left(\begin{array}{c|c} \mathrm{Alt}_p(\mathbb{C}) & 0 \\ \hline 0 & \mathrm{Alt}_q(\mathbb{C}) \end{array}\right) \oplus \left(\begin{array}{c|c} 0 & M_{p,q}(\mathbb{C}) \\ \hline {}^tM_{p,q}(\mathbb{C}) & 0 \end{array}\right) = \mathfrak{k} \oplus \mathfrak{p},$$

$$\mathfrak{g}' = \mathfrak{sp}(2n, \mathbb{C}) = \left(\begin{array}{c|c} M_n(\mathbb{C}) & 0 \\ \hline 0 & -{}^tM_n(\mathbb{C}) \end{array}\right) \oplus \left(\begin{array}{c|c} 0 & \mathrm{Sym}_n(\mathbb{C}) \\ \hline \mathrm{Sym}_n(\mathbb{C}) & 0 \end{array}\right) = \mathfrak{k}' \oplus \mathfrak{p}'.$$

Thus, we can identify $\mathfrak{p} = M_{p,q}(\mathbb{C})$ and $\mathfrak{p}' = \mathfrak{p}'_+ \oplus \mathfrak{p}'_- = \mathrm{Sym}_n(\mathbb{C}) \oplus \mathrm{Sym}_n(\mathbb{C})$. To define the lifting, we consider the following double fibration map

$$\begin{array}{ccc} & W = M_{p,n} \oplus M_{q,n} & \\ & \varphi \swarrow \quad \searrow \psi & \\ \mathfrak{p} = M_{p,q} & & \mathrm{Sym}_n \oplus \mathrm{Sym}_n = \mathfrak{p}' \end{array}$$

where the moment maps φ and ψ are explicitly given by

$$(A, B) \in M_{p,n} \oplus M_{q,n} = W,$$
$$\begin{cases} \varphi(A, B) = A\,{}^tB \in M_{p,q} = \mathfrak{p}, \\ \psi(A, B) = ({}^tAA, {}^tBB) \in \mathrm{Sym}_n \oplus \mathrm{Sym}_n = \mathfrak{p}'. \end{cases}$$

These maps are equivariant quotient maps onto their images. For example, φ is a quotient map by $GL_n(\mathbb{C})$ onto its image (rank $\le n$ matrices in $M_{p,q}(\mathbb{C})$), and it is $K_{\mathbb{C}}$-equivariant. Note that ψ is surjective by our assumption that the pair is strictly in the stable range.

The following theorem is established in [14]. It is also obtained by Ohta [16] and Daszkiewicz-Kraśkiewicz-Przebinda [1] independently.

Theorem 5.1. *Take a nilpotent $K'_{\mathbb{C}}$-orbit $\mathcal{O}'$ in $\mathfrak{p}'$. The push-down of the inverse image $\varphi(\psi^{-1}(\overline{\mathcal{O}'}))$ of the closure of $\mathcal{O}'$ is equal to the closure of a nilpotent $K_{\mathbb{C}}$-orbit $\overline{\mathcal{O}}$. This gives a one-to-one correspondence from the set of nilpotent $K'_{\mathbb{C}}$-orbits in $\mathfrak{p}'$ to the set of nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}$.*

We write this correspondence as $\mathcal{O} = \theta(\mathcal{O}')$, and call it the *theta lift* of $\mathcal{O}'$. For associated cycles we can extend the theta lifting by

$$\theta(\textstyle\sum_i m_i[\overline{\mathcal{O}'_i}]) = \textstyle\sum_i m_i[\overline{\theta(\mathcal{O}'_i)}]. \tag{5.1}$$

We can now state our main theorem.

Theorem 5.2 (N-Zhu)**.** *Let (G, G') be a reductive dual pair of type I. We assume that the pair is strictly in the stable range with G' the smaller member, and that G' is of Hermitian type* (*see Table* 1 *in* §3)*. Let π' be a genuine unitary highest weight representation of $\widetilde{G'}$ which appears in the Howe correspondence of a compact dual pair. Then the associated cycle is preserved by the theta lifting.*

$$\theta(\mathcal{AC}(\pi')) = \mathcal{AC}(\theta(\pi')) \tag{5.2}$$

More precisely, if the associated cycle of π' is given by $\mathcal{AC}(\pi') = m_{\pi'}[\overline{\mathcal{O}'}]$, then $\mathcal{AC}(\theta(\pi')) = m_{\pi'}[\overline{\theta(\mathcal{O}')}]$ with the same multiplicity.

Some remarks are in order.

First, for the pairs $(O(p,q), Sp(2n,\mathbb{R}))$ and $(U(p,q), U(m,n))$, all the unitary highest weight module of $\widetilde{G'}$ appears in the Howe correspondence for some compact dual pairs. This is proved in [7]. However, for the pair $(Sp(p,q), O^*(2n))$, there are small exception. See [2].

Second, $\mathcal{AC}(\pi')$ is well-understood; $\mathcal{AV}(\pi')$ is irreducible, and the multiplicity $m_{\pi'}$ can be given by the dimension of certain subspace of representations of compact groups. See Example 4.3.

Third, if π' is a singular unitary highest weight representation, the formula of $\mathcal{AC}(\pi')$ is also interpreted as the preservation of the associated cycle under the theta lifting. In fact, π' is the theta lift of a finite dimensional representation in the stable range. However, if π' is not singular, we can see that the associated cycle is no longer preserved by the theta lifting. Thus the assumption of the stable range condition is necessary.

For the proof of this theorem, we refer to [15].

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Principal series Whittaker functions on $SL(3,\mathbf{R})$

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This is an extract from a preprint with the same title. The full proofs are contained in that. Here we write only the major results. *The numbering of the statement are the same as the original full paper. Some statements in the original are skipped.*

Introduction

The study of Whittaker models of algebraic groups over local fields has already some history. The Jacquet integral is named after the investigation of H.Jacquet [7]. Multiplicity free theorem by J.Shalika for quasi-split groups, was later enhanced for the case of the real field by N.Wallach. For redutive groups over the real field, this theme was investigated by M.Hashizume [5], B.Kostant, D. Vogan, H.Matsumoto, and the joint work of R.Goodman and N.Wallach [4].

More specifically $GL(n,\mathbf{R})$, explicit expressions for class 1 Whittaker functions are obtained, firstly for $n=3$ by D.Bump [2]. The main contributor for the case of general n seems to be E.Stade. Other related results will be find in the references of the papers of him ([9],[10]).

Let us explain the outline of this paper. The purpose of the master thesis [1] refered above is to investigate the Whittaker functions belonging to the non-spherical principal series representations of $SL(3,\mathbf{R})$. The minimal K-type of such representations is 3-dimensional. So we have to consider vector-valued functions. The main results are, firstly, to obtain the holonomic system of the A-radial part of such Whittaker functions with minimal K-type explicitly (§4), and secondly to have 6 formal solutions (§5, Theorem (5.5)), which are considered as examples of confluent hypergeometric series of two variables. We also have integral expressions of these 6 solutions(§5, Theorem (5.6)). In the subsequent section, the Jacquet integral (so to say, the primary Whittaker function) is written as a sum of these 6 *secondary* Whittaker functions (§6-8).

1 Preliminaries. Basic terminology

1.1 Whittaker model

Given an irreducible admissible representation (π,H) of $G=SL(3,\mathbf{R})$, we consider its model or realization in the space of Whittaker functions. This means, for a non-degenerate unitary character ψ of a maximal unipotent subgroup $N = \{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in G\}$ of G defined by

$$\psi(\begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{pmatrix}) = \exp\{2\pi\sqrt{-1}(c_1x_{12} + c_2x_{23})\}$$

with $c_1, c_2 \in \mathbf{R}$ being non-zero, we consider a smooth induction $C^\infty\text{-Ind}_N^G(\psi)$ to G, and the space of intertwining operators of smooth G-modules

$$\mathrm{Hom}_G(H_\infty, C^\infty\text{-Ind}_N^G(\psi))$$

with H_∞ the subspace consisting of C^∞-vectors in H. Or more algebraically speaking, we might consider the corresponding space in the context of $(\mathfrak{g}, K)$-modules (with $\mathfrak{g} = \mathrm{Lie}(G), K = SO(3)$):

$$\mathrm{Hom}_{(\mathfrak{g},K)}(H_\infty, C^\infty\text{-Ind}_N^G(\psi)).$$

1.2 Principal series representations

Let P_0 be a minimal parabolic subgroup of G given by the upper triangular matrices in G, and $P_0 = MAN$ be a Langlands decomposition of P_0 with $M = K \cap \{\text{diagonals in } G\}$, $A = \exp\mathfrak{a}$, with

$$\mathfrak{a} = \{\mathrm{diag}(t_1, t_2, t_3) | t_i \in \mathbf{R}, t_1 + t_2 + t_3 = 0\}.$$

In order to define a principal series representation with respect to the minimal parabolic subgroup P_0 of G, we firstly fix a character σ of the finite abelian group M of type $(2, 2)$ and a linear form $\nu \in \mathfrak{a}^* \otimes_{\mathbf{R}} \mathbf{C} = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{C})$. For such data, we can define a representation $\sigma \otimes e^\nu$ of MA, and extend this to P_0 by the identification $P_0/N \cong MA$. Then we set

$$\pi_{\sigma,\nu} = L^2\text{-Ind}_{P_0}^G(\sigma \otimes e^{\nu+\rho} \otimes 1_N).$$

Here $\nu(\mathrm{diag}(t_1, t_2, t_3)) = \sum_{i=1}^3 \nu_i t_i$ with $\nu_i \in \mathbf{C}$ and ρ is the half-sum of positive roots of $(\mathfrak{g}, \mathfrak{a})$ for P_0, given as follows. For $i < j$ $(1 \le i, j \le 3)$, we put $\eta_{ij}(a) = a_i/a_j$ for $a = \mathrm{diag}(a_1, a_2, a_3)$ $(a_1a_2a_3 = 1)$. Then we have $a^{2\rho} = \prod_{i<j} a_i/a_j = a_1^2/a_3^2 = a_1^4a_2^2$ by definition. Hence $a^\rho = a_1^2a_2$.

Here the characters σ_j of M are identified as follows. The group M consisting of 4 elements is a finite abelian group of $(2, 2)$ type, and its elements except for the unity is given by the matrices

$$m_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, m_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, m_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since M is commutative, all the irreducible unitary representations of it is 1-dimensional. For any $\sigma \in \widehat{M}$, we have $\sigma^2 = 1$. Therefore the set $\widehat{M}$ consisting of

4 characters $\{\sigma_j : j = 0, 1, 2, 3\}$, where each σ_j, except for the trivial character σ_0, is specified by the following table of values at the elements m_i.

	m_1	m_2	m_3
σ_1	1	-1	-1
σ_2	-1	1	-1
σ_3	-1	-1	1

Proposition (1.1) (i) *If σ is the trivial character of M, the representation $\pi_{\sigma,\nu}$ is spherical or class 1, i.e., it has a (unique) K-invariant vector in the representation space $H_{\sigma,\nu}$.*
(ii) *If σ is not trivial, then the minimal K-type of the restriction $\pi_{\sigma,\nu|K}$ to K is a 3-dimensional representation of $K = SO(3)$, which is isomorphic to the unique standard one (τ_2, V_2). The multiplicity of this minimal K-type is one:*

$$\dim_{\mathbf{C}} \mathrm{Hom}_K(\tau_2, H_{\sigma,\nu}) = 1,$$

namely there is a unique non-zero K-homomorphism

$$\iota : (\tau_2, V_2) \to (\pi_{\sigma,\nu|K}, H_{\sigma,\nu})$$

up to constant multiple.

2 Representations of $K = SO(3)$

2.1 The spinor covering

To describe the finite dimensional irreducible representations of $SO(3)$, the simplest way seems to utilize the double covering $s : SU(2) = Spin(3) \to SO(3)$, which is realized as follows.

The Hamilton quaternion algebra $\mathbf{H}$ is realized in $M_2(\mathbf{C})$ by

$$\mathbf{H} = \{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in M_2(\mathbf{C}) | a, b \in \mathbf{C}\}.$$

Then $SU(2)$ is the subgroup of the multiplicative group consisting of quaternions with reduced norm 1, i.e.,

$$SU(2) = \{x \in \mathbf{H} | \det x = 1\}.$$

Let $\mathbf{P} = \{x \in \mathbf{H} | \mathrm{tr} x = 0\}$ be the 3-dimensional real Euclidean space consisting of pure quaternions. Then for each $x \in SU(2)$, the map

$$p \in \mathbf{P} \mapsto x \cdot p \cdot x^{-1} \in \mathbf{P}$$

preserve the Euclid norm $p \mapsto \det p$ and the orientation, hence we have a homomorphism

$$s : SU(2) \to SO(\mathbf{P}, \det) = SO(3),$$

which is surjective, since the range is a connected group. The kernel of this homomorphism is given by $\{\pm 1_2\}$.

By the derivation of $s\ ds : \mathfrak{su}(2) \to \mathfrak{so}(3)$, the standard generators:

$$u_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

are mapped to $2K_1, 2K_2, 2K_3$ with

$$K_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, K_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{k},$$

respectively. Here $\mathfrak{k}$ is the Lie algebra of K.

2.2 Representations of $SU(2)$

The set of equivalence classes of the finite dimensional continuous representations of $SU(2)$ is exhaused by the symmetric tensor products τ_l $(l = 0, 1, \dots,)$ of the standard representation. These are realized as follows.

Let V_l be the subspace consisting of homogeneous polynomials of two variables x, y in the polynomial ring $\mathbf{C}[x, y]$. For $g \in SU(2)$ with $g^{-1} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, and $f(x, y) \in V_l$ we set

$$\tau_l(g)f(x, y) := f(ax + by, -\bar{b}x + \bar{a}y).$$

Passing to the Lie algebra $Lie(SU(2)) = \mathfrak{su}(2)$, the derivation of τ_l, denoted by the same symbol, is described as follows by using the standard basis $\{v_k = x^k y^{l-k}\ (0 \le k \le l)\}$ and the standard generators

$$u_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

Namely we have

$$\tau_l(u_1)v_k = \sqrt{-1}(l - 2k)v_k, \quad \tau_l(X_+)v_k = (l - k)v_{k+1}, \quad \tau_l(X_-)v_k = -k \cdot v_{k-1}.$$

Here we put $X_+ = \frac{1}{2}(u_2 + \sqrt{-1}u_3),\ X_- = \frac{1}{2}(u_2 - \sqrt{-1}u_3)$.

The condition that τ_l defines a representation of $SO(3)$ by passing to the quotient with respect $s : SU(2) \to SO(3)$ is that $\tau_l(-1_2) = (-1)^l = +1$, i.e., l is even. Therefore the dimension of V_l, $l + 1$ is odd in this case.

The representation τ_2 of $SU(2)$ is equivalent to the spinor homomorphism. Hence passing to the quotient, τ_2 is equivalent to the tautological representation $SO(3) \to GL(3, \mathbf{C})$.

2.3 Irreducible components of $\tau_2 \otimes \tau_4$ and $\tau_2 \otimes Ad_{\mathfrak{p}}$

For our later use, we want to specify the standard basis of the unique irreducible constituent τ_2 in the tensor product $\tau_2 \otimes \tau_4$.

Lemma (2.1) *Let* $\{v_i\ (i=0,1,2)\}$ *and* $\{w_j\ (0 \le j \le 4)\}$ *be the standard basis of* (τ_2, V_2) *and* (τ_4, V_4)*, respectively. Then the elements*

$$\begin{aligned} v'_0 &= v_0 \otimes w_2 - 2v_1 \otimes w_1 + v_2 \otimes w_0, \\ v'_1 &= v_0 \otimes w_3 - 2v_1 \otimes w_2 + v_2 \otimes w_1, \\ v'_2 &= v_0 \otimes w_4 - 2v_1 \otimes w_3 + v_2 \otimes w_2 \end{aligned}$$

define a set of standard basis in $\tau_2 \subset \tau_2 \otimes \tau_4$*, which is unique up to a common scalar multiple.*

2.4 The K-module isomorphism betwenn $\mathfrak{p}_{\mathbf{C}}$ and V_4

We denote by $\mathfrak{p}_{\mathbf{C}}$ the complexification of the orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ with respect to the Killing form, on which the group K acts via the adjont action $Ad_{\mathfrak{p}}$. We denote by E_{ij} the matrix unit with 1 at (i,j)-th entry and 0 at other entries. Then E_{ii} and $E_{ij}+E_{ji}$ are considered as elements in $\mathfrak{p}$. We set $H_{ij} = E_{ii} - E_{jj}$ for $i \ne j$.

Lemma (2.2) *Via the unique isomorphism* V_4 *and* $\mathfrak{p}_{\mathbf{C}}$ *as* K*-modules we have the identification*

$$\begin{aligned} w_0 &= -2\{H_{23} - \sqrt{-1}(E_{23}+E_{32})\}, \\ w_1 &= \sqrt{-1}\{(E_{12}+E_{21}) - \sqrt{-1}(E_{13}+E_{31})\}, \\ w_2 &= \tfrac{2}{3}(H_{12}+H_{13}), \\ w_3 &= \sqrt{-1}\{(E_{12}+E_{21}) + \sqrt{-1}(E_{13}+E_{31})\}, \\ w_4 &= -2\{H_{23} + \sqrt{-1}(E_{23}+E_{32})\}. \end{aligned}$$

3 Principal series $(\mathfrak{g}, K)$-modules

3.1 The case of the class one principal series

3.1.1 The Capelli elements

A set of generators for the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g} = \mathfrak{sl}_3$ is obtained as Capelli elements, because $\mathfrak{sl}_3$ is of type A_2.

Let

$$E'_{ii} = E_{ii} - \frac{1}{3}(\sum_{a=1}^{3} E_{aa}),\ E'_{ij} = E_{ij}\ (i \ne j).$$

Then $E'_{ij} \in \mathfrak{g}$. Define a matrix $\mathcal{C}$ of size 3 with entries in $\mathfrak{g}$ by

$$\mathcal{C} = \begin{pmatrix} E'_{11} & E'_{12} & E'_{13} \\ E'_{21} & E'_{22} & E'_{23} \\ E'_{31} & E'_{32} & E'_{33} \end{pmatrix} - \mathrm{diag}(-1,0,1).$$

Then for

$$\mathcal{A} = (A_{ij})_{1 \le i,j \le 3} = x \cdot 1_3 - \mathcal{C} \in M_3(\mathfrak{g}[x]) \subset M_3(U(\mathfrak{g})[x]),$$

we define its *vertical* determinant by

$$\det \downarrow (\mathcal{A}) = \sum_{\sigma \in \mathfrak{S}_3} \mathrm{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} A_{3\sigma(3)}.$$

Then it is written in the form $x^3+Cp_2x-Cp_3 \in U(\mathfrak{g})[x]$ with some elements Cp_2 and Cp_3 in $Z(\mathfrak{g})$.

Proposition (3.1) *The set $\{Cp_2, Cp_3\}$ is a system of independent generators of $Z(\mathfrak{g})$. Here are explicit formulae of Cp_2 and Cp_3:*

$$\begin{aligned} Cp_2 &= (E'_{11}-1)E'_{22}+E'_{22}(E'_{33}+1)+(E'_{11}-1)(E'_{33}+1) \\ &\qquad -E_{23}E_{32}-E_{13}E_{31}-E_{12}E_{21}, \end{aligned}$$

$$\begin{aligned} Cp_3 = &\quad (E'_{11}-1)E'_{22}(E'_{33}+1)+E_{12}E_{23}E_{31}+E_{13}E_{21}E_{32} \\ &-(E'_{11}-1)E_{23}E_{32}-E_{13}E'_{22}E_{31}-E_{12}E_{21}(E'_{33}+1). \end{aligned}$$

Eigenvalues of $Cp_2,\ Cp_3$

We compute the value $Cp_2f_0(e)$ and $Cp_3f_0(e)$. Let $S_2(a,b,c)=ab+bc+ca$ and $S_3(a,b,c)=abc$ be the elementary symmetric functions of three variables of degree 2 and 3, respectively. Then we have the following.

Proposition (3.2) *The infinitesimal character of $\pi_{\sigma_0,\nu}$ is given by*

$$Cp_2f_0 = S_2(\frac{1}{3}(2\nu_1-\nu_2), \frac{1}{3}(2\nu_2-\nu_1), \frac{1}{3}(\nu_1+\nu_2))f_0$$

and

$$Cp_3f_0 = S_3(\frac{1}{3}(2\nu_1-\nu_2), \frac{1}{3}(2\nu_2-\nu_1), \frac{1}{3}(\nu_1+\nu_2))f_0.$$

3.2 $(\mathfrak{g}, K)$-module structure of non-spherical principal series at the minimal K-type

3.2.1 Construction of K-equivariant differential operators

Lemma (3.3) *Let $\{f_i\ (i=0,1,2)\}$ be the set of the standard basis of the minimal K-type $\tau \subset \pi_{\sigma,\nu}$ of a non-spherical principal series representation $\pi_{\sigma,\nu}=\pi$. Define another three C^∞-elements $\{\varphi_i\ (i=0,1,2)\}$ by the formulae:*

$$\begin{aligned} \varphi_0 &= \tfrac{2}{3}\pi(2E_{11}-E_{22}-E_{33})f_0 \\ &\quad -2\sqrt{-1}\pi(E_{12}+E_{21}-\sqrt{-1}(E_{13}+E_{31}))f_1 \\ &\quad -2\pi(E_{12}+E_{21}-\sqrt{-1}(E_{23}+E_{32}))f_2, \\ \varphi_1 &= \sqrt{-1}\pi(E_{12}+E_{21}+\sqrt{-1}(E_{13}+E_{31}))f_0 \\ &\quad -\tfrac{4}{3}\pi(2E_{11}-E_{22}-E_{33})f_1 \\ &\quad +\sqrt{-1}\pi(E_{12}+E_{21}-\sqrt{-1}(E_{13}+E_{31}))f_2, \\ \varphi_2 &= -2\pi(E_{22}-E_{33}+\sqrt{-1}(E_{23}+E_{32}))f_0 \\ &\quad -2\sqrt{-1}\pi(E_{12}+E_{21})+\sqrt{-1}(E_{13}+E_{31}))f_1 \\ &\qquad +\tfrac{2}{3}\pi(2E_{11}-E_{22}-E_{33})f_2. \end{aligned}$$

Then $(\varphi_0, \varphi_1, \varphi_2)$ is a constant multiple of (f_0, f_1, f_2).

3.2.2 Computation of eigenvalues

The previous lemma tells that there exist a scalar $\lambda(\sigma,\nu)$ depending on σ and ν such that $\varphi_i = \lambda(\sigma,\nu) f_i$ $(i = 0,1,2)$. We determine this eigenvalue $\lambda(\sigma,\nu)$ by using explicit models of the principal series $\pi_{\sigma,\nu}$.

To do this, we have to find functions in

$$L^2\text{-}\mathrm{Ind}_M^K(\sigma_i) = L^2_{M,\sigma_i}(K) = \{f \in L^2(K) | f(mk) = \sigma(m)f(k) \text{ for all } m \in M, k \in K\}$$

corresponding to the standard basis in the minimal K-type for each i.

In the larger space $L^2(K)$, the τ_2-isotypic component is generated by the 9 matrix elements $s_{ij}(k)$ $(1 \le i,j \le 3)$ of the tautological representation

$$k \in K \mapsto S(k) = (s_{ab}(k))_{1\le a,b\le 3} \in SO(3).$$

It is directly confirmed that $s_{ib}(k)$ $(b = 0,1,2)$ belong to the subspace $L^2_{M,\sigma_i}(K)$ for each i.

Diagonalizing the action of u_1, we find that s_{i1} corresponds to v_1 for each i. And finally we find that the standard basis is given by

$$v_0 = \sqrt{-1}(s_{i2} - \sqrt{-1}s_{i3}),\ v_1 = s_{i1},\ \text{ and } v_2 = \sqrt{-1}(s_{i2} + \sqrt{-1}s_{i3}).$$

We need the values of these standard functions $f_a(k) = v_a$ $(a = 0,1,2)$ at the identity $e \in K$.

Lemma (3.4) *The values of the standard functions at $e \in K$ is given as follows.*

1. *If $\sigma = \sigma_1$, $(f_0(e), f_1(e), f_2(e)) = (0,1,0)$.*
2. *If $\sigma = \sigma_2$, $(f_0(e), f_1(e), f_2(e)) = (\sqrt{-1}, 0, \sqrt{-1})$.*
3. *If $\sigma = \sigma_3$, $(f_0(e), f_1(e), f_2(e)) = (1,0,-1)$.*

Now we can proceed to the compuation of the value $\lambda(\sigma_i,\nu)$.

Lemma (3.5)

$$\lambda(\sigma_1,\nu) = -\frac{4}{3}(2\nu_1 - \nu_2),\ \lambda(\sigma_2,\nu) = \frac{4}{3}(\nu_1 - 2\nu_2),\ \lambda(\sigma_3,\nu) = \frac{4}{3}(\nu_1 + \nu_2).$$

Summing up the lemmata in this section, we have the following.

Proposition (3.6) *Let $\{f_i\ (i = 0,1,2)\}$ be the set of the standard basis of the minimal K-type $\tau \subset \pi_{\sigma,\nu}$ of a non-spherical principal series representation $\pi_{\sigma,\nu} = \pi$. Define another three C^∞-elements $\{\varphi_i\ (i = 0,1,2)\}$ by the formulae:*

$$\begin{aligned}
\varphi_0 &= \tfrac{2}{3}\pi(H_{12} + H_{13})f_0 \\
&\quad -2\sqrt{-1}\pi(E_{12} + E_{21} - \sqrt{-1}(E_{13} + E_{31}))f_1 \\
&\quad -2\pi(H_{23} - \sqrt{-1}(2E_{23} + \tfrac{1}{2}u_1)f_2, \\
\varphi_1 &= \sqrt{-1}\pi(E_{12} + E_{21} + \sqrt{-1}(E_{13} + E_{31}))f_0 \\
&\quad -\tfrac{4}{3}\pi(H_{12} + H_{13})f_1 \\
&\quad +\sqrt{-1}\pi(E_{12} + E_{21} - \sqrt{-1}(E_{13} + E_{31}))f_2, \\
\varphi_2 &= -2\pi(H_{23} + \sqrt{-1}(2E_{23} + \tfrac{1}{2}u_1)f_0 \\
&\quad -2\sqrt{-1}\pi(E_{12} + E_{21}) + \sqrt{-1}(E_{13} + E_{31}))f_1 \\
&\quad +\tfrac{2}{3}\pi(H_{12} + H_{13})f_2.
\end{aligned}$$

Then we have

$$(\varphi_0, \varphi_1, \varphi_2) = \lambda(\sigma_i, \nu)(f_0, f_1, f_2)$$

with eigenvalue $\lambda(\sigma_i, \nu)$ *given by*

$$\lambda(\sigma_1, \nu) = -\frac{4}{3}(2\nu_1 - \nu_2),\ \lambda(\sigma_2, \nu) = \frac{4}{3}(\nu_1 - 2\nu_2),\ \lambda(\sigma_3, \nu) = \frac{4}{3}(\nu_1 + \nu_2).$$

In the next section, we consider the Whittaker realization of the equation of the above proposition. Then we need the following Iwasawa decomposition of standard elements of $\mathfrak{g}$.

Lemma (3.7) *We have the following decompostion of standard generators of* $\mathfrak{g}$ *with respect to the Iwasawa decomposition* $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$*. For* $H_{ij} \in \mathfrak{a}$ *we have*

$$H_{ij} = 0 + H_{ij} + 0.$$

Since $E_{ij} + E_{ji} = 2E_{ij} - (E_{ij} - E_{ji})$*, we have*

$$E_{12}+E_{21} = 2E_{12}+0+K_3, \quad E_{13}+E_{31} = 2E_{13}+0+(-K_2), \quad E_{23}+E_{23} = 2E_{23}+0+K_1.$$

4 The holonomic system for the A-radial part of the principal series Whittaker functions

4.1 The case of the class one principal series

Let I be a non-zero Whittaker functional from the class one principal series $\pi_{\sigma_0,\nu}$ to $C^\infty\text{-}\mathrm{Ind}_N^G(\psi)$. Let F be the restriction of the image $I(f_0)$ of the K-fixed vector f_0 to A. We write here the holonomic system for F with respect to the variables $y_1 = \eta_{12}(a) = a_1/a_2,\ y_2 = \eta_{23}(a) = a_2/a_3 = a_1/a_2^2$.

Proposition (4.1) *Put* $F(y_1, y_2) = y_1 y_2 G(y_1, y_2)$ *(note* $a^\rho = y_1 y_2$*). Then* $G(y_1, y_2)$ *satisfies the partial differential eqautions:*

$$\Delta_2 G = \frac{1}{3}(\nu_1^2 + \nu_2^2 - \nu_1\nu_2)G$$

and

$$\{\partial_1(\partial_1 - \partial_2)\partial_2 + 4\pi^2 c_2^2 y_2^2 \partial_1 - 4\pi^2 c_1^2 y_1^2 \partial_2\}G = -\frac{1}{27}(2\nu_1 - \nu_2)(2\nu_2 - \nu_1)(\nu_1 + \nu_2)G.$$

Here ∂_i *is the Euler operator* $y_i \frac{\partial}{\partial y_i}$ *for* $i = 1, 2$*. and we write*

$$\Delta_2 = (\partial_1^2 + \partial_2^2 - \partial_1\partial_2) - 4\pi^2(c_1^2 y_1^2 + c_2^2 y_2^2).$$

Remark From these equations for the monodromy exponents α_1, α_2 at the origin $y_1 = 0,\ y_2 = 0$, we have an equality of sets of complex numbers:

$$\{\alpha_1, -\alpha_1 + \alpha_2, -\alpha_2\} = \{\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(2\nu_2 - \nu_1), -\frac{1}{3}(\nu_1 + \nu_2)\}.$$

4.2 The holonomic system for the A-radial part of non-spherical Whittaker functions

Let I be a non-zero Whittaker functional from the principal series $\pi_{\sigma_i,\nu}$. For the set $\{f_i|(i=0,1,2)\}$ of standard functions, we put $F_i = I(f_i)$.

Theorem (4.4) *Let $F(a) = {}^t(F_0(a), F_1(a), F_2(a)) = (y_1y_2)^t(G_0(y), G_1(y), G_2(y))$ be the vector of the A-radial part of the standard Whittaker functions with minimal K-type of the principal series representation $\pi_{\sigma,\nu}$ with non-trivial $\sigma = \sigma_i$. Then it satisfies the following partial differential equations:*

(i):

$$\begin{pmatrix} \partial_1 & 4\pi c_1 y_1 & \partial_1 - 2\partial_2 - 4\pi c_2 y_2 \\ -2\pi c_1 y_1 & -2\partial_1 & -2\pi c_1 y_1 \\ \partial_1 - 2\partial_2 + 4\pi c_2 y_2 & 4\pi c_1 y_1 & \partial_1 \end{pmatrix} \begin{pmatrix} G_0(y) \\ G_1(y) \\ G_2(y) \end{pmatrix} = \frac{1}{2}\lambda_i \begin{pmatrix} G_0(y) \\ G_1(y) \\ G_2(y) \end{pmatrix},$$

(ii):

$$\Delta_2 \cdot 1_3 \cdot \begin{pmatrix} G_0(y) \\ G_1(y) \\ G_2(y) \end{pmatrix} - 2\pi c_2 y_2 \begin{pmatrix} G_0(y) \\ 0 \\ -G_2(y) \end{pmatrix} + 2\pi c_1 y_1 \begin{pmatrix} G_1(y) \\ \frac{1}{2}(G_0(y) + G_2(y)) \\ G_1(y) \end{pmatrix} = \frac{1}{3}\mu \begin{pmatrix} G_0(y) \\ G_1(y) \\ G_2(y) \end{pmatrix}.$$

Moreover the eigenvalues λ_i and μ depending on the representation $\pi_{\sigma,\nu}$ are given by

$$\begin{cases} \lambda_1 = -\frac{4}{3}(2\nu_1 - \nu_2) & (\sigma = \sigma_1) \\ \lambda_2 = \frac{4}{3}(\nu_1 - 2\nu_2) & (\sigma = \sigma_2) \\ \lambda_3 = \frac{4}{3}(\nu_1 + \nu_2) & (\sigma = \sigma_3) \end{cases} \quad \text{and } \mu = \nu_1^2 + \nu_2^2 - \nu_1\nu_2.$$

Remark We can write the differential equations (i) and (ii) of the above Theorem as

$$\text{(i)}: \quad \mathcal{D}_1 G = \lambda_i G \qquad \text{(ii)}: \quad \mathcal{D}_2 G = \mu G,$$

with $\mathcal{D}_i$ $(i = 1, 2)$ 3 by 3 matrix-valued differential operators. Then we have

$$\mathcal{D}_1 \cdot \mathcal{D}_2 - \mathcal{D}_2 \cdot \mathcal{D}_1 = 0.$$

4.3 The equations via the tautological basis

Let $k \in K \mapsto S(k) = (s_{ij}(k))_{1 \le i,j \le 3}$ be the tautological representation of $K = SO(3)$. Let $I \in \mathrm{Hom}_{\mathfrak{g},K}(\pi_{\sigma_i,\nu}, \mathrm{Ind}_N^G(\psi))$ be a Whittaker functional and define function T_{ij} on A by

$$I(s_{ij})_{|A} = y_1 y_2 T_{ij}(y) \quad (1 \le i, j, \le 3).$$

Then

$$\begin{pmatrix} G_0 \\ G_1 \\ G_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{-1} & 1 \\ 1 & 0 & 0 \\ 0 & \sqrt{-1} & -1 \end{pmatrix} \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix}.$$

Then for each i, the equation (i) of the above theorem is transformed to

$$\begin{pmatrix} -\partial_1 & -2\pi\sqrt{-1}c_1y_1 & 0 \\ -2\pi\sqrt{-1}c_1y_1 & \partial_1-\partial_2 & -2\pi\sqrt{-1}c_2y_2 \\ 0 & -2\pi\sqrt{-1}c_2y_2 & \partial_2 \end{pmatrix} \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix} = \frac{1}{2}\lambda_i \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix},$$

and the equation (ii) to

$$\left[\Delta_2 \cdot 1_3 + \begin{pmatrix} 0 & 2\pi\sqrt{-1}c_1y_1 & 0 \\ -2\pi\sqrt{-1}c_1y_1 & 0 & 2\pi\sqrt{-1}c_2y_2 \\ 0 & -2\pi\sqrt{-1}c_2y_2 & 0 \end{pmatrix}\right] \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix} = \frac{1}{3}\mu \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix}.$$

5 Power series solutions at the origin

We determine 6 linearly independent formal power series at the origin $(y_1, y_2) = (0, 0)$ for generic parameter ν in this section. These formal solutions converges because the singularity at the origin is a regular singularity. These solutions do not have exponential decay at infinity, different from the unique 'good' solution given by Jacquet integral. We refer to these solutions as *secondary* Whittaker functions sometimes.

5.1 The case of the class one principal series

This case is more or less discussed in the paper of Bump [2], up to some difference of notations. We omit its explicit formula.

An integral expression of this power series solution was found by Stade ([9, Lemma 3.10], [11, Theorem 2]) as an analogue of an integral formula for Jacquet integral by Vinogradov and Takhadzhyan [12]. The same as non-spherical case discussed later, we let $\{e_1, e_2, e_3\}$ be a permutation of the three complex numbers $\{-\frac{1}{3}(2\nu_1-\nu_2), -\frac{1}{3}(2\nu_2-\nu_1), \frac{1}{3}(\nu_1+\nu_2)\} = \{\frac{1}{4}\lambda_1, \frac{1}{4}\lambda_2, \frac{1}{4}\lambda_3\}$

Theorem (5.2) *For* $\mathrm{Re}(e_2 - e_1) > 2$,

$$\Phi(y_1, y_2) = \Gamma(\tfrac{e_2-e_1}{2}+1)\Gamma(\tfrac{e_3-e_1}{2}+1)\Gamma(\tfrac{e_2-e_3}{2}+1)(\pi c_1y_1)^{\frac{e_3}{2}}(\pi c_2y_2)^{-\frac{e_3}{2}}(\pi c_1)^{e_1}(\pi c_2)^{-e_2}$$
$$\cdot \frac{1}{2\pi\sqrt{-1}}\int_{|u|=1} I_{\frac{e_2-e_1}{2}}(2\pi c_1y_1\sqrt{1+1/u})I_{\frac{e_2-e_1}{2}}(2\pi c_2y_2\sqrt{1+u})\, u^{-\frac{3}{4}e_3}\frac{du}{u}.$$

5.2 The case of the non-spherical principal series

In this case also, the holonomic system obtained in Theorem (4.4) has regular singularities at the origin $(y_1, y_2) = (0, 0)$ with rank 6, i.e., the order of the Weyl group of $SL(3, \mathbf{R})$, for generic values of parameter ν. We determine the characteristic indices and the convergent formal power series solutions at $y = 0$. Here to abridge the notation, we write the set of variables (y_1, y_2) as y collectively.

By inspection we find that it is convenient to introduce scalar functions $\Phi_i(y_1, y_2)$ $(i = 0, 1, 2)$ by

$$F(y) = y_1y_2G(y) = y_1y_2\{\Phi_0(y)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \Phi_1(y)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \Phi_2(y)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\}.$$

5.3 The holonomic system for $\Phi_i(y)$

Now we can rewrite the holonomic system for G_i to that for Φ_i.

Proposition (5.3) *The holonomic system in Theorem (4.4) is equivalent to the following system for* $\Phi_i = \Phi_i(y_1, y_2)$ $(i = 0, 1, 2)$.

(1) (i) $[\partial_1 + \frac{1}{4}\lambda_i]\Phi_0 + (2\pi c_1 y_1)\Phi_1 = 0,$

(ii) $[\partial_1 - \partial_2 - \frac{1}{4}\lambda_i]\Phi_1 + (2\pi c_1 y_1)\Phi_0 + (2\pi c_2 y_2)\Phi_2 = 0,$

(iii) $[\partial_2 - \frac{1}{4}\lambda_i]\Phi_2 - (2\pi c_2 y_2)\Phi_1 = 0,$

(2) (i) $[\Delta_2 - \frac{1}{3}\mu]\Phi_0 + (2\pi c_1 y_1)\Phi_1 = 0,$

(ii) $[\Delta_2 - \frac{1}{3}\mu]\Phi_1 + (2\pi c_1 y_1)\Phi_0 - (2\pi c_2 y_2)\Phi_2 = 0,$

(iii) $[\Delta_2 - \frac{1}{3}\mu]\Phi_2 - (2\pi c_2 y_2)\Phi_1 = 0.$

5.4 The characteristic indices at the origin $(y_1, y_2) = (0, 0)$ and the recurrence formulae.

Let

$$\Phi_k(y) = y_1^{-e_1} y_2^{e_2} \sum_{n_1, n_2 \geq 0} c_{k;n_1,n_2} (\pi c_1 y_1)^{n_1} (\pi c_2 y_2)^{n_2}, \quad (k = 0, 1, 2)$$

be a system of formal power series solutions at the origin $y = 0$.

Now we can determine the 6 pairs $(-e_1, e_2)$ of characteristic indices at the origin, and the corresponding initial values conditions for F or Φ_i. the system at the origin and to determine the first coefficients Moreover we have the recurrence relations between the coefficients.

Lemma (5.4) *When* $\sigma = \sigma_i$ *for* $i = 1,\ 2$ *or* 3, *we have the following*:

(1) *The characteristic indices take the six values*:

$$(-e_1, e_2) = (-\tfrac{1}{4}\lambda_k, \tfrac{1}{4}\lambda_l) \quad (1 \leq k \neq l \leq 3).$$

(2) *For each case, the set of first coefficients, or the initial values at the origin are given as follows*:

(i) *If* $(-e_1, e_2) = (-\frac{1}{4}\lambda_i, \frac{1}{4}\lambda_k)$ $(k \neq i)$,

$(y_1^{e_1} y_2^{-e_2} G)(0,0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, *i.e.*, $(y_1^{e_1} y_2^{-e_2}\Phi_0)(0,0) = 1$, *and* $(y_1^{e_1} y_2^{-e_2}\Phi_j)(0,0) = 0$ *for other* j.

(ii) *If* $(-e_1, e_2) = (-\frac{1}{4}\lambda_k, \frac{1}{4}\lambda_l)$ $(k \neq i, l \neq i, k \neq l)$,

$(y_1^{e_1} y_2^{-e_2} G)(0,0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, *i.e.*, $(y_1^{e_1} y_2^{-e_2}\Phi_1)(0,0) = 1$, *and* $(y_1^{e_1} y_2^{-e_2}\Phi_j)(0,0) = 0$ *for other* j.

(iii) *If* $(-e_1, e_2) = (-\frac{1}{4}\lambda_k, \frac{1}{4}\lambda_i) \quad (k \neq i)$,

$(y_1^{e_1} y_2^{-e_2} G)(0,0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, *i.e.*, $(y_1^{e_1} y_2^{-e_2}\Phi_2)(0,0) = 1$, *and* $(y_1^{e_1} y_2^{-e_2}\Phi_j)(0,0) = 0$ *for other* j.

(3) *We have the following recurrence relations for the coefficients*:

(i) $(n_1 - e_1 + \frac{1}{4}\lambda_i)c_{0;n_1,n_2} + 2c_{1;n_1-1,n_2} = 0$;

(ii) $(n_1 - n_2 - e_1 - e_2 - \frac{1}{4}\lambda_i)c_{1;n_1,n_2} + 2c_{0;n_1-1,n_2} + 2c_{2;n_1,n_2-1} = 0$;

(iii) $(n_2 + e_2 - \frac{1}{4}\lambda_i)c_{2;n_1,n_2} - 2c_{1;n_1,n_2-1} = 0$.

5.5 Power series solutions at the origin

Now we can show the following formulae for the power series solutions.

Theorem (5.5) *Assume that* $\frac{1}{4}(\lambda_k - \lambda_l) \notin \mathbf{Z}$. *Then we have the following.*

(I) *When* $\sigma = \sigma_1$ *we have the following six independent solutions.*

$$
{}^t(\Phi_0^{1,I}, \Phi_1^{1,I}, \Phi_2^{1,I}) = y_1^{-\frac{\lambda_1}{4}} y_2^{\frac{\lambda_2}{4}}
$$

$$
\cdot \begin{pmatrix}
\displaystyle\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_1+m_2}}{(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_1}(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1}(\pi c_2 y_2)^{2m_2}}{m_1! m_2! (\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{m_1}(\frac{\lambda_2-\lambda_3}{8}+1)_{m_2}} \\
\displaystyle -\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_1+m_2+1}}{(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_1+1}(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1+1}(\pi c_2 y_2)^{2m_2}}{m_1! m_2! (\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{m_1+1}(\frac{\lambda_2-\lambda_3}{8}+1)_{m_2}} \\
\displaystyle -\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_1+m_2+1}}{(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_1+1}(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_2+1}} \cdot \frac{(\pi c_1 y_1)^{2m_1+1}(\pi c_2 y_2)^{2m_2+1}}{m_1! m_2! (\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{m_1+1}(\frac{\lambda_2-\lambda_3}{8}+1)_{m_2}}
\end{pmatrix},
$$

$$
{}^t(\Phi_0^{1,III}, \Phi_1^{1,III}, \Phi_2^{1,III}) = y_1^{-\frac{\lambda_2}{4}} y_2^{\frac{\lambda_3}{4}}
$$

$$
\cdot \begin{pmatrix}
\displaystyle\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1+m_2}}{(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1}(\frac{\lambda_3-\lambda_2}{8}+1)_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1+1}(\pi c_2 y_2)^{2m_2}}{m_1! m_2! (\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1+1}(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{m_2}} \\
\displaystyle -\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1+m_2}}{(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1}(\frac{\lambda_3-\lambda_2}{8}+1)_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1}(\pi c_2 y_2)^{2m_2}}{m_1! m_2! (\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1}(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{m_2}} \\
\displaystyle -\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1+m_2}}{(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1}(\frac{\lambda_3-\lambda_2}{8}+1)_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1}(\pi c_2 y_2)^{2m_2+1}}{m_1! m_2! (\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1}(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{m_2+1}}
\end{pmatrix},
$$

$$
{}^t(\Phi_0^{1,V}, \Phi_1^{1,V}, \Phi_2^{1,V}) = y_1^{-\frac{\lambda_2}{4}} y_2^{\frac{\lambda_1}{4}}
$$

$$
\cdot \begin{pmatrix}
\displaystyle -\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1+m_2+1}}{(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1+1}(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_2+1}} \cdot \frac{(\pi c_1 y_1)^{2m_1+1}(\pi c_2 y_2)^{2m_2+1}}{m_1! m_2! (\frac{\lambda_3-\lambda_2}{8}+1)_{m_1}(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})_{m_2+1}} \\
\displaystyle \sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1+m_2+1}}{(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1}(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_2+1}} \cdot \frac{(\pi c_1 y_1)^{2m_1}(\pi c_2 y_2)^{2m_2+1}}{m_1! m_2! (\frac{\lambda_3-\lambda_2}{8}+1)_{m_1}(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})_{m_2+1}} \\
\displaystyle \sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1+m_2}}{(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1}(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1}(\pi c_2 y_2)^{2m_2}}{m_1! m_2! (\frac{\lambda_3-\lambda_2}{8}+1)_{m_1}(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})_{m_2}}
\end{pmatrix},
$$

and other three solutions $\Phi_i^{1,II}, \Phi_i^{1,IV}$ *and* $\Phi_i^{1,VI}$ *are given by exchanging the role of* λ_2 *and* λ_3 *in the expression for* $\Phi_i^{1,I}, \Phi_i^{1,III}$ *and* $\Phi_i^{1,V}$, *respectively.*

(II) *When* $\sigma = \sigma_2$, *exchange* λ_1 *and* λ_2 *in the part* (I).
(III) *When* $\sigma = \sigma_3$, *exchange* λ_1 *and* λ_3 *in the part* (I).

5.6 Integral representations of the secondary Whittaker functions

In this subsection, we rewrite the power series solutions of the previous subsection by integral expressions.

Theorem (5.6) (I) *When* $\sigma = \sigma_1$ *we have*

$$
{}^t(\Phi_0^{1,I}, \Phi_1^{1,I}, \Phi_2^{1,I}) = (\pi c_1 y_1)^{\frac{\lambda_3}{8}+\frac{1}{2}} (\pi c_2 y_2)^{-\frac{\lambda_3}{8}+\frac{1}{2}}
$$

$$
\cdot (2\pi\sqrt{-1})^{-1} \Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2}) \Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2}) \Gamma(\tfrac{\lambda_2-\lambda_3}{8}+1) (\pi c_1)^{\frac{\lambda_1}{4}} (\pi c_2)^{-\frac{\lambda_2}{4}}
$$

$$
\cdot \begin{pmatrix}
\displaystyle\int_{|u|=1} I_{\frac{\lambda_2-\lambda_1}{8}-\frac{1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_2-\lambda_1}{8}-\frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u})\, u^{-\frac{3}{16}\lambda_3+\frac{1}{4}} \frac{du}{u} \\
\displaystyle(-1)\int_{|u|=1} I_{\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_2-\lambda_1}{8}-\frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u})\, u^{-\frac{3}{16}\lambda_3-\frac{1}{4}} (1+u)^{\frac{1}{2}} \frac{du}{u} \\
\displaystyle(-1)\int_{|u|=1} I_{\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u})\, u^{-\frac{3}{16}\lambda_3-\frac{1}{4}} \frac{du}{u}
\end{pmatrix}
$$

for $\mathrm{Re}(\frac{\lambda_2-\lambda_1}{8}) > \frac{3}{2}$,

$$
{}^t(\Phi_0^{1,III}, \Phi_1^{1,III}, \Phi_2^{1,III}) = (\pi c_1 y_1)^{\frac{\lambda_1}{8}} (\pi c_2 y_2)^{-\frac{\lambda_1}{8}}
$$

$$
\cdot (2\pi\sqrt{-1})^{-1} \Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2}) \Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2}) \Gamma(\tfrac{\lambda_3-\lambda_2}{8}+1) (\pi c_1)^{\frac{\lambda_2}{4}} (\pi c_2)^{-\frac{\lambda_3}{4}}
$$

$$
\cdot \begin{pmatrix}
\displaystyle(\pi c_1 y_1)\int_{|u|=1} I_{\frac{\lambda_3-\lambda_2}{8}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_3-\lambda_2}{8}}(2\pi c_2 y_2 \sqrt{1+u})\, u^{-\frac{3}{16}\lambda_3-\frac{1}{2}} \frac{du}{u} \\
\displaystyle(-1)\int_{|u|=1} \Big[\pi c_1 y_1 \sqrt{1+1/u}\, I_{\frac{\lambda_3-\lambda_2}{8}-1}(2\pi c_1 y_1 \sqrt{1+1/u}) + (\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2}) \\
\displaystyle\cdot I_{\frac{\lambda_3-\lambda_2}{8}}(2\pi c_1 y_1 \sqrt{1+1/u})\Big] I_{\frac{\lambda_3-\lambda_2}{8}}(2\pi c_2 y_2 \sqrt{1+u})\, u^{-\frac{3}{16}\lambda_3-\frac{1}{2}} \frac{du}{u} \\
\displaystyle(-1)(\pi c_2 y_2)\int_{|u|=1} I_{\frac{\lambda_3-\lambda_2}{8}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_3-\lambda_2}{8}}(2\pi c_2 y_2 \sqrt{1+u})\, u^{-\frac{3}{16}\lambda_3+\frac{1}{2}} \frac{du}{u}
\end{pmatrix}
$$

for $\mathrm{Re}(\frac{\lambda_3-\lambda_2}{8}) > 1$,

$$
{}^t(\Phi_0^{1,V}, \Phi_1^{1,V}, \Phi_2^{1,V}) = (\pi c_1 y_1)^{\frac{\lambda_3}{8}+\frac{1}{2}} (\pi c_2 y_2)^{-\frac{\lambda_3}{8}+\frac{1}{2}}
$$

$$
\cdot (2\pi\sqrt{-1})^{-1} \Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2}) \Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2}) \Gamma(\tfrac{\lambda_3-\lambda_2}{8}+1) (\pi c_1)^{\frac{\lambda_2}{4}} (\pi c_2)^{-\frac{\lambda_1}{4}}
$$

$$
\cdot \begin{pmatrix}
\displaystyle(-1)\int_{|u|=1} I_{\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u})\, u^{-\frac{3}{16}\lambda_3+\frac{1}{4}} \frac{du}{u} \\
\displaystyle\int_{|u|=1} I_{\frac{\lambda_1-\lambda_2}{8}-\frac{1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u})\, u^{-\frac{3}{16}\lambda_3-\frac{1}{4}} (1+u)^{\frac{1}{2}} \frac{du}{u} \\
\displaystyle\int_{|u|=1} I_{\frac{\lambda_1-\lambda_2}{8}-\frac{1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_1-\lambda_2}{8}-\frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u})\, u^{-\frac{3}{16}\lambda_3-\frac{1}{4}} \frac{du}{u}
\end{pmatrix}
$$

for $\mathrm{Re}(\frac{\lambda_1-\lambda_2}{8}) > \frac{3}{2}$.

To have the integral expression for $\Phi_i^{1,II}, \Phi_i^{1,IV}$ *and* $\Phi_i^{1,VI}$*, we have to exchange the role of* λ_2 *and* λ_3 *in the expression for* $\Phi_i^{1,I}, \Phi_i^{1,III}$ *and* $\Phi_i^{1,V}$*, respectively.*

(II) *When* $\sigma = \sigma_2$*, exchange* λ_1 *and* λ_2 *in* (I).
(III) *When* $\sigma = \sigma_3$*, exchange* λ_1 *and* λ_3 *in* (I).

6 Evaluation of Jacquet integrals

We give explicit descriptions of Jacquet integrals for non-spherical principal series Whittaker functions here. These are similar to the class one case ([12]).

6.1 Jacquet integrals

Let us denote by $g = n(g)a(g)k(g)$ the Iwasawa decomposition of $g \in G$. We define *Jacquet integral* J_{ij} for $\sigma_i \in \widehat{M}$ $(1 \le i, j \le 3)$ as

$$J_{ij}(g) = \int_N \psi(n)^{-1} a(s_0^{-1}ng) s_{ij}(k(s_0^{-1}ng)) dn$$

for $1 \le j \le 3$. Here

$$s_0 = \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}$$

the longest element in the Weyl group of $SL(3, \mathbf{R})$ and $s_{ij}(k)$ is the element of the tautological representation of K (cf. [4, (7.1)]).

Since

$$v_0 = \sqrt{-1}(s_{i2} - \sqrt{-1}s_{i3}),\ v_1 = s_{i1},\ v_2 = \sqrt{-1}(s_{i2} + \sqrt{-1}s_{i3})$$

(§3.2.2) and

$$\Phi_0 = G_1,\ 2\Phi_1 = G_0 + G_2,\ 2\Phi_2 = G_0 - G_2,$$

(§5.2) the vector of integrals ${}^t(J_{i1}, \sqrt{-1}J_{i2}, J_{i3})$ has the same K-type as ${}^t(\Phi_0, \Phi_1, \Phi_2)$.

For an element $a \in A$, we use the coordinates $(y_1, y_2) = (a_1/a_2, a_1 a_2^2)$. In the Iwasawa decomposition of the element $s_0^{-1}na$ its A-part $a(s_0^{-1}na)$ is given by

$$a(s_0^{-1}na) = \left(\frac{y_1^{\frac{1}{3}} y_2^{\frac{2}{3}}}{\sqrt{\Delta_1}}, \left(\frac{y_2}{y_1}\right)^{\frac{1}{3}} \sqrt{\frac{\Delta_1}{\Delta_2}} \right)$$

with

$$\Delta_1 = y_1^2 y_2^2 + y_1^2 n_2^2 + (n_1 n_2 - n_3)^2, \quad \Delta_2 = y_1^2 y_2^2 + y_2^2 n_1^2 + n_3^2.$$

Under the symbol above

$$J_{ij}(y) = y_1^{(2\nu_1-\nu_2)/3+1} y_2^{(\nu_1+\nu_2)/3+1} \cdot \int_{\mathbf{R}^3} \Delta_1^{(\nu_2-\nu_1-1)/2} \Delta_2^{(-\nu_2-1)/2} k_{ij} \exp\bigl(-2\pi\sqrt{-1}(c_1 n_1 + c_2 n_2)\bigr)\, dn_1 dn_2 dn_3.$$

Here $(k_{ij})_{1 \le i,j \le 3} = k(s_0^{-1}na)$.

6.2 Integral representations of Jacquet integrals

To write down our results, we use the following notation.

Notation.

$$K(\alpha,\beta,\gamma,\delta\,;y) := 4\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}(y_1y_2)(\pi|c_1|y_1)^{\frac{\lambda_2}{8}}(\pi|c_2|y_2)^{-\frac{\lambda_2}{8}}$$
$$\cdot\int_0^\infty K_{\frac{\lambda_3-\lambda_1}{8}+\alpha}(2\pi|c_1|y_1\sqrt{1+1/v})K_{\frac{\lambda_3-\lambda_1}{8}+\beta}(2\pi|c_2|y_2\sqrt{1+v})\,v^{-\frac{3}{16}\lambda_2+\gamma}(1+v)^{\delta}\,\frac{dv}{v}$$

with $K_\nu(z)$ the K-Bessel function.

6.2.1 The case of the class one principal series

In the case of class one, the Jacquet integral $J_0(y)$ is
Theorem (6.2) ([12]) *For* $\mathrm{Re}(\lambda_2-\lambda_1)>0$, $\mathrm{Re}(\lambda_3-\lambda_2)>0$,

$$J_0(y)=\frac{1}{\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})}\,K(0,0,0,0\,;y).$$

6.2.2 The case of the non-spherical principal series

Theorem (6.3) *For* $\mathrm{Re}(\lambda_2-\lambda_1)>0$, $\mathrm{Re}(\lambda_3-\lambda_2)>0$, *the Jacquet integrals* J_{ij} *can be written as follows.*

$$\begin{pmatrix}J_{11}(y)\\J_{12}(y)\\J_{13}(y)\end{pmatrix}=\frac{(\pi|c_1|)^{\frac12}(\pi|c_2|)^{\frac12}(\pi|c_1|y_1)^{\frac12}(\pi|c_2|y_2)^{\frac12}}{\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac12)\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)}\cdot\begin{pmatrix}\varepsilon_1\varepsilon_2K(-\frac12,-\frac12,\frac14,0\,;y)\\-\sqrt{-1}\,\varepsilon_2K(\frac12,-\frac12,-\frac14,\frac12\,;y)\\-K(\frac12,\frac12,-\frac14,0\,;y)\end{pmatrix},$$

$$\begin{pmatrix}J_{21}(y)\\J_{22}(y)\\J_{23}(y)\end{pmatrix}=\frac{1}{\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac12)}\cdot\begin{pmatrix}-\sqrt{-1}\,\varepsilon_1K(0,0,-\frac12,0\,;y)\\-K(0,0,\frac12,-1\,;y)\\\sqrt{-1}\,\varepsilon_2K(0,0,\frac12,0\,;y)\end{pmatrix},$$

$$\begin{pmatrix}J_{31}(y)\\J_{32}(y)\\J_{33}(y)\end{pmatrix}=\frac{(\pi|c_1|)^{\frac12}(\pi|c_2|)^{\frac12}(\pi|c_1|y_1)^{\frac12}(\pi|c_2|y_2)^{\frac12}}{\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac12)\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)}\cdot\begin{pmatrix}-K(\frac12,\frac12,\frac14,0\,;y)\\\sqrt{-1}\,\varepsilon_1K(-\frac12,\frac12,-\frac14,\frac12\,;y)\\\varepsilon_1\varepsilon_2K(-\frac12,-\frac12,-\frac14,0\,;y)\end{pmatrix}.$$

Here ε_i $(i=1,2)$ *means* 1 *if* $c_i>0$ *and* -1 *if* $c_i<0$.

7 Integral expression of Mellin-Barnes type

As in [9], we consider the Mellin-Barnes integral expression for $J_{ij}(y)$ to find linear relations between Jacquet integrals J_{ij} and power series solutions $\Phi_k^{i,*}$. We discuss only the non-spherical case.

Lemma (7.1) *For* $p,q\in\mathbf{C}$,

$$(\pi|c_1|y_1)^p(\pi|c_2|y_2)^q\int_0^\infty K_\alpha(2\pi|c_1|y_1\sqrt{1+1/v})K_\beta(2\pi|c_2|y_2\sqrt{1+v})\,v^\gamma(1+v)^\delta\,\frac{dv}{v}$$
$$=\frac{1}{2^4(2\pi\sqrt{-1})^2}\int_{\rho_1-\sqrt{-1}\infty}^{\rho_1+\sqrt{-1}\infty}\int_{\rho_2-\sqrt{-1}\infty}^{\rho_2+\sqrt{-1}\infty}V_0(s_1,s_2)(\pi|c_1|y_1)^{-s_1}(\pi|c_2|y_2)^{-s_2}ds_1ds_2,$$

with

$$V_0(s_1,s_2)=\frac{\Gamma(\frac{s_1+p+\alpha}{2})\Gamma(\frac{s_1+p-\alpha}{2})\Gamma(\frac{s_1+p+2\gamma}{2})\Gamma(\frac{s_2+q+\beta}{2})\Gamma(\frac{s_2+q-\beta}{2})\Gamma(\frac{s_2+q-2\gamma-2\delta}{2})}{\Gamma(\frac{s_1+s_2+p+q}{2}-\delta)}.$$

Here the lines of integration are taken as to the right of all poles of the integrand.

Proposition (7.2) *Let*

$$\begin{aligned}&M(a_1,a_2,a_3\,;b_1,b_2,b_3\,;c\,;y)\\&=\frac{1}{(2\pi\sqrt{-1})^2}\int_{\rho_1-\sqrt{-1}\infty}^{\rho_1+\sqrt{-1}\infty}\int_{\rho_2-\sqrt{-1}\infty}^{\rho_2+\sqrt{-1}\infty}V(s_1,s_2)(\pi|c_1|y_1)^{-s_1}(\pi|c_2|y_2)^{-s_2}ds_1ds_2,\end{aligned}$$

with

$$V(s_1,s_2)=\frac{\Gamma(\frac{s_1+a_1-\lambda_1}{2})\Gamma(\frac{s_1+a_2-\lambda_2}{2})\Gamma(\frac{s_1+a_3-\lambda_3}{2})\Gamma(\frac{s_1+b_1+\lambda_1}{2})\Gamma(\frac{s_1+b_2+\lambda_2}{2})\Gamma(\frac{s_1+b_3+\lambda_3}{2})}{\Gamma(\frac{s_1+s_2+c}{2})}.$$

Here the lines of integration are taken as to the right of all poles of the integrand. Then

$$\begin{pmatrix}J_{11}(y)\\J_{12}(y)\\J_{13}(y)\end{pmatrix}=\frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\,\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)}\cdot\begin{pmatrix}\varepsilon_1\varepsilon_2M(0,1,1\,;1,0,0\,;1\,;y)\\-\sqrt{-1}\,\varepsilon_2M(1,0,0\,;1,0,0\,;0\,;y)\\-M(1,0,0\,;0,1,1\,;1\,;y)\end{pmatrix},$$

$$\begin{pmatrix}J_{21}(y)\\J_{22}(y)\\J_{23}(y)\end{pmatrix}=\frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\,\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})}\cdot\begin{pmatrix}-\sqrt{-1}\,\varepsilon_1M(1,0,1\,;0,1,0\,;1\,;y)\\-M(0,1,0\,;0,1,0\,;0\,;y)\\\sqrt{-1}\,\varepsilon_2M(0,1,0\,;1,0,1\,;1\,;y)\end{pmatrix},$$

$$\begin{pmatrix}J_{31}(y)\\J_{32}(y)\\J_{33}(y)\end{pmatrix}=\frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\,\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)}\cdot\begin{pmatrix}-M(1,1,0\,;0,0,1\,;1\,;y)\\\sqrt{-1}\,\varepsilon_1M(0,0,1\,;0,0,1\,;0\,;y)\\\varepsilon_1\varepsilon_2M(0,0,1\,;1,1,0\,;1\,;y)\end{pmatrix}.$$

Proof. It is obvious from Lemma (7.1). □

Remark. In view of this proposition, we can see the following symmetry for J_{ij} with respect to the parameter $(\lambda_1,\lambda_2,\lambda_3)$. This is natural but is not immediately seen from the formulae for J_{ij} (Theorem 6.3). We denote

$$\tilde{J}_i(\lambda_1,\lambda_2,\lambda_3)=\left(\frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{\Gamma(\frac{\lambda_2-\lambda_1}{8}+p_i)\Gamma(\frac{\lambda_3-\lambda_2}{8}+q_i)\Gamma(\frac{\lambda_3-\lambda_1}{8}+r_i)}\right)^{-1}{}^t(J_{i1}(y),J_{i2}(y),J_{i3}(y))$$

with $(p_i,q_i,r_i)=(1,\frac{1}{2},1)$ $(i=1)$, $(1,1,\frac{1}{2})$ $(i=2)$, $(\frac{1}{2},1,1)$ $(i=3)$. Then

$$\tilde{J}_2(\lambda_1,\lambda_2,\lambda_3)=(-\sqrt{-1})\,\varepsilon_2\tilde{J}_1(\lambda_2,\lambda_1,\lambda_3),\quad \tilde{J}_3(\lambda_1,\lambda_2,\lambda_3)=-\varepsilon_1\varepsilon_2\tilde{J}_1(\lambda_3,\lambda_2,\lambda_1).$$

8 Relation between Jacquet integrals and power series solutions.

We omit the case of the class one principal series here, which is discussed by other people. In the same way of [9] for class one case, we move the lines of Mellin-Barnes integral expression in Proposition (7.2) to the left and sum up the residues at the poles. Then we obtain the following.

Theorem (8.2)

$$
\begin{aligned}
{}^t(J_{11}(y), J_{12}(y), J_{13}(y)) &= \frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\,\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)} \\
&\cdot\Big[\varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8})\,{}^t(\Phi_0^{1,I},\Phi_1^{1,I},\Phi_2^{1,I}) \\
&+\varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8})\,{}^t(\Phi_0^{1,II},\Phi_1^{1,II},\Phi_2^{1,II}) \\
&-\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8})\,{}^t(\Phi_0^{1,III},\Phi_1^{1,III},\Phi_2^{1,III}) \\
&-\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8})\,{}^t(\Phi_0^{1,IV},\Phi_1^{1,IV},\Phi_2^{1,IV}) \\
&-(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8})\,{}^t(\Phi_0^{1,V},\Phi_1^{1,V},\Phi_2^{1,V}) \\
&-(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8})\,{}^t(\Phi_0^{1,VI},\Phi_1^{1,VI},\Phi_2^{1,VI})\Big],
\end{aligned}
$$

$$
\begin{aligned}
{}^t(J_{21}(y), J_{22}(y), J_{23}(y)) &= \frac{-\sqrt{-1}\,\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\,\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})} \\
&\cdot\Big[\varepsilon_1(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8})\,{}^t(\Phi_0^{2,I},\Phi_1^{2,I},\Phi_2^{2,I}) \\
&+\varepsilon_1(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8})\,{}^t(\Phi_0^{2,II},\Phi_1^{2,II},\Phi_2^{2,II}) \\
&-(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8})\,{}^t(\Phi_0^{2,III},\Phi_1^{2,III},\Phi_2^{2,III}) \\
&-(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8})\,{}^t(\Phi_0^{2,IV},\Phi_1^{2,IV},\Phi_2^{2,IV}) \\
&-\varepsilon_2(\pi|c_1|)^{\frac{-\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8})\,{}^t(\Phi_0^{2,V},\Phi_1^{2,V},\Phi_2^{2,V}) \\
&-\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8})\,{}^t(\Phi_0^{2,VI},\Phi_1^{2,VI},\Phi_2^{2,VI})\Big],
\end{aligned}
$$

$$
\begin{aligned}
{}^t(J_{31}(y), J_{32}(y), J_{33}(y)) &= \frac{-\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\,\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)} \\
&\cdot\Big[(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_1}{8})\,{}^t(\Phi_0^{3,I},\Phi_1^{3,I},\Phi_2^{3,I}) \\
&+(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_2}{8})\,{}^t(\Phi_0^{3,II},\Phi_1^{3,II},\Phi_2^{3,II}) \\
&-\varepsilon_1(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_2}{8})\,{}^t(\Phi_0^{3,III},\Phi_1^{3,III},\Phi_2^{3,III}) \\
&-\varepsilon_1(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_1}{8})\,{}^t(\Phi_0^{3,IV},\Phi_1^{3,IV},\Phi_2^{3,IV}) \\
&-\varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_2}{8})\,{}^t(\Phi_0^{3,V},\Phi_1^{3,V},\Phi_2^{3,V}) \\
&-\varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_1}{8})\,{}^t(\Phi_0^{3,VI},\Phi_1^{3,VI},\Phi_2^{3,VI})\Big].
\end{aligned}
$$

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Fractional Weights and non-congruence subgroups

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Abstract

This note reviews the connection between the existence of fractional weight automorphic forms on real Lie groups, and the existence of non-congruence subgroups. It is intended to explain the simple results which are rarely even stated, and to avoid the complicated question of precisely where and why the congruence subgroup property fails. As a consequence, a new method is presented, for obtaining congruences between Eisenstein series and cusp forms in half-integral weight.

Let G be a (real) connected Lie group with a connected cyclic cover

$$1 \to \mu_n \to \tilde{G} \to G \to 1.$$

Here μ_n denotes the group of n-th roots of unity in $\mathbb{C}$. Suppose we have an arithmetic subgroup $\Gamma \subset G$. We shall discuss the following questions:

does Γ lift to a subgroup of $\tilde{G}$?

does Γ have a subgroup of finite index which lifts to $\tilde{G}$?

Example. Suppose the group G is $\mathrm{SL}_2(\mathbb{R})$. The fundamental group of G is $\mathbb{Z}$, and so for every $n \in \mathbb{N}$ there is a unique connected n-fold cover. For simplicity we shall assume that the arithmetic subgroup Γ is torsion-free.

A. If Γ has cusps then Γ is a free group. Therefore Γ lifts to every cover of G.

B. If Γ is cocompact then Peterson showed (see [7]) that Γ lifts to the n-fold cover if and only if n is a factor of the Euler characteristic $\chi(\Gamma)$. In particular for every n there is a Γ which lifts.

Very roughly speaking, Peterson's theorem is proved as follows. One finds a generator $\sigma \in H^2(G, \mathbb{Z})$ corresponding to the universal cover of G. A subgroup Γ lifts to the n-fold cover if and only if the image of σ in $H^2(\Gamma, \mathbb{Z}) \cong \mathbb{Z}$ is a multiple of n. The image of σ in $H^2(G, \mathbb{R})$ is represented by an invariant 2-form on the upper half-plane. This 2-form turns out to be the Euler form. To find the image of σ in $H^2(\Gamma, \mathbb{Z}) \cong \mathbb{Z}$ one integrates the 2-form over a fundamental domain for Γ. Hence by the Gauss-Bonnet theorem the image of σ in $H^2(\Gamma, \mathbb{Z})$ is $\chi(\Gamma)$. This implies the result.

1 Fractional weight multiplier systems

Let $\mathbb{C}^1$ denote the groups of complex numbers with absolute value 1. Suppose $w : G \times G \to \mu_n$ is a 2-cocycle representing the group extension $\tilde{G}$. By a weight w multiplier system on Γ, we shall mean a function $\chi : \Gamma \to \mathbb{C}^1$ such that

$$\chi(\gamma_1\gamma_2) = w(\gamma_1, \gamma_2)\chi(\gamma_1)\chi(\gamma_2).$$

In other words the image of w in $Z^2(\Gamma, \mathbb{C}^1)$ is the coboundary $\partial\chi$. If an arithmetic subgroup Γ lifts to $\tilde{G}$ then such a χ exists on Γ. We shall now prove a converse to this:

Proposition 1 *If there is a weight w multiplier system on an arithmetic subgroup $\Gamma \subset G$ then there is an arithmetic subgroup $\Gamma_0 \subset \Gamma$ which lifts to $\tilde{G}$.*

Proof. Suppose first that $\mathrm{rk}_{\mathbb{R}}(G) \geq 2$. In this case it is known (see [11]) that the commutator subgroup Γ' has finite index in Γ. From the exact sequence

$$1 \to \mu_n \to \mathbb{C}^1 \xrightarrow{n} \mathbb{C}^1 \to 1$$

we obtain a long exact sequence containing:

$$H^1(\Gamma, \mathbb{C}^1) \to H^2(\Gamma, \mu_n) \to H^2(\Gamma, \mathbb{C}^1).$$

The image of w in $H^2(\Gamma, \mathbb{C}^1)$ is trivial, so w is the image of an element $\varphi \in H^1(\Gamma, \mathbb{C}^1)$. However $\varphi : \Gamma \to \mathbb{C}^1$ is just a character. Let $\Gamma_0 = \ker(\varphi)$. It follows that the restriction of w to Γ_0 is trivial, so Γ_0 lifts to $\tilde{G}$. Since $\Gamma_0 \supset \Gamma'$, it follows that Γ_0 is an arithmetic subgroup of G.

The above argument fails when $\mathrm{rk}_{\mathbb{R}} G = 1$ since Γ/Γ' is often infinite in this case. However since Γ is finitely generated, Γ/Γ' is a finitely generated abelian group, and so is of the form $F \oplus \mathbb{Z}^r$, where F is a finite abelian group. We extend our sequence one step to the left to give:

$$H^1(\Gamma, \mathbb{C}^1) \xrightarrow{\times n} H^1(\Gamma, \mathbb{C}^1) \to H^2(\Gamma, \mu_n) \to H^2(\Gamma, \mathbb{C}^1).$$

This gives:

$$0 \to H^1(\Gamma, \mathbb{C}^1)/n \to H^2(\Gamma, \mu_n) \to H^2(\Gamma, \mathbb{C}^1).$$

Note that we have

$$H^1(\Gamma, \mathbb{C}^1)/n = \mathrm{Hom}(F \oplus \mathbb{Z}^r, \mathbb{C}^1)/n = \mathrm{Hom}(F, \mathbb{C}^1)/n.$$

This implies

$$0 \to \mathrm{Hom}(F, \mathbb{C}^1)/n \to H^2(\Gamma, \mu_n) \to H^2(\Gamma, \mathbb{C}^1).$$

We may therefore choose $\varphi : F \oplus \mathbb{Z}^r \to \mathbb{C}^1$ to be trivial on $\mathbb{Z}^r$. Hence $\ker(\varphi)$ again has finite index in Γ and the result follows as before. □

2 A trivial case

Suppose for a moment that the covering group $\tilde{G}$ is a linear group. In this case there is always some arithmetic subgroup Γ_0 of G which lifts to $\tilde{G}$. To see this, choose any arithmetic subgroup Γ of G and let $\tilde{\Gamma}$ be the preimage of Γ in $\tilde{G}$. Each element of the kernel μ_n is in $\tilde{\Gamma}$. For each of these elements apart from the identity, we can choose a congruence subgroup of $\tilde{\Gamma}$ not containing that element. Hence the intersection Γ_0 of all these congruence subgroups is a congruence subgroup with trivial intersection with μ_n. Thus Γ_0 is a lift to $\tilde{G}$ of a congruence subgroup of Γ.

3 A reformulation

In view of the above remark, it makes sense to assume that the group G is an (algebraically) simply connected linear group and that the covering group $\tilde{G}$ is non-linear. We shall make this restriction from now on.

In order to fix notation, we shall recall the definition of an arithmetic subgroup of the Lie group G. Suppose k is a totally real field with real places $v_1, \ldots, v_r$ and let $\mathcal{G}/k$ be an algebraic group such that

(i) $\mathcal{G}(k_{v_1})$ is isomorphic to G, and

(ii) $\mathcal{G}(k_{v_i})$ is compact for $i = 2, \ldots, r$.

We shall write $G(\mathfrak{O})$ for the projection of $\mathcal{G}(\mathfrak{O})$ onto G. By an arithmetic subgroup of G we mean a subgroup of G commensurable with some $G(\mathfrak{O})$. As usual we let $k_\infty = k \otimes_{\mathbb{Q}} \mathbb{R}$.

Proposition 2 *Let $G/\mathbb{R}$ and $\mathcal{G}/k$ be as above.*

(i) Every topological cover $\tilde{\mathcal{G}}(k_\infty)$ of $\mathcal{G}(k_\infty)$ is of the form

$$\tilde{G} \oplus \mathcal{G}(k_{v_2}) \oplus \ldots \oplus \mathcal{G}(k_{v_r}),$$

for some unique cover $\tilde{G} \to G$.

(ii) An arithmetic subgroup Γ lifts from $\mathcal{G}(k_\infty)$ to $\tilde{\mathcal{G}}(k_\infty)$ if and only if its projection in G lifts to $\tilde{G}$.

Proof. Part (ii) is immediate from (i). To prove (i), we must show that for $i > 1$, the compact group $\mathcal{G}(k_{v_i})$ is (topologically) simply connected. Note that $\mathcal{G}(k_{v_i})$ is a compact real form of $\mathcal{G}(\mathbb{C}) = G(\mathbb{C})$, and is hence a maximal compact subgroup of $G(\mathbb{C})$. By the Iwasawa decomposition of $G(\mathbb{C})$, we know that $G(\mathbb{C})$ is homotopic to $\mathcal{G}(k_{v_i})$. However as $G/\mathbb{R}$ is (algebraically) simply connected, we know that $G(\mathbb{C})$ is simply connected. □

4 Metaplectic covers

Let $\mathcal{G}$ be a linear algebraic group over an algebraic number field k. We shall write $\mathbb{A}$ for the adèle ring of k. Let A be a finite Abelian group. By a *metaplectic extension* of $\mathcal{G}$ by A, we shall mean a topological central extension:

$$\begin{array}{ccccccccc} 1 & \to & A & \to & \tilde{\mathcal{G}}(\mathbb{A}) & \to & \mathcal{G}(\mathbb{A}) & \to & 1 \\ & & & & & \nwarrow & \uparrow & & \\ & & & & & & \mathcal{G}(k) & & \end{array},$$

which splits on the subgroup $\mathcal{G}(k)$ of k-rational points of $\mathcal{G}$. Suppose we have such an extension and let $\tilde{\mathcal{G}}(k_\infty)$ be the pre-image of $\mathcal{G}(k_\infty)$ in $\tilde{\mathcal{G}(\mathbb{A})}$. We therefore have an extension of Lie groups:

$$1 \to A \to \tilde{\mathcal{G}}(k_\infty) \to \mathcal{G}(k_\infty) \to 1.$$

We shall show that this extension splits on a congruence subgroup of $\mathcal{G}(k_\infty)$.

To see this we let $\mathbb{A}_f$ denote the ring of finite adèles of k. As the map $\mathrm{pr} : \tilde{\mathcal{G}}(\mathbb{A}_f) \to \mathcal{G}(\mathbb{A}_f)$ is a topological covering, there is a neighbourhood U_1 of the identity in $\mathcal{G}(\mathbb{A}_f)$ such that $\mathrm{pr}^{-1}(U_1)$ is a disjoint union of homeomorphic copies of U_1. We may therefore choose a continuous section $\tau : U_1 \to \hat{U}_1$, where $\hat{U}_1$ is the copy of U_1 which contains the identity element of $\tilde{\mathcal{G}}(\mathbb{A}_f)$. Now define for $\alpha, \beta \in U_1$, $\sigma(\alpha,\beta) = \tau(\alpha)\tau(\beta)\tau(\alpha\beta)^{-1}$. Clearly σ is continuous on $U_1 \times U_1$ and has values in A. Furthermore $\sigma(1,1)$ is the identity element of A. Hence there is a neighbourhood U_2 of the identity in $\mathcal{G}(\mathbb{A}_f)$ such that σ is trivial on $U_2 \times U_2$. Now choose $U_3 \subset U_2$ to be a compact open subgroup of $\mathcal{G}(\mathbb{A}_f)$. On U_3 the section τ satisfies $\tau(\alpha\beta) = \tau(\alpha)\tau(\beta)$ and so the extension splits on U_3. Restricting the metaplectic extension we obtain:

$$1 \to A \to \tilde{\mathcal{G}}(k_\infty) \rtimes \tau(U_3) \to \mathcal{G}(k_\infty) \oplus U_3 \to 1 .$$

(Remark: it is widely believed that the local factors of metaplectic groups always commute. This belief is false; some counterexamples are described in [8].) As U_3 commutes with $\mathcal{G}(k_\infty)$, it follows that the action of $\tau(U_3)$ by conjugation on $\tilde{\mathcal{G}}(k_\infty)$ is trivial in a neighbourhood of the identity of $\tilde{\mathcal{G}}(k_\infty)$. Therefore $\tau(U_3)$ acts by permuting the connected components of $\tilde{\mathcal{G}}(k_\infty)$. It follows that there is a subgroup U_4 of finite index in U_3, such that $\tau(U_4)$ commutes with $\tilde{\mathcal{G}}(k_\infty)$. We therefore have

$$1 \to A \to \tilde{\mathcal{G}}(k_\infty) \oplus \tau(U_4) \to \mathcal{G}(k_\infty) \oplus U_4 \to 1 .$$

Now consider the congruence subgroup:

$$\Gamma = \mathcal{G}(k) \cap \Big(\mathcal{G}(k_\infty) \oplus U_4\Big).$$

As the metaplectic extension splits on $\mathcal{G}(k)$, we have by restriction:

$$\begin{array}{ccccccccc} 1 & \to & A & \to & \tilde{\mathcal{G}}(k_\infty) \oplus \tau(U_4) & \to & \mathcal{G}(k_\infty) \oplus U_4 & \to & 1 \\ & & & & & \nwarrow & \uparrow & & \\ & & & & & & \Gamma & & \end{array} .$$

Factoring out by U_4 and $\tau(U_4)$ in the above diagram, we obtain as required:

$$\begin{array}{ccccccccc} 1 & \to & A & \to & \tilde{\mathcal{G}}(k_\infty) & \to & \mathcal{G}(k_\infty) & \to & 1 \\ & & & & & \nwarrow & \uparrow & & \\ & & & & & & \Gamma & & \end{array} .$$

□

5 The congruence subgroup property

Let $\mathcal{G}/k$ be an absolutely simple and (algebraically) simply connected algebraic group over an algebraic number field k. We shall abbreviate $k_\infty = k \otimes_{\mathbb{Q}} \mathbb{R}$. Assume also that $\mathcal{G}(k_\infty)$ is not topologically simply connected. The group $\mathcal{G}$ will be said to satisfy the *congruence subgroup property* if every arithmetic subgroup of $\mathcal{G}(k)$ is a congruence subgroup.

The question of whether congruence subgroups exist or not has been reformulated by Serre as follows. By the strong approximation theorem, we have

$$\mathcal{G}(\mathbb{A}_f) = \lim_{\leftarrow (\Gamma\, congruence)} G(k)/\Gamma.$$

Now define

$$\hat{\mathcal{G}}(\mathbb{A}_f) = \lim_{\leftarrow (\Gamma arithmetic)} G(k)/\Gamma.$$

There is a surjective map $\hat{\mathcal{G}}(\mathbb{A}_f) \to \mathcal{G}(\mathbb{A}_f)$. The kernel $C(\mathcal{G})$ of this map is called the *congruence kernel*. The congruence kernel is trivial if and only if all arithmetic subgroups are congruence subgroups. Serre has conjectured ([15]), that $C(\mathcal{G})$ is a finite subgroup of the centre of $\hat{\mathcal{G}}(\mathbb{A}_f)$ if and only if $\mathrm{rk}_{\mathbb{R}}(\mathcal{G}(k_\infty)) \geq 2$. Serre's conjecture in known for most groups of real rank ≥ 2. In particular the conjecture is known for all isotropic groups apart from groups of type ${}^2E_{6,1}$.

If Serre's conjecture holds for $\mathcal{G}$ of real rank ≥ 2, then our assumption that $\mathcal{G}(k_\infty)$ is not simply connected implies that

$$C(\mathcal{G}) \cong \mathrm{Hom}(\overline{\mathcal{G}(k)'}/\mathcal{G}(k)', \mathbb{C}^1),$$

where $\mathcal{G}(k)'$ is the commutator subgroup of $\mathcal{G}(k)$ and $\overline{\mathcal{G}(k)'}$ is its closure with respect to the subspace topology on $\mathcal{G}(k)$ induced from $\mathcal{G}(\mathbb{A}_f)$. In particular, if $\mathcal{G}(k)$ is perfect then $C(\mathcal{G})$ is trivial. Furthermore the triviality of $C(\mathcal{G})$ would follow from a conjecture of Platonov and Margulis (see [14]). This Conjecture is known in most cases. More precisely we have:

Theorem 1 (Congruence Subgroup Property) *Suppose $\mathcal{G}/k$ is absolutely simple and (algebraically) simply connected, but $\mathcal{G}(k_\infty)$ is not topologically simply connected. Suppose also that $\sum_{v|\infty} \mathrm{rk}_v \mathcal{G} \geq 2$. If either $\mathcal{G}/k$ is isotropic but not of type ${}^2E_{6,1}$, or $\mathcal{G}/k$ is anisotropic but not of type, E_6 or ${}^{3,6}D_4$, and not an outer form of type ${}^2A_n$ then $\mathcal{G}$ satisfies the congruence subgroup property*

The results and conjectures referred to above are more fully described in the useful survey [14].

6 A partial converse

We shall now prove a partial converse of the result of §4.

Theorem 2 *Let let $\mathcal{G}/k$ be absolutely simple and simply connected. Suppose there is a topological central extension*

$$1 \to A \to \tilde{\mathcal{G}}(k_\infty) \to \mathcal{G}(k_\infty) \to 1,$$

which splits on some arithmetic subgroup Γ_0. If $\mathcal{G}$ satisfies the congruence subgroup property then this extension is the restriction to $\mathcal{G}(k_\infty)$ of a metaplectic extension of $\mathcal{G}$.

Remark 1 *In fact with some extra work one could replace the condition that all arithmetic subgroups are congruence subgroups by the weaker condition that the congruence kernel is finite. However, since $\mathcal{G}(k_\infty)$ is not topologically simply connected, it is conjectured that $C(\mathcal{G})$ is either infinite or trivial.*

Remark 2 *The theorem is essentially due to Deligne ([4]). Deligne makes the assumption that $\mathcal{G}(k)$ is perfect, which is slightly stronger than the congruence subgroup property here. However the assumptions are at least conjecturally equivalent.*

Proof. By the strong approximation theorem, $\mathcal{G}(k)$ is a dense subgroup of $\mathcal{G}(\mathbb{A}_f)$. We may therefore identify

$$\mathcal{G}(\mathbb{A}_f) = \lim_{\leftarrow} \mathcal{G}(k)/\Gamma,$$

where the limit is taken over the congruence subgroups, or equivalently over the arithmetic subgroups. We also define

$$\tilde{\mathcal{G}}(\mathbb{A}_f) = \lim_{\leftarrow} \tilde{\mathcal{G}}(k)/\tau(\Gamma),$$

where $\tilde{\mathcal{G}}(k)$ is the preimage of $\mathcal{G}(k)$ in $\tilde{\mathcal{G}}(k_\infty)$; Γ ranges over congruence subgroups of Γ_0 and $\tau : \Gamma_0 \to \tilde{\mathcal{G}}(k_\infty)$ is the splitting of the extension on Γ_0. For the moment we shall assume that $\tilde{\mathcal{G}}(\mathbb{A}(S))$ is a group.

The canonical projections $\tilde{\mathcal{G}}(k)/\tau(\Gamma) \to \mathcal{G}(k)/\Gamma$ induce a projection $\tilde{\mathcal{G}}(\mathbb{A}(S)) \to \mathcal{G}(\mathbb{A}(S))$. As $\tilde{\mathcal{G}}(\mathbb{A}(S))$ is a completion of $\tilde{\mathcal{G}}(k)$ it follows that we have a commutative diagramme:

$$\begin{array}{ccccccccc}
1 & \to & A & \to & \tilde{\mathcal{G}}(k_\infty) & \to & \mathcal{G}(k_\infty) & \to & 1 \\
 & & || & & \uparrow & & \uparrow & & \\
1 & \to & A & \to & \tilde{\mathcal{G}}(k) & \to & \mathcal{G}(k) & \to & 1 \\
 & & || & & \downarrow & & \downarrow & & \\
1 & \to & A & \to & \tilde{\mathcal{G}}(\mathbb{A}_f) & \to & \mathcal{G}(\mathbb{A}_f) & \to & 1.
\end{array}$$

Finally we define

$$\tilde{\mathcal{G}}(\mathbb{A}) = \left(\tilde{\mathcal{G}}(k_\infty) \oplus \tilde{\mathcal{G}}(\mathbb{A}_f)\right)/\Delta,$$

where $\Delta = \{(a,a) : a \in A\}$. As $(A \oplus A)/\Delta \cong A$, we have a central extension:

$$1 \to A \to \tilde{\mathcal{G}}(\mathbb{A}) \to \mathcal{G}(\mathbb{A}) \to 1.$$

The restriction of this extension to $\mathcal{G}(k_\infty)$ is our original extension. It remains show that this extension is metaplectic.

Choose any section $s : \mathcal{G}(k) \to \tilde{\mathcal{G}}(k)$ and define $t : \mathcal{G}(k) \to \left(\tilde{\mathcal{G}}(k) \oplus \tilde{\mathcal{G}}(k)\right)/\Delta$ by $t(\alpha) = (s(\alpha), s(\alpha))\Delta$. As the extensions are central we have $s(\alpha)s(\beta)s(\alpha\beta)^{-1} \in A$. Hence $t(\alpha)t(\beta)t(\alpha\beta)^{-1} \in \Delta$, so t is a homomorphism. This proves the theorem apart from the assertion that $\tilde{\mathcal{G}}(\mathbb{A}(S))$ is actually a group. □

Remark 3 *As the above theorem fails for the group* $\mathrm{SL}_2/\mathbb{Q}$*, and we have not yet used the congruence subgroup property, we may deduce that in this case the completion* $\widetilde{\mathrm{SL}}_2(\mathbb{A}_f)$ *is not a group.*

6.1 A remark on profinite limits

Suppose G is an abstract group and we have a directed system $\mathcal{F}$ of subgroups $\Gamma \subset G$. We shall call $\mathcal{F}$ *normal* if for every $g \in G$ and every $\Gamma \in \mathcal{F}$ the subgroup $g^{-1}\Gamma g$ contains an element of $\mathcal{F}$. If $\mathcal{F}$ is a normal filtration then the profinite limit

$$\bar{G} = \lim_{\leftarrow \Gamma \in \mathcal{F}} G/\Gamma.$$

is a group (with the group operation continuous and compatible with the canonical map $G \to \bar{G}$).

To complete the proof of the above theorem we must show that the system of subgroups

$$\mathcal{F} = \{\tau(\Gamma) : \Gamma \text{ is a congruence subgroup of } \Gamma_0\}$$

is normal in $\tilde{\mathcal{G}}(k)$. Choose any $\tilde{g} \in \tilde{\mathcal{G}}(k)$ and any congruence subgroup $\Gamma \subseteq \Gamma_0$. Let g be the projection of $\tilde{g}$ in $\mathcal{G}(k)$. We define a section $\tau^g : \Gamma^g \to \tilde{\mathcal{G}}(k)$ by $\tau^g(g^{-1}\gamma g) = \tilde{g}^{-1}\tau(\gamma)\tilde{g}$. Clearly the image of τ^g is $(\tau(\Gamma))^{\tilde{g}}$.

The intersection $\Gamma \cap \Gamma^g$ is a congruence subgroup. Furthermore on $\Gamma \cap \Gamma^g$ we have two splittings τ and τ^g. As our extension is central we easily verify that

$$\tau^g(\gamma) = \varphi(\gamma)\tau(\gamma), \quad \gamma \in \Gamma \cap \Gamma^g,$$

where $\varphi : \Gamma \cap \Gamma^g \to A$ is a homomorphism. Finally let $\Gamma_1 = \ker \varphi$. As A is finite, Γ_1 is an arithmetic subgroup of Γ_0. Hence, by the congruence subgroup property, Γ_1 is a congruence subgroup. The sections τ and τ^g coincide on Γ_1. Therefore $\tau(\Gamma_1) \subseteq \tau^g(\Gamma^{\tilde{g}}) = \tau(\Gamma)^{\tilde{g}}$. □

6.2 The classification of metaplectic extensions.

The above theorem is useful because the mataplectic extensions of absolutely simple, simply connected groups have been classified. For such a group G one defined the *metaplectic kernel* $M(\mathcal{G})$ to be the kernel of the restriction

$$H^2(\mathcal{G}(\mathbb{A}), \mathbb{C}^1) \to H^2(\mathcal{G}(k), \mathbb{C}^1).$$

This group is conjectured to be isomorphic to the Pontryagin dual of the group of roots on unity in the base field k. This conjecture is proved in almost all cases (see [13]). Thus if $\mathcal{G}(k)$ is not topologically simply connected then (in almost all cases) the metaplectic kernel has order 2. As a consequence we obtain the following.

Theorem 3 *Let $G/\mathbb{R}$ be absolutely simple and simply connected and let $\tilde{G} \to G$ be a connected n-fold cyclic cover. Let Γ be a congruence subgroup of G such that every subgroup of finite index in Γ is a congruence subgroup. Furthermore, in the case that G is a special unitary group, assume that the construction of Γ does not involve is a non-abelian division algebra. If Γ lifts to $\tilde{G}$ then $n \leq 2$.*

Proof. The special unitary case we have excluded is the only case in which the metaplectic kernel is not known. Let $\sigma \in H^2(G, \mu_n)$ correspond to the extension. As the extension is part of a metaplectic extension, we know that the image of σ in $H^2(G, \mathbb{C}^1)$ has order at most 2. However we have an exact sequence

$$H^1(G, \mathbb{C}^1) \to H^2(G, \mu_n) \to H^2(G, \mathbb{C}^1).$$

As G is perfect, it follows that σ has order at most 2 in $H^2(G, \mu_n)$. □

7 Examples

The descriptions of fundamental groups of Sp_{2n}, SU and SO given below are taken from [16]. The results for $\mathrm{Spin}(p, q)$ may be found in [6].

7.1 Symplectic groups

The symplectic group $\mathrm{Sp}_{2r}(\mathbb{R})$ of rank r is absolutely simple and algebraically simply connected. However it's topological fundamental group is $\mathbb{Z}$. Hence $\mathrm{Sp}_{2r}(\mathbb{R})$ has an n-fold cover for every $n \in \mathbb{N}$. If $r = 1$ then $\mathrm{Sp}_{2r}(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$ and it follows from Peterson's result that all fractional weights occur. However if $r \geq 2$, then we only have forms of half-integral weight. This was pointed out in [4].

7.2 Spin groups

Let $p \geq q \geq 1$. The spin group $\mathrm{Spin}(p, q)$ has rank q. The group $\mathrm{Spin}(2, 2)$ is isomorphic to $\mathrm{SL}_2(\mathbb{R}) \oplus \mathrm{SL}_2(\mathbb{R})$, so is not absolutely simple.

If $p \geq q \geq 3$ then the topological fundamental group of $\mathrm{Spin}(p, q)$ is μ_2, so we have only a double cover of $\mathrm{Spin}(p, q)$.

For $p \geq 3$ the group $\mathrm{Spin}(p, 2)$ is absolutely simple and simply connected. The fundamental group is $\mathbb{Z}$, so this group has an n-fold cover for every n. The congruence subgroup property holds in this case. Hence we have only half-integral weight forms on $\mathrm{Spin}(p, 2)$.

7.3 Orthogonal groups

Let $p \geq q \geq 1$. The special orthogonal group $\mathrm{SO}(p, q)$ has rank q. The group has two connected components. Let $O^+(p, q)$ denote the connected component of the identity. For $p \geq 3$ the fundamental group of $O^+(p, 2)^o$ is $\mathbb{Z}/2 \oplus \mathbb{Z}$.

The group $\mathrm{Spin}(p, 2)$ is the double cover of $O^+(p, 2)^o$ corresponding to the infinite cyclic subgroup of $\mathbb{Z} \oplus \mathbb{Z}/2$ generated by $(1, 1)$. Thus the unique double cover $\widetilde{\mathrm{Spin}}(p, 2)$ of $\mathrm{Spin}(p, 2)$ is the cover of $O^+(p, 2)$ corresponding to the subgroup generated by $(2, 0)$. This shows that $\widetilde{\mathrm{Spin}}(p, 2)$ is a $\mathbb{Z}/2 \oplus \mathbb{Z}/2$-cover of $O^+(p, 2)$ (rather than a $\mathbb{Z}/4$-cover).

If we had a form of fractional weight on $O^+(p, 2)$, then we could pull the form back to a fractional weight on $\mathrm{Spin}(p, 2)$. However this form would be a function on $\widetilde{\mathrm{Spin}}(p, 2)$. Hence the original form would have to be of half-integral weight.

7.4 Congruences between modular forms

We shall end by pointing out a consequence of the above result using Borcherds products. Recall that a *nearly holomorphic* modular form for $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f(q)$ on the upper half-plane, which has the usual transformation behaviour, but which may have a pole at ∞. In other words the Fourier expansion is allowed a finite number of negative terms:

$$f(q) = \sum_{n \gg -\infty} b_n q^n.$$

Let f be a nearly holomorphic form of weight $1 - l/2$, normalized so that $b_n \in \mathbb{Z}$ for all $n < 0$. Corresponding to such an f there is an automorphic form Ψ on $\mathrm{SO}(2, l)^o$ given by a Borcherds product (see [2],[3]). The weight of Ψ is $b_0/2$. As we know that there are only half-integral weight forms on $\mathrm{SO}(2, l)^o$ ($l \geq 3$), we deduce the following:

Corollary 1 *Let $f(q) = \sum b_n q^n$ be a nearly holomorphic form on $\mathrm{SL}_2(\mathbb{Z})$ negative weight. If $b_n \in \mathbb{Z}$ for $n < 0$ then $b_0 \in \mathbb{Z}$.*

For a nearly holomorphic form f, we shall call the negative part of its Fourier expansion the *principal part*. The following result is proved in [3].

Theorem 4 *Let $b_{-1}, \dots, b_{-n} \in \mathbb{C}$. There is a nearly holomorphic form of (integral) weight $2-k$ and principal part $b_{-1}q^{-1} + \dots + b_{-n}q^{-n}$ if and only if for every weight k cusp form $f(q) = \sum a_i q^i$, we have*

$$\sum_{i=1}^{n} a_i b_{-i} = 0.$$

If such a nearly holomorphic form exists then its constant term is given by

$$b_0 = \sum_{i=1}^{n} c_i b_i,$$

where $E(q) = 1 + \sum_{i=1}^{\infty} c_i q^i$ is the weight k Eisenstein series, normalized so as to have constant term 1.

Using this characterization, we may reformulate our corollary as follows.

Corollary 2 *Let E be the (integral) weight k level 1 Eisenstein series normalized so that the coefficients are integers with no common factor. Then there is a cusp form f such that the coefficients of f are congruent to those of E modulo the constant term of E.*

The above result can be obtained by much more elementary methods; in fact it follows immediately from the fact that E_4 and E_6 have constant term 1. One can however obtain a similar result for the vector-valued, half-integral weight forms studied in [3] in the same way. Such congruences have been proved for scalar valued forms of weight $\frac{3}{2}$ and prime level in [10]. However as far a I know for general half-integral weight, this is a new result.

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Standard L-functions attached to vector valued Siegel modular forms

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In this report, we study the analytic continuation of standard L-functions attached to vector valued Siegel modular forms. In Section 1, we define vector valued Siegel modular forms and standard L-functions. In Section 2, we describe the results in special cases and tools to prove. In Section 3, we describe one of the tools the differential operator generalized by Ibukiyama, and construct the operator explicity in the cases. In Section 4, we consider in general case.

§1. Vector valued Siegel modular forms and standard L- functions

Let n be a positive integer. Let

$$\mathbf{H}_n := \{Z \in M(n, \mathbf{C}) \mid Z = {}^tZ, \quad \mathrm{Im}(Z) > 0\}$$

be the Siegel upper half space of degree n, and

$$\Gamma_n := Sp(n, \mathbf{Z}) := \{\gamma \in GL(2n, \mathbf{Z}) \mid {}^t\gamma J \gamma = J\}$$

the Siegel modular group of degree n, where $J := \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$. Let (ρ, V_ρ) be an irreducible rational representation of $GL(n, \mathbf{C})$ on a finite-dimensional complex vector space V_ρ such that the highest weight of ρ is $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{Z}^n$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Furthermore, we fix an inner product $\langle \cdot, \cdot \rangle$ on V_ρ such that

$$\langle \rho(g)v, w \rangle = \langle v, \rho({}^t\overline{g})w \rangle \quad \text{for } g \in GL(n, \mathbf{C}),\ v,\ w \in V_\rho.$$

A C^∞-function $f : \mathbf{H}_n \to V_\rho$ is called a V_ρ-valued C^∞-modular form of type ρ if it satisfies

$$\rho(CZ + D)\, f(Z) = f((AZ + B)(CZ + D)^{-1}) \quad \text{for all } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n.$$

The space of all such functions is denoted by M_ρ^∞. The space of V_ρ-valued Siegel modular forms of type ρ is defined by

$$M_\rho := \{f \in M_\rho^\infty \mid f \text{ is holomorphic on } \mathbf{H}_n \text{ (and its cusps)}\},$$

and the space of cuspforms by

$$S_\rho := \left\{ f \in M_\rho \;\middle|\; \lim_{\lambda\to\infty} f(\begin{pmatrix} Z & 0 \\ 0 & i\lambda \end{pmatrix}) = 0 \quad \text{for all } Z \in \mathbf{H}_{n-1} \right\}.$$

Let $\mathcal{H}^n$ be the Hecke algebra for $(\Gamma_n, G^+Sp(n,\mathbf{Q}))$ over $\mathbf{C}$, where

$$G^+Sp(n,\mathbf{Q}) := \left\{ g \in GL(2n,\mathbf{Q}) \;\middle|\; {}^t gJg = rJ \text{ with some } r > 0 \right\}.$$

Then $\mathcal{H}^n$ has the following structure

$$\mathcal{H}^n = \bigotimes_{p:\text{prime}}{}' \mathcal{H}_p^n, \quad \mathcal{H}_p^n \simeq \mathbf{C}[X_0^{\pm 1}, \ldots, X_n^{\pm 1}]^W.$$

Here $\mathcal{H}_p^n$ is the Hecke algebra for $(\Gamma_n, G^+Sp(n,\mathbf{Q}) \cap GL(2n, \mathbf{Z}[1/p]))$ over $\mathbf{C}$, and W is the group generated by w_1, …, w_n and permutations in X_1, …, X_n, where w_1, …, w_n are automorphisms on $\mathbf{C}[X_0^{\pm 1}, \ldots, X_n^{\pm 1}]$ defined by

$$w_j(X_i) := \begin{cases} X_0 X_j & \text{if } i = 0, \\ X_i & \text{if } i \neq j, \\ X_i^{-1} & \text{if } i = j. \end{cases}$$

Suppose f is an eigenform, i.e., a non-zero common eigenfunction of the Hecke algebra $\mathcal{H}^n$. For $T \in \mathcal{H}^n$, let $\lambda(T)$ be the eigenvalue on f of T. Then for any prime number p, we determine $(\alpha_0(p), \ldots, \alpha_n(p)) \in (\mathbf{C}^\times)^{n+1}$ such that it gives the homomorphism

$$\lambda\colon \mathcal{H}_p^n \simeq \mathbf{C}[X_0^{\pm 1}, \ldots, X_n^{\pm 1}]^W \xrightarrow{X_j \mapsto \alpha_j(p)} \mathbf{C},$$

where $X_j \mapsto \alpha_j(p)$ means substituting $\alpha_j(p)$ into X_j ($j = 0$, …, n). The numbers $\alpha_0(p)$, …, $\alpha_n(p)$ are called the Satake p-parameters of f. Then we define the standard L- function attached to f by

$$L(s, f, \underline{\mathrm{St}}) := \prod_{p:\text{prime}} \left\{ (1 - p^{-s}) \prod_{j=1}^n (1 - \alpha_j(p) p^{-s})(1 - \alpha_j(p)^{-1} p^{-s}) \right\}^{-1}.$$

The right-hand side converges absolutely and locally uniformly for $\mathrm{Re}(s)$ sufficiently large.

§2. Problem and results

Problem. (Langlands [6])

The standard L-function $L(s, f, \underline{\mathrm{St}})$ has meromorphic continuation to the whole s-plane and satisfies a functional equation.

More precisely, we expect the following:

Conjecture. (Takayanagi [9])

We put

$$\Lambda(s, f, \underline{\mathrm{St}}) := \Gamma_\rho(s)\, L(s, f, \underline{\mathrm{St}}),$$

where

$$\Gamma_\rho(s) := \Gamma_{\mathbf{R}}(s+\varepsilon) \prod_{j=1}^{n} \Gamma_{\mathbf{C}}(s + \lambda_j - j)$$

with

$$\Gamma_{\mathbf{R}}(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right), \quad \Gamma_{\mathbf{C}}(s) := 2(2\pi)^{-s}\Gamma(s),$$

and

$$\varepsilon := \begin{cases} 0 & \text{if } n \text{ even,} \\ 1 & \text{if } n \text{ odd.} \end{cases}$$

Then $\Lambda(s, f, \underline{\mathrm{St}})$ satisfies the functional equation

$$\Lambda(s, f, \underline{\mathrm{St}}) = \Lambda(1-s, f, \underline{\mathrm{St}}).$$

We assume that k is a positive even integer and f is a cuspform.

For $\rho = \det^k$, this conjecture was solved by Andrianov and Kalinin [1], and Böcherer [2], and for $\rho = \det^k \otimes \mathrm{sym}^l$ and $\rho = \det^k \otimes \mathrm{alt}^{n-1}$ was solved by Takayanagi [9], [10].

Result.

We proved the conjecture in the following two cases:

Case 1. $\rho = \det^k \otimes \mathrm{alt}^l$ *(the highest weight* $(\underbrace{k+1, \ldots, k+1}_{l}, \underbrace{k, \ldots, k}_{n-l})$*).*

Case 2. *the highest weight of ρ is* $(k+2, \underbrace{k+1, \ldots, k+1}_{l-2}, \underbrace{k, \ldots, k}_{n-l+1})$.

To prove the above result, we use the non-holomorphic Eisenstein series and the differential operator generalized by Ibukiyama [4].

First, for $Z \in \mathbf{H}_n$ and a complex number s, we define the Eisenstein series $E_k^n(Z,s)$ by

$$E_k^n(Z,s) := \det(\operatorname{Im}(Z))^s \sum_{(C,D)} \det(CZ+D)^{-k} \left|\det(CZ+D)\right|^{-2s},$$

where (C,D) runs over a complete system of representatives of $\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \;\middle|\; C = 0 \right\} \backslash \Gamma_n$. Then $E_k^n(Z,s)$ converges absolutely and locally uniformly for $k + 2\operatorname{Re}(s) > n+1$. Furthermore the following properties are known:

(i) The Eisenstein series $E_k^n(Z,s)$ has meromorphic continuation to the whole s-plane and satisfies a functional equation. (Langlands [7], Kalinin [5] and Mizumoto [8])

(ii) Any partial derivative (in the entries of Z and $\overline{Z}$) of the Eisenstein series $E_k^n(Z,s)$ is slowly increasing (locally uniformly in s). (Mizumoto [8])

Next, we introduce the differential operator $\mathcal{D}$ which sends the Eisenstein series to the tensor product of two V_ρ-valued Siegel modular forms. Using Garrett decomposition [3], we compute $(\mathcal{D}E_k^{2n})\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}, s\right)$. Taking the Petersson inner product of f and $(\mathcal{D}E_k^{2n})\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}, s\right)$ in the variable W, we obtain the integral representation of the standard L-function $L(s, f, \underline{\mathrm{St}})$, i.e.,

$$\left(f, (\mathcal{D}E_k^{2n})\left(\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}, \overline{s}\right)\right) = (\Gamma\text{-factor}) \cdot L(2s+k-n, f, \underline{\mathrm{St}}) \cdot (\iota^{-1}(f))(Z).$$

Using the properties (i) and (ii) of the Eisenstein series, we prove the conjecture.

In the above cases, we can construct the differential operator explicitly and compute the integral representation of the standard L-function.

§3. Differential operator

In this section, we describe the differential operator generalized Ibukiyama and in the above cases we construct the operator explicitly.

Let (ρ'_j, V_j) $(j = 1, 2)$ be irreducible rational representations of $GL(n, \mathbf{C})$ such that ρ'_1 is equivalent to ρ'_2.

We assume $k \geq n$, and put $\rho_j := \det^k \otimes \rho'_j$.

If a polynomial P

$$P \colon M(n, 2k; \mathbf{C}) \times M(n, 2k; \mathbf{C}) \to V_1 \otimes V_2$$

satisfies

(C1) $P(a_1 X_1, a_2 X_2) = \rho'(a_1) \otimes \rho'(a_2)\, P(X_1, X_2)$ for all a_1, $a_2 \in GL(n, \mathbf{C})$,

(C2) $P(X_1 g, X_2 g) = P(X_1, X_2)$ for all $g \in O(2k)$

(C3) $P(X_1, X_2)$ is pluri-harmonic for each X_1, X_2,

then there exists a polynomial Q

$$Q \colon \mathrm{sym}(2n, \mathbf{C}) \to V_1 \otimes V_2$$

such that

$$P(X_1, X_2) = Q(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} {}^t\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}).$$

Here $O(2k)$ is the orthogonal group of degree $2k$, and $\mathrm{sym}(2n, \mathbf{C})$ the set of all $\mathbf{C}$-valued symmetric matrices of size $2n$. And for $j = 1, 2$, let $X_j = (x^{(j)}_{\mu\nu})$ be variables, then P is called pluri- harmonic for X_j if

$$\sum_{\kappa=1}^{2k} \frac{\partial}{\partial x^{(j)}_{\mu\kappa}} \frac{\partial}{\partial x^{(j)}_{\nu\kappa}} P = 0 \quad \text{for all } \mu,\ \nu.$$

We define the differential operator $\mathcal{D}$ by

$$\mathcal{D} := Q(\partial),$$

where

$$\partial := \left(\frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial z_{ij}} \right)_{1 \leq i,j \leq 2n}, \quad \mathcal{Z} = (z_{ij})_{1 \leq i,j \leq 2n} \in \mathbf{H}_{2n}.$$

Here δ_{ij} is the Kronecker's delta. Then

Theorem. (Ibukiyama)

If f is a C^∞-modular form (resp. a Siegel modular form) of degree $2n$ and type $\det^k$, then

$$(\mathcal{D}f)(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}) \in M^\infty_{\rho_1} \otimes M^\infty_{\rho_2} \quad (\textit{resp. } M_{\rho_1} \otimes M_{\rho_2}).$$

In the above cases, we construct the differential operators explicitly.

First we write (ρ'_j, V_j) $(j = 1, 2)$ explicity. We put

$$W_1 := \mathbf{C}e_1 \oplus \cdots \oplus \mathbf{C}e_n, \quad W_2 := \mathbf{C}e_{n+1} \oplus \cdots \oplus \mathbf{C}e_{2n}.$$

Let l be an even integer. Let $T^l(W_j)$ be the l-th tensor product of W_j, i.e.,

$$T^l(W_j) := \underbrace{W_j \otimes \cdots \otimes W_j}_{l},$$

and ρ'_j the standard representation of $GL(n, \mathbf{C})$ on $T^l(W_j)$. Let c_j be the Young symmetrizer of $(\lambda'_1, \ldots, \lambda'_n)$ on $T^l(W_j)$ such that $\lambda'_1 \geq \ldots \geq \lambda'_n$ and $\lambda'_1 + \cdots + \lambda'_n = l$. In Case 1, $(\lambda'_1, \ldots, \lambda'_n) = (\underbrace{1, \ldots, 1}_{l}, \underbrace{0, \ldots, 0}_{n-l})$, and in Case 2, $(\lambda'_1, \ldots, \lambda'_n) = (2, \underbrace{1, \ldots, 1}_{l-2}, \underbrace{0, \ldots, 0}_{n-l+1})$. We put $V_j := c_j(T^l(W_j))$. Then (ρ', V_j) is an irreducible representation of $GL(n, \mathbf{C})$.

On the other hand, let $e_i^{(\alpha)}$ $(i = 1, \ldots, 2n,\ \alpha = 1, \ldots, l)$ be indeterminants. And for a symmetric matrix A of size $2n$ and positive integers α, β $(1 \leq \alpha, \beta \leq l)$, we define

$$\begin{aligned} A^{\alpha\beta} &:= (e_1^{(\alpha)}, \ldots, e_n^{(\alpha)}, 0, \ldots, 0)\, A\, {}^t(e_1^{(\beta)}, \ldots, e_n^{(\beta)}, 0, \ldots, 0), \\ A^{\alpha}_{\beta} &:= (e_1^{(\alpha)}, \ldots, e_n^{(\alpha)}, 0, \ldots, 0)\, A\, {}^t(0, \ldots, 0, e_{n+1}^{(\beta)}, \ldots, e_{2n}^{(\beta)}), \\ A_{\alpha\beta} &:= (0, \ldots, 0, e_{n+1}^{(\alpha)}, \ldots, e_{2n}^{(\alpha)})\, A\, {}^t(0, \ldots, 0, e_{n+1}^{(\beta)}, \ldots, e_{2n}^{(\beta)}). \end{aligned}$$

We consider a product

$$A^{\alpha_1\alpha_2} \ldots A^{\alpha_{2\nu-1}\alpha_{2\nu}} A_{\beta_1\beta_2} \ldots A_{\beta_{2\nu-1}\beta_{2\nu}} A^{\alpha_{2\nu+1}}_{\beta_{2\nu+1}} \ldots A^{\alpha_l}_{\beta_l}$$

with $\{\alpha_1, \ldots, \alpha_l\} = \{\beta_1, \ldots, \beta_l\} = \{1, \ldots, l\}$. Then this product is

$$\sum_{\substack{1 \leq r_j \leq n \\ n+1 \leq s_j \leq 2n}} (\text{coefficient})\, e_{r_1}^{(1)} \ldots e_{r_l}^{(l)} e_{s_1}^{(1)} \ldots e_{s_l}^{(l)}.$$

Now we identify $e_{r_1}^{(1)} \ldots e_{r_l}^{(l)} e_{s_1}^{(1)} \ldots e_{s_l}^{(l)}$ with $e_{r_1} \otimes \ldots \otimes e_{r_l} \otimes e_{s_1} \otimes \ldots \otimes e_{s_l} \in T^l(W_1) \otimes T^l(W_2)$. Then this product belongs to $T^l(W_1) \otimes T^l(W_2)$.

We call a linear combination of such products a "homogeneous polynomial" of A. If $Q\colon \mathrm{sym}(2n, \mathbf{C}) \to V_1 \otimes V_2$ is "homogeneous polynomial", then $Q(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} {}^t\begin{pmatrix} X_1 \\ X_2 \end{pmatrix})$ satisfies (C1), (C2). Therefore if $Q(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} {}^t\begin{pmatrix} X_1 \\ X_2 \end{pmatrix})$

is pluri- harmonic for each X_1, X_2, then we obtain the differential operator $\mathcal{D}$.

We put $S := \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}^t \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$. Then in Case 1,

$$c_1 c_2 S_1^1 \dots S_l^l$$

is pluri-harmonic for each X_1, X_2, and in Case 2,

$$c_1 c_2 (S_1^1 \dots S_l^l - \frac{l}{2(2k-(l-2))} S^{12} S_{12} S_3^3 \dots S_l^l)$$

is pluri-harmonic for each X_1, X_2. Therefore we can compute $(\mathcal{D}E_k^{2n})$ $\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}, s\right)$. And we obtain the integral representation of the standard L-function $L(s, f, \underline{\mathrm{St}})$.

§4. Supplement

In general case, there exist three difficulties in proving the conjecture, i.e.,

(i) to construct the differential operator $\mathcal{D}$ explicitly,

(ii) to compute $(\mathcal{D}E_k^{2n}) \left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}, s\right)$,

(iii) to compute the Petersson inner product $\left(f, (\mathcal{D}E_k^{2n})\left(\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}, \overline{s}\right)\right)$.

However, if we cannot construct the differential operator explicitly, the following holds:

Proposition 1.

If $Q(S)$ is a "homogeneous polynomial" of $S := \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}^t \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ and pluri-harmonic for each X_1, X_2, then there exists a "homogeneous polynomial" $\mathcal{P}(X, s)$ of X such that

$$\mathcal{D}(\delta^{-k} |\delta|^{-2s} \varepsilon^s)|_{\mathcal{Z}=\mathcal{Z}_0} = (\delta^{-k} |\delta|^{-2s} \varepsilon^s \cdot \mathcal{P}(\Delta - \mathrm{E}, s))|_{\mathcal{Z}=\mathcal{Z}_0}.$$

Here for $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{2n}$ *and* $\mathcal{Z} \in \mathbf{H}_{2n}$*, we put* $\delta := \det(C\mathcal{Z}+D)$, $\varepsilon := \det(\mathrm{Im}(\mathcal{Z}))$, $\Delta := (C\mathcal{Z}+D)^{-1}C$, *and* $\mathrm{E} := \frac{1}{2i}(\mathrm{Im}(\mathcal{Z}))^{-1}$. *And we put* $\mathcal{Z}_0 := \begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}$.

For example, in Case 1, the "homogeneous polynomial" $\mathcal{P}(X,s)$ is

$$\mathcal{P}(X,s) = c_1c_2 \prod_{j=1}^{l} \Big(-k-s+\frac{j-1}{2}\Big)\, X_1^1 \dots X_l^l,$$

and in Case 2,

$$\begin{aligned}\mathcal{P}(X,s) \;&=\; c_1c_2 \prod_{j=1}^{l-1} \Big(-k-s+\frac{j-1}{2}\Big) \\ &\times\Big\{\Big(-k-s-\frac{1}{2}+\frac{l}{2(2k-(l-2))}\Big)X_1^1X_2^2\dots X_l^l \\ &\qquad +\frac{ls}{2(2k-(l-2))}X^{12}X_{12}X_3^3\dots X_l^l\Big\}.\end{aligned}$$

Furthermore, using the "homogeneous polynomial" $\mathcal{P}(X,s)$, we obtain the following:

Proposition 2.

Under the assumption of Proposition 1, the Petersson inner product $\Big(f, (\mathcal{D}E_k^{2n})\Big(\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}, \overline{s}\Big)\Big)$ *is equal to*

$$\begin{aligned}&(\Gamma\text{-factor}) \cdot L(2s+k-n, f, \underline{\mathrm{St}}) \\ &\times \frac{1}{\langle v,v\rangle}\Big\langle \int_{\mathbf{S}_n} \langle \rho_2(1_n-\overline{S}S)\,\iota(v), \mathcal{P}(R,\overline{s})\rangle \,\det(1_n-\overline{S}S)^{s-n-1}\, dS, v\Big\rangle \\ &\times(\iota^{-1}(f))(Z),\end{aligned}$$

where $v \in V_1$,

$$\mathbf{S}_n := \{S \in M(n,\mathbf{C}) \mid S = {}^tS, \quad 1_n - S\overline{S} > 0\},$$

$$R := -\frac{1}{2i}\begin{pmatrix} S & -2i\,1_n \\ -2i\,1_n & 2^2\overline{S}(1_n-S\overline{S})^{-1}\end{pmatrix},$$

and $\iota\colon V_1 \to V_2$ *is the isomorphism defined by* $\iota(e_j) = e_{n+j}$ *for* $j = 1, \dots, n$.

And if

$$\frac{1}{\langle v,v\rangle}\Big\langle\int_{\mathbf{S}_n}\langle\rho_2(1_n-\overline{S}S)\,\iota(v),\mathcal{P}(R,\overline{s})\rangle\,\det(1_n-\overline{S}S)^{s-n-1}\,dS,v\Big\rangle$$

is equal to

$$(\text{constant})\times\prod_{j=1}^{n}\frac{\Gamma(2s+k-n+\lambda_j-j)}{\Gamma(2s+2k+1-2j)},$$

then the conjecture holds.

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Spherical functions on certain spherical homogeneous spaces over p-adic fields

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§0 Introduction.

Throughout this paper, let k be a $\mathfrak{p}$-adic field. Let $\mathbb{G}$ be an algebraic group defined over k, $G = \mathbb{G}(k)$, K a special good maximal bounded subgroup of G, $\mathbb{X}$ a $\mathbb{G}$-homogeneous affine algebraic variety defined over k, and $X = \mathbb{X}(k)$. We write the action of $\mathbb{G}$ on $\mathbb{X}$ by $(g, x) \longmapsto g \star x$. Denote by $\mathcal{C}^\infty(K\backslash X)$ the set of left K-invariant $\mathbb{C}$-valued functions on X. The Hecke algebra $\mathcal{H}(G, K)$ acts on $\mathcal{C}^\infty(K\backslash X)$ from the left by the convolution product, which we write $(f, \Psi) \longmapsto f * \Psi$. A nonzero function $\Psi \in \mathcal{C}^\infty(K\backslash X)$ is called *a spherical function* if it is an $\mathcal{H}(G, K)$-common eigenfunction, which means, there exists a $\mathbb{C}$-algebra map $\lambda : \mathcal{H}(G, K) \longrightarrow \mathbb{C}$ satisfying

$$f * \Psi = \lambda(f)\Psi \quad \text{for } f \in \mathcal{H}(G, K).$$

Spherical functions are very interesting objects to investigate. The explicit expressions of spherical functions on p-adic groups have been given by I.G.Macdonald [Mac]. Later on, W.Casselman has reformulated them by representation theoretical method ([Cas]), for which there is an interpretative article written by P.Cartier([Car]). W. Casselman and J.Shalika carried forward this method to obtain explicit expressions of Whittaker functions associated to p-adic reductive group ([CasS]).

F.Sato and the author have investigated spherical functions on certain symmetric spaces; the space of alternating forms ([HS1]) and the spaces of hermitian and symmetric forms ([H1]-[H3]). In these cases, spherical functions can be regarded as generating functions of local densities of representations of forms by forms of the same kind. Hence, as an application, explicit formulas of local densities have been given([HS1], [HS2], [H3], [H4]).

In a similar method to [CasS], S. Kato has announced explicit expressions for spherical functions on certain spherical homogeneous spaces obtained by general linear groups([K2]), and S.Kato, A.Murase and T.Sugano have obtained explicit expressions for Whittaker-Shintani functions (spherical functions) of certain spherical homogeneous spaces obtained by special orthogonal groups([KMS]). For the spaces which they investigated, the

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A full version of this paper will be appear in elsewhere.

space of spherical functions attached to each Satake parameter, in other words, corresponding to each eigenvalue, is of dimension 1.

On the other hand, in a similar method to [Cas], the author has given an expression of spherical functions of certain spherical homogeneous spaces for which the dimension of the space of spherical functions is not necessarily one ([H3, Proposition 1.9]), and applied it to the space of unramified hermitian forms and given the explicit expression of spherical functions (the dimension is 2^n according to the size n of forms). This result has also used by K.Takano and S.Kato to give an explicit expression of spherical functions for the space $GL(n,k')/GL(n,k)$, where k' is an unramified quadratic extension of k. In this case the space of spherical functions has dimension one([Tak]).

In the following, we investigate spherical functions on the following space:

$$\mathbb{G} = Sp_2 \times (Sp_1)^2, \qquad \mathbb{X} = Sp_2,$$

where $(Sp_1)^2$ is imbedded into Sp_2 and the action is given by

$$\widetilde{g} \star x = g_1 x^t g_2, \quad \text{for } \widetilde{g} = (g_1, g_2) \in Sp_2 \times (Sp_1)^2,\ x \in Sp_2,$$

(for the precise definition, see the beginning of Section 1). This $\mathbb{X}$ is a spherical homogeneous $\mathbb{G}$-space, which means $\mathbb{X}$ has a Zariski open orbit for a Borel subgroup $\mathbb{B}$ of $\mathbb{G}$, and $\mathbb{X}$ is not a $\mathbb{G}$-symmetric space.

For this case, we will use the same result in [H3] in order to obtain a explicit formula of spherical functions. The space of spherical functions attached to each Satake parameter is of dimension 4. In [KMS], $SO(n) \times SO(n-1)$-space $SO(n)$ is considered, which is spherical and has an open Borel orbit over k for every n, and the case when $n = 5$ is isogeneous to the present case. But there seems to have no direct correspondence between respective explicit formulas of spherical functions. Finally, $Sp_{2n} \times (Sp_n)^2$-space Sp_{2n} is no longer spherical for $n \geq 2$.

We shall give a brief summary of our results. Taking a set $\{d_i \mid 1 \leq i \leq 4\}$ of basic relative $\mathbb{B}$-invariants (cf. (1.5)) and characters χ of $k^\times/(k^\times)^2$, we construct typical spherical functions (cf. (1.6))

$$\omega(x;\chi;s) = \int_K \chi(\prod_{i=1}^4 d_i(k\star x)) \prod_{i=1}^4 |d_i(k \star x)|^{s_i}\, dk, \qquad (x \in X,\ s \in \mathbb{C}^4),$$

where $|\ |$ is the absolute value on k and dk is the Haar measure on K, and the integral of the right hand side is absolutely convergent if $\mathrm{Re}(s_i) \geq 0$ $(1 \leq i \leq 4)$ and analytically continued to a rational function in $q^{s_1}, \ldots, q^{s_4}$, where q is the residual number of k. We introduce a new variable z related to s by

$$\begin{aligned} z_1 &= s_1 + s_2 + s_3 + s_4 + 2, & z_2 &= s_3 + s_4 + 1,\\ z_3 &= s_1 + s_3 + 1, & z_4 &= s_2 + s_3 + 1, \end{aligned}$$

and write $\omega(x;\chi;z)$ in stead of $\omega(x;\chi;s)$.

These $\omega(x;\chi;z)$ are $\mathcal{H}(G,K)$-common eigenfunctions correspond to the same $\mathbb{C}$-algebra homomorphism $\lambda_z : \mathcal{H}(G,K) \longrightarrow \mathbb{C}$, which gives the Satake transform

$$\lambda_z : \mathcal{H}(G,K) \xrightarrow{\sim} \mathbb{C}[q^{\pm z_1}, q^{\pm z_2}, q^{\pm z_3}, q^{\pm z_4}]^W \quad \text{(Proposition 1.1)},$$

where W is the Weyl group of $\mathbb{G}$.

Under the assumption that k has odd residual characteristic, our main results are the following.

[1] To give a complete se of representatives of K-orbits in X (Theorem 1).

[2] For each χ, to give a rational function $F_\chi(z)$ for which$F_\chi(z) \cdot \omega(x;\chi;z)$ belongs to $\mathbb{C}[q^{\pm\frac{z_1}{2}}, q^{\pm\frac{z_2}{2}}, q^{\pm\frac{z_3}{2}}, q^{\pm\frac{z_4}{2}}]$ and W-invariant (Theorem 2).

[3] To give an explicit formula for $\omega(x;\chi;z)$ (Theorem 3).

[4] Employing spherical functions as kernel function, we give an $\mathcal{H}(G,K)$-module isomorphism (spherical transform)

$$\mathcal{S}(K\backslash X) \xrightarrow{\sim} \left(\mathbb{C}[q^{\pm z_1}, q^{\pm z_2}, q^{\pm z_3}, q^{\pm z_4}]^W \oplus \prod_{i=1}^{4} (q^{\frac{z_i}{2}} + q^{-\frac{z_i}{2}}) \cdot \mathbb{C}[q^{\pm z_1}, q^{\pm z_2}, q^{\pm z_3}, q^{\pm z_4}]^W \right)^2.$$

Especially, $\mathcal{S}(K\backslash X)$ is a free $\mathcal{H}(G,K)$-module of rank 4, and we give a free basis (Theorem 4).

[5] Eigenvalues for spherical functions are parametrized by $z \in \left(\mathbb{C}/\frac{2\pi\sqrt{-1}}{\log q}\mathbb{Z}\right)^4/W$. The space of spherical functions on X corresponding to $z \in \mathbb{C}^4$ has dimension 4 and a basis is given explicitly (Theorem 5).

Professor S. Böcherer has suggested to the author the significance of the investigation of this space Sp_2 from the view point of its relation to the global Gross-Prasad conjecture for $SO(5)$ (cf. [GR]). The explicit Hecke module structure of the Schwartz space of it would be helpful for the question whether the vanishing of the period integral on spherical vectors implies the vanishing of the period integral on the full modular representation space. The author would like to express her gratitude to him for these useful discussion.

Notation: Throughout this paper, we denote by k a nonarchimedian local field of characteristic 0. Denote by $\mathcal{O}$ the ring of integers in k, $\mathfrak{p}$ the maximal ideal in $\mathcal{O}$, π a fixed prime element of k, q the cardinality of $\mathcal{O}/\mathfrak{p}$ and $|\ |$ the normalized absolute value on k. For convenience of notation, we understand $|0|^s = 0$ for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. For an algebraic set $\mathbb{Y}$ defined over k, we use the corresponding letter Y for the set of k-rational points $\mathbb{Y}(k)$.

As usual, we denote by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$, respectively, the complex number field, the real number field, the rational number field, and the set of natural numbers.

§1 The spherical homogeneous space Sp_2.

Set

$$Sp_n = \left\{ x \in GL_{2n} \;\middle|\; {}^t x J_n x = J_n \right\}, \qquad J_n = \left(\begin{array}{c|c} & 1_n \\ \hline -1_n & \end{array} \right), \tag{1.1}$$

and let $\mathbb{G} = Sp_2 \times (Sp_1)^2$ and we embed $(Sp_1)^2 = (SL_2)^2$ into Sp_2 by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right) \longmapsto \left(\begin{array}{cc|cc} a & & b & \\ & e & & f \\ \hline c & & d & \\ & g & & h \end{array} \right).$$

Hereafter, we understand empty places in matrices mean 0-entries.

Take $\mathbb{X} = Sp_2$, and consider the action of $\mathbb{G}$ on $\mathbb{X}$ defined by

$$\widetilde{g} \star x = g_1 x {}^t g_2, \qquad \widetilde{g} = (g_1, g_2) \in \mathbb{G},\ x \in \mathbb{X}.$$

We set the Borel subgroup $\mathbb{B} = \mathbb{B}_1 \times \mathbb{B}_2$ of $\mathbb{G}$ by

$$\mathbb{B}_1 = \begin{pmatrix} * & * & & \\ 0 & * & & * \\ & & * & 0 \\ \multicolumn{2}{c}{0} & * & * \end{pmatrix} \subset Sp_2, \quad \mathbb{B}_2 = \left(\begin{array}{cc|cc} * & & 0 & \\ & * & & 0 \\ \hline * & & * & \\ & * & & * \end{array} \right) \subset (Sp_1)^2. \tag{1.2}$$

Let us write an element $\mathbf{b} \in \mathbb{B}$ as

$$\mathbf{b} = \left(\left(\begin{array}{cc|cc} * & * & & \\ & * & & \\ \hline & & b_1 & 0 \\ & & c & b_2 \end{array} \right) \left(\begin{array}{cc|cc} 1 & & x_1 & x_2 \\ & 1 & x_2 & x_3 \\ \hline & & 1 & \\ \multicolumn{2}{c|}{0} & & 1 \end{array} \right), \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline y_1 & & 1 & \\ & y_2 & & 1 \end{array} \right) \left(\begin{array}{cc|cc} b_3 & & & \\ & b_4 & & \\ \hline & & * & \\ & & & * \end{array} \right) \right),$$

where the entries at marked $*$ are automatically determined. Then the left invariant Haar measure on $\mathbb{B}(k)$ is given by

$$d\mathbf{b} = \frac{|b_3|\,|b_4|}{|b_1|\,|b_2|^2} \cdot |db_1|\,|db_2|\,|dc|\,|dx_1|\,|dx_2|\,|dx_3|\,|db_3|\,|db_4|\,|dy_1|\,|dy_2| \tag{1.3}$$

and the modulus character δ ($d(bb') = \delta^{-1}(b')db$) is $\delta(b) = |b_1|^{-4}\,|b_2|^{-2}\,|b_3|^{-2}\,|b_4|^{-2}$.

Let $W = W_1 \times W_2$ be the Weyl group of $\mathbb{G}$ with respect to the maximal torus consisting of diagonal matrices in $\mathbb{G}$, which is isomorphic to $(C_2 \ltimes (C_2)^2) \times (C_2)^2$, and we fix generators $\{ w_i \mid 1 \le i \le 4 \}$ of W by their action on the maximal torus

$$w_i : (b_1, b_2, b_3, b_4) \longmapsto \begin{cases} (b_2, b_1, b_3, b_4) & \text{if } i = 1 \\ (b_1, b_2^{-1}, b_3, b_4) & \text{if } i = 2 \\ (b_1, b_2, b_3^{-1}, b_4) & \text{if } i = 3 \\ (b_1, b_2, b_3, b_4^{-1}) & \text{if } i = 4. \end{cases} \tag{1.4}$$

A set of basic relative $\mathbb{B}$-invariants and corresponding characters of $\mathbb{B}$ is given as follows. Let $x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{X}$ with 2 by 2 matrices A, B, C and D and we write $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in M_2$ for simplicity. Set

$$\begin{aligned} d_1(x) &= C_1, & \phi_1(\mathbf{b}) &= b_1 b_3 \\ d_2(x) &= C_2, & \phi_2(\mathbf{b}) &= b_1 b_4 \\ d_3(x) &= \det C = C_1C_4 - C_2C_3, & \phi_3(\mathbf{b}) &= b_1b_2b_3b_4 \\ d_4(x) &= (\det C\,(C^{-1}D))_3 = C_1D_3 - C_3D_1, & \phi_4(\mathbf{b}) &= b_1 b_2, \end{aligned} \tag{1.5}$$

then $\{d_i \mid 1 \le i \le 4\}$ forms a basis for relative $\mathbb{B}$-invariants and $\mathfrak{X}(\mathbb{B}) =< \phi_i \mid 1 \le i \le 4 >$ becomes the group of rational characters of $\mathbb{B}$ which corresponds to relative $\mathbb{B}$-invariants.

Let $K = \mathbb{G}(\mathcal{O})$ and $\mathcal{H}(G, K)$ be the Hecke algebra of $G = \mathbb{G}(k)$ with respect to K. We consider the following integral. For $x \in X$, $s \in \mathbb{C}^4$ and a character χ of $k^\times \big/ (k^\times)^2$,

$$\omega(x; s; \chi) = \int_K \chi(\prod_{i=1}^{4} d_i(k \star x)) \prod_{i=1}^{4} |d_i(k \star x)|^{s_i}\ dk, \tag{1.6}$$

where dk is the normalized Haar measure on K. The right hand of (1.6) is absolutely convergent for $\mathrm{Re}(s_i) \ge 0$ $(1 \le i \le 4)$ and analytically continued to rational functions in $q^{s_1}, \ldots, q^{s_4}$, which is a $\mathcal{H}(G, K)$-common eigenfunction with respect to the convolution product (cf. [H3, Remark 1.1, Proposition 1.1]).

It is convenient to introduce a new variable z which is related to s as follows

$$\begin{cases} z_1 = s_1 + s_2 + s_3 + s_4 + 2 \\ z_2 = s_3 + s_4 + 1 \\ z_3 = s_1 + s_3 + 1 \\ z_4 = s_2 + s_3 + 1, \end{cases} \qquad \begin{cases} s_1 = \frac{1}{2}(z_1 - z_2 + z_3 - z_4 - 1) \\ s_2 = \frac{1}{2}(z_1 - z_2 - z_3 + z_4 - 1) \\ s_3 = \frac{1}{2}(-z_1 + z_2 + z_3 + z_4 - 1) \\ s_4 = \frac{1}{2}(z_1 + z_2 - z_3 - z_4 - 1), \end{cases} \tag{1.7}$$

and we write also

$$\omega(x; \chi; s) = \omega(x; \chi; z),$$

if there is no danger of confusion. It is easy to see

$$\prod_{i=1}^{4} |d_i(bg \star x)|^{s_i} = (\xi\delta^{\frac{1}{2}})(b) \cdot \prod_{i=1}^{4} |d_i(g \star x)|^{s_i}, \qquad (b \in B, g \in G, x \in X),$$

where

$$\xi(b) = |b_1|^{s_1+s_2+s_3+s_4+2}\, |b_2|^{s_3+s_4+1}\, |b_3|^{s_1+s_3+1}\, |b_4|^{s_2+s_3+1} = |b_1|^{z_1}\, |b_2|^{z_2}\, |b_3|^{z_3}\, |b_4|^{z_4}$$

for $b = \left(\begin{pmatrix} * & \begin{array}{cc} * & \end{array} \\ \hline 0 & \begin{array}{cc} b_1 & \\ & b_2 \end{array} \end{pmatrix}, \begin{pmatrix} \begin{array}{cc} b_3 & \\ & b_4 \end{array} & 0 \\ \hline * & * \end{pmatrix} \right) \in B$. The Weyl group W acts on the set $\{z_1, z_2, z_3, z_4\}$ through its action on the character ξ of B, and we have

$$w_i(z_1, z_2, z_3, z_4) = \begin{cases} (z_2, z_1, z_3, z_4) & \text{for } i = 1 \\ (z_1, -z_2, z_3, z_4) & \text{for } i = 2 \\ (z_1, z_2, -z_3, z_4) & \text{for } i = 3 \\ (z_1, z_2, z_3, -z_4) & \text{for } i = 4. \end{cases} \tag{1.8}$$

The following statements can be calculated directly, though they are a special case of Satake transform of algebraic groups [Si] and spherical functions on homogeneous spaces [H3, Proposition 1.1].

Proposition 1.1 *For every $f \in \mathcal{H}(G, K)$, let*

$$\widetilde{f}(z) = \int_G f(g)\xi^{-1}\delta^{\frac{1}{2}}(p(g))dg,$$

where dg is the Haar measure on G normalized by $\int_K dg = 1$ and $g = p(g)k \in G = BK$. Then, by the map $f \longmapsto \widetilde{f}(z)$, we have

$$\mathcal{H}(G, K) \cong \mathbb{C}[q^{z_1} + q^{-z_1} + q^{z_2} + q^{-z_2},\ (q^{z_1} + q^{-z_1})(q^{z_2} + q^{-z_2}),\ q^{z_3} + q^{-z_3},\ q^{z_4} + q^{-z_4}],$$

and for every $f \in \mathcal{H}(G, K)$

$$(f * \omega(\ ; \chi; z))(x) = \widetilde{f}(z) \cdot \omega(x; \chi; z) \qquad (x \in X).$$

We recall the Bruhat decomposition of $\mathbb{X} = Sp_2$

$$\mathbb{X} = \bigsqcup_{w \in W_1} \mathbb{B}_1 w \mathbb{B}_1, \tag{1.9}$$

where W_1 is the Weyl group of Sp_2 and the symbol $\sqcup$ means disjoint union. It is easy to see that

$$\mathbb{B}_1 = \bigsqcup_{s,t} E_{s,t}\mathbb{B}_S, \qquad \text{with } \mathbb{B}_S = {}^t\mathbb{B}_2,\ E_{s,t} = \left(\begin{array}{cc|cc} 1 & s & st & t \\ & 1 & t & \\ \hline & & 1 & \\ & & -s & 1 \end{array} \right),$$

where s, t runs over the algebraic closure $\overline{k}$ of k, so we get for each $w \in W_1$ that

$$\mathbb{B}_1 w \mathbb{B}_1 = \bigcup_{s,t} \mathbb{B}_1 w E_{s,t} \mathbb{B}_S = \bigcup_{s,t} \mathbb{B} \star w E_{s,t}\ . \tag{1.10}$$

Set

$$w_0 = \left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ & -1 & & \end{array}\right) \quad (= w_2 w_1 w_2 w_1 \in W).$$

The following Proposition tells us that our space is spherical homogeneous.

Proposition 1.2 *The set*

$$\mathbb{Y} = \left\{ x \in \mathbb{X} \;\middle|\; \prod_{i=1}^{4} d_i(x) \neq 0 \right\}$$

is an open $\mathbb{B}$*-orbit over the algebraic closure of* k

$$\mathbb{Y} = \mathbb{B} \star x_0 \qquad \text{with} \qquad x_0 = \left(\begin{array}{cc|cc} & & 1 & 0 \\ & & 1 & 1 \\ \hline -1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right) (= w_0 E_{-1,-1}).$$

Further, the B*-orbit decomposition of the set of* k*-rational points in* $\mathbb{Y}$ *is given by*

$$\mathbb{Y}(k) = \bigsqcup_{u \in k^\times/(k^\times)^2} Y_u,$$

where

$$Y_u = \left\{ x \in X \;\middle|\; \prod_{i=1}^{4} d_i(x) \equiv u \bmod (k^\times)^2 \right\} \ni w_0 E_{-1,-u} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 & 0 \\ & & 1 & 1 \\ \hline -1 & 1 & -u & u \\ 0 & -1 & u & 0 \end{array}\right).$$

Remark. By Proposition 1.2 and the injectivity of Poisson integral (cf. [K1]), we see that $\omega(x; \chi; z)$ is not identically zero for generic z and linearly independent for characters χ. Indeed, we will see that the space of spherical functions has dimension 4 and we give a basis by modifying $\omega(x; \chi; z)$ for various χ (cf. Theorem 5 in Section 5).

Before closing this section, we confirm the assumption (A2) of [H3]. Denote by $\mathbb{H}$ the stabilizer $\mathbb{G}_{x_0}$ of x_0 in $\mathbb{G}$ and consider the action of $\mathbb{B} \times \mathbb{H}$ on $\mathbb{G}$ by

$$(b, h) * g = bgh^{-1} \quad (b, h) \in \mathbb{B} \times \mathbb{H}, \; g \in \mathbb{G},$$

then $\mathbb{X} \cong \mathbb{G}/\mathbb{H}$ as $\mathbb{G}$-sets. Further, we see that $\mathbb{B}\mathbb{H} = (\mathbb{B} \times \mathbb{H}) * 1$ is an open orbit in $\mathbb{G}$ and $\mathbb{G}$ is decomposed into a finite number of $\mathbb{B} \times \mathbb{H}$-orbits.

For $g \in \mathbb{G}$, denote by $\mathbb{B}_{(g)}$ the image of the stabilizer $(\mathbb{B} \times \mathbb{H})_g$ by the projection $\mathbb{B} \times \mathbb{H} \longrightarrow \mathbb{B}$. Then we have

Lemma 1.3 *For each* $g \in \mathbb{G}$, $g \notin \mathbb{B}\mathbb{H}$, *there exists a rational character in* $\mathfrak{X}(\mathbb{B})$ *which is nontrivial on* $\mathbb{B}_{(g)}$.

§2 Cartan decomposition

Hereafter we assume that k has odd residual characteristic. In this section we consider "Cartan decomposition" of X, that is we give a complete set of representatives of K-orbits in X.

To state the result, we introduce some notation: Let

$$\begin{aligned} \Lambda &= \left\{ (\lambda_1,\lambda_2,\lambda_3,\lambda_4) \in \mathbb{Z}^4 \cup (\tfrac{1}{2}+\mathbb{Z})^4 \;\middle|\; \lambda_1 \geq \lambda_2 \geq 0,\ \lambda_3 \geq 0,\ \lambda_4 \geq 0 \right\}, \\ \Lambda_* &= \{ \lambda \in \Lambda \mid \lambda_1 > \lambda_2 > 0,\ \lambda_3 > 0,\ \lambda_4 > 0 \}, \end{aligned} \tag{2.1}$$

and for $\lambda \in \Lambda$ and $\xi \in \mathcal{O}^\times$ set

$$\begin{aligned} \pi_{(\lambda;\xi)} &= \left(\begin{array}{cc|cc} & & -\pi^{\lambda_1+\lambda_3} & \\ & & \xi\pi^{\lambda_2+\lambda_3} & -\pi^{\lambda_2+\lambda_4} \\ \hline \pi^{-\lambda_1-\lambda_3} & \xi\pi^{-\lambda_1-\lambda_4} & \xi\pi^{-\lambda_1+\lambda_3} & \pi^{-\lambda_1+\lambda_4} \\ & \pi^{-\lambda_2-\lambda_4} & \pi^{-\lambda_2+\lambda_3} & \end{array} \right) \\ &= \left(\begin{array}{cc|cc} \pi^{\lambda_1} & & & \\ & \pi^{\lambda_2} & & \\ \hline & & \pi^{-\lambda_1} & \\ & & & \pi^{-\lambda_2} \end{array} \right) \left(\begin{array}{cc|cc} & & -1 & \\ & & \xi & -1 \\ \hline 1 & \xi & \xi & 1 \\ & 1 & 1 & \end{array} \right) \left(\begin{array}{cc|cc} \pi^{-\lambda_3} & & & \\ & \pi^{-\lambda_4} & & \\ \hline & & \pi^{\lambda_3} & \\ & & & \pi^{\lambda_4} \end{array} \right) \end{aligned}$$

Then our main result is the following.

Theorem 1 *Let*

$$\widetilde{\mathcal{R}} = \left\{ \pi_{(\lambda;\xi)} \;\middle|\; \begin{array}{l} \lambda \in \Lambda,\ \xi \in \mathcal{O}^\times/(\mathcal{O}^\times)^2 \\ \xi = 1 \ \textit{unless}\ \lambda \in \Lambda_* \end{array} \right\},$$

then $\widetilde{\mathcal{R}}$ makes a complete set of representatives of K-orbits in X.

In order to prove Theorem 1, we first construct another complete set of representatives. We introduce some more notation. Set $K_1 = Sp_2(\mathcal{O})$ and $K_2 = (Sp_1(\mathcal{O}))^2 (\subset K_1)$, then it suffices to consider the representatives of double cosets in the space $K_1\backslash X/K_2$. Set

$$\begin{aligned} T_{(x,y,z,w)} &= \left(\begin{array}{cc|cc} x^{-1} & -x^{-1}y^{-1}z & & \\ & y^{-1} & & \\ \hline & & x & \\ & & z & y \end{array} \right) \left(\begin{array}{c|cc} 1_2 & & w \\ & w & \\ \hline & & 1_2 \end{array} \right) \\ &= \left(\begin{array}{cc|cc} x^{-1} & -x^{-1}y^{-1}z & -x^{-1}y^{-1}zw & x^{-1}w \\ & y^{-1} & y^{-1}w & \\ \hline & & x & \\ & & z & y \end{array} \right) \end{aligned}$$

and for $a,b,c,d \in \mathbb{Z}$ and $\varepsilon \in \mathcal{O}^\times$, set

$$\begin{aligned} A_{(a,b)} &= T_{(\pi^a,\pi^b,0,0)}, & B_{(a,b,c)} &= T_{(\pi^a,\pi^b,\pi^c,0)}, \\ C_{(a,b,d)} &= T_{(\pi^a,\pi^b,0,\pi^d)}, & D_{(a,b,c,d;\varepsilon)} &= T_{(\pi^a,\pi^b,\varepsilon\pi^c,\pi^d)}. \end{aligned}$$

Proposition 2.1 *The set* $\mathcal{R} = \bigsqcup_{i=1}^{4} \mathcal{R}_i$ *is a complete set of representatives of* $K\backslash X$*, where*

$$\begin{aligned}
\mathcal{R}_1 &= \left\{ A_{(a,b)} \;\middle|\; a \geq 0,\ b \geq 0 \right\}, \quad \mathcal{R}_2 = \left\{ B_{(a,b,c)} \;\middle|\; a > c \geq 0,\ b \geq 0 \right\}, \\
\mathcal{R}_3 &= \left\{ C_{(a,b,d)} \;\middle|\; \begin{array}{l} a > 0,\ b > 0,\ a+b > d \geq 0 \\ a \geq b \ \ \textit{if} \ \ d = 0 \end{array} \right\}, \\
\mathcal{R}_4 &= \left\{ D_{(a,b,c,d;\varepsilon)} \;\middle|\; \begin{array}{l} a > c,\ b+c > d,\ b+d > c,\ c+d > b \\ \varepsilon \in \mathcal{O}^\times \big/ (\mathcal{O}^\times)^2 \end{array} \right\}.
\end{aligned}$$

Remark 2.1. (1) One proves that every K-orbit has a representative in the set $\mathcal{R}$ by Lemmas 2.2 and 2.3. It is possible but tedious to show directly that there occurs no K-equivalence within $\mathcal{R}$, so we take another way.

We will see (in Corollary 5.3) that spherical functions $\omega(x, \chi, z)$ take different values at each element of $\mathcal{R}$, by using their explicit formulas. Since spherical functions are K-invariant function, it means that each element in $\mathcal{R}$ belongs to the different K-orbit in X, and we see that $\mathcal{R}$ is a complete set of representatives of K-orbit of X. Thus we establish Proposition 2.1.

(2) The set $\mathcal{R}_4$ corresponds bijectively to the set

$$\widetilde{\mathcal{R}_*} = \left\{ \pi_{(\lambda;\xi)} \;\middle|\; \lambda \in \Lambda_*,\ \xi \in \mathcal{O}^\times/(\mathcal{O}^\times)^2 \right\}. \tag{2.2}$$

(3) In a direct calculation, the assumption on the residual characteristic is needed only for the proof that there occurs no K-equivalence within $\mathcal{R}_4$. For the even residual characteristic case, we have to choose a suitable subset within $\mathcal{R}_4$ (or within $\widetilde{\mathcal{R}_*}$).

Lemma 2.2 *Set* $\mathcal{R}' = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}'_3 \cup \mathcal{R}'_4$ *with*

$$\begin{aligned}
\mathcal{R}'_3 &= \left\{ C_{(a,b,d)} \;\middle|\; a \geq 0,\ b \geq 0,\ d \geq 0 \right\}, \\
\mathcal{R}'_4 &= \left\{ D_{(a,b,c,d;\varepsilon)} \;\middle|\; a > c \geq 0,\ b \geq 0,\ d \geq 0,\ \varepsilon \in \mathcal{O}^\times \big/ (\mathcal{O}^\times)^2 \right\}.
\end{aligned}$$

Then every K*-orbit in* X *has a representative in* $\mathcal{R}'$.

Lemma 2.3 *Because of the following relations, one can replace* $\mathcal{R}'_3$ *and* $\mathcal{R}'_4$ *by* $\mathcal{R}_3$ *and* $\mathcal{R}_4$*, respectively.*

$$C_{(a,b,d)} \sim_K A_{(a,b)} \qquad \textit{if } d \geq a+b. \tag{2.3}$$

$$C_{(a,0,d)} \sim_K B_{(a,0,d)}. \tag{2.4}$$

$$C_{(0,b,d)} \sim_K B_{(b-d,d,0)} \qquad \textit{if } b \geq d. \tag{2.5}$$

$$C_{(a,b,0)} \sim_K C_{(b,a,0)}. \tag{2.6}$$

$$D_{(a,b,c,d;\varepsilon)} \sim_K B_{(a,b,d)} \qquad \textit{if } d \geq b+c. \tag{2.7}$$

$$D_{(a,b,c,d;\varepsilon)} \sim_K C_{(c,a+b-c,d)} \qquad \textit{if } b \geq c+d. \tag{2.8}$$

$$D_{(a,b,c,d;\varepsilon)} \sim_K C_{(a,b,d)} \qquad \textit{if } c \geq b+d. \tag{2.9}$$

Now we make each element of $\mathcal{R}$ correspond systematically to an element in $\widetilde{\mathcal{R}}$. Set

$$\widetilde{D}_{(a,b,c,d;x)} = \left(\begin{array}{c|c} 0 & -1_2 \\ \hline 1_2 & 0 \end{array}\right) \cdot D_{(a,b,c,d;\varepsilon)} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & -\pi^{a} & 0 \\ & & -\varepsilon\pi^{c} & -\pi^{b} \\ \hline \pi^{-a} & -\varepsilon\pi^{-a-b+c} & -\varepsilon\pi^{-a-b+c+d} & \pi^{-a+d} \\ 0 & \pi^{-b} & \pi^{-b+d} & 0 \end{array}\right),$$

then

$$\pi_{(\lambda;\xi)} = \widetilde{D}_{(a,b,c,d;\varepsilon)}$$

for

$$\begin{aligned} & a = \lambda_1 + \lambda_3,\ b = \lambda_2 + \lambda_4,\ c = \lambda_2 + \lambda_3,\ d = \lambda_3 + \lambda_4, \\ & \lambda_1 = \frac{2a+b-c-d}{2},\ \lambda_2 = \frac{b+c-d}{2},\ \lambda_3 = \frac{-b+c+d}{2},\ \lambda_4 = \frac{b-c+d}{2}, \\ & \varepsilon = -\xi. \end{aligned}$$

Then $\mathcal{R}$ corresponds bijectively to $\widetilde{\mathcal{R}}$, in particular $\mathcal{R}_4$ corresponds to $\widetilde{\mathcal{R}_*}$.

§3 Functional equations and rationality of spherical functions

The functional equations for $\omega(x;z;\chi)$ and $\omega(x;z;w_i(\chi))$ for $w_i \in W,\ 1 \leq i \leq 4$ can be obtained by taking suitable parabolic subgroup $\mathbb{P}_i$ containing $\mathbb{B}$ and prehomogeneous space $(\mathbb{P}_i \times GL_1, \mathbb{X} \times M_{2,1})$, for the details see [H5, §3]. Then we have the following theorem, which gives us some information on the location of poles and zeros of spherical functions.

Theorem 2 *For each character χ of $k^\times/(k^\times)^2$, set*

$$F_\chi(z) = G_\chi(z)\big/G(z),$$

where

$$G(z) = (1 - q^{-z_1+z_2-1})(1 - q^{-z_1-z_2-1})\prod_{i=1}^{4}(1 - q^{-z_i-1}),$$

$$G_\chi(z) = \begin{cases} \{(+---)(-++-)(-+-+)(-+--)(--++)(--+-) & \\ \qquad \times (---+)(----)\}_\varepsilon & \text{if } \chi(\mathcal{O}^\times) = 1 \text{ and } \chi(\pi) = \varepsilon \\ q^{-\frac{3z_1+z_2+z_3+z_4}{2}} & \text{if } \chi(\mathcal{O}^\times) \neq 1, \end{cases}$$

and

$$(\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4)_\varepsilon = 1 - \varepsilon q^{\frac{1}{2}(\varepsilon_1 z_1 + \varepsilon_2 z_2 + \varepsilon_3 z_3 + \varepsilon_4 z_4 - 1)} \quad (\varepsilon_i = +, -,\ \varepsilon = 1, -1).$$

Then $F_\chi(z) \cdot \omega(x;z;\chi)$ belongs to $\mathbb{C}[q^{\pm\frac{z_1}{2}}, q^{\pm\frac{z_2}{2}}, q^{\pm\frac{z_3}{2}}, q^{\pm\frac{z_4}{2}}]$ and is invariant under the action of the Weyl group W of G.

§4 Explicit expressions of spherical functions

In this section we give explicit expressions of spherical functions $\omega(x;\chi;z)$ for each element in $\widetilde{\mathcal{R}}$ following the method of [H3, §1]. Since spherical functions are K-invariant, it is enough to give such formulas for the representatives of $K\backslash X$. In Section 2, we have given a set $\widetilde{\mathcal{R}}$ of representatives of $K\backslash X$ and left the proof that there is no K-equivalence within $\widetilde{\mathcal{R}}$, which will be proved through the explicit formula $\omega(x;\chi;z)$ in Corollary 5.5.

Set

$$\mathcal{P}(x;\chi;z) \quad = \quad \int_U \chi(\prod_{i=1}^4 d_i(u \star x)) \prod_{i=1}^4 |d_i(u\star x)|^{s_i}\, du, \tag{4.1}$$

where the variable $z \in \mathbb{C}^4$ is related to $s \in \mathbb{C}^4$ by (1.7), U is the Iwahori subgroup of G compatible with B and du is the Haar measure on U normalized by $\int_U du = 1$. The right hand side of (4.1) is absolutely convergent for $\mathrm{Re}(s_i) \geq 0$ $(1 \leq i \leq 4)$ and analytically continued to a rational function in $q^{s_1}, \ldots, q^{s_4}$.

Applying [H3, Proposition 1.9] to our case, we have the following.

Proposition 4.1 *Let $G(z)$ and $G_\chi(z)$ be as in Theorem 2, and set*

$$H(z) = (1-q^{-z_1+z_2})(1-q^{-z_1-z_2}) \cdot \prod_{i=1}^4 (1-q^{-z_i}),$$

where the variable $z \in \mathbb{C}^4$ is related to $s \in \mathbb{C}^4$ by (1.7). Then we have

$$\omega(x;\chi;z) = \frac{1}{(1+q^{-1})^4(1+q^{-2})} \cdot \frac{G(z)}{G_\chi(z)} \cdot \sum_{\sigma\in W} \sigma\left(\frac{G_\chi(z)}{H(z)} \cdot \mathcal{P}(x;\chi;z)\right).$$

We set

$$\widetilde{\mathcal{R}_+} = \left\{ \pi_{(\lambda;\xi)} \;\middle|\; \lambda \in \Lambda,\ \xi \in \mathcal{O}^\times/(\mathcal{O}^\times)^2 \right\},$$

and calculate $\mathcal{P}(x;\chi;z)$ for $x \in \widetilde{\mathcal{R}_+}$.

Proposition 4.2 *For $\pi_{(\lambda;\xi)} \in \widetilde{\mathcal{R}_+}$, we have*

$$\mathcal{P}(\pi_{(\lambda;\xi)};\chi;z) = \chi(\xi)\chi(\pi)^{2\lambda_1} q^{-\|\lambda\|-\lambda_1} \cdot q^{<\lambda,z>},$$

where $\|\lambda\| = \sum_{i=1}^4 \lambda_i$ and $<\lambda, z> = \sum_{i=1}^4 \lambda_i z_i$.

The following Proposition is an easy consequence of Propositions 4.1 and 4.2.

Proposition 4.3 *Let χ be nontrivial on $\mathcal{O}^\times$ and $x \in X$ be K-equivalent to some element in $\widetilde{\mathcal{R}} \backslash \widetilde{\mathcal{R}_*}$. Then $\omega(x;\chi;z) = 0$.*

For an element σ of the Weyl group W, we set $\varepsilon(\sigma) = 1$ (resp. -1) if σ is expressed by a product of even (resp. odd) numbers of $\{w_1, w_2, w_3, w_4\}$.

By Proposions 4.1, 4.2 and 4.3, we obtain our main results on explicit expressions of spherical functions.

Theorem 3 *For each $\lambda \in \Lambda$, $\xi \in \mathcal{O}^\times$ and character χ of $k^\times \big/ (k^\times)^2$, set*

$$c_{\lambda,\xi,\chi}(z) = \frac{\chi(\xi)\chi(\pi)^{2\lambda_1} q^{-\|\lambda\|-\lambda_1}}{(1+q^{-1})^4(1+q^{-2})} \cdot \frac{G(z)}{G_\chi(z)} \cdot \frac{1}{H_0(z)},$$

where $G(z)\big/G_\chi(z) = F_\chi(z)^{-1}$ is given in Theorem 2 and

$$H_0(z) \;=\; (q^{z_1} - q^{z_2})(1 - q^{-z_1-z_2}) \cdot \prod_{i=1}^{4} (q^{\frac{z_i}{2}} - q^{\frac{-z_i}{2}}) \left(= q^{\frac{3z_1+z_2+z_3+z_4}{2}} \cdot H(z) \right);$$

so if χ is nontrivial on $\mathcal{O}^\times$, $G(z)\big/G_\chi(z)H_0(z)$ coincides with the c-function $G(z)\big/H(z)$ of G. Then the explicit formulas of spherical functions are given in the following.

(i) *If χ is trivial on $\mathcal{O}^\times$, we have*

$$\omega(\pi_{(\lambda,\xi)}; \chi; z) \;=\; c_{\lambda,1,\chi}(z) \cdot \sum_{\sigma \in W} \varepsilon(\sigma) \cdot \sigma \left(G_\chi(z) \cdot q^{<\widetilde{\lambda}, z>} \right),$$

where $\widetilde{\lambda} = (\lambda_1 + \frac{3}{2}, \lambda_2 + \frac{1}{2}, \lambda_3 + \frac{1}{2}, \lambda_4 + \frac{1}{2}) (\in \Lambda_*)$.

(ii) *Let χ be nontrivial on $\mathcal{O}^\times$. Then $\omega(\pi_{(\lambda;\xi)}; \chi; z) = 0$ unless $\lambda \in \Lambda_*$, and if $\lambda \in \Lambda_*$, we have*

$$\begin{aligned} &\omega(\pi_{(\lambda;\xi)}; \chi; z) \\ &\quad = c_{\lambda,\xi,\chi}(z) \cdot \left(\left(q^{\lambda_1 z_1} - q^{-\lambda_1 z_1}\right)\left(q^{\lambda_2 z_2} - q^{-\lambda_2 z_2}\right) - \left(q^{\lambda_2 z_1} - q^{-\lambda_2 z_1}\right)\left(q^{\lambda_1 z_2} - q^{-\lambda_1 z_2}\right) \right) \\ &\quad\quad \times \prod_{i=3,4} \left(q^{\lambda_i z_i} - q^{-\lambda_i z_i}\right). \end{aligned}$$

§5 Spherical Fourier transform

Let $\mathcal{S}(K\backslash X)$ be set of K-invariant Schwartz-Bruhat functions on X:

$$\mathcal{S}(K\backslash X) = \{\varphi \in \mathcal{C}^\infty(K\backslash X) \mid \textit{compactly supported}\},$$

and we introduce the spherical transform on $\mathcal{S}(K\backslash X)$ in the following. Set

$$\Psi_1(x; z) = F_1(z) \cdot \omega(x; 1; z), \qquad \Psi_2(x; z) = F_{\chi^*}(z) \cdot \omega(x; \chi^*; z),$$

where 1 is the trivial character and χ^* is the character for which $\chi^*(\pi) = 1$ and $\chi^*(\varepsilon) = \left(\frac{\varepsilon}{\mathfrak{p}}\right)$ for $\varepsilon \in \mathcal{O}^\times$, and $F_\chi(z)$ is the function defined in Theorem 2. By Theorem 2, we know that $\Psi_i(x; z),\ i = 1, 2$ belong to

$$\mathbb{C}[q^{\pm\frac{z_1}{2}}, q^{\pm\frac{z_2}{2}}, q^{\pm\frac{z_3}{2}}, q^{\pm\frac{z_4}{2}}]^W (= \mathcal{C}_0, \text{say}).$$

On the other hand, as we saw in Proposition 1.1, $\mathcal{H}(G, K)$ is isomorphic to $\mathcal{C}_0$ by Satake isomorphism.

Now we define the spherical Fourier transform on $\mathcal{S}(K\backslash X)$ for $i = 1, 2$

$$\begin{array}{rccl} \mathcal{F}_i: & \mathcal{S}(K\backslash X) & \longrightarrow & \mathbb{C}[q^{\pm\frac{z_1}{2}}, q^{\pm\frac{z_2}{2}}, q^{\pm\frac{z_3}{2}}, q^{\pm\frac{z_4}{2}}]^W (= \mathcal{C}_0, \text{say}) \\ & \varphi & \longmapsto & \mathcal{F}_i(\varphi)(z) \end{array}$$

by

$$\mathcal{F}_i(\varphi)(z) = \int_X \varphi(x) \cdot \Psi_i(x; z) dx,$$

where dx is the normalized G-invariant measure on X. Since $\mathcal{F}_i$ satisfies for every $f \in \mathcal{H}(G, K)$

$$\mathcal{F}_i(f * \varphi)(z) = \widetilde{\check{f}}\,(z) \cdot \mathcal{F}_i(\varphi)(z), \qquad \check{f}(g) = f(g^{-1}),$$

$\mathcal{F}_i$ is an $\mathcal{H}(G, K)$-module homomorphism, $i = 1, 2$.

Let us recall the sets Λ and Λ_* defined in the beginning of Section 2. Set $\Lambda_0 = \Lambda \setminus \Lambda_*$. For $\lambda \in \Lambda$, denote by φ_λ the characteristic function of the K-orbit containing $\pi_{(\lambda;1)}$ and by $\varphi_{\lambda *}$ the characteristic function of the K-orbit containing $\pi_{(\lambda;\xi)}$ for $\xi \in \mathcal{O}^\times,\ \xi \notin (\mathcal{O}^\times)^2$. Then $\mathcal{S}(K\backslash X)$ is generated by $\{\varphi_\lambda \mid \lambda \in \Lambda_0\} \cup \{\varphi_\lambda, \varphi_{\lambda *} \mid \lambda \in \Lambda_*\}$.

For simplicity, we set

$$\eta(z) = \prod_{i=1}^{4} \left(q^{\frac{z_i}{2}} + q^{-\frac{z_i}{2}}\right), \qquad \mathcal{C} = \mathcal{C}_0 \oplus \eta(z) \cdot \mathcal{C}_0,$$

here we regard $\mathcal{C}_0$ and $\mathcal{C}$ as free $\mathcal{H}(G, K)$-modules through the Satake transform.

Our main theorem is the following.

Theorem 4 *Set*

$$\begin{aligned} \mathcal{S}_1 &= <\varphi_\lambda \mid \lambda \in \Lambda_0>_\mathbb{C} + <\varphi_\lambda + \varphi_{\lambda *} \mid \lambda \in \Lambda_*>_\mathbb{C}, \\ \mathcal{S}_2 &= <\varphi_\lambda - \varphi_{\lambda *} \mid \lambda \in \Lambda_*>_\mathbb{C}. \end{aligned}$$

Then $\mathcal{S}(K\backslash X) = \mathcal{S}_1 \oplus \mathcal{S}_2$ as an $\mathcal{H}(G, K)$-module, and $\mathcal{F}_j$ induces the $\mathcal{H}(G, K)$-module isomorphism $\mathcal{S}_j \cong \mathcal{C}$ for $j = 1, 2$.

In particular, $\mathcal{S}(K\backslash X)$ is a free $\mathcal{H}(G, K)$-module of rank 4 with basis

$$\left\{\varphi_\lambda \,\middle|\, \lambda = (0,0,0,0),\ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\right\} \cup \left\{\varphi_\lambda - \varphi_{\lambda *} \,\middle|\, \lambda = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (2,1,1,1)\right\}.$$

It is clear that $\mathrm{Ker}\mathcal{F}_1 \supset \mathcal{S}_2$, $\mathrm{Ker}\mathcal{F}_2 \supset \mathcal{S}_1$ and $\mathcal{F}_2$ is injective on $\mathcal{S}_2$. Theorem 5 follows from Propositions 5.1 and 5.2 below.

Proposition 5.1 *For $\lambda \in \Lambda_*$, set*

$$\widetilde{m_\lambda}(z) = \sum_{\sigma \in W} \sigma\left(\frac{q^{<\lambda,z>}}{H_0(z)}\right).$$

Then

$$\mathcal{F}_2(\varphi_\lambda - \varphi_{\lambda*}) \equiv \widetilde{m_\lambda(z)} \ (\mathrm{mod}\, \mathbb{C}^\times),$$

$\widetilde{m_\lambda}(z) \in \mathcal{C}_0$ (resp. $\eta(z)\mathcal{C}_0$) if $\lambda_1 \in \frac{1}{2} + \mathbb{Z}$ (resp. $\lambda_1 \in \mathbb{Z}$), and

$$\widetilde{m_\lambda}(z) = \begin{cases} 1 & \text{if } \lambda = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \\ \eta(z) & \text{if } \lambda = (2, 1, 1, 1). \end{cases}$$

In Particular, $\mathcal{F}_2$ gives an $\mathcal{H}(G,K)$-module isomorphism $\mathcal{S}_2 \cong \mathcal{C}$.

Proposition 5.2 *For $\lambda \in \Lambda$, set*

$$K_\lambda(z) = \sum_{\sigma \in W} \sigma\left(\frac{G_1(z) \cdot q^{<\lambda,z>}}{H_0(z)}\right).$$

Then,

$$\mathcal{F}_1(\varphi_\lambda) = \mathcal{F}_1(\varphi_{\lambda*}) \equiv K_{\widetilde{\lambda}}(z) \ (\mathrm{mod}\, \mathbb{C}^\times), \quad \widetilde{\lambda} = (\lambda_1 + \frac{3}{2}, \lambda_2 + \frac{1}{2}, \lambda_3 + \frac{1}{2}, \lambda_4 + \frac{1}{2}),$$

and $\lambda \in \Lambda_*$, $K_\lambda(z)$ can be expressed as

$$K_\lambda(z) = c_\lambda \widetilde{m_\lambda}(z) + \sum_{\substack{\mu \in \Lambda_* \\ \lambda \succ \mu}} c_\mu \widetilde{m_\mu}(z), \ \text{with some } c_\lambda \in \mathbb{C}^\times, \ c_\mu \in \mathbb{C},$$

where $\lambda \succ \mu$ means that $\|\lambda\| > \|\mu\|$ or $\|\lambda\| = \|\mu\|$, $\lambda_1 > \mu_1$. In Particular, $\mathcal{F}_1$ gives an $\mathcal{H}(G,K)$-module isomorphism $\mathcal{S}_1 \cong \mathcal{C}$. In particular

Since $\omega(x;\chi^*;z)$ vanishes on $\widetilde{\mathcal{R}_0} = \widetilde{\mathcal{R}} \backslash \widetilde{\mathcal{R}_*}$ and takes a different value at each element of $\widetilde{\mathcal{R}_*}$ and $\omega(x;1;z)$ takes a different value at each element of $\widetilde{\mathcal{R}_0}$, we conclude the proof of Cartan decomposition given in Section 2.

Corollary 5.3 *The set $\widetilde{\mathcal{R}}$, as well as $\mathcal{R}$, is a complete set of representatives of K-orbit in X.*

Finally, we give a parametrization of spherical functions. The characters on $k^\times/(k^\times)^2$ are given by $\{1, \ \chi^*, \chi_\pi, \chi_\pi^*\}$, where $\chi_\pi(\pi) = -1$, $\chi_\pi(\mathcal{O}^\times) = 1$ and $\chi_\pi^* = \chi^*\chi_\pi$. We set for each χ

$$\Psi_\chi(x;z) = F_\chi(z) \cdot \omega(x;\chi;z),$$

so $\Psi_{\chi^*}(x;z) = \Psi_2(x;z)$ in the previous notation.

Theorem 5 *Eigenvalues for spherical functions are parametrized by* $z \in \left(\mathbb{C}/\frac{2\pi\sqrt{-1}}{\log q}\mathbb{Z}\right)^4/W$ *through the Satake transform* $\mathcal{H}(G,K) \longrightarrow \mathbb{C},\ f \longmapsto \widetilde{f}(z)$ *(cf. Proposition 1.1). The set*

$$\left\{ \Psi_1(x;z) + \Psi_{\chi_\pi}(x;z),\ \Psi_{\chi^*}(x;z) - \Psi_{\chi^*_\pi}(x;z),\ \frac{\Psi_1(x;z) - \Psi_{\chi_\pi}(x;z)}{\eta(z)},\ \frac{\Psi_{\chi^*}(x;z) + \Psi_{\chi^*_\pi}(x;z)}{\eta(z)} \right\}$$

forms a basis of the space of spherical functions on X *corresponding to* $z \in \mathbb{C}^4$.

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New vectors for GSp(4): a conjecture and some evidence

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In this paper we present and state evidence for a conjecture on the existence and properties of new vectors for generic irreducible admissible representations of $\mathrm{GSp}(4, F)$ with trivial central character for F a nonarchimedean field of characteristic zero. To summarize the conjecture, let $\mathcal{O}$ be the ring of integers of F and let $\mathcal{P}$ be the prime ideal of $\mathcal{O}$. We define, by a simple formula, a sequence of compact open subgroups $\mathrm{K}(\mathcal{P}^n)$ of $\mathrm{GSp}(4, F)$ indexed by nonnegative integers n. The first group $\mathrm{K}(\mathcal{O})$ is $\mathrm{GSp}(4, \mathcal{O})$. The second group $\mathrm{K}(\mathcal{P})$ is the other maximal compact subgroup of $\mathrm{GSp}(4, F)$, up to conjugacy, and is called the paramodular group. Automorphic forms for the global version of this group have been considered by T. Ibukiyama and his collaborators in a number of papers dealing with a genus two version of Eichler's correspondence and old and new forms. In general, we refer to $\mathrm{K}(\mathcal{P}^n)$ as the paramodular group of level $\mathcal{P}^n$. Given a generic irreducible admissible representation π of $\mathrm{GSp}(4, F)$ with trivial central character, we consider the space of vectors fixed by each $\mathrm{K}(\mathcal{P}^n)$. The conjecture for π makes three assertions. First, for some nonnegative n, the space of $\mathrm{K}(\mathcal{P}^n)$ fixed vectors is nonzero; second, if N_π is the smallest such n, then the space of $\mathrm{K}(\mathcal{P}^{N_\pi})$ fixed vectors is one dimensional; and third, this one dimensional space contains a vector W_π whose Novodvorsky zeta integral gives the Novodvorsky L-factor of the representation:

$$Z(s, W_\pi) = L(s, \pi).$$

We call W_π the new vector of π. Zeta integrals depend on a choice of Whittaker model, which depends on a choice of nondegenerate character: we make a choice independent of π.

Evidently, the conjecture is similar to the theory of new vectors for generic irreducible admissible representations of $\mathrm{GL}(2, F)$ with trivial central character. Just as for $\mathrm{GL}(2, F)$, there is a simple relation between new vectors and

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ϵ-factors. Assume the conjecture holds for π. There exists an Atkin-Lehner type element u_{N_π} in $\mathrm{GSp}(4,F)$ which normalizes $\mathrm{K}(\mathcal{P}^{N_\pi})$ and whose square is in the center. Thus, $\pi(u_{N_\pi})W_\pi = \epsilon_\pi W_\pi$ for some $\epsilon_\pi = \pm 1$. Moreover, it is easy to show that

$$\epsilon(s,\pi) = \epsilon_\pi q^{-N_\pi(s-1/2)}$$

so that $\epsilon(1/2,\pi) = \epsilon_\pi$. Here, q is the order of $\mathcal{O}/\mathcal{P}$, and we use the mentioned nondegenerate character in the definition of the ϵ-factor.

We state three pieces of evidence for the conjecture. First, the first two parts of the conjecture are true for all π containing a nonzero vector fixed by the Iwahori subgroup. As evidence for the third part of the conjecture for such π one also has

$$\epsilon(s,\varphi_\pi,\psi,\mathrm{d}x_\psi) = \epsilon_\pi q^{-N_\pi(s-1/2)}$$

where φ_π is the L-parameter assigned to π by [KL]. Second, the first two parts of the conjecture are true for many π induced from the Siegel or Klingen parabolic subgroups, and for these π, the level $\mathcal{P}^{N_\pi}$ is as expected. Finally, in proving the analogue for $\mathrm{GSp}(4)$ of the dihedral case of the global Langlands-Tunnell theorem, [R1] defined certain local L-packets $\Pi(\tau)$ and L-parameters $\varphi(\tau)$ for $\mathrm{GSp}(4,F)$ which depend on a generic tempered irreducible admissible representation τ of $\mathrm{GL}(2,E)$ with trivial central character, where E is either a quadratic extension of F, or $F\times F$. The work [R1] gave strong global evidence that $\Pi(\tau)$ is the L-packet of $\varphi(\tau)$. Assuming q is odd, we show that if E/F is unramified or $E = F\times F$, then the generic element π of $\Pi(\tau)$ contains a nonzero vector W fixed by $\mathrm{K}(\mathcal{P}^N)$, where N is defined by $\epsilon(s,\varphi(\tau),\psi,\mathrm{d}x_\psi) = cq^{-N(s-1/2)}$, and c is a constant. Moreover, $Z(s,W) = L(s,\pi)$.

To end this introduction, we emphasis that our conjecture is for generic irreducible admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character. In gathering evidence we have encountered various related cases and questions, as mentioned below. But, for example, currently we are not in a position to state a conjecture for the case of nontrivial central character.

Notation

In this paper $\mathrm{GSp}(4,F)$ is the group of g in $\mathrm{GL}(4,F)$ such that

$${}^t g \begin{bmatrix} 0 & 1_2 \\ -1_2 & 0 \end{bmatrix} g = \lambda(g) \begin{bmatrix} 0 & 1_2 \\ -1_2 & 0 \end{bmatrix}$$

for some $\lambda(g)$ in $F^\times$. Fix a continuous character ψ of F with conductor $\mathcal{O}$ and a generator ϖ for $\mathcal{P}$. Let $|\cdot|$ be the valuation on F such that if μ is a Haar measure on F, then $\mu(xA) = |x|\mu(A)$ for x in F and measurable sets A in F. If π is an irreducible admissible representation of a group of td-type [Car], let ω_π denote the central character of π. Let $\mathrm{L}_F = \mathrm{W}_F \times \mathrm{SU}(2,\mathbb{R})$ be the Langlands group of F, where W_F is the Weil group of F. A GSp(4) L-parameter over F is a continuous homomorphism $\varphi : \mathrm{L}_F \to \mathrm{GSp}(4,\mathbb{C})$ such that $\varphi(x)$ is semisimple for all $x \in \mathrm{W}_F$ and $\varphi|_{1\times \mathrm{SU}(2,\mathbb{R})}$ is a smooth representation. We denote the ϵ-factor of φ with respect to ψ and the Haar measure $\mathrm{d}x_\psi$ self-dual with respect to ψ by $\epsilon(s,\varphi,\psi,\mathrm{d}x_\psi)$. One has $\epsilon(s,\varphi,\psi,\mathrm{d}x_\psi) = cq^{-N(s-1/2)}$ for some nonnegative integer N and constant c.

1 The conjecture

To state the conjecture we need some definitions and results. First, we recall the fundamentals of the theory of Novodvorsky zeta integrals for $\mathrm{GSp}(4,F)$, as proven in [T-B]. Fix $c_1, c_2 \in F^\times$. Let π be an irreducible admissible representation of $\mathrm{GSp}(4,F)$. We say that π is **generic** if $\mathrm{Hom}_U(\pi,\psi_{c_1,c_2}) \neq 0$, where U is the group of all elements

$$u = \begin{bmatrix} 1 & u_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -u_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & u_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and $\psi_{c_1,c_2}(u) = \psi(c_1u_1 + c_2u_2)$. Whether π is generic does not depend on the choice of c_1 and c_2. Assume π is generic. Consider the space of functions $W : \mathrm{GSp}(4,F) \to \mathbb{C}$ such that $W(ug) = \psi_{c_1,c_2}(u)W(g)$ for u in U and g in $\mathrm{GSp}(4,F)$, and W is right invariant under some compact open subgroup of $\mathrm{GSp}(4,F)$. There exists a unique $\mathrm{GSp}(4,F)$ subspace $W(\pi,\psi_{c_1,c_2})$ of this space which is isomorphic to π [Rod]. This subspace is called the Whittaker model of π with respect to ψ_{c_1,c_2}. Fix Haar measures on $F^\times$ and F. Let $\mu : F^\times \to \mathbb{C}^\times$ be a continuous quasi-character. If W is in $W(\pi,\psi_{c_1,c_2})$, the Novodvorsky **zeta integral** associated to W and μ is

$$Z(s,W,\mu) = \int_{F^\times}\int_F W(\begin{bmatrix} y & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & x & 0 & 1 \end{bmatrix})\mu(y)|y|^{s-3/2}\,\mathrm{d}x\,\mathrm{d}^\times y.$$

The $Z(s,W,\mu)$ for W in $W(\pi,\psi_{c_1,c_2})$ converge absolutely in some right half plane and are elements of $\mathbb{C}(q^{-s})$. There exists $\gamma(s,\pi,\mu,\psi_{c_1,c_2})$ in $\mathbb{C}(q^{-s})$ such that the following **functional equation**

$$Z(1-s,\pi(\begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix})W,(\omega_\pi\mu)^{-1}) = \gamma(s,\pi,\mu,\psi_{c_1,c_2})Z(s,W,\mu)$$

holds for W in $W(\pi, \psi_{c_1,c_2})$. This γ-factor does not depend on the choices of Haar measure on F and $F^\times$. Here,

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The $\mathbb{C}[q^s, q^{-s}]$ module generated by the $Z(s, W, \mu)$ for W in $W(\pi, \psi_{c_1,c_2})$ is a fractional ideal of $\mathbb{C}(q^{-s})$ with generator of the form $1/Q(q^{-s})$ with $Q(0) = 1$, where $Q(X)$ is in $\mathbb{C}[X]$. We define

$$L(s, \pi, \mu) = 1/Q(q^{-s}).$$

This L-factor does not depend on the choices of Haar measures or c_1 and c_2. We also define

$$\epsilon(s, \pi, \mu, \psi_{c_1,c_2}) = \gamma(s, \pi, \mu, \psi_{c_1,c_2}) \frac{L(s, \pi, \mu)}{L(1 - s, \pi, (\omega_\pi \mu)^{-1})}.$$

The function $\epsilon(s, \pi, \mu, \psi_{c_1,c_2})$ is a nonzero monomial in q^{-s} (e.g., see the top of p. 65 of [J]). The work [R2] verifies that $L(s, \pi, \mu) = L(s, \varphi, \mu)$, and $\epsilon(s, \pi, \mu, \psi_{1,-1}) = \epsilon(s, \varphi, \mu, \psi, \mathrm{d}x_\psi)$ for the generic element π in $\Pi(\chi, \tau)$ and $\varphi = \varphi(\chi, \tau)$, where $\Pi(\chi, \tau)$ and $\varphi(\chi, \tau)$ are the local L-packets and parameters defined in [R1]. We take $c_1 = 1$ and $c_2 = -1$ in the remainder of this paper, and write $W(\pi) = W(\pi, \psi_{1,-1})$, $\gamma(s, \pi, \mu) = \gamma(s, \pi, \mu, \psi_{1,-1})$ and $\epsilon(s, \pi, \mu) = \epsilon(s, \pi, \mu, \psi_{1,-1})$. If $\mu = 1$ we drop μ from our notation.

Next, we define the paramodular group of level $\mathcal{P}^n$. This requires that we first define the Klingen congruence subgroup of level $\mathcal{P}^n$. Let n be a nonnegative integer. The **Klingen congruence subgroup** $\mathrm{Kl}(\mathcal{P}^n)$ of level $\mathcal{P}^n$ is the subgroup of $\mathrm{GSp}(4, F)$ of all elements k such that $\lambda(k)$ is in $\mathcal{O}^\times$ and

$$k \in \begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{P}^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{P}^n & \mathcal{P}^n & \mathcal{O} & \mathcal{P}^n \\ \mathcal{P}^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{bmatrix}.$$

Define the **Atkin-Lehner element** of level $\mathcal{P}^n$ in $\mathrm{GSp}(4, F)$ to be

$$u_n = \begin{bmatrix} 0 & J \\ -\varpi^n J & 0 \end{bmatrix}.$$

Evidently, $u_n^2 = \varpi^n$ is in the center of $\mathrm{GSp}(4, F)$. We now define the **paramodular group** $\mathrm{K}(\mathcal{P}^n)$ of level $\mathcal{P}^n$ to be the subgroup of $\mathrm{GSp}(4, F)$

generated by $\mathrm{Kl}(\mathcal{P}^n)$ and $u_n \mathrm{Kl}(\mathcal{P}^n) u_n^{-1} = u_n^{-1} \mathrm{Kl}(\mathcal{P}^n) u_n$. Equivalently, $\mathrm{K}(\mathcal{P}^n)$ is the subgroup of $\mathrm{GSp}(4, F)$ of all elements k such that $\lambda(k)$ is in $\mathcal{O}^\times$ and

$$k \in \begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{P}^{-n} & \mathcal{O} \\ \mathcal{P}^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{P}^n & \mathcal{P}^n & \mathcal{O} & \mathcal{P}^n \\ \mathcal{P}^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{bmatrix}.$$

Conjecture 1.1 *Let π be a generic irreducible admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character. For each nonnegative integer n, let $\pi(\mathcal{P}^n)$ be the subspace of π of vectors fixed by $\mathrm{K}(\mathcal{P}^n)$.*

1. *For some nonnegative integer n the space $\pi(\mathcal{P}^n)$ is nonzero.*
2. *If N_π is the smallest n such that $\pi(\mathcal{P}^n)$ is nonzero, then*
$$\dim \pi(\mathcal{P}^{N_\pi}) = 1.$$
3. *There exists W_π in $\pi(\mathcal{P}^{N_\pi})$ such that*
$$Z(s, W_\pi) = L(s, \pi).$$

In (3) of the conjecture we use the Whittaker model $W(\pi)$ for π as defined above. If the conjecture holds for π, we call $\mathcal{P}^{N_\pi}$ the **level** of π and W_π the **new vector** of π.

The reader will note that while the conjecture is quite similar to the theory of new vectors for generic irreducible admissible representations of $\mathrm{GL}(2, F)$ with trivial central character, there is a significant difference: $\mathrm{K}(\mathcal{P}^n)$ is not contained in $\mathrm{K}(\mathcal{P}^{n+1})$! Thus, the theory of old vectors for $\mathrm{GSp}(4, F)$ will not be strictly analogous to that for $\mathrm{GL}(2, F)$. Nevertheless, we have some evidence, which we will not discuss here, that a coherent theory of old vectors for $\mathrm{GSp}(4, F)$ does exist.

2 A formal heuristic

Before stating implications for ϵ-factors and our evidence, we will give some formal motivation for the conjecture. As far as we know, there does not exist a conjectural conceptual theory of new vectors for representations of the F points of an arbitrary reductive algebraic group defined over F. The situation seems to be that, given a particular group like $\mathrm{GSp}(4)$, a theory of new vectors would be useful, but one has no reason to believe it exists. Groups for which new vectors have been considered include $\mathrm{GL}(n)$ (see [Cas], [D], [J-PS-S]) and SL(2) (see [LR]); for GSp(4) see also [S] for the case of square-free level. In our considerations we mostly have been guided by empirical facts. Still, for GSp(4) we can offer the following formal motivation.

Suppose one wants to derive the statement for a conjectural simple theory of new vectors for generic irreducible admissible representations of GSp(4) with trivial central character, and let π be one such representation. In π one might consider the space of Klingen vectors of level $\mathcal{P}^n$, i.e., the subspace $\pi_{\mathrm{Kl}}(\mathcal{P}^n)$ of vectors fixed by $\mathrm{Kl}(\mathcal{P}^n)$. Alternatively, one might consider vectors fixed by $\Gamma_0(\mathcal{P}^n)$, the Siegel congruence subgroup of level $\mathcal{P}^n$. However, without going into details, examples show that these vectors will not give a simple theory. One might hope, then, that Klingen vectors work, so that if N is the smallest n such that $\pi_{\mathrm{Kl}}(\mathcal{P}^n)$ is nonzero, then $\dim \pi_{\mathrm{Kl}}(\mathcal{P}^N) = 1$, and there exists a W in $\pi_{\mathrm{Kl}}(\mathcal{P}^N)$ such that $Z(s, W) = L(s, \pi)$. One might also hope, as a consequence, that $\epsilon(s, \pi) = cq^{-N(s-1/2)}$ for some constant c. Examples show, however, for the smallest n such that $\pi_{\mathrm{Kl}}(\mathcal{P}^n)$ is nonzero one can have $\dim \pi_{\mathrm{Kl}}(\mathcal{P}^n) > 1$: being a Klingen vector at the smallest nontrivial level is not enough to give uniqueness. It seems an enlargement of the Klingen congruence subgroup is required.

How can one arrive at such an enlargement? One might start with a Klingen vector W of level $\mathcal{P}^N$ for which $Z(s, W) = L(s, \pi)$ and $\epsilon(s, \pi) = cq^{-N(s-1/2)}$ and see if W reasonably might be fixed by a natural larger compact open subgroup. Using $Z(s, W) = L(s, \pi)$, the functional equation gives

$$\gamma(s, \pi)L(s, \pi) = Z(1 - s, \pi(\begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix})W).$$

Dividing by $L(1 - s, \pi)$, one obtains the ϵ-factor:

$$\epsilon(s, \pi) = Z(1 - s, \pi(\begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix})W)/L(1 - s, \pi).$$

Now $\epsilon(s, \pi) = cq^{-N(s-1/2)}$; how can one make the right hand side look like this? A bit of algebra yields

$$\epsilon(s, \pi) = \frac{Z(1 - s, \pi(u_N)W)}{L(1 - s, \pi)} \cdot q^{-N(s-1/2)}.$$

It follows that $Z(s, \pi(u_N)W)$ is a constant multiple of $L(s, \pi)$, or equivalently, $Z(s, \pi(u_N)W)$ is a constant multiple of $Z(s, W)$. What condition on W can guarantee this? It would hold if $\pi(u_N)W$ is a constant multiple of W; and if $\pi(u_N)W$ is a constant multiple of W, then $\pi(u_N)W$ is fixed by $\mathrm{Kl}(\mathcal{P}^N)$. Thus, one might consider, for nonnegative integers n, vectors W such that W and

u_nW are both fixed by $\mathrm{Kl}(\mathcal{P}^n)$, or equivalently, vectors fixed by $\mathrm{K}(\mathcal{P}^n)$. Note that if W is fixed by $\mathrm{Kl}(\mathcal{P}^n)$ then one has no reason to expect $\pi(u_n)W$ to also be fixed by $\mathrm{Kl}(\mathcal{P}^n)$, as u_n does not normalize $\mathrm{Kl}(\mathcal{P}^n)$. On the other hand, u_n does normalize the Borel congruence subgroup $B(\mathcal{P}^n) = \mathrm{Kl}(\mathcal{P}^n) \cap \Gamma_0(\mathcal{P}^n)$ of level $\mathcal{P}^n$, so if W is fixed by $\mathrm{Kl}(\mathcal{P}^n)$, then at least $\pi(u_n)W$ will be fixed by $\mathrm{B}(\mathcal{P}^n)$.

3 The connection to ϵ-factors

As mentioned in the introduction, the new vector and level of a representation satisfying the conjecture are closely connected to its ϵ-factor. This is useful in providing evidence for the conjecture.

Proposition 3.1 *Let π be a generic irreducible admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. Assume (1) and (2) of the conjecture for π hold. Then W_π is an eigenvector for $\pi(u_{N_\pi})$ with eigenvalue $\epsilon_\pi = \pm 1$:*

$$\pi(u_{N_\pi})W_\pi = \epsilon_\pi W_\pi.$$

Assume (3) of the conjecture for π also holds. Then

$$\epsilon(s,\pi) = \epsilon_\pi q^{-N_\pi(s-1/2)},$$

so that $\epsilon_\pi = \epsilon(1/2,\pi)$.

Proof. Assume (1) and (2) of the conjecture for π hold. A computation shows u_{N_π} normalizes $\mathrm{K}(\mathcal{P}^{N_\pi})$. This implies that $\pi(u_{N_\pi})W_\pi$ is in $\pi(\mathcal{P}^{N_\pi})$; since this space is one dimensional, $\pi(u_{N_\pi})W_\pi = \epsilon_\pi W_\pi$ for some $\epsilon_\pi \in \mathbb{C}^\times$. As $u_{N_\pi}^2 = \varpi^{N_\pi}$, and π has trivial central character, we have $\pi(u_{N_\pi})^2 = 1$, so that $\epsilon_\pi^2 = 1$. Next, assume (3) of the conjecture for π also holds. Applying the functional equation to W_π, we obtain

$$Z(1-s, \pi(\begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix})W_\pi) = \gamma(s,\pi) Z(s, W_\pi).$$

The definitions of the zeta integral and u_{N_π} imply

$$Z(1-s, \pi(\begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix})W_\pi) = \epsilon_\pi q^{-N_\pi(s-1/2)} Z(1-s, W_\pi).$$

Substituting this into the functional equation and using $Z(s,W_\pi) = L(s,\pi)$, we obtain

$$\epsilon_\pi q^{-N_\pi(s-1/2)} L(1-s,\pi) = \gamma(s,\pi) L(s,\pi),$$

so that $\epsilon(s,\pi) = \epsilon_\pi q^{-N_\pi(s-1/2)}$. □

This proposition can be used to supply evidence for the conjecture. For example, suppose π is a generic irreducible admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, and parts (1) and (2) of the conjecture for π are known. To obtain evidence for (3) of the conjecture for π we may proceed as follows. Suppose that it is believed that a certain L-parameter φ is the L-parameter associated to π via the conjectural local Langlands correspondence, so that it is believed that $\epsilon(s,\varphi,\psi,\mathrm{d}x_\psi) = \epsilon(s,\pi)$ (or even suppose this equality is known). Then, in light of Proposition 3.1, verifying

$$\epsilon(1/2,\varphi,\psi,\mathrm{d}x_\psi) = \epsilon_\pi q^{-N_\pi(s-1/2)}$$

gives evidence that (3) of the conjecture for π holds.

4 Evidence

We currently have three different pieces of evidence for the conjecture. Our evidence considers a wide variety of generic irreducible admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character, and includes all representations of lower level and several families of induced and supercuspidal representations.

To state the first piece of evidence, define the **Iwahori subgroup** I of $\mathrm{GSp}(4,F)$ to be the subgroup of all k in $\mathrm{GSp}(4,F)$ with $\lambda(k)$ in $\mathcal{O}^\times$ and

$$k \in \begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{P} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{P} & \mathcal{P} & \mathcal{O} & \mathcal{P} \\ \mathcal{P} & \mathcal{P} & \mathcal{O} & \mathcal{O} \end{bmatrix}.$$

Then we have the following theorem. The number ϵ_π is defined in Proposition 3.1.

Theorem 4.1 *Parts (1) and (2) of the conjecture are true for all generic irreducible admissible representations of* $\mathrm{GSp}(4,F)$ *with trivial central character which contain a nonzero vector fixed by the Iwahori subgroup. Moreover, suppose* π *is a generic irreducible admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character which contains a nonzero vector fixed by the Iwahori subgroup, and let* φ *be the* L*-parameter associated to* π *by [KL]. Then*

$$\epsilon(1/2,\varphi,\psi,\mathrm{d}x_\psi) = \epsilon_\pi q^{-N_\pi(s-1/2)},$$

which gives evidence that (3) of the conjecture for π *holds, as explained in section 3.*

In fact, we have computed the spaces of vectors fixed by $\mathrm{K}(\mathcal{P}^0)$, $\mathrm{K}(\mathcal{P}^1)$, $\mathrm{K}(\mathcal{P}^2)$ and $\mathrm{K}(\mathcal{P}^3)$ in all the, possibly nongeneric, irreducible admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character which contain a nonzero vector fixed by the Iwahori subgroup. This information is displayed in the table in the next section, which also includes information on how to understand the table. It is interesting to observe that (1) and (2) of the conjecture and $\epsilon(1/2,\varphi,\psi,\mathrm{d}x_\psi) = \epsilon_\pi q^{-N_\pi(s-1/2)}$ hold, with one exception, for all irreducible admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character which contain a nonzero vector fixed by the Iwahori subgroup. This exception is the representation VIb, which does not admit a nonzero vector fixed by $\mathrm{K}(\mathcal{P}^0)$, $\mathrm{K}(\mathcal{P}^1)$, $\mathrm{K}(\mathcal{P}^2)$ or $\mathrm{K}(\mathcal{P}^3)$; we would expect a nonzero vector fixed by $\mathrm{K}(\mathcal{P}^2)$. However, the representations VIa and VIb form an L-packet, and the conjecture holds for the representation VIa. This suggests that (1) and (2) of the conjecture and the equality $\epsilon(1/2,\varphi,\psi,\mathrm{d}x_\psi) = \epsilon_\pi q^{-N_\pi(s-1/2)}$ may be true for all irreducible admissible representations of $\mathrm{GSp}(4,F)$ with trivial central character at the level of L-packets.

Our second parcel of evidence concerns certain induced representations. For the representations considered in the following theorem there is a naturally associated L-parameter φ, which should be the L-parameter associated to π by the conjectural local Langlands conjecture; define the nonnegative integer N by $\epsilon(s,\varphi,\psi,\mathrm{d}x_\psi) = cq^{-N(s-1/2)}$, where c is a constant. We use the notation of [ST] for induced representations.

Theorem 4.2 *Let τ be a generic irreducible admissible representation of $\mathrm{GL}(2,F)$. Assume ω_τ is unramified.*

1. *(Siegel parabolic) Let σ be an unramified quasi-character of $F^\times$ such that $\omega_\tau\sigma^2 = 1$. Assume*
$$\pi = \tau \rtimes \sigma$$
is irreducible. Then π is a generic irreducible admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, and (1) and (2) of the conjecture for π are true. Moreover, $N_\pi = N$.

2. *(Klingen parabolic) Assume*
$$\pi = \omega_\tau^{-1} \rtimes \tau$$
is irreducible. Then π is a generic irreducible admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, and (1) and (2) of the conjecture for π are true. Moreover, $N_\pi = N$.

Our final piece of evidence considers a broad distribution of representations of $\mathrm{GSp}(4, F)$, including supercuspidals. Recall that [R1] proved an analogue for GSp(4) of the global Langlands-Tunnell theorem. In doing so, [R1] defined certain local L-packets of representations of $\mathrm{GSp}(4, F)$. Let $\Pi(\tau) = \Pi(1, \tau)$ be such a local L-packet which happens to occur in a global situation as in Theorem 8.6 of [R1]. Thus, in particular, τ is a tempered generic irreducible admissible representation of $\mathrm{GL}(2, E)$ with trivial central character, where E is either a quadratic extension of F, or $E = F \times F$. The packet $\Pi(\tau)$ has one or two elements, and all elements are tempered irreducible admissible representations of $\mathrm{GSp}(4, F)$ with trivial central character. In [R2] it is shown that exactly one element π of $\Pi(\tau)$ is generic. The paper [R1] also associates to τ an L-parameter $\varphi(\tau) = \varphi(1, \tau)$, and Theorem 8.6 of [R1] provides evidence that $\Pi(\tau)$ is the L-packet associated to $\varphi(\tau)$ by the conjectural local Langlands correspondence for $\mathrm{GSp}(4, F)$. Again, define the nonnegative integer N by $\epsilon(s, \varphi(\tau), \psi, \mathrm{d}x_\psi) = cq^{-N(s-1/2)}$, where c is a constant.

Theorem 4.3 *Assume q is odd. If E is unramified or $E = F \times F$, then π contains a vector W fixed by $\mathrm{K}(\mathcal{P}^N)$ such that $Z(s, W) = L(s, \pi)$.*

In writing $Z(s, W) = L(s, \pi)$ we are, as in the conjecture, using the Whittaker model $W(\pi)$ defined in section 1.

5 The table

The table gives information relevant to the conjecture about all the irreducible admissible representations of $\mathrm{GSp}(4, F)$ with trivial central character which contain a nonzero vector fixed by the Iwahori subgroup.

The first column

By [Bo], an irreducible admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character contains a nonzero vector fixed by I if and only if it is an irreducible subquotient of a representation of $\mathrm{GSp}(4, F)$ with trivial central character induced from an unramified quasi-character of the Borel subgroup. The basic reference on representations of $\mathrm{GSp}(4, F)$ induced from a quasi-character of the Borel subgroup is section 3 of [ST], and we will use the notation of that paper. Thus, St is the Steinberg representation, $\mathbf{1}$ is the trivial representation, and $\nu = |\cdot|$. The reader will have to consult [ST] for more details. It is also useful to consult section 4.1 of [T-B]. Let χ_1, χ_2 and σ be unramified quasi-characters of $F^\times$ with $\chi_1\chi_2\sigma^2 = 1$, so that the representation $\chi_1 \times \chi_2 \rtimes \sigma$ of $\mathrm{GSp}(4, F)$ induced from the quasi-character $\chi_1 \otimes \chi_2 \otimes \sigma$ has trivial central character. Of course, $\chi_1 \times \chi_2 \rtimes \sigma$ may be reducible. It turns out that by section 3 of [ST], there are six types of $\chi_1 \times \chi_2 \rtimes \sigma$ such that every irreducible admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character which contains a nonzero vector fixed by I is an irreducible subquotient of a representative of one of these six types, and that no two representatives of two different types share a common irreducible subquotient. The first column gives the name of the type. In the table we choose a representative for a type with the notation as below, and in subsequent columns we give information about the irreducible subquotients of that representative. The types are described as follows:

Type I

These are the $\chi_1 \times \chi_2 \rtimes \sigma$ where χ_1, χ_2 and σ are unramified quasi-characters of $F^\times$ such that $\chi_1\chi_2\sigma^2 = 1$ and $\chi_1 \times \chi_2 \rtimes \sigma$ is irreducible. See Lemma 3.2 of [ST].

Type II

These are the $\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$ where χ and σ are unramified quasi-characters of $F^\times$ such that $\chi^2\sigma^2 = 1$. See Lemmas 3.3 and 3.7 of [ST].

Type III

These are the $\chi \times \nu \rtimes \nu^{-1/2}\sigma$ where χ and σ are unramified quasi-characters of $F^\times$ such that $\chi\sigma^2 = 1$. See Lemmas 3.4 and 3.9 of [ST].

Type IV

These are the $\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$ where σ is an unramified quasi-character of $F^\times$ such that $\sigma^2 = 1$. See Lemma 3.5 of [ST].

Type V

These are the $\nu\xi_0 \times \xi_0 \rtimes \nu^{-1/2}\sigma$ where ξ_0 and σ are unramified quasi-characters of $F^\times$ such that ξ_0 has order two and $\sigma^2 = 1$. See Lemma 3.6 of [ST].

Type VI

These are the $\nu \times 1 \rtimes \nu^{-1/2}\sigma$ where σ is an unramified quasi-character of $F^\times$ such that $\sigma^2 = 1$. See Lemma 3.8 of [ST].

The second column

Choose a type as in the first column, and choose a representative $\chi_1 \times \chi_2 \rtimes \sigma$ of that type. Then $\chi_1 \times \chi_2 \rtimes \sigma$ admits a finite number of irreducible subquotients, and this number depends only on the type of $\chi_1 \times \chi_2 \rtimes \sigma$. We index the irreducible subquotients by lower case Roman letters. The letter "a" is reserved for the generic irreducible subquotient.

The third column

This column lists the irreducible subquotients of the representative of the type of the first column. We use the specific notation as in the discussion of the first column.

The fourth column

Suppose π is an entry of the third column, and let φ be the L-parameter associated to π by [KL]. We define N by the equation

$$\epsilon(s, \varphi, \psi, \mathrm{d}x_\psi) = cq^{-N(s-1/2)},$$

where c is a constant.

The fifth column

Using the notation of the explanation of the fourth column, this is $\epsilon = c = \epsilon(1/2, \varphi, \psi, \mathrm{d}x_\psi)$.

The sixth, seventh, eighth and ninth columns

The numbers in the columns give the dimensions of the $\mathrm{K}(\mathcal{P}^n)$ fixed vectors for the representations in the third column for $n = 0, 1, 2$ and 3. Note that to save space we have abbreviated $\mathrm{K}(\mathcal{P}^n)$ by $\mathrm{K}(n)$. The signs under the numbers indicate how these spaces of $\mathrm{K}(\mathcal{P}^n)$ fixed vectors split under the action of the Atkin-Lehner operator $\pi(u_n)$. The signs are correct if in the type II case, where the central character of π is $\chi^2\sigma^2$, the character $\chi\sigma$ is trivial, and in the type IV, V, and IV cases, where the central character of π is σ^2, the character σ is trivial. If these assumptions are not met, then the plus and minus signs must be interchanged to obtain the correct signs.

		representation	N	ϵ	$K(0)$	$K(1)$	$K(2)$	$K(3)$
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irred.)	0	1	$\mathbf{1}$ $+$	2 $+-$	4 $+++-$	6 $+++$ $---$
II	a	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	1	$-\sigma\chi(\varpi)$	0	$\mathbf{1}$ $-$	2 $+-$	4 $++--$
	b	$\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	0	1	$\mathbf{1}$ $+$	1 $+$	2 $++$	2 $++$
III	a	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GL}(2)}$	2	1	0	0	$\mathbf{1}$ $+$	2 $+-$
	b	$\chi \rtimes \sigma \mathbf{1}_{\mathrm{GL}(2)}$	0	1	$\mathbf{1}$ $+$	2 $+-$	3 $++-$	4 $++--$
IV	a	$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$	3	$-\sigma(\varpi)$	0	0	0	$\mathbf{1}$ $-$
	b	$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GL}(2)})$	2	1	0	0	$\mathbf{1}$ $+$	1 $+$
	c	$L(\nu^{\frac{3}{2}} \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-\frac{3}{2}}\sigma)$	1	$-\sigma(\varpi)$	0	$\mathbf{1}$ $-$	2 $+-$	3 $+--$
	d	$\sigma \mathbf{1}_{\mathrm{GSp}(4)}$	0	1	$\mathbf{1}$ $+$	1 $+$	1 $+$	1 $+$
V	a	$\delta([\xi_0, \nu\xi_0], \nu^{-\frac{1}{2}}\sigma)$	2	-1	0	0	$\mathbf{1}$ $-$	2 $+-$
	b	$L(\nu^{\frac{1}{2}}\xi_0 \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-\frac{1}{2}}\sigma)$	1	$\sigma(\varpi)$	0	$\mathbf{1}$ $+$	1 $+$	2 $++$
	c	$L(\nu^{\frac{1}{2}}\xi_0 \mathrm{St}_{\mathrm{GL}(2)}, \xi_0\nu^{\frac{1}{2}}\sigma)$	1	$-\sigma(\varpi)$	0	$\mathbf{1}$ $-$	1 $+$	2 $--$
	d	$L(\nu\xi_0, \xi_0 \rtimes \nu^{-\frac{1}{2}}\sigma)$	0	1	$\mathbf{1}$ $+$	0	1 $+$	0
VI	a	$\tau(S, \nu^{-\frac{1}{2}}\sigma)$	2	1	0	0	$\mathbf{1}$ $+$	2 $+-$
	b	$\tau(T, \nu^{-\frac{1}{2}}\sigma)$	2	1	0	0	0	0
	c	$L(\nu^{\frac{1}{2}} \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-\frac{1}{2}}\sigma)$	1	$-\sigma(\varpi)$	0	$\mathbf{1}$ $-$	1 $-$	2 $--$
	d	$L(\nu, \mathbf{1}_{F^\times} \rtimes \nu^{-\frac{1}{2}}\sigma)$	0	1	$\mathbf{1}$ $+$	1 $+$	2 $++$	2 $++$

Table 1: Representations containing a nonzero vector fixed by the Iwahori subgroup. Consult section 5 for definitions and comments.

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Supercuspidal Representations Attached to Symmetric Spaces

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§**1. Some motivation.**– The purpose of this lecture is to survey some recent results related to harmonic analysis on $H\backslash G$, where (G,H) is a symmetric space over a nonarchimedean local field. Harmonic analysis on symmetric spaces over $\mathbb{R}$ and $\mathbb{C}$ has been developed extensively by many authors over many years. By contrast, the p-adic theory is relatively undeveloped and new.

The impetus for much of the research in this field has come from Jacquet's relative trace formulas (starting with **[15]**) which were designed to study those automorphic representations of a given adelic group which satisfy a specific period condition. Without going into details about the global theory and what we mean by a "period condition," suffice it to say that the set of automorphic representations associated to a period condition tends to be an important set for a variety of reasons. For example, it may be the image of an important (automorphic or theta) lifting. It may be set of representations for which a certain automorphic L-function has a pole. It may be the set which determines when an induced representation is irreducible. Or it may be all of these things (and some other things as well). The original point of developing the local theory was that it described which representations could arise as local components of automorphic representations satisfying a period condition.

At first, most of the results in this area involved a combination of known techniques from: (a) the theory of harmonic analysis on p-adic groups, (b) global theory, and (c) the archimedean theory of symmetric spaces. Recently, more innovative techniques have been developed and we are seeing phenomena which have no archimedean analogues. I have been especially interested in finding techniques which exploit the special features of supercuspidal representations. Below I will indicate various local applications which are similar to the global applications mentioned above.

§**2. Basic concepts.**– We start by recalling the notion of a "symmetric space over a nonarchimedean field." Let F be a finite extension of some p-adic field $\mathbb{Q}_p$. For simplicity, we assume p is odd. Assume $\mathbf{G}$ is a connected reductive group over a field F and let $G = \mathbf{G}(F)$. Assume τ is an automorphism of $\mathbf{G}$ of order 2 which is defined over F. Let $\mathbf{G}^\tau$ denote the group of fixed points of τ and let $(\mathbf{G}^\tau)^\circ$ be the identity component of $\mathbf{G}^\tau$. Assume $\mathbf{H}$ is an F-subgroup of $\mathbf{G}$ such that $(\mathbf{G}^\tau)^\circ \subset \mathbf{H} \subset \mathbf{G}^\tau$. Now let $H = \mathbf{H}(F)$. Then the pair (G,H) (or the quotient $H\backslash G$) is called a *symmetric space over* F.

The terminology *harmonic analysis on* $H\backslash G$ may mean different things to different people. Classically, one might think of the decomposition of $L^2(H\backslash G)$ or some other induced representation $\mathrm{Ind}_H^G(1)$. For our purposes, it is appropriate to take $\mathrm{Ind}_H^G(1)$ to be the space $C^\infty(H\backslash G)$ of smooth (that is, locally constant) functions on $H\backslash G$.

Suppose $\pi : G \to \mathrm{Aut}(V)$ is an irreducible, admissible complex representation of G. Then we say π is H-*distinguished* if it occurs in $\mathrm{Ind}_H^G(1)$ in the sense that $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(1))$ is nonzero. A specific embedding $\Lambda : \pi \hookrightarrow \mathrm{Ind}_H^G(1)$ will be called an H-*model for* π.

Frobenius Reciprocity gives a canonical bijection between $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(1))$ and the space $\mathrm{Hom}_H(\pi, 1)$ of linear forms $\lambda : V \to \mathbb{C}$ satisfying $\lambda(\pi(h)v) = \lambda(v)$, for all $h \in H$ and $v \in V$. Such linear forms λ are called *H-invariant functionals.* The explicit relation between Λ and λ is $\Lambda(v)(g) = \lambda(\pi(g)v)$, where $g \in G$ and $v \in V$.

The relation between H-models and H-invariant functionals is entirely analogous to the relation between Whittaker models and Whittaker functionals. One can hope for an analogue of the uniqueness property of Whittaker models in the symmetric space setting.

Definition. *We say that (G, H) has the multiplicity one property (or is a Gelfand pair) if* $\dim \mathit{Hom}_H(\pi, 1) \leq 1$ *for all irreducible, admissible representations π.*

Note that not everyone uses the terminology "Gelfand pair" in this way.

Definition. *We say (G, H) is a geometric Gelfand pair if there exists an anti-automorphism σ of G of order two such that $Hg^\sigma H = HgH$ for all $g \in G$.*

The Gelfand/Kazhdan Lemma [6]. *If there exists an anti-automorphism σ of G of order two which fixes all bi-H-invariant distributions on G then (G, H) is a Gelfand pair.*

The problem with this result is that, in principle, one needs to study all of the bi-H-invariant distributions on G in order to satisfy the hypotheses of the lemma. However, if (G, H) is a geometric Gelfand pair then the hypothese are automatically satisfied and hence we have the following:

Corollary. *If (G, H) is a geometric Gelfand pair then it is a Gelfand pair.*

§**3. The example** $(GL(n, E), GL(n, F))$**.–** Assume E is a quadratic extension of F and use the notation $x \mapsto \bar{x}$ for the nontrivial Galois automorphism of E/F. We consider the pair (G, H), with $G = GL(n, E)$ and $H = GL(n, F)$. This is a symmetric space over F. If $g \in G$ let $\bar{g}$ be the matrix obtained by applying $x \mapsto \bar{x}$ to each entry of g. Then τ is an automorphism of G of order two and H is the group of fixed points. It is easy to show $H\bar{g}^{-1}H = HgH$, for all $g \in G$. Hence, (G, H) is a geometric Gelfand pair.

The prototype example is the case in which $n = 2$ which I studied in my Ph.D. thesis and in some subsequent papers motivated by the work of Jacquet/Lai [**15**] and Harder/Langlands/Rapoport [**13**]. Flicker [**2**] generalized some of these results for arbitrary n. In some cases, he arrived at the appropriate conjectures relating distinguishedness with base change from unitary groups and the existence of a pole for the Asai L-function (a.k.a., twisted tensor L-function). For $n = 2$, there are two base change maps from $U(2, E/F)$ to $GL(n, E)$, each characterized by character relations analogous to Shintani's character relations which characterize quadratic base change for $GL(2)$. Flicker showed that the H-distinguished representations of G are precisely the representations which unstable lifts from $U(2, E/F)$. We also note that representations which are base change lifts from $U(2, E/F)$ are characterized by the symmetry condition $\tilde{\pi} \simeq \bar{\pi}$, where $\bar{\pi}(g) = \pi(\bar{g})$. The connection with Asai L-functions for general n has recently been firmly established in unpublished work of Kable [**17**] and, independently, Anandavardhanan and Tandon [**1**]. Their work builds on [**13**] and results developed by Flicker in several papers (starting with [**3**]).

A natural problem, which we will call the "classification problem," is to explicitly determine which irreducible, admissible representations of G are H-distinguished. Assume for a moment longer that $n = 2$. For the nonsupercuspidal representations, it is fairly easy to give explicit conditions on the inducing data for these representations which correspond to distinguishedness. This was probably first done by Clozel in unpublished notes. (See **[2]**, **[4]** and **[9]** for more details.) For supercuspidal representations, a characterization of distinguishedness in terms of Jacquet-Langlands ϵ-factors was given in **[9]**:

Proposition 1 [9]. *Let ψ be a nontrivial character of E which is trivial on F. Then an irreducible, supercuspidal representation π of $G = GL(2, E)$ is H-distinguished if and only if $\epsilon(1/2, \pi \otimes \chi, \psi) = 1$ for all quasicharacters χ of $E^\times$ which are trivial on $F^\times$.*

The result in **[9]** is stated only under the assumption that the central character of π is trivial, however, this assumption is totally unnecessary. Note that the criterion in Proposition 1 is closely related to Corollary 2.4 in Saito's paper **[24]** on Tunnell's formula.

According to the work of Howe **[14]** (in the tame case) and Kutzko (in general), the supercuspidal representations of G may be realized via compactly supported induction from compact-mod-center subgroups. To give a satisfactory solution to the classification problem for distinguished supercuspidal representations requires giving conditions on the inducing data which corresponds to distinguishedness. This is partially done in the tame case for general n in **[12]**. (Note that if $p > n$ then all representations are tame.) According to Howe's construction, each irreducible tame supercuspidal representation π of G corresponds to a certain equivalence class of quasicharacters $\chi : L^\times \to \mathbb{C}^\times$ where L is a degree n tamely ramified extension of E. The quasicharacter χ must be E-admissible in the sense of Kutzko. If $\tilde{\pi} \simeq \bar{\pi}$, as is the case whenever π is H-distinguished, then it is a basic fact that there must exist an automorphism σ of order two of L/F such that $\sigma(x) = \bar{x}$ for all $x \in E$ and $\chi^{-1} = \chi \circ \sigma$. Let L' be the fixed field of σ. We say that the pair $(L/E, \sigma)$ is *odd* if the ramification degree $e(L/E)$ is odd, L/L' is unramified and E/F is ramified. Otherwise, $(L/E, \sigma)$ is *even*. Let $\chi_{_{L/L'}}$ and $\chi_{_{E/F}}$ be the class field theory characters associated to L/L' and E/F, respectively. The following result was proved in collaboration with Fiona Murnaghan:

Theorem 2 [12]. *Assume and $\chi = \chi^{-1} \circ \sigma$ is an E-admissible character of $L^\times$ and π is the associated irreducible, tame supercuspidal representation of G such that $\tilde{\pi} \simeq \pi \circ \tau$. If $(L/E, \sigma)$ is even and $\chi | L'^\times = 1$ or if $(E/F, \sigma)$ is odd and $\chi | L'^\times = \chi_{_{L/L'}}$ then π is H-distinguished. If π is not H-distinguished and χ' is a character of $E^\times$ such that $\chi^L | L'^\times = \chi_{_{L/L'}}$ then $\pi \otimes \chi'$ is H-distinguished. Such characters χ' always exist, for example, one may take any character of $E^\times$ whose restriction to $F^\times$ is $\chi_{_{E/F}}$.*

A closely related result in the case in which E/F is unramified was obtained by Dipendra Prasad **[22]** by totally different methods.

Murnaghan's initial interest in such problems resulted from her joint work with Repka **[21]** on the reduciblity of induced representations of unitary groups, following the approach of Goldberg **[7]** and Shahidi **[25]**. Roughly speaking, G may be embedded as a Levi component of a maximal parabolic subgroup of the quasisplit unitary group $U(2n, E/F)$. If π is an irreducible, admissible representation of G then there is an associated induced representation $I(\pi)$ of $U(2n, E/F)$. Then $I(\pi)$ is irreducible if and only if π is H-distinguished.

When $n = 2$, this is evident in the work of Kazuko Konno [**18**], where all of the non-supercuspidal representations of the unitary group are computed.

The H-distinguished representations of G also arise in connection with the generic packet conjecture for unitary groups. A relative trace formula approach to this problem is developed for $n = 3$ in [**5**]. An alternate approach to the generic packet conjecture is given by Takuya Konno [**19**].

§**4. The example** $(GL(n), U(n))$.– Let E/F be a quadratic extension and $G = GL(n, E)$, as in the previous example. Now fix $\eta \in G$ which is hermitian in the sense that ${}^t\eta = \bar{\eta}$. Let $H = \{h \in G : h\eta{}^t\bar{h} = \eta\}$ be the associated unitary group. One may expect that (G, H) is Gelfand pair, since the analogous pair over a finite field is. Unfortunately, it is not a Gelfand pair, though we will see that it comes very close.

Theorem 3 [11]. *If π is an irreducible, tame supercuspidal representation of G then the dimension of $Hom_H(\pi, 1)$ is at most one.*

Again, it is natural to ask whether distinguishedness can be characterized in terms of a simple condition on the inducing data. We have:

Theorem 4 [11]. *Let L be a tamely ramified degree n extension of E which is embedded, via an E-embedding, in the ring M of n-by-n matrices with entries in E. Assume that ι is the automorphism of M given by applying the nontrivial Galois automorphism of E/F to the entries of each matrix in M. Let $G = M^\times = GL(n, E)$ and $T = L^\times$. Suppose χ is an admissible character of T and let π be the irreducible, supercuspidal representation of G associated to χ by Howe's construction. Let H be a unitary group in G associated to some hermitian matrix $\eta \in G$. Then the following conditions are equivalent:*

i. The space $Hom_H(\pi, 1)$ is nonzero.
ii. $\pi \sim \pi \circ \iota$.
iii. π is a base change lift from $GL(n, F)$.
iv. There exists an automorphism σ of L which agrees with ι on E and satisfies $\theta = \theta \circ \sigma$.
v. θ is trivial $U(1, L/L')$, where L' is the fixed field of an automorphism of L of order two which agrees with ι on E.

The method we use to solve the classification problem for tame supercuspidal representations for $(GL(n), U(n))$ has worked, with some modifications, for other pairs (G, H), as well. Using Jiu-Kang Yu's building theoretic extension [**26**] of Howe's construction, we hope to extend our methods to essentially arbitrary pairs (G, H).

The situation for $(GL(n), U(n))$ motivates the following:

Definition. *A pair (G, H) is a supercuspidal Gelfand pair if $\dim Hom_H(\pi, 1) \leq 1$ for all irreducible supercuspidal representations π of G.*

Fiona Murnaghan has recently found some examples of symmetric spaces which are not supercuspidal Gelfand pairs. Before this, there was a general suspicion that such pairs might not exist.

§**5. The example** $(GL(n), GL(n/2) \times GL(n/2))$.– Assume $n = 2m$ is even and let $G = GL(n, F)$, where we write the elements of G as block matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, with $a, b, c, d \in$

$M(m,F)$. Let $H \cong GL(m,F)\times GL(m,F)$ be the subgroup of G consisting of block diagonal matrices. Jacquet and Rallis **[16]** have shown in this case that (G,H) is a Gelfand pair. However, since (G,H) is not a geometric Gelfand pair, it was necessary for Jacquet and Rallis to conduct a very difficult 50-page analysis of the bi-H-invariant distributions on G in order to show that the hypotheses of the Gelfand/Kazhdan Lemma are satisfied.

We have the following block matrix identity:

$$\begin{pmatrix} bd^{-1}c-a & 0 \\ 0 & d-ca^{-1}b \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} -a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which is only valid when a and d are invertible. This shows that $Hg^{-1}H = HgH$ for almost all $g \in G$.

Definition. *(G,H) is almost a Gelfand pair if there exists an anti-automorphism σ of order two such that $Hg^{\sigma}H = HgH$, for almost all $g \in G$.*

Theorem 5 [10]. *Suppose α is an automorphism of order two of G such that $Hg^{\alpha}H = Hg^{-1}H$ for almost all $g \in G$. If π is an irreducible, H-distinguished supercuspidal representation of G then the contragredient $\tilde{\pi}$ of π is equivalent to the representation $\pi^{\alpha}(g) = \pi(g^{\alpha})$ and $\dim Hom_H(\pi,1) = \dim Hom_H(\tilde{\pi},1) = 1$.*

Corollary. *If (G,H) is almost a Gelfand pair then it must be a supercuspidal Gelfand pair.*

So, for $(GL(n), GL(n/2)\times GL(n/2))$, this reduces Jacquet/Rallis' lengthy argument to the above matrix identity. Of course, Jacquet/Rallis' result applies to arbitrary irreducible, admissible representations and not just supercuspidal representations. We will discuss some of the ingredients in the proof in the next section.

In the present context, Murnaghan and I **[12]** have an analogue of Theorem 2 which gives a weak solution to the classification problem. Since it is rather technical to state, we will not state it here.

We remark that distinguishedness may be correlated to the existence of a pole of the exterior square L-function, in much the same way that distinguishedness for $(GL(n,E), GL(n,F))$ is related to the existence of a pole of the Asai L-function. There also is a relation with reducibility of induced representations of classical groups and it is well known that the self-contragredient representations are expected to be lifts from classical groups. We refer to **[12]** for details and references for these things.

§**6. Character theory and the proof of Theorem 5.**– If V is the space of π and $\widetilde{V}$ is the space of $\tilde{\pi}$, then we note that V embeds in the space $\widetilde{V}^*$ of linear forms on $\widetilde{V}$. In particular, $v \in V$ corresponds to the linear form $v \mapsto \langle v, -\rangle$ on $\widetilde{V}$. The pairing $\langle -,-\rangle$ is the natural pairing on $V\times\widetilde{V}$ and it extends in an obvious way to a pairing on $(\widetilde{V}^*\times\widetilde{V})\cup(V\times V^*)$. The elements of $\widetilde{V}^*$ are sometimes referred to as "generalized vectors" associated to π. Similarly, V^* is the space of generalized vectors for $\tilde{\pi}$. If $f\in C_c^{\infty}(G)$ and $\lambda\in\widetilde{V}^*$ then we may define $\pi(f)\lambda\in\widetilde{V}^*$ by

$$\langle \pi(f)\lambda, \tilde{v}\rangle = \langle \lambda, \tilde{\pi}(\check{f})\tilde{v}\rangle,$$

where $\check{f}(g) = f(g^{-1})$ and $\tilde{v} \in \widetilde{V}$. In fact, $\pi(f)\lambda$ lies in V. Consequently, given generalized vectors $\lambda \in \widetilde{V}^*$ and $\tilde{\lambda} \in V^*$ there is an associated distribution

$$\Theta_{\lambda,\tilde{\lambda}}(f) = \langle \pi(f)\lambda, \tilde{\lambda} \rangle.$$

It is natural to refer to such distributions as *generalized matrix coefficients* because they generalize the matrix coefficients $f_{v,\tilde{v}}(g) = \langle \pi(g)v, \tilde{v} \rangle$, where $g \in G$, $v \in V$ and $\tilde{v} \in \widetilde{V}$.

For harmonic analysis on $H\backslash G$, the generalized matrix coefficients of most interest are the coefficients $\Theta_{\lambda,\tilde{\lambda}}$ for which $\lambda \in \mathrm{Hom}_H(\tilde{\pi}, 1)$ and $\tilde{\lambda} \in \mathrm{Hom}_H(\pi, 1)$. We call these *spherical matrix coefficients*.

If (G, H) is a Gelfand pair and π and $\tilde{\pi}$ are distinguished then, up to scalar multiples, there is a unique nonzero spherical matrix coefficient of π. This spherical matrix coefficient should be viewed as a symmetric space analogue of the character distributuion $\mathrm{tr}\pi(f)$ of π. One can ask whether these objects enjoy the same analytic properties (such as local integrability and smoothness on the regular set) established for the character distributions by Harish-Chandra (using various results of Howe). Indeed this is the case for pairs of the form $(\mathbf{H}(E), \mathbf{H}(F))$, where $\mathbf{H}$ is a connected reductive F-group and E/F is quadratic. (See [**8**]) However, Rader and Rallis [**23**] have studied this problem for general pairs (G, H) and they have shown the precise extent to which Harish-Chandra's results fail to generlize nicely.

Let us now give a sketch of the formal argument which underlies the proof of the theorem. For the sake of convenience and to simplify our exposition, we now assume that G has trivial center. Assume π is supercuspidal, as in the hypothesis of the theorem. Note that if $f_{v,\tilde{v}}$ is a matrix coefficient of π then, since π is supercuspidal, we have $f_{v,\tilde{v}} \in C_c^\infty(G)$. In addition, $\check{f}_{v,\tilde{v}} = f_{\tilde{v},v}$ is a matrix coefficient of $\tilde{\pi}$. So if π is a supercuspidal H-distinguished representation of G with spherical matrix coefficient $\Theta_{\lambda,\tilde{\lambda}}$ and if $f_{\tilde{v},v}$ is a matrix coefficient of $\tilde{\pi}$ then the quantity $\Theta_{\lambda,\tilde{\lambda}}(f_{\tilde{v},v})$ is well defined. A straightforward generalization of the Schur orthogonality relations shows that

$$\Theta_{\lambda,\tilde{\lambda}}(f_{\tilde{v},v}) = d(\pi)^{-1} \langle \lambda, \tilde{v} \rangle \langle v, \tilde{\lambda} \rangle,$$

where $d(\pi)$ is the formal degree of π.

Unfortunately, $\Theta_{\lambda,\tilde{\lambda}}$ is not a true matrix coefficient, however, it may be realized, in a suitable sense, as a limit of matrix coefficients $f_{w_n,\tilde{w}_n}$. For the moment, in order to provide a formal heuristic, we will pretend that $\Theta_{\lambda,\tilde{\lambda}}$ coincides with a matrix coefficient $f_{w,\tilde{w}}$, where w and $\tilde{w}$ are H-fixed vectors. To legitimize this heuristic, one must engage in various technical manipulations involving approximations of $\Theta_{\lambda,\tilde{\lambda}}$ by matrix coefficients.

Proceeding formally, we now let $\varphi = f_{w,\tilde{w}} f_{\tilde{v},v} \in C_c^\infty(G)$. Rader and Rallis have produced a symmetric space analogue of the Weyl integration formula which formally looks like:

$$\int_G \varphi(g)\, dg = \sum_T \frac{1}{w_T} \int |\Delta(t)|^{1/2} f_{w,\tilde{w}}(t) \Phi^T_{f_{\tilde{v},v}}(t)\, dt,$$

where: (i) we are summing over classes of "Cartan subsets" T of $H\backslash G$, (ii) Δ is a symmetric space analogue of the Weyl discriminant, and (iii) $\Phi^T_{f_{\tilde{v},v}}(t)$ is a type of orbital integral of

$f_{\tilde{v},v}(t)$ which represents an average over the double coset HtH. So we have a fundamental identity

$$d(\pi)^{-1}\langle\lambda,\tilde{v}\rangle\langle v,\tilde{\lambda}\rangle = \sum_T \frac{1}{w_T}\int |\Delta(t)|^{1/2} f_{w,\tilde{w}}(t)\Phi^T_{f_{\tilde{v},v}}(t)\;dt.$$

This identity, though we have obtained it by dubious means, is actually valid if $f_{w,\tilde{w}}$ is interpreted as the smooth function, given by Rader and Rallis, which represents $\Theta_{\lambda,\tilde{\lambda}}$ on the (G,H)-regular set.

Now let σ be the anti-involution $g^\sigma = (g^\alpha)^{-1}$, where α is as in the hypothesis of the theorem. We observe that $f_{\tilde{v},v}(g^\sigma) = \langle v,\tilde{\pi}(g^\sigma)\tilde{v}\rangle = \langle\pi(g^\alpha)v,\tilde{v}\rangle$ is a matrix coefficient of $\pi^\alpha(g) = \pi(g^\alpha)$. Since

$$d(\pi)^{-1}\langle\lambda,\tilde{v}\rangle\langle v,\tilde{\lambda}\rangle = \int_G \varphi(g)\;dg = \int_G \varphi(g^\sigma)\;dg$$

is nonzero for suitable v and $\tilde{v}$ and since this is an average of a matrix coefficient of π against a matrix coefficient of π^α, Schur orthogonality implies that π must be equivalent to $\tilde{\pi}^\alpha$. Thus we may choose a nonzero intertwining operator $I : V \to \widetilde{V}$ such that $I(\pi(g)v) = \tilde{\pi}^\alpha(g)I(v)$ for all $g\in G$ and $v\in V$. Consequently,

$$f_{\tilde{v},v}(g^\sigma) = \langle\pi(g^\alpha)v,\tilde{v}\rangle = \langle I^{-1}(\tilde{v}),\tilde{\pi}(g)I(v)\rangle = f_{I(v),I^{-1}(\tilde{v})}(g).$$

It follows that

$$\int_G \varphi(g^\sigma)\;dg = d(\pi)^{-1}\langle\lambda,I(v)\rangle\langle I^{-1}(\tilde{v}),\tilde{\lambda}\rangle.$$

This yields the identity

$$\langle\lambda,\tilde{v}\rangle\langle v,\tilde{\lambda}\rangle = \langle\lambda,I(v)\rangle\langle I^{-1}(\tilde{v}),\tilde{\lambda}\rangle.$$

The theorem follows immediately from this identity, though this may not be obvious. Indeed, fix $\tilde{v}$ such that $\langle\lambda,\tilde{v}\rangle\neq 0$. Since we know that v may be chosen so that $\langle v,\tilde{\lambda}\rangle\neq 0$, we see that $\langle I^{-1}(\tilde{v}),\tilde{\lambda}\rangle\neq 0$. Now letting v vary, we deduce that $I(\ker\tilde{\lambda}) = \ker\lambda$. This seems to contradict the fact that λ and $\tilde{\lambda}$ were chosen independently. The only explanation of this is that both $\mathrm{Hom}_H(\pi,1)$ and $\mathrm{Hom}_H(\tilde{\pi},1)$ have dimension one and thus we essentially have no choice when choosing λ and $\tilde{\lambda}$. This completes the formal argument. The precise details of the proof of the theorem are in **[10]**.

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ON FUNCTORIALITY OF ZELEVINSKI INVOLUTIONS

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Let F be a p-adic field and G a connected reductive algebraic group defined over F. For simplicity, we assume that G is quasi-split. We denote by W_F the Weil group of F. Let ${}^LG = \hat{G} \rtimes W_F$ be the L-group of G. We denote by $\mathcal{L}^G$ the set of standard Levi subgroups of G. For $M \in \mathcal{L}^G$, we denote by $r(M)$ the semisimple split F-rank of M. Let $\Pi(G)$ be the set of equivalence classes of irreducible admissible representations of $G(F)$ and $\mathbb{C}[\Pi(G)]$ the space of virtual characters of $G(F)$. The parabolic induction defines a homomorphism $i_M^G : \mathbb{C}[\Pi(M)] \longrightarrow \mathbb{C}[\Pi(G)]$ and the (normalized) Jacquet functor defines a homomorphism $r_M^G : \mathbb{C}[\Pi(G)] \longrightarrow \mathbb{C}[\Pi(M)]$. Following S. Kato [11], we define the Zelevinski involution $\mathbf{D}_G$ by

$$\mathbf{D}_G = \sum_{M \in \mathcal{L}^G} (-1)^{r(M)} i_M^G \circ r_M^G.$$

Let $\{M\}$ be the set of associate standard Levi subgroups of M. We say that $\pi \in \Pi(G)$ is of type $\{M_\pi\}$ if $r_{M_\pi}^G(\pi)$ is a non-zero linear combination of supercuspidal representations of $M_\pi(F)$. We put $r_\pi = r(M_\pi)$. For $\pi \in \Pi(G)$, we define

$$\mathbf{d}_G(\pi) = (-1)^{r_\pi} \mathbf{D}_G(\pi).$$

A.-M. Aubert [4, 5] proved that $\mathbf{d}_G(\pi)$ is irreducible. Thus the Zelevinski involution preserves the irreducibility. It seems natural to consider the relation between the Zelevinski involution and the conjectural Langlands functoriality. Nevertheless the Zelevinski involution does not preserve the L-packets. We consider the A-packets conjectured by J. Arthur [3, Conjecture 6.1]. For a Langlands parameter $\phi : W_F \times SU_2(\mathbb{C}) \longrightarrow {}^LG$, we denote by $\Pi_\phi(G)$ the corresponding conjectural L-packet. Although $SU_2(\mathbb{C})$ is isomorphic to $SL_2(\mathbb{C})$, we denote the second factor of this group by $SU_2(\mathbb{C})$ in order to distinguish it from the factor $SL_2(\mathbb{C})$ used to define the Arthur parameters in [3]. Let

$$\psi : W_F \times SU_2(\mathbb{C}) \times SL_2(\mathbb{C}) \longrightarrow {}^LG$$

be an Arthur parameter of G. We put

$$S_\psi = \mathrm{Cent}(\psi, \hat{G}),$$
$$\mathbb{S}_\psi = S_\psi / S_\psi^0 \cdot Z_{\hat{G}}^\Gamma,$$

where S_ψ^0 is the identity component of S_ψ and $Z_{\hat{G}}^\Gamma$ is the subgroup of the center $Z_{\hat{G}}$ of $\hat{G}$ consisting of the elements fixed by $\Gamma = \mathrm{Gal}(\overline{F}/F)$.

Let $\Pi_\psi(G)$ be the conjectural A-packet of ψ and $\Pi_{\phi_\psi}(G)$ the L-packet corresponding to ψ. We fix Whittaker data χ of $G(F)$. This determines the base point $\pi_\chi \in \Pi_{\phi_\psi}(G)$ as in [3, §6]. For $\overline{s} \in \mathbb{S}_\psi$ and $\pi \in \Pi_\psi(G)$, we define $\langle \overline{s}, \pi | \pi_\chi \rangle$ as in [3, Conjecture 6.1]. Then it is conjectured that $\langle \cdot\, , \pi | \pi_\chi \rangle$ is an irreducible character of $\mathbb{S}_\psi$. We say that a virtual character $\theta \in \mathbb{C}[\Pi(G)]$ is stable if θ is stable as a distribution on $G(F)$. Let $\mathbb{C}[\Pi(G)]^{st}$ be the space of stable virtual characters of $G(F)$ and $\mathbb{C}[\Pi_\psi(G)]$ the subspace of $\mathbb{C}[\Pi(G)]$ generated by $\Pi_\psi(G)$. We put $\mathbb{C}[\Pi_\psi(G)]^{st} = \mathbb{C}[\Pi(G)]^{st} \cap \mathbb{C}[\Pi_\psi(G)]$. As F is a p-adic field, the following hypothesis is believed.

Hypothesis 1. *The map*

$$\pi \in \Pi_\psi(G) \longrightarrow \langle \cdot\, , \pi | \pi_\chi \rangle \in \Pi(\mathbb{S}_\psi)$$

is injective, where $\Pi(\mathbb{S}_\psi)$ is the set of irreducible characters of $\mathbb{S}_\psi$, and

$$\dim \mathbb{C}[\Pi_\psi(G)]^{st} = 1.$$

In this article, we assume the Arthur conjecture [3, Conjecture 6.1] and this hypothesis.

Now we turn to the Zelevinski involution. We identify $SU_2(\mathbb{C})$ with $SL_2(\mathbb{C})$ and define $d(\psi)$ by

$$d(\psi)(w \times t \times u) = \psi(w \times u \times t),$$
$$w \times t \times u \in W_F \times SU_2(\mathbb{C}) \times SL_2(\mathbb{C}).$$

Then $d(\psi)$ is an Arthur parameter of G constructed from ψ by interchanging the role of $SU_2(\mathbb{C})$ and $SL_2(\mathbb{C})$.

Conjecture 2.

$$\mathbf{d}_G(\Pi_\psi(G)) = \Pi_{d(\psi)}(G).$$

Since $S_\psi = S_{d(\psi)}$, we may identify $\mathbb{S}_\psi$ with $\mathbb{S}_{d(\psi)}$. We denote the base point in $\Pi_{\phi_{d(\psi)}}(G)$ by $\pi_{d,\chi}$.

Conjecture 3. *There exists a one-dimensional character μ of $\mathbb{S}_\psi$ which satisfies*

$$\langle \overline{s}, \mathbf{d}_G(\pi) | \pi_{d,\chi} \rangle = \mu(\overline{s}) \langle \overline{s}, \pi | \pi_\chi \rangle,$$

for all $\overline{s} \in \mathbb{S}_\psi$.

If $\mathbb{S}_\psi = \{1\}$, then $\Pi_\psi(G) = \{\pi_\chi\}$ and $\Pi_{d(\psi)}(G) = \{\pi_{d,\chi}\}$. The following conjecture is a special case of Conjecture 2 .

Conjecture 4. *If ψ satisfies $\mathbb{S}_\psi = \{1\}$, then*

$$\mathbf{d}_G(\pi_\chi) = \pi_{d,\chi}.$$

In general, nevertheless, $\mathbf{d}_G(\pi_\chi)$ may not be equivalent to $\pi_{d,\chi}$. If $G = SL_2$ and if ψ corresponds to an induced representation of G which is a direct sum of two irreducible tempered representations, then $\mathbf{d}_G$ interchanges these two representations. Thus $\mathbf{d}_G(\pi_\chi) \neq \pi_{d,\chi}$.

In the case that $G = GL_n$, Conjecture 2 follows from the results of C. Moeglin and J.-L. Waldspurger [20]. Recently, K. Konno and T. Konno have checked that Conjecture 2 is compatible with their candidates for the A-packets of $G = U(2,2)$.

Conjecture 3 implies that the Zelevinski involutions behave well under the endoscopic transfers. Thus it turns our attention to the relation between the Zelevinski involutions and the endoscopic transfers. Since $i_M^G(\mathbb{C}[\Pi(M)]^{st}) \subset \mathbb{C}[\Pi(G)]^{st}$ and $r_M^G(\mathbb{C}[\Pi(G)]^{st}) \subset \mathbb{C}[\Pi(M)]^{st}$, we have

$$\mathbf{D}_G(\mathbb{C}[\Pi(G)]^{st}) = \mathbb{C}[\Pi(G)]^{st}.$$

Let $(\mathcal{H}, H, s, \xi)$ be (standard) endoscopic data. For the sake of brevity, we assume that $\mathcal{H} \cong {}^L H$. Unfortunately the existence of the endoscopic transfer is still hypothetical. In this article, to define the endoscopic transfer of virtual characters, *we assume the fundamental lemma for groups* [1, Hypothesis 3.1] *and for Lie algebras* [21, Conjecture 1.3]. Let

$$\mathrm{Tran}_H^G : \mathbb{C}[\Pi(H)]^{st} \longrightarrow \mathbb{C}[\Pi(G)]$$

be the endoscopic transfer from H to G. Let A_0 (resp. $A_{H,0}$) be a maximal split torus of G (resp. H). We put $a(G) = \dim(A_0)$ and $a(H) = \dim(A_{H,0})$. Then we have the following theorem.

Theorem 5. *Assume the fundamental lemma for groups and for Lie algebras. Then we have*

$$\mathbf{D}_G \circ \mathrm{Tran}_H^G = (-1)^{a(G)-a(H)} \mathrm{Tran}_H^G \circ \mathbf{D}_H.$$

By using this theorem, we can reduce Conjecture 2 to Conjecture 4. Moreover, we can show that Conjecture 4 implies the following formula;

$$\langle \overline{s}, \mathbf{d}_G(\pi) | \pi_{d,\chi} \rangle = \langle \overline{s}, \mathbf{d}_G(\pi_\chi) | \pi_{d,\chi} \rangle \langle \overline{s}, \pi | \pi_\chi \rangle,$$

where $\langle \cdot, \mathbf{d}_G(\pi_\chi) | \pi_{d,\chi} \rangle$ is a one-dimensional character of $\mathbb{S}_\psi$. This is Conjecture 3.

To prove Theorem 5, we show some properties of the double cosets of the Weyl groups (a generalization of [7, Proposition 2.7.7]) and an analogue of the geometric lemma [6, Lemma 2.12].

We fix an F-splitting $(B_0, T_0, \{X_\alpha\})$ of G, an F-splitting $(B_{H,0}, T_{H,0}, \{Y_\alpha\})$ of H, a Γ-splitting $(\mathcal{B}, \mathcal{T}, \{\mathcal{X}_{\check{\alpha}}\})$ of $\hat{G}$ and a Γ-splitting $(\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{Y}_{\check{\alpha}}\})$ of $\hat{H}$. Then we may identify $\hat{T}_0$ (resp. $\hat{T}_{H,0}$) with $\mathcal{T}$ (resp. $\mathcal{T}_H$). We may assume that $A_0 \subset T_0$ and that $A_{H,0} \subset T_{H,0}$. We say that a subtorus of A_0 is standard if it is equal to the split component of the center of a standard Levi subgroup of G. We assume that $s \in \mathcal{T}$, $\xi(\mathcal{T}_H) = \mathcal{T}$ and $\xi(\mathcal{B}_H) \subset \mathcal{B}$. Let $i_0 : T_{H,0} \longrightarrow T_0$ be the dual homomorphism of $\xi^{-1} : \mathcal{T} \longrightarrow \mathcal{T}_H$. We may assume that $i_0(A_{H,0})$ is a standard subtorus of A_0. We identify $A_{H,0}$ with the image $i_0(A_{H,0})$ in A_0. Put $M_H = \mathrm{Cent}(A_{H,0}, G)$.

We discuss the properties of the double cosets of the Weyl groups with respect to the endoscopic groups. Let

$$\Omega(G) = \mathrm{Norm}(A_0, G)/\,\mathrm{Cent}(A_0, G),$$
$$\Omega(H) = \mathrm{Norm}(A_{H,0}, H)/\,\mathrm{Cent}(A_{H,0}, H),$$

be the Weyl groups. We denote the set of roots of (G, A_0) (resp. $(H, A_{H,0})$) by $R(G) = R(G, A_0)$ (resp. $R(H) = R(H, A_{H,0})$). For $\omega_H \in \Omega(H)$, there exists a unique $\omega_G \in \Omega(G)$ which satisfies the following three conditions.

1) $\omega_G(A_{H,0}) = A_{H,0}$,
2) $\omega_G|_{A_{H,0}} = \omega_H$,
3) $\omega_G(R^+(M_H)) > 0$.

By identifying ω_H with ω_G, we may regard $\Omega(H)$ as a subgroup of $\Omega(G)$. For $M \in \mathcal{L}^G$, we put

$$\Omega(G)_{M,H} = \{\omega \in \Omega(G) |\, \omega(A_{H,0}) \supset A_M\},$$

where A_M is the split component of the center of M. We also put

$$\tilde{D}_M = \{\omega \in (\Omega(G)_{M,H})^{-1} |\, \omega(R^+(M)) > 0\}.$$

Let $\alpha \in R^+(H)$ and $\omega \in (\tilde{D}_M)^{-1}$. Choose $\tilde{\alpha} \in R^+(G)$ whose restriction to $A_{H,0}$ is α. We say that $\omega\alpha$ *is positive* (and write $\omega\alpha > 0$) if $\omega\tilde{\alpha}$ is contained in $R^+(G)$. It is not hard to show that the positivity of $\omega\alpha$ does not depend on the choice of $\tilde{\alpha}$. We define $D_{M,H}$ by

$$D_{M,H} = \{\omega \in (\tilde{D}_M)^{-1} |\, \omega(R^+(H)) > 0\}.$$

Lemma 6. *(1) The set $D_{M,H}$ is a system of representatives for*

$$\Omega(M)\backslash\Omega(G)_{M,H}/\Omega(H).$$

(2) For $\omega \in D_{M,H}$, put

$$M_\omega = \mathrm{Cent}((\omega \circ i_0)^{-1}(A_M), H),$$

then M_ω is a standard Levi subgroup of H.

For $L \in \mathcal{L}^H$, we put

$$D_{M,H,L} = \{\omega \in D_{M,H} |\, M_\omega = L\}$$

and

$$a_{M,H,L} = \sharp D_{M,H,L}.$$

Then we have the following formula, which is a generalization of [7, Proposition 2.7.7].

Proposition 7.

$$\sum_{M \in \mathcal{L}^G} (-1)^{r(M)} a_{M,H,L} = (-1)^{a(G)-a(H)} \cdot (-1)^{r(L)}.$$

Let ${}^LM_\omega$ be the L-group of M_ω. Then we may regard ${}^LM_\omega$ as a subgroup of ${}^LH$. Since G is quasi-split, we may regard $\Omega(G)$ as a subgroup of $\Omega(G, T_0)$. The choice of the splittings defines an isomorphism $\Omega(G, T_0) \longrightarrow \Omega(\hat{G}, \mathcal{T})$. We choose a representative $\hat{n}_\omega \in \mathrm{Norm}(\mathcal{T}, \hat{G})$ of

$$\omega \in \Omega(G) \subset \Omega(G, T_0) \cong \Omega(\hat{G}, \mathcal{T}).$$

We put $s_\omega = \mathrm{Int}\, \hat{n}_\omega(s)$ and $\xi_\omega = \mathrm{Int}\, \hat{n}_\omega \circ \xi$. Then $({}^LM_\omega, M_\omega, s_\omega, \xi_\omega)$ is endoscopic data of M. We choose absolute transfer factors of these endoscopic data and choose Haar measures of standard Levi subgroups and tori suitably. The following formula is an analogue of the formula of Bernstein–Zelevinski [6, Lemma 2.12].

Proposition 8. *Assume the fundamental lemma for groups and for Lie algebras. Then we have*

$$r_M^G \circ \mathrm{Tran}_H^G = \sum_{\omega \in D_{M,H}} \mathrm{Tran}_{M_\omega}^M \circ r_{M_\omega}^H.$$

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CAP automorphic representations of low rank groups *

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Abstract

In this talk, I report my recent joint work with K. Konno on non-tempered automorphic representations on low rank groups [KK]. We obtain a fairly complete classification of such automorphic representations for the quasisplit unitary groups in four variables.

1 CAP forms

The term CAP in the title is a short hand for the phrase "Cuspidal but Associated to Parabolic subgroups". This is the name given by Piatetski-Shapiro [PS83] to those cuspidal automorphic representations which apparently contradict the generalized Ramanujan conjecture. More precisely, let G be a connected reductive group defined over a number field F, and G^* be its quasisplit inner form. We write $\mathbb{A} = \mathbb{A}_F$ for the adéle ring of F. An irreducible cuspidal representation $\pi = \bigotimes_v \pi_v$ is a *CAP form* if there exists a residual discrete automorphic representation $\pi^* = \bigotimes_v \pi_v^*$ such that, at all but finite number of v, π_v and π_v^* share the same absolute values of Hecke eigenvalues.

It is a consequence of the result of Jacquet-Shalika [JS81a], [JS81b] and Moeglin-Waldspurger [MW89] that there are no CAP forms on the general linear groups. On the other hand, for a central division algebra D of dimension n^2 over $F^\times$, the trivial representation of $D^\times(\mathbb{A})$ is clearly a CAP form which shares the same local component, at any place v where D is unramified, with the residual representation $\mathbf{1}_{GL(n,\mathbb{A})}$. On the other hand, a quasisplit unitary group $U_{E/F}(3)$ of 3-variables already have non-trivial CAP forms, which can be obtained as θ-lifts of some automorphic characters of $U_{E/F}(1)$ [GR90], [GR91]. But the first and the most well-known example of CAP forms are the analogues of the θ_{10} representation by Howe-Piatetski-Shapiro [Sou88] and the Saito-Kurokawa representations of Sp_4 [PS83]. Also Gan-Gurevich-Jiang obtained very interesting example

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of CAP forms on the split group of type G_2 [GGJ02] (see also the article by Gan in this volume).

In any case, the local components of CAP forms at almost all places are non-trivial Langlands quotients by definition, and hence non-tempered in an apparent way. To put such forms into the framework of Langlands' conjecture, J. Arthur proposed a series of conjectures [Art89]. The conjectural description is through the so-called *A-parameters*, homomorphisms ψ from the direct product of the hypothetical Langlands group $\mathcal{L}_F$ of F with $SL(2,\mathbb{C})$ to the L-group ${}^LG$ of G [Bor79]:

$$\psi : \mathcal{L}_F \times SL(2,\mathbb{C}) \longrightarrow {}^LG,$$

considered modulo $\widehat{G}$-conjugation. We write $\Psi(G)$ for the set of $\widehat{G}$-conjugacy classes of A-parameters for G. By restriction, we obtain the local component

$$\psi_v : \mathcal{L}_{F_v} \times SL(2,\mathbb{C}) \to {}^LG_v$$

of ψ at each place v. Here the local Langlands group $\mathcal{L}_{F_v}$ is defined in [Kot84, §12], and ${}^LG_v$ is the L-group of the scalar extension $G_v = G \otimes_F F_v$. The local conjecture, among other things, associates to each ψ_v a finite set $\Pi_{\psi_v}(G_v)$ of isomorphism classes of irreducible unitarizable representations of $G(F_v)$, called an *A-packet*. At all but finite number of v, $\Pi_{\psi_v}(G_v)$ is expected to contain a unique unramified element π_v^1. Using such elements, we can form the global A-packet associated to ψ

$$\Pi_\psi(G) := \left\{ \bigotimes_v \pi_v \;\middle|\; \begin{array}{ll} \text{(i)} & \pi_v \in \Pi_{\psi_v}(G_v),\ \forall v; \\ \text{(ii)} & \pi_v = \pi_v^1,\ \forall' v \end{array} \right\}.$$

Arthur's conjecture predicts the multiplicity of each element in $\Pi_\psi(G)$ in the discrete spectrum of the right regular representation of $G(\mathbb{A})$ on $L^2(G(F)\mathfrak{A}_G\backslash G(\mathbb{A}))$. Here $\mathfrak{A}_G$ is the maximal $\mathbb{R}$-vector subgroup in the center of the infinite component $G(\mathbb{A}_\infty)$ of $G(\mathbb{A})$.

We say an A-parameter ψ is of *CAP type* if

(i) ψ is *elliptic*. This is the condition for $\Pi_\psi(G)$ to contain an element which occurs in the discrete spectrum.

(ii) $\psi|_{SL(2,\mathbb{C})}$ is non-trivial.

According to the conjecture, the CAP automorphic representations of $G(\mathbb{A})$ is contained in some of the global A-packets associated to such A-parameters. In this talk, we shall classify the CAP forms by such parameters along the line of Arthur's conjecture, in the case of the quasisplit unitary group $U_{E/F}(4)$ of four variables. Although our description of such forms tells nothing about the character relations conjectured in [Art89], it is quite explicit and fairly complete. We hope to apply this to certain analysis of the cohomology of the Shimura variety attached to $GU_{E/F}(4)$.

2 Parameter consideration

Global case Take a quadratic extension E/F of number fields and write σ for the generator of the Galois group of this extension. Let $G = G_n := U_{E/F}(n)$ be the quasisplit unitary groups in n variables associated to E/F. Later we shall mainly be concerned with the case $n = 4$. The L-group ${}^LG$ is the semi-direct product of $\widehat{G} = GL(n, \mathbb{C})$ by the absolute Weil group W_F of F, where W_F acts through $W_F/W_E \simeq \mathrm{Gal}(E/F)$ by

$$\rho_G(\sigma)g = \mathrm{Ad}(I_n)^t g^{-1}, \quad I_n := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ (-1)^{n-1} & & & \end{pmatrix}.$$

Thus an A-parameter ψ for G is determined by its restriction to $\mathcal{L}_E \times SL(2, \mathbb{C})$, which is just a completely reducible representation:

$$\psi|_{\mathcal{L}_E \times SL(2,\mathbb{C})} = \bigoplus_{i=1}^{r} \varphi_{\Pi_i} \otimes \rho_{d_i}.$$

Here Π_i is an irreducible cuspidal representation of $GL(m_i, \mathbb{A}_E)$ enjoying the following properties:

- $\sigma(\Pi_i) := \Pi_i \circ \sigma$ is isomorphic to the contragredient $\Pi_i^\vee$.
- Its central character ω_{Π_i} restricted to $\mathbb{A}^\times$ equals $\omega_{E/F}^{n-d_i-m_i+1}$, where $\omega_{E/F}$ is the quadratic character associated to E/F by the classfield theory.
- Some condition on the order of its twisted Asai L-functions at $s = 1$.

ρ_d is the d-dimensional irreducible representation of $SL(2, \mathbb{C})$. We note that ψ is elliptic if and only if its irreducible components $\varphi_{\Pi_i} \otimes \rho_{d_i}$ are distinct to each other. The S-group

$$\mathcal{S}_\psi(G) := \pi_0(\mathrm{Cent}(\psi, \widehat{G})/Z(\widehat{G}))$$

is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{r-1}$, where $\pi_0(\bullet)$ stands for the group of connected components. This plays a central role in the conjectural multiplicity formula.

Local case Similar description for the A-packets of the unitary group $G = G_n$ associated to a quadratic extension E/F of local fields is also valid. For each A-parameter ψ, we have the associated non-tempered Langlands parameter

$$\phi_\psi : \mathcal{L}_F \ni w \longmapsto \psi\big(w, \begin{pmatrix} |w|_F^{1/2} & 0 \\ 0 & |w|_F^{-1/2} \end{pmatrix}\big) \in {}^LG.$$

Here the "absolute value" $|\ |_F$ on $\mathcal{L}_F$ is the composite $|\ |_F : \mathcal{L}_F \twoheadrightarrow W_F^{\mathrm{ab}} \xrightarrow[\sim]{\mathrm{rec}} F^\times \xrightarrow{|\ |_F} \mathbb{R}_+^\times$. (rec denotes the reciprocity map in the local classfield theory.) In Arthur's conjecture, it was imposed that the L-packet $\Pi_{\phi_\psi}(G)$ associated to ϕ_ψ should be contained in $\Pi_\psi(G)$. We also have the S-group $\mathcal{S}_\psi(G)$ as in the global case. We postulate the following:

Assumption 2.1. *There exists a bijection* $\Pi_\psi(G) \ni \pi \longmapsto (\bar{s} \mapsto \langle \bar{s}, \pi \rangle_\psi) \in \Pi(\mathcal{S}_\psi(G))$. *Here* $\Pi(\mathcal{S}_\psi(G))$ *is the set of isomorphism classes of irreducible representations of* $\mathcal{S}_\psi(G)$.

Now for $n = 4$, the possibilities of $\{(d_i, m_i)\}_i$ for elliptic A-parameters with non-trivial $SL(2, \mathbb{C})$-component are given as follows.

(1) Stable cases. $\{(4, 1)\}$, $\{(2, 2)\}$.

(2) Endoscopic cases.

   (a) $\{(3, 1), (1, 1)\}$;
   (b) $\{(2, 1), (1, 2)\}$;
   (c) $\{(2, 1), (2, 1)\}$;
   (d) $\{(2, 1), (1, 1), (1, 1)\}$.

In the cases (1), (2.a), it follows from Assumption 2.1 that $\Pi_{\phi_\psi}(G) = \Pi_\psi(G)$, and we know from [Kon98] that all the contribution of the corresponding global A-packets belong to the residual spectrum. On the other hand, $\Pi_\psi(G) \setminus \Pi_{\phi_\psi}(G)$ is expected to be non-empty in the rest cases. We shall use the local θ-correspondence to construct the missing members.

3 Local θ-correspondence

Local Howe duality First let us recall the local θ-correspondence. We consider an m-dimensional (non-degenerate) hermitian space $(V, (\ ,\ ))$ and n-dimensional skew-hermitian space $(W, \langle\ ,\ \rangle)$ over E. We write $G(V)$ and $G(W)$ for the unitary groups of V and W, respectively. If we define the symplectic space $(\mathbb{W}, \langle\langle\ ,\ \rangle\rangle)$ by

$$\mathbb{W} := V \otimes_E W, \quad \langle\langle v \otimes w, v' \otimes w' \rangle\rangle := \frac{1}{2}\mathrm{Tr}_{E/F}[(v, v')\sigma(\langle w, w' \rangle)],$$

Then $(G(V), G(W))$ form a so-called *dual reductive pair* in the symplectic group $Sp(\mathbb{W})$ of this symplectic space:

$$\iota_{V,W} : G(V) \times G(W) \ni (g, g') \longmapsto g \otimes g' \in Sp(\mathbb{W}).$$

Fixing a non-trivial character ψ_F of F, we have the metaplectic group of $\mathbb{W}$ which is a central extension

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow Mp_{\psi_F}(\mathbb{W}) \longrightarrow Sp(\mathbb{W}) \longrightarrow 1.$$

This admits a unique Weil representation ω_{ψ_F} on which $\mathbb{C}^1$ acts by the multiplication [RR93]. For each pair $\underline{\xi} = (\xi, \xi')$ of characters of $E^\times$ satisfying $\xi|_{F^\times} = \omega_{E/F}^n$, $\xi'|_{F^\times} = \omega_{E/F}^m$, we have the corresponding lifting $\widetilde{\iota}_{V,W,\underline{\xi}} : G(V) \times G(W) \to Mp_{\psi_F}(\mathbb{W})$ of $\iota_{V,W}$:

$$\begin{array}{ccc} G(V) \times G(W) & \xrightarrow{\widetilde{\iota}_{V,W,\underline{\xi}}} & Mp_{\psi_F}(\mathbb{W}) \\ \| & & \downarrow \\ G(V) \times G(W) & \xrightarrow{\iota_{V,W}} & Sp(\mathbb{W}) \end{array}$$

The composite $\omega_{V,W,\underline{\xi}} := \omega_\psi \circ \widetilde{\iota}_{V,W,\underline{\xi}}$ is the *Weil representation* of the dual reductive pair $(G(V), G(W))$ associated to $\underline{\xi}$. It is the product of the Weil representations $\omega_{W,\underline{\xi}}$ of $G(V)$ and $\omega_{V,\xi'}$ of $G(W)$.

We write $\mathscr{R}(G(V),\omega_{W,\xi})$ for the set of isomorphism classes of irreducible admissible representations of $G(V)$ which appear as quotients of $\omega_{W,\xi}$. For $\pi_V\in\mathscr{R}(G(V),\omega_{W,\xi})$, the maximal π_V-isotypic quotient of $\omega_{V,W,\underline{\xi}}$ is of the form $\pi_V\otimes\Theta_{\underline{\xi}}(\pi_V,W)$ for some smooth representation $\Theta_{\underline{\xi}}(\pi_V,W)$ of $G(W)$. Similarly we have $\mathscr{R}(G(W),\omega_{V,\xi'})$ and $\Theta_{\underline{\xi}}(\pi_W,V)$ for each $\pi_W\in\mathscr{R}(G(W),\omega_{V,\xi'})$. The local Howe duality conjecture, which was proved by R. Howe himself if F is archimedean [How89] and by Waldspurger if F is a non-archimedean local field of odd residual characteristic [Wal90], asserts the following:

(i) $\Theta_{\underline{\xi}}(\pi_V,W)$ (resp. $\Theta_{\underline{\xi}}(\pi_W,V)$) is an admissible representation of finite length of $G(W)$ (resp. $G(V)$), so that it admits an irreducible quotient.

(ii) Moreover its irreducible quotient $\theta_{\underline{\xi}}(\pi_V,W)$ (resp. $\theta_{\underline{\xi}}(\pi_W,V)$) is unique.

(iii) $\pi_V\mapsto\theta_{\underline{\xi}}(\pi_V,W)$, $\pi_W\mapsto\theta_{\underline{\xi}}(\pi_W,V)$ are bijections between $\mathscr{R}(G(V),\omega_{W,\xi})$ and $\mathscr{R}(G(W),\omega_{V,\xi'})$ converse to each other.

Adams' conjecture A link between the local θ-correspondence and A-packets is given by the following conjecture of J. Adams [Ada89]. Suppose $n\geq m$. Then we have an L-embedding $i_{V,W,\underline{\xi}}:{}^LG(V)\to{}^LG(W)$ given by

$$i_{V,W,\underline{\xi}}(g\rtimes w):=\begin{cases}\xi'\xi^{-1}(w)\begin{pmatrix}g & \\ & \mathbf{1}_{n-m}\end{pmatrix}\times w & \text{if } w\in W_E,\\ \begin{pmatrix} & g\\ J_{n-m}^{n-m-1} & \end{pmatrix}\rtimes w_\sigma & \text{if } w=w_\sigma,\end{cases}$$

where w_σ is a fixed element in $W_F\setminus W_E$ and

$$J_n:=\begin{pmatrix}1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & (-1)^{n-1}\end{pmatrix}$$

Let $\mathrm{T}:SL(2,\mathbb{C})\to\mathrm{Cent}(i_{V,W,\underline{\xi}},\widehat{G}(W))$ be the homomorphism which corresponds to a regular unipotent element in $\mathrm{Cent}(i_{V,W,\underline{\xi}},\widehat{G}(W))\simeq GL(n-m,\mathbb{C})$ (the *tail representation* of $SL(2,\mathbb{C})$). Using this, we define the θ-lifting of A-parameters by

$$\theta_{V,W,\underline{\xi}}:\Psi(G(V))\ni\psi\longmapsto(i_{V,W,\underline{\xi}}\circ\psi^\vee)\cdot T\in\Psi(G(W)).$$

Conjecture 3.1 ([Ada89] Conj.A). *The local θ-correspondence should be subordinated to the map of A-packets: $\Pi_\psi(G(V))\mapsto\Pi_{\theta_{V,W,\underline{\xi}}(\psi)}(G(W))$.*

Here we have said subordinated because $\mathscr{R}(G(V),\omega_{W,\xi})$ is not compatible with A-packets, that is, $\Pi_\psi(G(V))\cap\mathscr{R}(G(V),\omega_{W,\xi})$ is often strictly smaller than $\Pi_\psi(G(V))$. But when these two are assured to coincide, we can expect more:

Conjecture 3.2 ([Ada89] Conj.B). *For V, W in the stable range, that is, the Witt index of W is larger than m, we have*

$$\Pi_{\theta_{V,W,\underline{\xi}}(\psi)}(G(W)) = \bigcup_{V;\, \dim_E V=m} \theta_{\underline{\xi}}(\Pi_\psi(G(V)), W).$$

Now we note that our situation is precisely that of Conj. 3.2 with $m = 2$ and $W = V \oplus -V$. Moreover, we find that the A-parameters in the cases (2.b), (2.c), (2.d) in § 2 are exactly those of the form

$$\theta_{V,W,\underline{\xi}}(\psi), \quad \psi \in \Psi(G(V)).$$

ε-dichotomy We explain the construction of the A-packets when F is non-archimedean. We need one more ingredient.

Proposition 3.3 (ε-dichotomy). *Suppose $\dim_E V = 2$ and write W_1 for the hyperbolic skew-hermitian space $(E^2, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))$. Take an L-packet Π of $G_2(F) = G(W)$ and $\tau \in \Pi$ [Rog90, Ch.11].*
(i) $\tau \in \mathscr{R}(G(W), \omega_{V,\xi'})$ if and only if

$$\varepsilon(1/2, \Pi \times \xi\xi'^{-1}, \psi_F)\omega_\Pi(-1)\lambda(E/F, \psi_F)^{-2} = \omega_{E/F}(-\det V).$$

Here the ε-factor on the right hand side is the standard ε-factor for G_2 twisted by $\xi\xi'^{-1}$ defined by the Langlands-Shahidi theory *[Sha90]. ω_Π is the central character of the elements of Π and $\lambda(E/F, \psi_F)$ is Langlands' λ-factor [Lan70].*
(ii) If this is the case, we have $\theta_{\underline{\xi}}(\tau, V) = (\xi^{-1}\xi')_{G(V)}\tau_V^\vee$. Here $(\xi^{-1}\xi')_{G(V)}$ denotes the character of $G(V)$ given by the composite

$$G(V) \stackrel{\det}{\to} U_{E/F}(1, F) \ni z/\sigma(z) \mapsto \xi^{-1}\xi'(z) \in \mathbb{C}^\times.$$

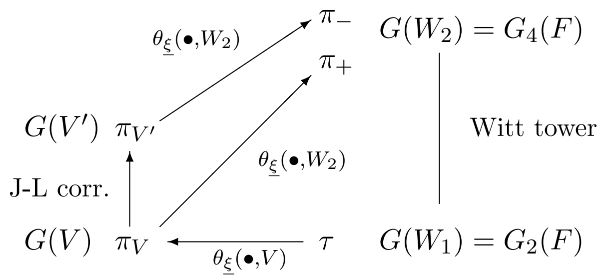
τ_V stands for the Jacquet-Langlands correspondent[1] of τ.

This is a special case of the ε-dichotomy of the local θ-correspondence for unitary groups over p-adic fields, which was proved for general unitary groups (at least for supercuspidal representations) in [HKS96]. But since we need to combine this with our description of the residual spectrum [Kon98], we have to use the Langlands-Shahidi ε-factors instead of Piatetski-Shapiro-Rallis's doubling ε-factors adopted by them. By this reason, we deduced this proposition from the analogous result for the unitary similitude groups [Har93] combined with the following description of the base change for G_2.

Lemma 3.4. *Let $\widetilde{\pi} = \omega \otimes \pi'$ be an irreducible admissible representation of the unitary similitude group $GU_{E/F}(2) \simeq (E^\times \times GL(2, F))/\Delta F^\times$, and write $\Pi(\widetilde{\pi})$ for the associated L-packet of $G_2(F)$ consisting of the irreducible components of $\widetilde{\pi}|_{G_2(F)}$. Then the standard base change of $\Pi(\widetilde{\pi})$ to $GL(2, E)$ [Rog90, 11.4] is given by $\omega(\det)\pi'_E$, where π'_E is the base change lift of π' to $GL(2, E)$ [Lan80].*

[1] In fact, the Jacquet-Langlands correspondence for unitary groups in two variables is defined only for L-packets and not for each member of the packets [LL79]. We know that $\tau \mapsto \tau_V$ certainly defines a bijection between Π and its Jacquet-Langlands correspondent. But we do not specify the bijection explicitly here. See Rem. 3.6 also.

Now we construct the A-packets. Our construction is summarized in the following picture.



Each A-parameter of our concern is of the form

$$\psi|_{\mathcal{L}_E \times SL(2,\mathbb{C})} = \psi_1|_{\mathcal{L}_E \times SL(2,\mathbb{C})} \oplus (\xi'\xi^{-1} \otimes \rho_2),$$

where ψ_1 is some A-parameter for G_2. Take $\tau \in \Pi_{\psi_1}(G_2)$ and let $(V, (\ ,\ ))$ be the 2-dimensional hermitian space such that the condition of Prop. 3.3 (i) holds. If we write $\pi_V := \theta_{\underline{\xi}}(\tau, V) \simeq (\xi\xi'^{-1})_{G(V)}\tau_V^\vee$, then the result of [Kud86] tells us $\pi_+ := \theta_{\underline{\xi}}(\pi_V, W_2)$, $(\tau \in \Pi_{\psi_1}(G_2))$ form the local residual L-packet $\Pi_{\phi_\psi}(G_4)$. We now suppose that there exists a Jacquet-Langlands corresondent $\pi_{V'} \simeq (\xi\xi'^{-1})_{G(V')}\tau_{V'}^\vee$ of π_V on the unitary group $G(V')$ of the other (isometry class of) 2-dimensional hermitian space. Then Prop. 3.3 (i) tells us that $\pi_{V'} \notin \mathscr{R}(G(V'), \omega_{W_1,\xi})$. Yet its local θ-lifting $\pi_- := \theta_{\underline{\xi}}(\pi_{V'}, W_2)$ to the larger group $G_4(F)$ still exists. This is the so-called *early lift* or the *first occurrence*. Following Conj. 3.2, we define

$$\Pi_\psi(G_4) := \{\pi_\pm \,|\, \tau \in \Pi_\psi(G_2)\}.$$

This gives sufficiently many members of the packet as predicted by Assumption 2.1.

Example 3.5. *(i) Suppose $\Pi_{\psi_1}(G_2)$ is an L-packet consisting of supercuspidal elements. For $\tau \in \Pi_{\psi_1}(G_2)$, π_+ is the Langlands quotient $J_{P_1}^{G_4}(\xi'\xi^{-1}|\ |_E^{1/2} \otimes \tau)$, where P_1 is a parabolic subgroup with the Levi factor $\mathrm{R}_{E/F}\mathbb{G}_m \times G_2$. On the other hand the early lift π_- of the supercuspidal τ is again supercuspidal. Thus $\Pi_\psi(G_4)$ consists of non-tempered members and supercuspidal elements.*
(ii) On the contrary, we take $\xi = \xi'$ and consider $\Pi_{\psi_1}(G_2)$ consists of either the Steinberg representation δ_{G_2} or the trivial representation $\mathbf{1}_{G_2}$.

- *δ_{G_2} lifts to $\pi_V = \mathbf{1}_{G(V)}$, where V is anisotropic. $\pi_{V'} = \delta_{G_2}$. $\pi_+ = J_{P_1}^{G_4}(|\ |_E^{1/2} \otimes \delta_{G_2})$ and π_- is an irreducible tempered but not square integrable representation.*

- *$\mathbf{1}_{G_2}$ lifts to $\pi_V = \mathbf{1}_{G(V)}$ but V is hyperbolic this time. $\pi_{V'}$ is again $\mathbf{1}_{G(V')}$ but this should be viewed as the Jacquet-Langlands correspondent of the A-packet $\{\mathbf{1}_{G(V)}\}$. We have $\pi_+ = J_{P_2}^{G_4}(I_{\mathbf{B}}^{GL(2)_E}(\mathbf{1} \otimes \mathbf{1})|\det|_E^{1/2})$, where P_2 is the so-called Siegel parabolic subgroup with the Levi factor $GL(2, E)$. Obviously $\pi_- = J_{P_1}^{G_4}(|\ |_E^{1/2} \otimes \delta_{G_2})$. This last representation is shared by the two packets considered here.*

Real case We end this section by some comments on the case $E/F = \mathbb{C}/\mathbb{R}$. Similar results are obtained by applying the argument of Adams-Barbasch [AB95]. In fact, the local θ-correspondence between unitary groups *of the same size* is described quite explicitly and in full generality in [Pau98]. Their argument also works in the present case. Let me explain some example.

We write $G_{p,q} = U(p,q)$. For a regular integral infinitesimal character $\lambda = (\lambda_1, \lambda_2)$ for $G_{1,1}$, consider the extended L-packet:

$$\Pi_\lambda = \{\delta_{1,1}^+, \delta_{1,1}^-, \delta_{2,0}, \delta_{0,2}\}$$

consisting of the discrete series representation of various $G_{p,q}$ with the infinitesimal character λ. The subscript p,q indicates that $\delta_{p,q}^\bullet$ lives on $G_{p,q}$. We can write $\xi'\xi^{-1}(z) = (z/\bar{z})^n$, $\forall z \in \mathbb{C}$ for some $n \in \mathbb{Z}$. An analogue of Prop. 3.3 in the real case asserts that the local θ-correspondence under the Weil representation $\omega_{V,W,\underline{\xi}}$ gives a bijection

$$\theta_{\underline{\xi}} : \Pi_\lambda \xrightarrow{\sim} \Pi_{n-\lambda},$$

where $n - \lambda = (n - \lambda_2, n - \lambda_1)$.

If λ is sufficiently regular, by which we mean $|\lambda_i - n| > 1$, then it is proved by J.-S. Li [Li90] that $\theta_{\underline{\xi}}(\theta_{\underline{\xi}}(\delta_{1,1}^\pm), W_2)$ is a non-tempered cohomological representation $A_{\mathfrak{q}}(\lambda')$, where the Levi factor of the θ-stable parabolic subalgebra $\mathfrak{q}$ is $\mathfrak{u}(1,1) \oplus \mathfrak{u}(1)^2$. As for the other elements $\delta_{p,q} \in \Pi_{n-\lambda}$, $\theta_{\underline{\xi}}(\delta_{p,q}, W_2)$ is a discrete series representation $A_{\mathfrak{q}}(\lambda')$. This time $\mathfrak{q}$ has the Levi factor $\mathfrak{u}(2) \oplus \mathfrak{u}(1)^2$. The resulting A-packet $\theta_{\underline{\xi}}(\Pi_{n-\lambda})$ is exactly the cohomological A-packet defined by Adams-Johnson [AJ87].

For the complete list of the packets both in the archimedean and non-archimedean case, see our paper [KK].

One can easily check that the S-groups in the cases (2.b), (2.c), (2.d) satisfy $\mathcal{S}_\psi(G_4) \simeq \mathcal{S}_{\psi_1}(G_2) \times \mathbb{Z}/2\mathbb{Z}$. Now we define the bijection in Assumption 2.1 by

- $\langle \bar{s}, \pi_\pm \rangle_\psi := \langle \bar{s}, \tau \rangle_{\psi_1}$ on $\bar{s} \in \mathcal{S}_{\psi_1}(G_2)$;
- $\langle\ ,\pi_\pm \rangle_\psi$ on $\mathbb{Z}/2\mathbb{Z}$ equals the sign character if π_- and trivial character otherwise.

For the other cases, only the first one in this definition is enough to give a complete bijection. This finishes our local task.

Remark 3.6. *In the above, we do not mention the definition of the pairing $\langle\ ,\ \rangle_{\psi_1}$. There are several choices for this, and we can choose one by fixing a non-trivial character ψ_F of F [LL79]. Also we did not specify the correspondence $\pi_V \mapsto \pi_{V'}$, which is again a subtle problem. In fact, we need to make a choice of (absolute) transfer factor as in [LL79] which again involves a choice of ψ_F (appearing in $\lambda(E/F, \psi_F)$ in the transfer factor). Using this specific transfer, we label the members of endoscopic L-packets of anisotropic unitary group. The correspondence $\pi_V \mapsto \pi_{V'}$ can be described in terms of these data, but we do not go into details here.*

4 Multiplicity formula

We now go back to the global situation where E/F is a quadratic extension of number fields. We note that there always exists a homomorphism $\mathcal{S}_\psi(G_4) \ni \bar{s} \mapsto \bar{s}(v) \in \mathcal{S}_{\psi_v}(G_{4,v})$. We can now state the main result of this talk. Although we treat only the number field case, we believe the result holds also over function fields of one variable over a finite field of odd characteristic.

Theorem 4.1. *Let ψ be an A-parameter of CAP type for $G_4 = U_{E/F}(4)$. As was explained in § 1, we form the global A- packet $\Pi_\psi(G_4) := \bigotimes_v \Pi_{\psi_v}(G_{4,v})$. Then the multiplicity $m(\pi)$ of $\pi = \bigotimes_v \pi_v \in \Pi_\psi(G_4)$ in $L^2(G(F)\backslash G(\mathbb{A}))$ is given by*

$$m(\pi) = \frac{1}{|\mathcal{S}_\psi(G_4)|} \sum_{\bar{s} \in \mathcal{S}_\psi(G_4)} \epsilon_\psi(\bar{s}) \prod_v \langle \bar{s}(v), \pi_v \rangle_{\psi_v},$$

where the sign character ϵ_ψ is defined by

$$\epsilon_\psi = \begin{cases} \mathrm{sgn}_{\mathcal{S}_\psi(G_4)} & \text{if } \psi_1 \text{ is a stable } L\text{-parameter and } \varepsilon(1/2, \psi_1 \otimes \xi\xi'^{-1}) = -1, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Here $\varepsilon(s, \psi_1 \otimes \xi\xi'^{-1})$ is the Artin root number attached to ψ_1, which equals the standard ε-function for $\Pi_{\psi_1}(G_2) \times \xi\xi'^{-1}$.

The proof divides into two parts. Our local construction together with the global θ-correspondence shows that the multiplicity is no less than the right hand side. Note that we also relies on the multiplicity formula of Labesse-Langlands for unitary groups in two variables [LL79], [Rog90]. Then we prove a characterization of the image of such θ-lifts by poles of certain L-functions, which gives the converse inequality. This also shows that all the CAP forms for $U_{E/F}(4)$ are obtained in the above as the contribution of the A-packets we constructed. In particular the A-packets contains the sufficiently many members at least for global purposes, so that our Assumption 2.1 is justified.

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Parabolic induction and parahoric induction

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1 Introduction

In the same way Eisenstein series theory is a masterpiece of the description of the automorphic spectrum, the so-called parabolic induction and restriction functors are prominent tools in the study of smooth representations of a p-adic group G. Given a parabolic subgroup P of G with Levi component M, we will note i_P^G and r_G^P respectively these functors. These are *a priori* functors between categories of all smooth representations of G and M, but it is well known that these functors restrict to (or respect) the subcategories of admissible, resp. finite length, smooth representations. And actually it is generally believed that only the latter category is relevant for automorphic applications. For example the first interesting question for someone interested in automorphic spectral problems is the study of reducibility (and of composition factors) of representations of G parabolically induced from irreducible ones of M, especially when the latter are local components of some automorphic representation. On this question we will say almost nothing.

But among all automorphic aspects, especially thinking to the links with Galois representations, is the study of congruences between automorphic forms as in the pioneering works of Serre and Ribet. This leads naturally to studying not only *complex* but *finite fields-valued* and even *ring-valued* smooth representations. For example one might be interested in studying stable $\overline{\mathbb{Z}}_l$-lattices in $\overline{\mathbb{Q}}_l$-representations. In this respect, the most prominent work is that of Vignéras for GL_n : she classified the finite coefficients smooth dual *à la* Bernstein-Zelevinski and *à la* Bushnell-Kutzko, she also could thoroughly study lattices as above, and eventually she got a beautiful local Langlands' type correspondance modulo a prime l and compatible with Harris-Taylor-Henniart's one through reduction of lattices. Unfortunately all this was possible only by Gelfand's derivatives theory and Bushnell-Kutzko's types theory which at present only exist for GL_n.

In this note we want to explain a general and systematic approach to the study of ring-valued smooth representations. The proofs may be found in [3]. Our general motivation is a possible further application to finite coefficients local Langland's functoriality.

The first systematic algebraic approach to smooth representation theory was that of Bernstein ; he recognized very soon the interest of working with more general smooth representations than just admissible ones. In this respect, he proved highly non trivial abstract (finiteness and cohomological) properties of parabolic functors and relevant categories. However his results work only for complex coefficients (more generally for coefficients in an algebraically closed field of *banal* characteristic). Our first task has been thus to try and extend his results to general ring coefficients. His approach hinges on a good "spectral" understanding of the parabolic functors, ours hinges rather on a tentative of "geometric" understanding. We use Bruhat-Tits' bulding theory and especially the parahoric groups they have defined after Iwahori's pioneering work. These are compact open subgroups in contrast with parabolic subgroups which are closed non-compact.

2 Problems arising from Bernstein's theory

Let R be a ring such that $p \in R^*$. Let us write $Mod_R(G)$ for the category of all smooth R-valued representations (recall that this merely means that any vector is fixed by an open subgroup). We will sum up Bernstein's theory [2], [1] in the following

Theorem 2.1 *(Bernstein)*

i) There is a categorical decomposition $\mathrm{Mod}_{\mathbb{C}}(G) = \bigoplus_{[M,\pi]} \mathrm{Mod}_{\mathbb{C}}(G)_{M,\pi}$ *where by definition* $\mathrm{Mod}_{\mathbb{C}}(G)_{M,\pi}$ *is the full subcategory of all objects all irreducible subquotients of which have cuspidal support conjugate to some unramified twist of* (M,π) *(and thus the sum runs over conjugacy-unramified-twisting classes of such pairs).*

ii) The category $\mathrm{Mod}_{\mathbb{C}}(G)$ *is noetherian. In particular, for any compact open subgroup* H *of* G*, the Hecke algebra* $\mathcal{H}_{\mathbb{C}}(G,H)$ *of compactly supported bi-*H*-invariant distributions is a noetherian algebra.*

iii) Parabolic induction functors send finitely generated complex *representations on finitely generated representations (the corresponding statement for restriction is also true and easy).*

iv) Parabolic restriction r_G^P *is right adjoint to opposite parabolic induction* $\mathrm{i}_{\bar{P}}^G$ *for* complex *representations (highly non-trivial fact, not to be confused with usual Frobenius reciprocity).*

Bernstein's arguments for the proofs of these statements rest heavily on the following

Fact 2.2 *Let* π *be a* complex *irreducible smooth representation of* G*, the following assumptions are equivalent*

i) π *is cuspidal (meaning that its matrix coefficients are compact-modulo-center).*

ii) π *never appears as a subquotient of a parabolically induced representation* $\mathrm{i}_P^G(\sigma)$*.*

iii) π *is a projective object in* $\mathrm{Mod}_{\mathbb{C}}(G)$ *("modulo center").*

Replacing $\mathbb{C}$ by a general algebraically closed field, the three above assumptions may be distinct as soon as the characteristic divides the order of some compact subgroup of G. As a consequence, point i) of the theorem is definitely not true over this kind of fields and no substitute is even conjectured in general. However, points ii), iii) and iv) are expected to hold true in general, even on (noetherian) rings of coefficients.

3 Buildings and parahoric subgroups

3.1 Assume $G = \mathfrak{G}(F)$ for some reductive algebraic group $\mathfrak{G}$ over the p-adic field F. Bruhat and Tits have attached to the pair $(\mathfrak{G}, F)$ an euclidean "extended" building $\mathcal{I}_G$. This is a metric space isomorphic to a product of a euclidean space and a polysimplicial complex with isometric polysimplicial action of G.

Example : In the case of SL_n, the euclidean part is trivial and the polysimplicial part is just simplicial of dimension $n-1$. The set of vertices is in bijection with the homothetic classes of lattices in F^n, while d-simplices correspond to collections of lattices $(\omega_i)_{i=0,\cdots,d-1}$ such that $\omega_0 \subset \omega_1 \subset \cdots \subset \omega_{d-1} \subset \varpi_F^{-1}\omega_0$. This together with obvious incidence relations give the data of a combinatorial polysimplex, and $\mathcal{I}_{SL_n}$ is the standard geometric realisation of this combinatorial polysimplex. One can then identify $\mathcal{I}_{SL_n}$ with the spaces of homothetic classes of norms on F^n. When $n = 2$ we get a homogeneous tree, each vertex belonging to $q+1$ segments.

In the case of a torus T, the simplicial part is trivial and the euclidean part is just $X_*^F(T) \otimes \mathbb{R}$ (rational cocharacters).

When $x \in \mathcal{I}_G$, we note G_x its fixator in G. It is a compact open subgroup, and it is well known that any compact open subgroup is contained in such a fixator. This group G_x has a pro-p-radical noted G_x^+. In general G_x/G_x^+ is isomorphic to the group of rational points of some reductive group over the residue field k_F of F.

Example : For SL_n, the stabilizer of some vertex is always $GL_n(F)$-conjugated to $SL_n(O_F)$ where O_F is the ring of integers of F. The reduction map to k_F sets up a bijection between parabolic subgroups of $SL_n(k_F)$ and fixators of points in the simplicial star of the vertex (*i.e.* the union of all facets whose closure contains the vertex).

3.2 Let M be a F-Levi subgroup of G. Bruhat and Tits have also shown the existence of a (non-unique) isometric and M-equivariant embedding $\mathcal{I}_M \hookrightarrow \mathcal{I}_G$. We will fix such an embedding and consider $\mathcal{I}_M$ as a subset of $\mathcal{I}_G$. Taking up the foregoing notations with M in place of G, it is obvious that $M_x = G_x \cap M$ and it is also true that $M_x^+ = G_x^+ \cap M$. This allows us to use the following general notation : if H is a subgroup of G, we will note $H_x := H \cap G_x$ and $H_x^+ := H \cap G_x^+$.

Example : If T the diagonal torus of SL_n and $\mu \in X_*(T)$ is a rational cocharacter, we can attach to μ the class of the lattice $\sum_{i=1}^n \mu(\varpi_F)_{ii} O_F e_i$ where e_i is the standard basis of F^n. This extends to an embedding of $X_*(T) \otimes \mathbb{R} \hookrightarrow \mathcal{I}_{SL_n}$, and the simplicial structure which is drawn on $X_*(T) \otimes \mathbb{R}$ by the ambient building is that attached to the hyperplane arrangement of $X_*(T) \otimes \mathbb{R}$ given by equations $\{\alpha(x) = k\}_{\alpha,x}$ for all roots α and $k \in \mathbb{Z}$.

3.3 Let P be a parabolic subgroup of G with Levi component M, and let $\overline{P}$ be the opposed parabolic subgroup . It is known that the group G_x^+ has a so-called Iwahori decomposition, meaning that the product map $U_x^+ \times M_x^+ \times \overline{U}_x^+ \longrightarrow G_x^+$ is a bijection, whatever ordering is chosen to make the product. We will briefly account for such decompositions by the simple notation $G_x^+ = U_x^+ M_x^+ \overline{U}_x^+$. Notice that G_x^+ by definition is a normal subgroup of G_x, so that the set $G_{x,P} := P_x G_x^+$ is a group. This group will be called a *parahoric subgroup* of G ; this differs slightly from the Bruhat-Tits definition. It also has a Iwahori decomposition $G_{x,P} = U_x M_x \overline{U}_x^+$.

3.4 Given x, M and P, we would like to construct functors $Mod_R(M_x) \longrightarrow Mod_R(G_{x,P}) \longrightarrow Mod_R(G_x)$ with model the classical construction of parabolic induction $Mod_R(M) \longrightarrow Mod_R(P) \longrightarrow Mod_R(G)$ where the first functor is inflation and the second one is induction. The problem in the parahoric situation is the inflation stage which is impossible since M_x *is not* a quotient of $G_{x,P}$. Next lemma is intended to solve this problem. We need some notations ; for any subgroup H of G we will note $\mathbb{Z}[\frac{1}{p}][H]$ the algebra of all $\mathbb{Z}[\frac{1}{p}]$-values compactly supported distributions. If K is pro-p-subgroup of H, we will note e_K the element of $\mathbb{Z}[\frac{1}{p}][H]$ given by the normalized Haar measure on K.

Lemma 3.5 *There is a central and invertible element $z_{x,P} \in \mathbb{Z}[\frac{1}{p}][G_{x,P}]$ such that $\varepsilon_{x,P} := z_{x,P}^{-1} e_{U_x} e_{\overline{U}_x^+}$ is an idempotent in $\mathbb{Z}[\frac{1}{p}][G_{x,P}]$.*

Notice that by our assumption $p \in R^*$, the algebra $\mathbb{Z}[\frac{1}{p}][G_{x,p}]$ naturaly acts on any smooth R-valued representation of $G_{x,P}$, in particular on the space $\mathcal{C}_R^\infty(G_x)$ of smooth R-valued functions on G_x. Thus we may define $E_{x,P} := \varepsilon_{x,P}.\mathcal{C}_R^\infty(G_x)$. This R-module is endowed with smooth action of G_x on the right and M_x on the left, since M_x normalizes $\varepsilon_{x,P}$. We may thus define functors

$$\begin{array}{rrcl} R_{x,P}: & Mod_R(G_x) & \rightarrow & Mod_R(M_x) \\ & V & \mapsto & E_{x,P} \otimes_{RG_x} V \end{array}$$

and

$$\begin{array}{rrcl} I_{x,P}: & Mod_R(M_x) & \rightarrow & Mod_R(G_x) \\ & W & \mapsto & E_{x,P} \otimes_{RM_x} W \end{array}$$

where tensor products are taken with respect to adequate (right or left) actions. The above lemma implies that $I_{x,P}$ is left adjoint to $R_{x,P}$.

3.6 Given x and M, next question is to what extend these functors rely on the choice of P. As already said, for any parabolic subgroup P containing M, $G_{x,P}$ is a parahoric subgroup of G_x. But the map $P \mapsto G_{x,P}$ is not injective in general : for example if x is inside a maximal simplex, all $G_{x,P}$ are equal to G_x which in this case is a Iwahori subgroup. But when one proves the former lemma, one can also prove that the above functors actually depend only on $G_{x,P}$ and not on P. By the way this justifies the name "parahoric induction/restriction".

But the following question remains open : does parahoric induction really depend on the parahoric subgroup $G_{x,P}$?

Thinking to the parabolic analog, it is well known that even for complex coefficients, the parabolic functors heavily depend on the choice of a parabolic subgroup. In contrast, for a finite group of Lie type, it was shown by Howlett and Lehrer [4] that the parabolic functors don't depend on this choice. Inspired by their work, we can restate our question of dependance in purely algebraic terms :

Question 3.7 *Fix x, M and let P be a parabolic subgroup with Levi component M. Do we have $\varepsilon_{x,P} \in \mathbb{Z}[\frac{1}{p}][G_x]\varepsilon_{x,\overline{P}}\varepsilon_{x,P}$ and $\varepsilon_{x,\overline{P}} \in \mathbb{Z}[\frac{1}{p}][G_x]\varepsilon_{x,P}\varepsilon_{x,\overline{P}}$?*

Next section will justify our interest in answering this question. The only cases we can treat at present are summed up in

Proposition 3.8 *i) If M is a minimal Levi subgroup, then the answer is positive for any parabolic P with Levi component M.*

ii) In general, we have $\varepsilon_{x,P}e_{M_x^+} \in \mathbb{Z}[\frac{1}{p}][G_x]\varepsilon_{x,\overline{P}}\varepsilon_{x,P}e_{M_x^+}$.

The second point is a direct consequence of Howlett and Lehrer's results.

4 Applications of parahoric functors

Theorem 4.1 *Fix a parabolic subgroup P with Levi component M and assume that question 3.7 has a positive answer for any $x \in \mathcal{I}_M$. Then the map*

$$\begin{array}{rcl} \varepsilon_{x,P}.\mathcal{C}_R^{\infty,c}(G) & \rightarrow & \mathcal{C}_R^{\infty,c}(U\backslash G) \\ f & \mapsto & (g \mapsto \int_U f(ug)du) \end{array}$$

is an isomorphism of $M_x \times G$ smooth R-representations, for any $x \in \mathcal{I}_M$.

In order to stress up the scope of the displayed statement in the theorem, let us explain some consequences. First for any x, M, P as above we get an isomorphism of functors on R-representations

$$Res_M^{M_x} \circ r_G^P \simeq R_{x,P} \circ Res_G^{G_x}.$$

Notice that this immediately implies that parabolic restriction respects admissibility, which is generally not known on non-Artinian rings of coefficients. On another hand we get after little further work an isomorphism of functors, still on R-representations,

$$ind_{G_x}^G \circ I_{x,P} \simeq i_P^G \circ ind_{M_x}^M.$$

As an immediate application, this clearly shows that parabolic induction sends finitely generated objects on finitely generated objects.

Next consequence rests on ideas of Bernstein and deserves a special treatment

Corollary 4.2 *Under the same hypothesis as in previous theorem, the functor* i_P^G *is left adjoint to the functor* r_G^P.

As an immediate application, we see that parabolic induction preserves projective objects while parabolic restriction preserves injective ones.

Resting on these results, we can then prove

Proposition 4.3 *Assume now that the answer to 3.7 is positive for any* x, M, P*. Then*

- *i) For any compact open* H*, there is a compact-modulo-center subset* $S_H \subset G$ *supporting all cuspidal bi-*H*-invariant functions on* G*,* regardless of the ring of coefficients.
- *ii) The category* $Mod_{\mathbb{Z}[\frac{1}{p}]}(G)$ *is noetherian.*

Other applications, to shape of reducibility points and to K-theory are given in [3], under the same assumptions as in this proposition.

Recall now that our theorem rests on a basic assumption we cannot grant in full generality. By the proposition in the former section, this assumption is fulfilled when M is minimal, and in this case our theorem gives a real result and the former proposition applies for any relative rank 1 group G. By the same proposition we can also state results on the "level 0 subcategory". We mention first :

Fact 4.4 *(Moy-Prasad-Vigneras [6] +ε) There is a decomposition*

$$Mod_{\mathbb{Z}[\frac{1}{p}]}(G) = Mod_{\mathbb{Z}[\frac{1}{p}]}(G)_0 \bigoplus Mod_{\mathbb{Z}[\frac{1}{p}]}(G)^0$$

where by definition $Mod_{\mathbb{Z}[\frac{1}{p}]}(G)_0$ *is the full subcategory of all objects generated by their* G_x^+*-invariants,* x *running through* $\mathcal{I}_G$ *(called the level* 0 *subcategory). Moreover, the parabolic functors preserve level* 0 *subcategories.*

For level 0 representations, our theorem and its consequences are listed in

Proposition 4.5 *i) For any* x, M, P*, the morphism*

$$\begin{array}{rcl} e_{M_x^+}\varepsilon_{x,P}.\mathcal{C}_R^{\infty,c}(G) & \to & e_{M_x^+}\mathcal{C}_R^{\infty,c}(U\backslash G) \\ f & \mapsto & (g \mapsto \int_U f(ug)du) \end{array}$$

is an isomorphim of $M_x \times G$ *representations.*

ii) On the level 0 *subcategories, the functor* i_P^G *is left adjoint to* r_G^P.

iii) The level 0 *subcategory* $Mod_{\mathbb{Z}[\frac{1}{p}]}(G)_0$ *is noetherian.*

About proofs in [3]: that of the lemma is elementary algebra, that of the theorem rests on a dynamical argument on the building inpired by work of Moy-Prasad [5], that of the corollary rests on “completions” as in Bernstein’s unpublished work [1], that of noetheriannity requires new other arguments.

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On Siegel modular forms of degree 2 with square-free level

RALF SCHMIDT

Introduction

For representations of GL(2) over a p–adic field F there is a well-known theory of local newforms due to CASSELMAN, see [Cas]. This local theory together with the global strong multiplicity one theorem for cuspidal automorphic representations of GL(2) is reflected in the classical Atkin–Lehner theory for elliptic modular forms.

In contrast to this situation, there is currently no satisfactory theory of local newforms for the group $\mathrm{GSp}(2, F)$. As a consequence, there is no analogue of Atkin–Lehner theory for Siegel modular forms of degree 2. In this paper we shall present such a theory for the "square-free" case. In the local context this means that the representations in question are assumed to have non-trivial Iwahori–invariant vectors. In the global context it means that we are considering congruence subgroups of square-free level.

We shall begin by reviewing some well known facts from the classical theory of elliptic modular forms. Then we shall give a definition of a space $S_k(\Gamma_0(N)^{(2)})^{\rm new}$ of newforms in degree 2, where N is a square-free positive integer. Table 1 on page 8 lies at the heart of our theory. It contains the dimensions of the spaces of fixed vectors under each parahoric subgroup in every irreducible Iwahori–spherical representation of GSp(2) over a p–adic field F.

Section 4 deals with a global tool, namely a suitable L–function theory for certain cuspidal automorphic representations of PGSp(2). Since none of the existing results on the spin L–function seems to fully serve our needs, we have to make certain assumptions at this point. Having done so, we shall present our main result in the final section 5. It essentially says that given a cusp form $f \in S_k(\Gamma_0(N))^{\rm new}$, assumed to be an eigenform for almost all unramified Hecke algebras and also for certain Hecke operators at places $p|N$, we can attach a *global L–packet* π_f of automorphic representations of $\mathrm{PGSp}(2, \mathbb{A}_\mathbb{Q})$ to f. This allows us to associate with f a global (spin) L–function with a nice functional equation. We shall describe the local factors at the bad places explicitly in terms of certain Hecke eigenvalues.

1 Review of classical theory

We recall some well-known facts for classical holomorphic modular forms. Let $f \in S_k(\Gamma_0(N))$ be an elliptic cuspform, and let $G = \mathrm{GL}(2)$, considered as an algebraic $\mathbb{Q}$–group. It follows from strong approximation for $\mathrm{SL}(2)$ that there is a unique associated adelic function $\Phi_f : G(\mathbb{A}) \to \mathbb{C}$ with the following properties:

i) $\Phi_f(\rho g z) = \Phi_f(g)$ for all $g \in G(\mathbb{A})$, $\rho \in G(\mathbb{Q})$ and $z \in Z(\mathbb{A})$. Here Z is the center of $\mathrm{GL}(2)$.

ii) $\Phi_f(gh) = \Phi_f(g)$ for all $g \in G(\mathbb{A})$ and $h \in \prod_{p<\infty} K_p(N)$. Here $K_p(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}_p) : c \in N\mathbb{Z}_p \right\}$ is the local analogue of $\Gamma_0(N)$.

iii) $\Phi_f(g) = (f\big|_k g)(i) := \det(g)^{k/2} j(g,i)^{-k} f(g\langle i \rangle)$ for all $g \in \mathrm{GL}(2,\mathbb{R})^+$ (the identity component of $\mathrm{GL}(2,\mathbb{R})$).

Since f is a cusp form, Φ_f is an element of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/Z(\mathbb{A}))$. Let π_f be the unitary $\mathrm{PGL}(2,\mathbb{A})$–subrepresentation of this L^2–space generated by Φ_f.

1.1 Theorem. *With the above notations, the representation π_f is irreducible if and only if f is an eigenform for the Hecke operators $T(p)$ for almost all primes p. If this is the case, then f is automatically an eigenform for $T(p)$ for <u>all</u> $p \nmid N$.*

Idea of Proof: We decompose the representation π_f into irreducibles, $\pi_f = \bigoplus_i \pi_i$. Each π_i can be written as a restricted tensor product of local representations,

$$\pi_i \simeq \bigotimes_{p \le \infty} \pi_{i,p}, \qquad \pi_{i,p} \text{ a representation of } \mathrm{PGL}(2, \mathbb{Q}_p).$$

Assuming that f is an eigenform, one can show easily that for almost all p we have $\pi_{i,p} \simeq \pi_{j,p}$. But *Strong Multiplicity One* for $\mathrm{GL}(2)$ says that two cuspidal automorphic representations coincide (as spaces of automorphic forms) if their local components are isomorphic at almost every place. It follows that π_f must be irreducible. ■

Thus to each eigenform f we can attach an *automorphic representation* $\pi_f = \otimes \pi_p$. A natural problem is to identify the local representations π_p given only the classical function f. This is easy at the archimedean place: π_∞ is the discrete series representation of $\mathrm{PGL}(2,\mathbb{R})$ with a lowest weight vector of weight k. It is also easy for finite primes p not dividing N. At such places π_p is an unramified principal series representation, i.e., π_p is an infinite-dimensional representation containing a non-zero $\mathrm{GL}(2,\mathbb{Z}_p)$–fixed vector. These representations are characterized by their Satake parameter $\alpha \in \mathbb{C}^*$, and the relationship between α and the Hecke–eigenvalue λ_p is $\lambda_p = p^{(k-1)/2}(\alpha + \alpha^{-1})$.

In general it is not easy to identify the local components π_p at places $p|N$. But if N is square-free, we have the following result.

1.2 Theorem. *Assume that N is a square-free positive integer, and let $f \in S_k(\Gamma_0(N))$ be an eigenform. Further assume that f is a* <u>*newform*</u>*. Then the local component π_p of the associated automorphic representation π_f at a place $p|N$ is given as follows:*

$$\pi_p = \begin{cases} \mathrm{St}_{\mathrm{GL}(2)} & \text{if } a_1 f = -f, \\ \xi\,\mathrm{St}_{\mathrm{GL}(2)} & \text{if } a_1 f = f. \end{cases}$$

Here $\mathrm{St}_{\mathrm{GL}(2)}$ is the Steinberg representation of $\mathrm{GL}(2,\mathbb{Q}_p)$, and ξ is the unique non-trivial unramified quadratic character of $\mathbb{Q}_p^*$. The operator a_1 is the Atkin–Lehner involution at p.

Idea of Proof: It follows from the fact that f is a modular form for $\Gamma_0(N)$ that π_p contains non-trivial vectors invariant under the Iwahori subgroup

$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}_p) : \; c \in p\mathbb{Z}_p \right\}.$$

The following is a complete list of all such Iwahori-spherical representations together with the dimensions of their spaces of fixed vectors under I and under $K = \mathrm{GL}(2,\mathbb{Z}_p)$.

representation	K	I
$\pi(\chi,\chi^{-1}),\ \chi$ unramified, $\chi^2 \neq \lvert\ \rvert^{\pm 1}$	1	2
$\mathrm{St}_{\mathrm{GL}(2)}$ or $\xi\,\mathrm{St}_{\mathrm{GL}(2)}$	0	1

(1)

We recall the definition of newforms, for notational simplicity assuming that $N = p$. We have two operators

$$T_0, T_1 : \; S_k(\mathrm{SL}(2,\mathbb{Z})) \longrightarrow S_k(\Gamma_0(p)), \qquad (2)$$

where T_0 is simply the inclusion and T_1 is given by $(T_1 f)(\tau) = f(p\tau)$. Then the space of *oldforms* is defined as

$$S_k(\Gamma_0(p))^{\rm old} = {\rm im}(T_0) + {\rm im}(T_1), \tag{3}$$

and the space of *newforms* $S_k(\Gamma_0(p))^{\rm new}$ is by definition the orthogonal complement of $S_k(\Gamma_0(p))^{\rm old}$ with respect to the Petersson inner product. Now it is easily checked that *locally*, in an unramified principal series representation $\pi(\chi, \chi^{-1})$ realized on a space V, we have

$$V^I = T_0 V^K + T_1 V^K. \tag{4}$$

Hence the fact that f is a newform means precisely that π_p cannot be an unramified principal series representation $\pi(\chi, \chi^{-1})$. Therefore $\pi_p = {\rm St}_{{\rm GL}(2)}$ or $\pi_p = \xi\, {\rm St}_{{\rm GL}(2)}$, and easy computations show the connection with the Atkin–Lehner eigenvalue (cf. [Sch], section 3). ■

Knowing the local components π_p allows to correctly attach local factors to the modular form f. For example, if f is a newform as in Theorem 1.2, one would define for $p|N$

$$L_p(s, f) = L_p(s, \pi_p) = \begin{cases} (1 - p^{-1/2-s})^{-1} & \text{if } a_1 f = -f, \\ (1 + p^{-1/2-s})^{-1} & \text{if } a_1 f = f. \end{cases}$$

$$\varepsilon_p(s, f) = \varepsilon_p(s, \pi_p) = \begin{cases} -p^{1/2-s} & \text{if } a_1 f = -f, \\ p^{1/2-s} & \text{if } a_1 f = f. \end{cases}$$

With these definitions, and unramified and archimedean factors as usual, the functional equation $L(s, f) = \varepsilon(s, f) L(1 - s, f)$ holds for $L(s, f) = \prod_p L_p(s, f)$ and $\varepsilon(s, f) = \prod_p \varepsilon_p(s, f)$.

2 Newforms in degree 2

It is our goal to develop a similar theory as outlined in the previous section for the space of Siegel cusp forms $S_k(\Gamma_0(N)^{(2)})$ of degree 2 and square-free level N. Here we are facing several difficulties.

- Strong multiplicity one fails for the underlying group ${\rm GSp}(2)$, and even weak multiplicity one is presently not known. Thus it is not clear how to attach an automorphic representation of ${\rm GSp}(2, \mathbb{A})$ to a classical cusp form f.

- The local representation theory of $\mathrm{GSp}(2,\mathbb{Q}_p)$ is much more complicated than that of $\mathrm{GL}(2,\mathbb{Q}_p)$. In particular, there are 13 different types of infinite-dimensional representations containing non-trivial vectors fixed under the local Siegel congruence subgroup, while in the $\mathrm{GL}(2)$ case we had only 2 (see table (1)).
- There is currently no generally accepted notion of newforms for Siegel modular forms of degree 2.

The last two problems are of course related. Let P_1 be the Siegel congruence subgroup of level p, i.e.,

$$P_1 = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}(2,\mathbb{Z}_p) : \ C \equiv 0 \bmod p \right\}. \tag{5}$$

Every classical definition of newforms with respect to P_1 must in particular be designed to exclude K–spherical representations, where $K = \mathrm{GSp}(2,\mathbb{Z}_p)$. Since an unramified principal series representation of $\mathrm{GSp}(2,\mathbb{Q}_p)$ contains a four–dimensional space of P_1–invariant vectors (see Table 1 below), we expect *four* operators

$$T_0, T_1, T_2, T_3 : \ S_k(\mathrm{Sp}(2,\mathbb{Z})) \longrightarrow S_k(\Gamma_0(p))$$

whose images would span the space of oldforms. (From now on, when we write $\Gamma_0(N)$, we mean groups of 4×4–matrices.) For this purpose we are now going to introduce four endomorphisms $T_0(p), \ldots, T_3(p)$ of the space $S_k(\Gamma_0(N))$, where N is square-free and $p|N$.

- $T_0(p)$ is simply the identity map.
- $T_1(p)$ is the Atkin–Lehner involution at p, defined as follows. Choose integers α, β such that $p\alpha - \frac{N}{p}\beta = 1$. Then the matrix

  $$\eta_p = \begin{pmatrix} p\alpha & & 1 & \\ & p\alpha & & 1 \\ N\beta & & p & \\ & N\beta & & p \end{pmatrix}$$

  is in $\mathrm{GSp}(2,\mathbb{R})^+$ with multiplier p. It normalizes $\Gamma_0(N)$, hence the map $f \mapsto f\big|_k \eta_p$ defines an endomorphism of $S_k(\Gamma_0(N))$. Since $\eta_p^2 \in p\Gamma_0(N)$, this endomorphism is an involution (we always normalize the slash operator as

  $$(f\big|_k g)(Z) = \mu(g)^k j(g,Z)^{-k} f(g\langle Z\rangle) \qquad (\mu \text{ is the multiplier}),$$

  which makes the center of $\mathrm{GSp}(2,\mathbb{R})^+$ act trivially). This is the *Atkin–Lehner involution* at p. It is independent of the choice of α and β.

- We define $T_2(p)$ by

$$(T_2(p)f)(Z) = \sum_{g\in\Gamma_0(N)\backslash\Gamma_0(N)\left(\begin{smallmatrix}\mathbf{1} & \\ & p\mathbf{1}\end{smallmatrix}\right)\Gamma_0(N)} \big(f\big|_k g\big)(Z)$$
$$= \sum_{x,\mu,\kappa\in\mathbb{Z}/p\mathbb{Z}} \left(f\Big|_k \begin{pmatrix}1&&&\\&1&&\\&&p&\\&&&p\end{pmatrix}\begin{pmatrix}1&&x&\mu\\&1&\mu&\kappa\\&&1&\\&&&1\end{pmatrix}\right)(Z). \qquad (6)$$

This is a well-known operator in the classical theory. In terms of Fourier expansions, if $f(Z) = \sum_{n,r,m} c(n,r,m)e^{2\pi i(n\tau+rz+m\tau')}$ with $Z = \begin{pmatrix}\tau & z\\ z & \tau'\end{pmatrix}$, then

$$(T_2(p)f)(Z) = \sum_{n,r,m} c(np,rp,mp)e^{2\pi i(n\tau+rz+m\tau')}. \qquad (7)$$

- Finally, we define $T_3(p) := T_1(p)\circ T_2(p)$.

Now we are ready to define newforms in degree 2.

2.1 Definition. *Let N be a square-free positive integer. In $S_k(\Gamma_0(N))$ we define the subspace of* **oldforms** $S_k(\Gamma_0(N))^{\rm old}$ *to be the sum of the spaces*

$$T_i(p)S_k\big(\Gamma_0(Np^{-1})\big), \qquad i = 0,1,2,3, \quad p|N.$$

The subspace of **newforms** $S_k(\Gamma_0(N))^{\rm new}$ *is defined as the orthogonal complement of $S_k(\Gamma_0(N))^{\rm old}$ inside $S_k(\Gamma_0(N))$ with respect to the Petersson scalar product.*

Note that this definition is analogous to the definition of oldforms in the degree 1 case. The operator T_1 given in (2) has the same effect as the Atkin–Lehner involution on modular forms for $\mathrm{SL}(2,\mathbb{Z})$.

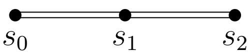
See [Ib] for more comments on the topic of old and new Siegel modular forms.

3 Local newforms

Let us realize $G = \mathrm{GSp}(2)$ using the symplectic form $\begin{pmatrix} & \mathbf{1}\\ -\mathbf{1} & \end{pmatrix}$. In this section we shall consider G as an algebraic group over a p–adic field F. Let $\mathfrak{o}$ be the ring of integers of F and $\mathfrak{p}$ its maximal ideal. Let $K = G(\mathfrak{o})$ be the standard special maximal compact subgroup of $G(F)$. As an Iwahori subgroup we choose

$$I = \left\{ g \in K : \; g \equiv \begin{pmatrix} * & & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \bmod \mathfrak{p} \right\}$$

The *parahoric subgroups* of $G(F)$ correspond to subsets of the simple Weyl group elements in the Dynkin diagram of the affine Weyl group C_2:



The Iwahori subgroup corresponds to the empty subset of $\{s_0, s_1, s_2\}$. The numbering is such that s_1 and s_2 generate the usual 8–element Weyl group of GSp(2). The corresponding parahoric subgroup is $P_{12} = K$. The Atkin–Lehner element

$$\eta = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \varpi & & \\ \varpi & & & \end{pmatrix} \in \mathrm{GSp}(2, F) \qquad (\varpi \text{ a uniformizer}) \tag{8}$$

induces an automorphism of the Dynkin diagram. The parahoric subgroup P_{01} corresponding to $\{s_0, s_1\}$ is therefore conjugate to K via η. We further have the Siegel congruence subgroup P_1 (see (5)), the Klingen congruence subgroup P_2, its conjugate $P_0 = \eta P_2 \eta^{-1}$, and the *paramodular group*

$$P_{02} = \left\{ g \in G(F) : g, g^{-1} \in \begin{pmatrix} \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-1} \\ \mathfrak{o} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{o} \end{pmatrix} \right\}.$$

K and P_{02} represent the two conjugacy classes of maximal compact subgroups of $\mathrm{GSp}(2, F)$. By a well-known result of Borel (see [Bo]) the Iwahori–spherical irreducible representations are precisely the constituents of representations induced from an unramified character of the Borel subgroup. For GSp(2), such representations were first classified by Rodier, see [Rod], but in the following we shall use the notation of Sally–Tadic [ST]. The following Table 1 gives a complete list of all the irreducible representations of $\mathrm{GSp}(2, F)$ with non-trivial I–invariant vectors. Behind each representation we have listed the dimension of the spaces of vectors fixed under each parahoric subgroup (modulo conjugacy). The last column gives the exponent of the conductor of the local parameter of each representation.

		representation	K	P_{02}	P_2	P_1	I	a
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	**1**	2 +−	4	4 ++ −−	8 ++++ −−−−	0
II	a	$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	0	**1** −	2	**1** −	4 +−−−	1
	b	$\chi \mathbf{1}_{\mathrm{GL}(2)} \rtimes \sigma$	**1**	1 +	2	3 ++−	4 +++−	0
III	a	$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(1)}$	0	0	**1**	**2** +−	4 ++−−	2
	b	$\chi \rtimes \sigma \mathbf{1}_{\mathrm{GSp}(1)}$	**1**	2 +−	3	2 +−	4 ++−−	0
IV	a	$\sigma \mathrm{St}_{\mathrm{GSp}(2)}$	0	0	0	0	**1** −	3
	b	$L((\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(1)}))$	0	0	**1**	**2** +−	3 ++−	2
	c	$L((\nu^{3/2} \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma))$	0	**1** −	2	**1** −	3 +−−	1
	d	$\sigma \mathbf{1}_{\mathrm{GSp}(2)}$	**1**	1 +	1	1 +	1 +	0
V	a	$\delta([\xi_0, \nu\xi_0], \nu^{-1/2}\sigma)$	0	0	**1**	0	2 +−	2
	b	$L((\nu^{1/2}\xi_0 \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$	0	**1** +	1	**1** +	2 ++	1
	c	$L((\nu^{1/2}\xi_0 \mathrm{St}_{\mathrm{GL}(2)}, \xi_0\nu^{-1/2}\sigma))$	0	**1** −	1	**1** −	2 −−	1
	d	$L((\nu\xi_0, \xi_0 \rtimes \nu^{-1/2}\sigma))$	**1**	0	1	2 +−	2 +−	0
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	0	0	**1**	**1** −	3 +−−	2
	b	$\tau(T, \nu^{-1/2}\sigma)$	0	0	0	**1** +	1 +	2
	c	$L((\nu^{1/2} \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma))$	0	**1** −	1	0	1 −	1
	d	$L((\nu, \mathbf{1}_{F^*} \rtimes \nu^{-1/2}\sigma))$	**1**	1 +	2	2 +−	3 ++−	0

Table 1: Dimensions of spaces of invariant vectors in Iwahori–spherical representations of $\mathrm{GSp}(2, F)$.

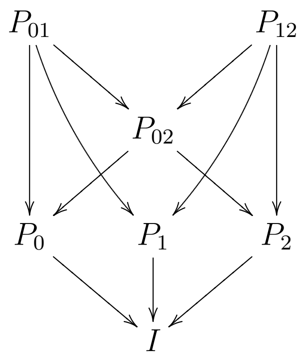
The signs under the entries for the "symmetric" subgroups P_{02}, P_1 and I indicate how these spaces of fixed vectors split into Atkin–Lehner eigenspaces, provided the central character of the representation is trivial. The signs listed in Table 1 are correct if one assumes that

- in Group II, where the central character is $\chi^2\sigma^2$, the character $\chi\sigma$ is trivial.
- in Groups IV, V and VI, where the central character is σ^2, the character σ itself is trivial.

If these assumptions are not met, then one has to interchange the plus and minus signs in Table 3 to get the correct dimensions.

The information in Table 1 is essentially obtained by computations in the standard models of these induced representations. Details will appear elsewhere.

Imitating the classical theory, one can define *oldforms* by introducing natural operators from fixed vectors for bigger to fixed vectors for smaller parahoric subgroups. Here "bigger" not always means inclusion, since we also consider K "bigger" than P_{02}. More precisely, we consider R' bigger than R, and shall write $R' \succ R$, if there is an arrow from R' to R in the following diagram.



(9)

Whenever $R' \succ R$, one can define natural operators from $V^{R'}$ to V^R, where V is any representation space. For example, our previously defined global operators $T_0(p)$ and $T_2(p)$ correspond to two natural maps $V^K \to V^{P_1}$. Our $T_1(p)$ and $T_3(p)$ correspond to two natural maps $V^{P_{01}} \to V^{P_1}$, composed with the Atkin–Lehner element $V^K \to V^{P_{01}}$.

This can be done for any parahoric subgroup, and it is natural to call any fixed vector that can be obtained from any bigger parahoric subgroup an *oldform*. Everything else would naturally be called a *newform*, but the meaning of "everything else" has to be made precise. Let it suffice to say that if the representation is unitary one can work with orthogonal complements as in the classical theory.

Once these notions of oldforms and newforms are defined, one can verify the decisive fact that *each space of fixed vectors listed in Table 1 consists either completely of oldforms or completely of newforms.* If this were not true, our notions of oldforms and newforms would make little sense. In Table 1 we have indicated the spaces of newforms by writing their dimensions in bold face. We see that they are not always one-dimensional.

4 L–functions

For the applications we have in mind we need the spin L–function of cuspidal automorphic representations of $\mathrm{GSp}(2,\mathbb{A})$ as a global tool. There are several results on this L–function, see [No], [PS] or [An]. Unfortunately none of these results fully serves our needs. What we need is the following.

4.1 L–Function Theory for $\mathrm{GSp}(2)$.

i) To every cuspidal automorphic representation π of $\mathrm{PGSp}(2,\mathbb{A})$ is associated a global L–function $L(s,\pi)$ and a global ε–factor $\varepsilon(s,\pi)$, both defined as Euler products, such that $L(s,\pi)$ has meromorphic continuation to all of $\mathbb{C}$ and such that a functional equation

$$L(s,\pi)=\varepsilon(s,\pi)L(1-s,\pi)$$

of the standard kind holds.

ii) For Iwahori–spherical representations, the local factors $L_v(s,\pi_v)$ and $\varepsilon_v(s,\pi_v,\psi_v)$ coincide with the spin local factors defined via the local Langlands correspondence as in [KL].

Of course such an L–function theory is predicted by general conjectures over any number field. For our classical applications we shall only need it over $\mathbb{Q}$. Furthermore, we can restrict to the archimedean component being a lowest weight representation with scalar minimal K–type (a discrete series representation if the weight is ≥ 3). All we need to know about ε–factors is in fact that they are of the form cp^{ms} with a constant $c\in\mathbb{C}^*$ and an integer m.

The local Langlands correspondence is not yet a theorem for $\mathrm{GSp}(2)$ (but see [Pr], [Rob]), but for Iwahori–spherical representations it is known by [KL].

In fact, the local parameters (four–dimensional representations of the Weil–Deligne group) of all the representations in Table 1 can easily be written down explicitly. Hence we know all their local factors. There is one case of L–indistinguishability in Table 1, namely, the representations VIa and VIb constitute an L–packet. The representation Va also lies in a two-element L–packet. Its partner is a θ_{10}–type supercuspidal representation.

4.2 Theorem. *We assume that an L–function theory as in 4.1 exists. Let $\pi_1 = \otimes\pi_{1,p}$ and $\pi_2 = \otimes\pi_{2,p}$ be two cuspidal automorphic representations of* $\mathrm{PGSp}(2, \mathbb{A}_\mathbb{Q})$. *Let S be a finite set of prime numbers such that the following holds:*

- *i) $\pi_{1,p} \simeq \pi_{2,p}$ for each $p \notin S$.*
- *ii) For each $p \in S$, both $\pi_{1,p}$ and $\pi_{2,p}$ possess non-trivial Iwahori–invariant vectors.*

Then, for each $p \in S$, the representations $\pi_{1,p}$ and $\pi_{2,p}$ are constituents of the same induced representation (from an unramified character of the Borel subgroup).

Idea of proof: We divide the two functional equations for $L(s, \pi_1)$ and $L(s, \pi_2)$ and obtain *finite* Euler products by hypothesis i). Since we are over $\mathbb{Q}$, and since the expressions p^{-s} for different p can be treated as independent variables, it follows that we get equalities

$$\frac{L_p(s, \pi_{1,p})}{L_p(s, \pi_{2,p})} = cp^{ms}\frac{L_p(1-s, \pi_{1,p})}{L_p(1-s, \pi_{2,p})}, \qquad c \in \mathbb{C}^*,\ m \in \mathbb{Z},$$

for each $p \in S$. But we have the complete list of all possible local Euler factors. One can check that such a relation is only possible if $\pi_{1,p}$ and $\pi_{2,p}$ are constituents of the same induced representation. ■

Remark: In Table 1, for two representations to be constituents of the same induced representation means that they are in the same group I–VI.

With some additional information on the representation this result sometimes allows to attach a unique *equivalence class* of automorphic representations to a classical cuspform f. For example, if N is square-free and $f \in S_k(\Gamma_0(N))^{\mathrm{new}}$ is an eigenform for almost all the unramified Hecke algebras and also an eigenvector for the Atkin–Lehner involutions for all $p|N$, then Theorem 4.2 together with the information in Table 1 show that the associated adelic function Φ_f generates a multiple of an automorphic representation π_f of $\mathrm{PGSp}(2, \mathbb{A})$.

5 The main result

Let N be a square-free positive integer. In the degree 1 case, given an eigenform $f \in S_k(\Gamma_0(N))^{\text{new}}$, knowing the Atkin–Lehner eigenvalues for $p|N$ was enough to identify the local representations and attach the correct local factors. In the degree 2 case, since there are more possibilities for the local representations, and since some of them have parameters, we need more information than just the Atkin–Lehner eigenvalues. For example, the representations IIa or IIIa, both of which have local newforms with respect to P_1, depend on characters χ and σ. Hence there are additional Satake parameters which enter into the L–factor. What we need are suitable Hecke operators on $S_k(\Gamma_0(N))^{\text{new}}$ to extract this information from the modular form f. It turns out that the previously defined operator $T_2(p)$ works well, but we need even more information. We are now going to define an additional endomorphism $T_4(p)$ of $S_k(\Gamma_0(N))^{\text{new}}$.

For notational simplicity assume $N = p$ is a prime and consider the following linear maps:

$$S_k(\Gamma_0(p))^{\text{new}} \underset{d_1}{\overset{d_{02}}{\rightleftarrows}} S_k(\Gamma^{\text{para}}(p))^{\text{new}} \tag{10}$$

Here d_1 and d_{02} are *trace operators* which always exist between spaces of modular forms for commensurable groups. Explicitly,

$$d_{02}f = \frac{1}{(\Gamma^{\text{para}}(p) : \Gamma_0(p) \cap \Gamma^{\text{para}}(p))} \sum_{\gamma \in (\Gamma_0(p) \cap \Gamma^{\text{para}}(p)) \backslash \Gamma^{\text{para}}(p)} f\big|_k \gamma.$$

It is obvious from Table 1 that these operators indeed map newforms to newforms. The additional endomorphism of $S_k(\Gamma_0(p))^{\text{new}}$ we require is

$$T_4(p) := (1+p)^2 \, d_1 \circ d_{02}. \tag{11}$$

Similarly we can define endomorphisms $T_4(p)$ of $S_k(\Gamma_0(N))^{\text{new}}$ for each $p|N$. Looking at local representations, the following is almost trivial.

5.1 Proposition. *Let N be square-free. The space $S_k(\Gamma_0(N))^{\text{new}}$ has a basis consisting of common eigenfunctions for the operators $T_2(p)$ and $T_4(p)$, all $p|N$, and for the unramified Hecke algebras at all good places $p \nmid N$.*

We can now state our main result.

5.2 Theorem. *We assume that an L–function theory as in 4.1 exists. Let N be a square-free positive integer, and let $f \in S_k(\Gamma_0(N))^{\rm new}$ be a newform in the sense of Definition 2.1. We assume that f is an eigenform for the unramified local Hecke algebras $\mathcal{H}_p$ for almost all primes p. We further assume that f is an eigenfunction for $T_2(p)$ and $T_4(p)$ for all $p|N$,*

$$T_2(p)f = \lambda_p f, \qquad T_4(p)f = \mu_p f \qquad \text{for } p|N. \tag{12}$$

Then:

i) *f is an eigenfunction for the local Hecke algebras $\mathcal{H}_p$ for all primes $p \nmid N$.*

ii) *Only the combinations of λ_p and μ_p as given in the following table can occur. Here ε is ± 1.*

λ	μ	rep.	$L_p(s,f)^{-1}$	$\varepsilon_p(s,f)$
$-\varepsilon p$	$\notin\{0,2p\}$	IIa	$(1+\varepsilon(p+1)(p-\mu)p^{-3/2-s}+p^{-2s})(1+\varepsilon p^{-1/2-s})$	$\varepsilon p^{1/2-s}$
$\neq\pm p$	0	IIIa	$(1-\lambda p^{-3/2-s})(1-\lambda^{-1}p^{1/2-s})$	p^{1-2s}
$-\varepsilon p$	$2p$	Vb,c	$(1-\varepsilon p^{1/2-s})(1-p^{-1/2-s})(1+p^{-1/2-s})$	$\varepsilon p^{1/2-s}$
$-\varepsilon p$	0	VIa,b	$(1+\varepsilon p^{-1/2-s})^2$	p^{1-2s}

(We omit some indices p.)

iii) *We define archimedean local factors according to our L–function theory and unramified spin Euler factors for $p \nmid N$ as usual. For places $p|N$ we define L– and ε–factors according to the table in ii). Then the resulting L–function has meromorphic continuation to the whole complex plane and satisfies the functional equation*

$$L(s,f) = \varepsilon(s,f)L(1-s,f), \tag{13}$$

where $L(s,f) = \prod_{p\le\infty} L_p(s,f)$ and $\varepsilon(s,f) = \prod_{p|N\infty} \varepsilon_p(s,f)$.

Sketch of proof: Statement i) follows from Theorem 4.2. Statement ii) follows by explicitly computing the possible eigenvalues of $T_2(p)$ and $T_4(p)$ in local representations. In the present case we cannot conclude that in the global representation $\pi_f = \oplus\pi_i$ all the irreducible components π_i must be isomorphic, because the eigenvalues in (12) cannot tell apart local representations VIa and VIb. This is however the only ambiguity, so that we can at least associate a *global L–packet* with f. (As mentioned before, VIa and VIb constitute a local L–packet.) The table in ii) indicates the possible representations depending on the Hecke eigenvalues.

The L–factors given in the table are those coming from the local Langlands correspondence. By hypothesis they coincide with the factors in our L–function theory. Hence the L–function in (13) coincides with the L–function of any one of the automorphic representations in our global L–packet. By our L–function theory we get the functional equation. ■

5.3 Corollary. *If a cusp form $f \in S_k(\mathrm{Sp}(2,\mathbb{Z}))$ is an eigenfunction for the unramified Hecke algebras $\mathcal{H}_p$ for almost all primes p, then it is an eigenfunction for those Hecke algebras for all p.*

Remarks:

i) The corollary does not claim that f generates an irreducible automorphic representation of $\mathrm{PGSp}(2,\mathbb{A})$, but a multiple of such a representation. Without knowing multiplicity one for PGSp(2) we cannot conclude that f is determined by all its Hecke eigenvalues.

ii) The local factors given in Theorem 5.2 are the Langlands L– and ε–factors for the *spin* (degree 4) L–function. The following table lists the Langlands factors for the *standard* (degree 5) L–function.

λ	μ	rep.	$L_p(s,f,\mathrm{st})^{-1}$	$\varepsilon_p(s,f,\mathrm{st})$
$-\varepsilon p$	$\notin\{0,2p\}$	IIa	$(1-(p+1)(p-\mu)p^{-2-s}+p^{-1-2s})(1-p^{-s})$	p^{1-2s}
$\neq\pm p$	0	IIIa	$(1-\lambda^2p^{-2-s})(1-\lambda^{-2}p^{2-s})(1-p^{-1-s})$	p^{1-2s}
$-\varepsilon p$	$2p$	Vb,c	$(1+p^{-1-s})(1+p^{-s})(1-p^{-s})$	p^{1-2s}
$-\varepsilon p$	0	VIa,b	$(1-p^{-s})^2(1-p^{-1-s})$	p^{1-2s}

iii) There is a statement analogous to Theorem 5.2 for modular forms with respect to the paramodular group $\Gamma^{\mathrm{para}}(N)$. Instead of $T_4(p)$ as defined in (11) this result makes use of the "dual" endomorphism $T_5(p) := (1+p)^2\, d_{02}\circ d_1$ of $S_k(\Gamma^{\mathrm{para}}(N))^{\mathrm{new}}$.

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Global base change identity and Drinfeld's shtukas

Ngô Bao Châu

This is the text of my talk at the conference "Automorphic forms and representation theory of p-adic groups" in Kyoto, January 2003. It summarizes my preprint [7] which will be published elsewhere. In loc. cit. we propose a new approach to prove the global base change identity which arises in the comparison of the Lefschetz trace formula on moduli space of Drinfeld's shtukas and the Selberg's trace formula, without using the fundamental lemma for base change.

I would like to thank the organizers Professors H. Saito and T. Takahashi for this very instructive conference. I am also grateful to Professor L. Breen for linguistic helps in the preparation of this manusript.

1 Drinfeld's shtukas with multiples modifications

Let X be a geometrically connected, smooth and projective curve over $\mathbb{F}_q$. Let $\bar{X} = X \otimes_{\mathbb{F}_q} k$ where k is an algebraic closure of $\mathbb{F}_q$. Let σ denote the geometric Frobenius element of $\mathrm{Gal}(k/\mathbb{F}_q)$.

Let F denote the function field of X. For every closed point $x \in |X|$, let F_x be the completion of F at x and $\mathcal{O}_x$ be the ring of integers of F_x.

Let $d \geq 2$ be an integer and $G = \mathrm{GL}_d$. According to Drinfeld, one has the notion of G-shtukas with multiples modifications which we are going to review in a moment. Let $\bar{x}_1, \ldots, \bar{x}_n \in X(k)$ be n mutually distinct geometric points of X. Let $\bar{T} = \{\bar{x}_1, \ldots, \bar{x}_n\}$. A $\bar{T}$-modification is an isomorphism

$$t : \mathcal{V}'^{\bar{T}} \tilde{\longrightarrow} \mathcal{V}^{\bar{T}}$$

between the restrictions $\mathcal{V}'^{\bar{T}}$ an $\mathcal{V}^{\bar{T}}$ of vector bundles of rank d $\mathcal{V}'$ and $\mathcal{V}$ over $\bar{X}$ to the $\bar{X} - \bar{T}$.

Let $\bar{x} \in \bar{T}$ and let denote $\mathcal{V}'_{\bar{x}}$ and $\mathcal{V}_{\bar{x}}$ the completions of $\mathcal{V}'$ and $\mathcal{V}$ at $\bar{x}$. These are free $\mathcal{O}_{\bar{x}}$-modules of rank d whose generic fibers are identified with $t_x : V'_{\bar{x}} \tilde{\longrightarrow} V_x$. By the theory of elementary divisors, two $\mathcal{O}_{\bar{x}}$-lattices within the same

$F_{\bar{x}}$-vector spaces can be given an invariant

$$\mathrm{inv}(t_{\bar{x}}) \in \mathbb{Z}^d_+ = \{(\lambda^1, \dots, \lambda^d) \in \mathbb{Z}^d \mid \lambda^1 \geq \cdots \geq \lambda^d\}.$$

For general reductive group G, $\mathbb{Z}^d_+$ must be replaced by the set of dominant coweights of G and and this set comes equipped with a natural partial order : $\lambda \geq \lambda'$ if and only if $\lambda - \lambda'$ is a sum of positive coroots. This partial order has geometric origin since an $\bar{x}$-modification with invariant λ can only degenerate to a $\bar{x}$-modification with some invariant $\lambda' \leq \lambda$. It will be convenient to write formally

$$\mathrm{inv}(t) = \sum_{i=1}^{n} \mathrm{inv}(t_{\bar{x}_i})\bar{x}_i.$$

We will say

$$\sum_{i=1}^{n} \mathrm{inv}(t_{\bar{x}_i})\bar{x}_i \leq \sum_{i=1}^{n} \lambda_i \bar{x}_i$$

if for every $i = 1, \dots, n$, we have $\mathrm{inv}(t_{\bar{x}_i}) \leq \lambda_i$.

Definition 1 (Drinfeld) *Let $\underline{x} = (\bar{x}_1, \dots, \bar{x}_n)$ be a collection of mutually distinct k-points of X and let $\underline{\lambda}$ a collection of dominant coweights $\lambda_1, \dots, \lambda_n \in \mathbb{Z}^d_+$. A $\underline{\lambda}$-shtuka over $\underline{x}$ is a pair $(\mathcal{V}, t)$ where $\mathcal{V}$ is a vector bundle of rank d over $\bar{X}$ and t is a $\bar{T}$-modification with $\bar{T} = \{\bar{x}_1, \dots, \bar{x}_n\}$*

$$t : {}^{\sigma}\mathcal{V}^{\bar{T}} \xrightarrow{\sim} \mathcal{V}^{\bar{T}}$$

with $\mathrm{inv}(t) \leq \sum_{i=1}^n \lambda_i \bar{x}_i$*. Here ${}^{\sigma}\mathcal{V}$ denotes the pull-back of $\mathcal{V}$ by the endomorphism* $\mathrm{id}_X \otimes_{\mathbb{F}_q} \sigma$ *of $X \otimes_{\mathbb{F}_q} k$*

These data have a moduli stack

$$c'_{\underline{\lambda}} : \mathcal{S}'_{\underline{\lambda}} \to X^n - \Delta$$

where Δ is the union of all diagonals in X^n. This moduli space can be continued over the diagonals at the price of a small break of symmetry. Let $\underline{x} = (\bar{x}_1, \dots, \bar{x}_n) \in X^n(k)$ with possibly $\bar{x}_i = \bar{x}_j$. Then a $\underline{\lambda}$-shtuka over $\underline{x}$ is a collection of vector bundles of rank d

$$\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_d$$

over $\bar{X}$ equipped with

- a collection of modifications

  $$t_1 : \mathcal{V}_1^{\bar{x}_1} \xrightarrow{\sim} \mathcal{V}_0^{\bar{x}_1}, \dots, t_n : \mathcal{V}_n^{\bar{x}_n} \xrightarrow{\sim} \mathcal{V}_{n-1}^{\bar{x}_n}$$

  such that for every $i = 1, \dots, n$, $\mathrm{inv}(t_i) \leq \lambda_i \bar{x}_i$,

- and an isomorphisme ${}^{\sigma}\mathcal{V}_0 \xrightarrow{\sim} \mathcal{V}_n$.

For a point $\underline{x}$ away from the diagonals Δ, this definition is equivalent to Definition 1.1. Therefore the above c'_{λ} can be continued in a natural way to a obtain a smooth morphism

$$c_{\underline{\lambda}} : \mathcal{S}_{\underline{\lambda}} \to X^n.$$

For every finite subscheme I of X, one can define the notion of an I-level structure of a shtuka. We also have a moduli space of $\underline{\lambda}$-shtukas with I-level structure

$$c^I_{\underline{\lambda}} : \mathcal{S}^I_{\underline{\lambda}} \to (X - I)^n.$$

This morphism is smooth, locally of finite type but in general not of finite type. This lack of finiteness is one of the main difficulties that Lafforgue had to overcome in his solution of Langlands' correspondence for GL_d over function fields [5]. Since we want to focus into another aspect of moduli spaces of shtukas, we prefer for the moment to avoid this difficulty by restricting ourself to the case of $\mathcal{D}$-shtukas associated to a division algebra.

Let D be a division algebra over F and let $\mathcal{D}$ be a maximal O_X-algebra with generic fiber D. Let X' be the open of X where D is unramified. Let $G = D^\times$ as F-group. For every place $v \in |X'|$, G_v is isomorphic to GL_d. We can define the moduli space of G-shtukas in completely similar way to shtukas for GL_d and obtain a morphism

$$c^I_{\underline{\lambda},a} : (\mathcal{D} - \mathcal{S}^I_{\underline{\lambda}})/a^{\mathbb{Z}} \to (X' - I)^n$$

which is a separated, proper and smooth morphism under the assumption $I \neq \emptyset$. Here $a \in \mathbb{A}_F^\times$ is an idele with $\deg(a) \neq 0$ and the group $a^{\mathbb{Z}}$ acts freely on the moduli space of shtukas by $(\mathcal{V}, t) \mapsto (\mathcal{V} \otimes \mathcal{L}(a), \mathrm{id}_{\mathcal{L}(a)})$ where $\mathcal{L}(a)$ is the line bundle on X associated to the idele a.

Let $\mathcal{F}_{\underline{\lambda}}$ be the intersection complex of $\mathcal{S}^I_{\underline{\lambda}}$. As usual, the restricted tensor product

$$\mathcal{H}^I = \bigotimes_{v \in |X' - I|} \mathcal{H}_v$$

where $\mathcal{H}_v$ is the unramified Hecke algebra of G_v, acts by correspondences on

$$\mathrm{R}^i (c^I_{\underline{\lambda},a})_* \mathcal{F}^I_{\underline{\lambda}}$$

which is a local system on $(X' - I)^n$ for all integer i.

Theorem 2 *We have the following equality in the Grothendieck group of local systems on $(X' - I)^n$ equipped with action of $\mathcal{H}^I$*

$$\sum_i (-1)^i [\mathrm{R}^i (c^I_{\underline{\lambda},a})_* \mathcal{F}_{\underline{\lambda}}] = \bigoplus_{\pi} m(\pi) \pi^I \otimes \bigotimes_{i=1}^{n} \mathrm{pr}_i^* \mathcal{L}_{\lambda_i}(\pi)$$

where π runs over the set of automorphic representation of $G(\mathbb{A}_F)$ where $a^{\mathbb{Z}}$ acts trivially, $m(\pi)$ its multiplicity, $\mathcal{L}_{\lambda_i}(\pi)$ is the local system on $X' - I$ such that the equality of L-functions holds

$$L(\mathcal{L}_{\lambda_i}(\pi), s) = L(\pi, \lambda_i; s)$$

where $L(\pi, \lambda_i; s)$ is the automorphic L-function associated to π and to the representation of $\hat{G}$ of highest weight λ_i.

This statement is what one can expect from the cohomology of moduli space of shtukas, according to Langlands' philosophy.

2 Outline of the proof

In order ro simplify the exposition, we will restrict ourself to the case $n = 1$ and $\lambda = (\lambda^1 \geq \cdots \geq \lambda^d)$ with $\sum_j \lambda^j = 0$.

Let $\bar{x} \in (X' - I)(k)$ with $\sigma^s(x) = x$ where σ denotes the action of the geometric Frobenius on $(X' - I)(k)$. Let x be the closed point of $X' - I$ supporting $\bar{x}$.

Let $T' \subset X' - I - \{x\}$ be a finite reduced subscheme and let $\lambda'_{T'} : |T'| \to \mathbb{Z}^d_+$ be an arbitrary function. Let

$$\Phi_{T', \lambda'_{T'}} = \bigotimes_{v \in |T'|} \phi_{\lambda'(v)} \otimes \bigotimes_{v \notin |T'|} 1_v \in \mathcal{H}^I$$

where $\phi_{\lambda'(v)}$ is the characteristic function of the double coset $G(\mathcal{O}_v)\lambda'_v G(\mathcal{O}_v)$ in $G(F_v)$, and 1_v is the unit function.

One can use a similar method for counting points, due to Langlands and Kottwitz [3], in order to prove the following formula

$$\mathrm{Tr}(\sigma^s \circ \Phi_{T', \lambda'_{T'}}) = \sum_{(\gamma_0, \delta_x)} \mathrm{vol}(J_{\gamma_0, \delta_x}(F) a^{\mathbb{Z}} \backslash J_{\gamma_0, \delta_x}(\mathbb{A}_F)) \prod_{v \in |X - T' - \{x\}|} \mathbf{O}_{\gamma_0}(1_v) \prod_{v \in |X'|} \mathbf{O}_{\gamma_0}(\phi_{\lambda'(v)}) \mathbf{TO}_{\delta_x}(\psi_{\lambda, \bar{x}}) \quad (1)$$

where

- γ_0 is a conjugacy class of $G(F)$, δ_x is a σ-conjugacy class of $G(F_x \otimes_{\mathbb{F}_q} \mathbb{F}_{q^s})$ whose norm down to $G(F_x)$ is the class of γ_0.
- $J_{(\gamma_0, \delta_x)}$ is the F-group which is an inner form of the centralizer G_{γ_0} of γ_0 such that at a place $v \neq x$, $(J_{(\gamma_0, \delta_x)})_v$ is isomorphic to $(G_{\gamma_0})_v$ and at x, $(J_{(\gamma_0, \delta_x)})_x$ is isomorphic to the twisted centralizer of δ_x. This inner form is well defined up to isomorphism.

- The function $\psi_{\lambda,\bar{x}} \in \mathcal{H}(G(F_x \otimes_{\mathbb{F}_q} \mathbb{F}_{q^s}))$ is defined as follows. Let $y_1, \dots, y_r$ be the places of $F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^s}$ over x. Assume the geometric point $\bar{x}$ lies over y_1. Then we define

  $$\psi_{\lambda,\bar{x}} = \psi_{\lambda(y_1)} \otimes 1_{y_2} \otimes \cdots \otimes \cdots 1_{y_r}$$

  where $1_{y_2}, \dots, 1_{y_r}$ are the unit functions of $\mathcal{H}(G_{y_2}), \dots, \mathcal{H}(G_{y_r})$ respectively. The function $\psi_{\lambda(y_1)} \in \mathcal{H}(G_{y_1})$ is the unique function whose the Satake transform is the function on $\hat{G}(\mathbb{C})$ given by

  $$\hat{g} \mapsto \mathrm{Tr}(\hat{g}, V_\lambda)$$

  where V_λ is the irreducible representation of $\hat{G}$ of highest weight λ.

I refer to [7] for the detailed proof of this counting point formula.

To prove the theorem, we need to transform (1) in to a sum without twisted orbital integral. Namely, we want to prove that (1) is equal to the following sum

$$\sum_{\gamma_0} \mathrm{vol}(G_{\gamma_0}(F)a^{\mathbb{Z}}\backslash G_{\gamma_0}(\mathbb{A}_F)) \prod_{v\in|X-T'-\{x\}|} \mathbf{O}_{\gamma_0}(1_v) \prod_{v\in|X'|} \mathbf{O}_{\gamma_0}(\phi_{\lambda'(v)})\mathbf{O}_{\gamma_0}(\mathbf{b}(\psi_{\lambda,\bar{x}})) \quad (2)$$

where

$$\mathbf{b} : \mathcal{H}(G(F_x \otimes_{\mathbb{F}_q} \mathbb{F}_{q^s})) \to \mathcal{H}(G(F_x))$$

is the base change homomorphism. Once the equality (1) = (2) has been etablished, it remains to apply Selberg to obtain the equality between the sum (2) and the following

$$\mathrm{Tr}\left(\bigotimes_{v\in|X-T'-\{x\}|} 1_v \otimes \bigotimes_{v\in|T'|} \phi_{\lambda'(v)} \otimes \mathbf{b}(\psi_{\lambda,\bar{x}}), \mathrm{L}^2\left(a^{\mathbb{Z}}G(F)\backslash G(\mathbb{A}_F)\right)\right) \quad (3)$$

and the theorem follows by a standard argument.

The above strategy is well known and goes back to Langlands and Kottwitz's work on Shimura varieties [2]. For the moduli space of shtukas, this is also done by Drinfeld and Lafforgue with maybe some technical differences. The only new point in our work concerns the proof of the identity (1) = (2). Usually, one needs the fundamental lemma for base changein order to convert a twisted orbital integral into orbital integral, which is known in p-adic case due to works of Kottwitz, Clozel and Labesse. In positive characteristic, the fundamental lemma for base change was not written down except for the function associated to the minuscule coweight which is proved by a direct calculation due to Drinfeld [6], but it is known to Henniart.

Our point is that one can prove the global base change identity (1) = (2) without using local harmonic analysis but rather a combination of counting of points, local model theory, a geometric interpretation of the base change homomorphism in terms of perverse sheaves and Tchebotarev's density theorem. We hope that our method can be generalized to other situations.

3 Global base change identity

Equality (1) = (2) will be proved by counting points on two different moduli spaces called A and B.

3.1 Situation A

The moduli space A is a scalar restriction à la Weil. Consider the s-fold product

$$(c^I_{\lambda,a})^s : (\mathcal{D} - \mathcal{S}^I_\lambda/a^{\mathbb{Z}})^s \to (X' - I)^s$$

of $c^I_{\lambda,a} : (\mathcal{D} - \mathcal{S}^I_\lambda)/a^{\mathbb{Z}} \to X' - I$. This morphism comes with an action of the symmetric group $\mathfrak{S}_s$ and of the action by correspondences of $(\mathcal{H}^I)^{\otimes s}$. Let denote

$$[A] := \sum_i (-1)^i \mathrm{R}(c^I_{\lambda,a})^s_* \mathcal{F}_\lambda^{\boxtimes s}$$

the class in the Grothendieck group of local system on $(X' - I)^s$ equipped with an action of $(\mathcal{H})^s$ and with a compatible action of $\mathfrak{S}_s$. By the Kunneth formula, $[A]$ should be

$$\bigoplus_{\pi_1,\ldots,\pi_s} \prod_{i=1}^s m(\pi_i) \bigotimes_{i=1}^s \pi_i^I \otimes \bigotimes_{i=1}^s \mathrm{pr}_i^* \mathcal{L}_\lambda(\pi_i) \qquad (4)$$

where $\pi_1, \ldots, \pi_s$ are automorphic representations of G with trivial action of $a^{\mathbb{Z}}$. It's clear how $\mathfrak{S}_s$ and $(\mathcal{H}^I)^s$ should act on (4).

Assume for simplicity that the closed point x supporting $\bar{x}$ is of degree 1. Let $\underline{x} = (\bar{x}, \ldots, \bar{x})$ be the correponding point in the small diagonal of $(X' - I)^s$. By usual properties of Weil's scalar restriction, (1) is equal to

$$\mathrm{Tr}(\tau \circ \sigma \circ (1 \otimes \cdots \otimes 1 \otimes \Phi_{T',\lambda'_{T'}}), [A]_{\underline{x}}) \qquad (5)$$

where $\tau \in \mathfrak{S}_s$ is the cyclic permutation.

3.2 Situation B

Let us consider a particular collection of coweights

$$\underline{s\lambda} = (\underbrace{\lambda, \ldots, \lambda}_{s})$$

and the associated moduli space of shtukas with "symmetric modifications"

$$c^I_{\underline{s\lambda}} : (\mathcal{D} - \mathcal{S}^I_{\underline{s\lambda}})/a^{\mathbb{Z}} \to (X' - I)^s.$$

By the very definition, for every $\tau \in \mathfrak{S}_s$, the fiber of $c^I_{\underline{s\lambda}}$ over a point $(\overline{x}_1, \ldots, \overline{x}_s)$ away from the union Δ of all diagonals, is canonically isomorphic with the fiber over $\tau(\overline{x}_1, \ldots, \overline{x}_s)$. This gives rises to a compatible action of $\mathfrak{S}_s$ on the restriction of

$$\mathrm{R}(c^I_{\underline{s\lambda}})_* \mathcal{F}_{\underline{s\lambda}}$$

to $(X'-I)^s - \Delta$. Since this direct image is a local system, we can extend canonically the action of $\mathfrak{S}_s$ over the diagonals. Let denote

$$[B] = \sum_i (-1)^i \mathrm{R}(c^I_{\underline{s\lambda}})_* \mathcal{F}_{\underline{s\lambda}}$$

the class in the Grothendieck group of local systems equipped with an action of $\mathcal{H}^I$ and a compatible action of $\mathfrak{S}_s$.

Assuming Theorem 2, $[B]$ should be

$$\bigoplus_{\pi} m(\pi)\pi^I \otimes \bigotimes_{i=1}^{s} \mathrm{pr}_i^* \mathcal{L}_\lambda(\pi) \tag{6}$$

Let $\underline{x} = (\overline{x}, \ldots, \overline{x})$ in the small diagonal as in 3.1. We want to compute

$$\mathrm{Tr}(\tau \circ \sigma \circ \Phi_{T',\lambda'_{T'}}, [B]_{\underline{x}}) \tag{7}$$

where τ is the cyclic permutation like in 3.1. A priori, it is not obvious how to compute this trace by counting points, since the action of the symmetric group is not concretely defined over the diagonals. This is however possible using local model theory and the geometric interpretation of the base change homomorphism in terms of perverse sheaves on the affine Grassmannian. What we get finally is (2) = (7).

3.3 Main observation

To prove (1) = (2) is now equivalent to proving (5) = (7). We can in fact prove a more general equality.

Theorem 3 *For all $\xi \in \pi_1((X' - I)^s)$ and $\phi \in \mathcal{H}^I$ and for the cyclic permutation $\tau \in \mathfrak{S}_r$, we have*

$$\mathrm{Tr}(\tau \circ \xi \circ (\underbrace{1 \otimes \cdots \otimes 1}_{s-1} \otimes \Phi), [A]_{\underline{x}}) = \mathrm{Tr}(\tau \circ \xi \circ \Phi, [B]_{\underline{x}}) \tag{8}$$

Heuristically, assuming Theorem 2, equality (8) can be proved as follows. In comparing (6) with (4) one can observe that (6) consists essentially in the diagonal terms $\pi_1 = \cdots = \pi_s$ of (4), up to multiplicity. But the non-diagonal terms of (4) are permuted around by τ and therefore don't contribute to the trace. The diagonals terms of (4) give now the same trace as (6) according to the following general linear algebra lemma which is implicit in papers of Saito and Shintani on base change.

Lemma 4 *Let V be a finite dimensional vector space over some field K. Let f be any endomorphism of V. Then*

$$\mathrm{Tr}(f, V) = \mathrm{Tr}(\tau \circ (\underbrace{1 \otimes \cdots \otimes 1}_{s-1} \otimes f), V^{\otimes s})$$

where τ is the cyclic permutation.

3.4 Tchebotarev's density theorem

The rigourous proof of Theorem 3 makes essential use of Tchebotarev's density theorem. Let $U = (X' - I)^s - \Delta$ be the complement of the union of all diagonals. Let $\tilde{U} = U/\langle\tau\rangle$ the free quotient of U by the cyclic group $\langle\tau\rangle = \mathbb{Z}/s\mathbb{Z}$ generated by τ. One has the exact sequence of fundamental groupoids

$$1 \to \pi_1(U) \to \pi_1(\tilde{U}) \to \mathbb{Z}/s\mathbb{Z} \to 1.$$

Any closed point $\tilde{u} \in |\tilde{U}|$ gives rises to a conjugacy class $\mathrm{Frob}_{\tilde{u}}$ of $\pi_1(\tilde{U})$. A closed point $\tilde{u} \in |\tilde{U}|$ is called **cyclic** if the image of $\mathrm{Frob}_{\tilde{u}}$ in $\mathbb{Z}/s\mathbb{Z}$ is the generator τ. By Tchebotarev's theorem, it is enough to prove

$$\mathrm{Tr}(\mathrm{Frob}_{\tilde{u}} \circ (\underbrace{1 \otimes \cdots \otimes 1}_{s-1} \otimes \Phi), [A]) = \mathrm{Tr}(\mathrm{Frob}_{\tilde{u}} \circ \Phi, [B])$$

for all cyclic closed points $\tilde{u} \in |\tilde{U}|$ and for all $\Phi \in \mathcal{H}^I$.

Since we are away from the diagonals one can compute the above traces by counting points without using local model theory. The nice feature of cyclic points is that in the expressions of traces of cyclic points on $[A]$ and $[B]$, there are no twisted orbital integrals. The expressions we get for the traces of cyclic points on $[A]$ and on $[B]$, are in fact identical.

Note that even outside the diagonals, if we take $\mathrm{Frob}_{\tilde{u}}^2$ instead of $\mathrm{Frob}_{\tilde{u}}$, the expressions we gets for the traces on $[A]$ and $[B]$ are no longer identical due to the apperance of twisted orbital integrals on both side. Therefore our proof relies heavily on Tchebotarev's theorem.

The proof in the case $n > 1$ is a little more complicated since the closed points of $X^{ns}/(\mathbb{Z}/s\mathbb{Z})$ are not as nice as those of $X^s/(\mathbb{Z}/s\mathbb{Z})$. For that case, we made essential use of a theorem of Drinfeld asserting that the representations of $\pi_1((X'-I)^{ns})$ on $[A]$ and on $[B]$ factor through $\pi_1(X'-I)^{ns}$. Consequently, instead of closed points $X^{ns}/(\mathbb{Z}/s\mathbb{Z})$ we can take collections of n cyclic closed points of $X^s/(\mathbb{Z}/s\mathbb{Z})$. We refer again to [7] for more details.

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RESTRICTION OF HERMITIAN MAASS LIFTS AND THE GROSS-PRASAD CONJECTURE (JOINT WITH T. IKEDA)

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This note is a report on a joint work with Tamotsu Ikeda [12].

After the discovery of the integral representation of triple product L-functions by Garrett [5], Harris and Kudla [10] determined the transcendental parts of the central critical values of triple product L-functions. The transcendental parts behaves differently according to whether the weights are "balanced" or not. In the "balanced" case, the critical values of triple product L-functions have also been studied by Garrett [5], Orloff [18], Satoh [20], Garrett and Harris [6], Gross and Kudla [7], Böcherer and Schulze-Pillot [4], and so on. By contrast, in the "imbalanced" case, there are no results on the critical values of triple product L-functions except [10] to our knowledge. We express certain period integrals of Maass lifts which appear in the Gross-Prasad conjecture [8], [9], as the algebraic parts of the central critical values in the "imbalanced" case.

1. The Gross-Prasad conjecture

In [8], [9], Gross and Prasad suggested that the central values of certain L-functions control a global obstruction of blanching rules for automorphic representations of special orthogonal groups. Let V be a non-degenerate quadratic space of dimension n over a number field k and $H = \mathrm{SO}(V)$ the special orthogonal group of V. Take a non-degenerate quadratic subspace V' of V of dimension $n-1$ and regard $H' = \mathrm{SO}(V')$ as a subgroup of H. Let $\tau \simeq \otimes_v \tau_v$ (resp. $\tau' \simeq \otimes_v \tau'_v$) be an irreducible cuspidal automorphic representation of $H(\mathbb{A}_k)$ (resp. $H'(\mathbb{A}_k)$).

Conjecture 1.1 (Gross-Prasad)**.** *Assume that τ and τ' are both tempered. Then the period integral*

$$\langle G|_{H'}, F\rangle = \int_{H'(k)\backslash H'(\mathbb{A}_k)} G(h)\overline{F(h)}\, dh$$

does not vanish for some $G \in \tau$ and some $F \in \tau'$ if and only if

(i) $\mathrm{Hom}_{H'(k_v)}(\tau_v, \tau'_v) \neq 0$ *for all places v of k,*

(ii) $L(1/2, \tau \times \tau') \neq 0$.

Remark that a meromorphic continuation of the L-function $L(s, \tau \times \tau')$ has not been established in general, however, it could be described in terms of L-functions of general linear groups by the functoriality.

We also note that the conjecture is supported by the results of Waldspurger [22] for $n = 3$, Harris and Kudla [10], [11] for $n = 4$, Böcherer, Furusawa, and Schulze-Pillot [3] for $n = 5$.

Gross and Prasad restricted their conjecture to the tempered cases. According to the Arthur conjecture [2], non-tempered cuspidal automorphic representations exist, and if τ or τ' is non-tempered, then the L-function $L(s, \tau \times \tau')$ could have a pole at $s = 1/2$. Hence a modification to the condition (ii) would be inevitable if one consider the Gross-Prasad conjecture in general (see [3] for $n = 5$). Our result provides an example for $n = 6$ when τ, τ' are both non-tempered. Remark that the triple product L-function considered in this note is only of degree 8 and is a part of the L-function $L(s, \tau \times \tau')$ of degree 24.

2. Saito-Kurokawa lifts

First, we review the notion of Saito-Kurokawa lifts [16], [17], [1], [23]. Let k be a positive even integer. Let

$$F(Z) = \sum_{B>0} A(B) e^{2\pi\sqrt{-1}\,\mathrm{tr}(BZ)} \in S_k(\mathrm{Sp}_2(\mathbb{Z})), \quad Z \in \mathfrak{h}_2$$

be a Siegel modular form of degree 2. Here $\mathfrak{h}_2$ is the Siegel upper half plane given by

$$\mathfrak{h}_2 = \left\{ Z = {}^tZ \in \mathrm{M}_2(\mathbb{C}) \mid \mathrm{Im}(Z) > 0 \right\}.$$

We say that F satisfies the Maass relation if there exists a function $\beta_F^* : \mathbb{N} \to \mathbb{C}$ such that

$$A\left(\begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}\right) = \sum_{d|(n,r,m)} d^{k-1} \beta_F^* \left(\frac{4nm - r^2}{d^2}\right).$$

We denote by $S_k^{\mathrm{Maass}}(\mathrm{Sp}_2(\mathbb{Z}))$ the space of Siegel cusp forms which satisfy the Maass relation.

Kohnen [13] introduced the plus subspace $S_{k-1/2}^+(\Gamma_0(4))$ given by

$$S_{k-1/2}^+(\Gamma_0(4)) = \{h(\tau) = \sum_{N>0} c(N) q^N \in S_{k-1/2}(\Gamma_0(4)) \mid c(N) = 0 \text{ if } -N \not\equiv 0, 1 \bmod 4\}.$$

For $F \in S_k^{\mathrm{Maass}}(\mathrm{Sp}_2(\mathbb{Z}))$, put

$$\Omega^{\mathrm{SK}}(F)(\tau) = \sum_{\substack{N \geq 0 \\ -N \equiv 0,1 \bmod 4}} \beta_F^*(N) q^N.$$

Then $\Omega^{\mathrm{SK}}(F) \in S_{k-1/2}^+(\Gamma_0(4))$, and the linear map

$$\Omega^{\mathrm{SK}} : S_k^{\mathrm{Maass}}(\mathrm{Sp}_2(\mathbb{Z})) \longrightarrow S_{k-1/2}^+(\Gamma_0(4))$$

is an isomorphism.

3. Hermitian Maass lifts

Next, we recall an analogue of Saito-Kurokawa lifts for hermitian modular forms by Kojima [14], Sugano [21], and Krieg [15]. Let $K = \mathbb{Q}(\sqrt{-\mathbf{D}})$ be an imaginary quadratic field with discriminant $-\mathbf{D} < 0$, $\mathcal{O}$ the ring of integers of K, w_K the number of roots of unity contained in K, and χ be the primitive Dirichlet character corresponding to $K/\mathbb{Q}$. Write

$$\chi = \prod_{q \in Q_{\mathbf{D}}} \chi_q,$$

where $Q_{\mathbf{D}}$ is the set of all primes dividing $\mathbf{D}$ and χ_q is a primitive Dirichlet character mod $q^{\mathrm{ord}_q \mathbf{D}}$ for each $q \in Q_{\mathbf{D}}$.

Let k be a positive integer such that $w_K \mid k$. Let

$$G(Z) = \sum_{H \in \Lambda_2(\mathcal{O})^+} A(H) e^{2\pi\sqrt{-1}\,\mathrm{tr}(HZ)} \in S_k(U(2,2)), \quad Z \in \mathcal{H}_2$$

be a hermitian modular form of degree 2. Here $\mathcal{H}_2$ is the hermitian upper half plane given by

$$\mathcal{H}_2 = \left\{ Z \in \mathrm{M}_2(\mathbb{C}) \;\middle|\; \frac{1}{2\sqrt{-1}}(Z - {}^t\bar{Z}) > 0 \right\},$$

and

$$\Lambda_2(\mathcal{O})^+ = \left\{ H = {}^t\bar{H} \in \frac{1}{\sqrt{-\mathbf{D}}}\mathrm{M}_2(\mathcal{O}) \;\middle|\; \mathrm{diag}(H) \in \mathbb{Z}^2,\, H > 0 \right\}.$$

We say that G satisfies the Maass relation if there exists a function $\alpha_G^* : \mathbb{N} \to \mathbb{C}$ such that

$$A(H) = \sum_{d \mid \varepsilon(H)} d^{k-1} \alpha_G^* \left(\frac{\mathbf{D}\det(H)}{d^2} \right),$$

where

$$\varepsilon(H) = \max\{ n \in \mathbb{N} \,|\, n^{-1}H \in \Lambda_2(\mathcal{O})^+ \}.$$

We denote by $S_k^{\mathrm{Maass}}(U(2,2))$ the space of hermitian cusp forms which satisfy the Maass relation.

Krieg [15] introduced the space $S_{k-1}^*(\Gamma_0(\mathbf{D}), \chi)$ which is an analogue of the Kohnen plus subspace and is given by

$$S_{k-1}^*(\Gamma_0(\mathbf{D}), \chi) = \{ g^*(\tau) = \sum_{N>0} a_{g^*}(N) q^N \in S_{k-1}(\Gamma_0(\mathbf{D}), \chi) \mid$$
$$a_{g^*}(N) = 0 \text{ if } \mathbf{a_D}(N) = 0 \},$$

where

$$\mathbf{a_D}(N) = \prod_{q \in Q_{\mathbf{D}}} (1 + \chi_q(-N)).$$

Let

$$g(\tau) = \sum_{N>0} a_g(N) q^N \in S_{k-1}(\Gamma_0(\mathbf{D}), \chi)$$

be a primitive form. For each $Q \subset Q_{\mathbf{D}}$, set

$$\chi_Q = \prod_{q \in Q} \chi_q, \quad \chi'_Q = \prod_{q \in Q_{\mathbf{D}} - Q} \chi_q.$$

Then there exists a primitive form

$$g_Q(\tau) = \sum_{N \geq 0} a_{g_Q}(N) q^N \in S_{k-1}(\Gamma_0(\mathbf{D}), \chi)$$

such that

$$a_{g_Q}(p) = \begin{cases} \chi_Q(p) a_g(p) & \text{if } p \notin Q, \\ \chi'_Q(p) \overline{a_g(p)} & \text{if } p \in Q, \end{cases}$$

for each prime p. Put

$$g^* = \sum_{Q \subset Q_{\mathbf{D}}} \chi_Q(-1) g_Q. \tag{3.1}$$

Then $g^* \in S^*_{k-1}(\Gamma_0(\mathbf{D}), \chi)$. When g runs over primitive forms in $S_{k-1}(\Gamma_0(\mathbf{D}), \chi)$, the forms g^* span $S^*_{k-1}(\Gamma_0(\mathbf{D}), \chi)$.

For $G \in S_k^{\mathrm{Maass}}(U(2,2))$, put

$$\Omega(G)(\tau) = \sum_{N > 0} \mathbf{a}_{\mathbf{D}}(N) \alpha^*_G(N) q^N.$$

Then $\Omega(G) \in S^*_{k-1}(\Gamma_0(\mathbf{D}), \chi)$, and the linear map

$$\Omega : S_k^{\mathrm{Maass}}(U(2,2)) \longrightarrow S^*_{k-1}(\Gamma_0(\mathbf{D}), \chi)$$

is an isomorphism.

4. Statement of the main theorem

Let k be a positive integer such that $w_K \mid k$. Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a primitive form and $h(\tau) = \sum_{N>0} c(N) q^N \in S^+_{k-1/2}(\Gamma_0(4))$ a Hecke eigenform which corresponds to f by the Shimura correspondence. Note that h is unique up to scalars. Let $F = (\Omega^{\mathrm{SK}})^{-1}(h) \in S_k^{\mathrm{Maass}}(\mathrm{Sp}_2(\mathbb{Z}))$ be the Saito-Kurokawa lift of f. Define the Petersson norms of f and F by

$$\langle f, f \rangle = \int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}_1} |f(\tau)|^2 y^{2k-4} d\tau,$$

$$\langle F, F \rangle = \int_{\mathrm{Sp}_2(\mathbb{Z}) \backslash \mathfrak{h}_2} |F(Z)|^2 |\det \mathrm{Im}(Z)|^{k-3} dZ,$$

respectively.

Let $g(\tau) = \sum_{N>0} a_g(N) q^N \in S_{k-1}(\Gamma_0(\mathbf{D}), \chi)$ be a primitive form and $G = \Omega^{-1}(g^*) \in S_k^{\mathrm{Maass}}(U(2,2))$ the hermitian Maass lift of g, where $g^* \in S^*_{k-1}(\Gamma_0(\mathbf{D}), \chi)$ is given by (3.1). Observe that $\mathfrak{h}_2 \subset \mathcal{H}_2$, and by [15], the restriction $G|_{\mathfrak{h}_2}$ belongs to $S_k^{\mathrm{Maass}}(\mathrm{Sp}_2(\mathbb{Z}))$.

The completed triple product L-function $\Lambda(s, g \times g \times f)$ is given by

$$\Lambda(s, g \times g \times f) = (2\pi)^{-4s+4k-8} \Gamma(s) \Gamma(s-2k+4) \Gamma(s-k+2)^2 L(s, g \times g \times f),$$

and satisfies a functional equation which replaces s with $4k-6-s$.

Our main result is as follows.

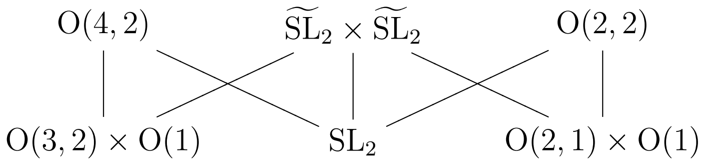
Theorem 4.1.

$$\frac{\Lambda(2k-3, g\times g\times f)}{\langle f, f\rangle^2} = -2^{4k-6}\mathbf{D}^{-2k+3}c(\mathbf{D})^2\frac{\langle G|_{\mathfrak{h}_2}, F\rangle^2}{\langle F, F\rangle^2}$$

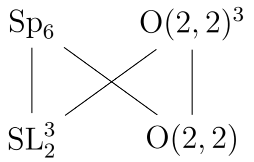
5. Proof

Theorem 4.1 follows from the following seesaws.

(5.1)



(5.2)



To explain these seesaws more precisely, we introduce some notation. In [13], Kohnen defined a linear map

$$\mathcal{S}^+_{-\mathbf{D}} : S^+_{k-1/2}(\Gamma_0(4)) \longrightarrow S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})),$$

$$\sum_{N>0} c(N)q^N \longmapsto \sum_{N>0}\sum_{d|N}\chi(d)d^{k-2}c\left(\frac{N^2}{d^2}\mathbf{D}\right)q^N.$$

If $h(\tau) = \sum_{N>0} c(N)q^N \in S^+_{k-1/2}(\Gamma_0(4))$ is a Hecke eigenform and corresponds to $f\in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ by the Shimura correspondence, then

$$\mathcal{S}^+_{-\mathbf{D}}(h) = c(\mathbf{D})f.$$

Let $\mathrm{Tr}^{\mathbf{D}}_1$ denote the trace operator given by

$$\mathrm{Tr}^{\mathbf{D}}_1 : S_{2k-2}(\Gamma_0(\mathbf{D})) \longrightarrow S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})),$$

$$f \longmapsto \sum_{\gamma\in\Gamma_0(\mathbf{D})\backslash\mathrm{SL}_2(\mathbb{Z})} f|\gamma.$$

The seesaw (5.1) accounts for the following identity.

Proposition 5.1.

$$\mathcal{S}^+_{-\mathbf{D}}(\Omega^{\mathrm{SK}}(G|_{\mathfrak{h}_2})) = a_g(\mathbf{D})^2\,\mathrm{Tr}^{\mathbf{D}}_1(g^2).$$

This identity is proved by computing the Fourier coefficients of the both sides explicitly.

The seesaw (5.2) accounts for the following refinement of the main identity by Harris and Kudla [10].

Proposition 5.2.

$$\Lambda(2k-3, g\times g\times f) = -2^{4k-6}\mathbf{D}^{-2k+3}a_g(\mathbf{D})^4\langle \mathrm{Tr}_1^{\mathbf{D}}(g^2), f\rangle^2$$

This identity is proved by computing the local zeta integrals which arise in the integral representation of triple product L-functions by Garrett [5], Piatetski-Shapiro and Rallis [19] at bad primes.

Now Theorem 4.1 follows from Propositions 5.1 and 5.2.

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MULTIPLICITIES OF CUSP FORMS

WEE TECK GAN

1. Introduction

Let G be a connected simple linear algebraic group defined over a number field F. It is a basic problem in the theory of automorphic forms to describe the spectral decomposition of the unitary representation $L^2(G(F)\backslash G(\mathbb{A}))$ of $G(\mathbb{A})$. Such a unitary representation possesses an orthogonal decompostion

$$L^2(G(F)\backslash G(\mathbb{A})) = L^2_{disc} \oplus L^2_{cont}$$

into the direct sum of its discrete spectrum and its continuous spectrum. Let us write:

$$L^2_{disc} = \bigoplus_{\pi} m_{disc}(\pi) \cdot \pi.$$

It is known that the discrete multiplicities $m_{disc}(\pi)$ are finite. The discrete spectrum has a further orthogonal decomposition

$$L^2_d(G(F)\backslash G(\mathbb{A})) = L^2_{cusp} \oplus L^2_{res}$$

where L^2_{cusp} is the subspace of cusp forms, and L^2_{res} is the so-called residual spectrum. Let us write:

$$L^2_{cusp} = \hat{\oplus}_{\pi} m_{cusp}(\pi) \cdot \pi \quad \text{and} \quad L^2_{res} = \hat{\oplus}_{\pi} m_{res}(\pi) \cdot \pi.$$

In this talk, we consider the following two simple-minded questions:

(A) Does there exist π such that $m_{cusp}(\pi) \cdot m_{res}(\pi) \neq 0$?

(B) Can the collection of non-negative integers $\{m_{cusp}(\pi)\}$ be unbounded?

Here are some prior results on these questions:

(i) When $G = PGL_n$, the results of Jacquet-Shalika [JS] and the multiplicity one theorem imply that $m_{disc}(\pi) \leq 1$ and thus the answers are negative for both questions.

(ii) When $G = SL_2$, it is a recent result of Ramakrishnan [R] that $m_{disc}(\pi) \leq 1$.

(iii) For a more general classical group G , it is known that $m_{cusp}(\pi)$ can be > 1. Examples of such failure of multiplicity one were constructed by Labesse-Langlands [LL] for the inner forms of SL_2, by Blasius [B] for SL_n (with $n \geq 3$) and by Li [L] for quaternionic unitary groups. However, in these examples, the multiplicities are bounded above by a number depending only on the given G.

In this talk, I will discuss the following theorem, which was obtained jointly with N. Gurevich and D.-H. Jiang in [GGJ]:

A talk given at the conference "Automorphic forms and representations of algebraic groups over local fields" at RIMS, Kyoto, 20-24th January 2003.

Theorem 1.1. *When $G = G_2$, both questions A and B have positive answers. More precisely, for each finite set S of places of F, with $\#S \geq 2$, there is an irreducible unitary representation π_S of $G_2(\mathbb{A})$ with*

$$\begin{cases} m_{res}(\pi_S) = 1, \\ m_{disc}(\pi_S) \geq \frac{1}{6}(2^{\#S} + (-1)^{\#S}2). \end{cases}$$

The representations π_S of the theorem are very degenerate: their local components are non-tempered and non-generic. They are the so-called unipotent representations. This may lead one to think that the phenomenon of unbounded cuspidal multiplicities only happens for very degenerate representations. However, as we explain in Section 3, it should already occur for representations in tempered L-packets. We shall discuss in Section 5 how we intend to construct these tempered representations of arbitrarily high cuspidal multiplicities.

In fact, the unboundedness of discrete multiplicities for G_2 is a consequence of a famous conjecture of J. Arthur (see [A1] and [A2]). Hence, we shall begin by reviewing his conjecture in the following section.

2. **My Understanding of Arthur's Conjecture**

In this section, we shall briefly discuss Arthur's conjecture on $L^2_{disc}(G(F)\backslash G(\mathbb{A}))$. For simplicity, we assume that G is split, simple and simply-connected, so that the dual group $\widehat{G}$ is adjoint. We begin by introducing some notations.

Let L_F denote the Langlands group of F (whose existence is still conjectural). For the purpose of understanding Arthur's conjecture, there is no loss in pretending that L_F is the absolute Galois group of F. For each place v of F, one also has a local group L_{F_v}, and there should be a natural class of embeddings $L_{F_v} \hookrightarrow L_F$. The group L_{F_v} is actually known to exist: it is the Weil group if v is archimedean and the Weil-Deligne group if v is finite.

By an Arthur parameter for G, we mean a $\widehat{G}$-conjugacy class of homomorphisms

$$\psi : L_F \times SL_2(\mathbb{C}) \longrightarrow \widehat{G}$$

so that the following conditions hold:

- $\psi(L_F)$ is bounded in $\widehat{G}$;
- the centralizer $\mathcal{S}_\psi$ of the image of ψ is finite.

Given ψ, Arthur defined a quadratic character ϵ_ψ of $\mathcal{S}_\psi$. In the examples we will look at later, ϵ_ψ turns out to be the trivial character. Hence we will not bother to go into the general definition here.

We will describe the conjecture in the statements A, B and C below.

(A) There is a decomposition:

$$L^2_{disc}(G(F)\backslash G(\mathbb{A})) = \bigoplus_\psi L^2[\psi],$$

indexed by the Arthur parameters for G.

Fix a parameter ψ. We must now describe the $G(\mathbb{A})$-module $L^2[\psi]$. Using the embedding $L_{F_v} \hookrightarrow L_F$, we obtain local parameters

$$\psi_v : L_{F_v} \times SL_2(\mathbb{C}) \hookrightarrow \widehat{G}.$$

Let us set:

- $\mathcal{S}_{\psi_v}$ = the finite group of components of the centralizer of the image of ψ_v.
- $\mathcal{S}_{\psi,\mathbb{A}} = \prod_v \mathcal{S}_{\psi_v}$, a compact group.
- $\Delta : \mathcal{S}_\psi \longrightarrow \mathcal{S}_{\psi,\mathbb{A}}$, the natural diagonal map.

(B) For each place v of F, there is a finite subset A_{ψ_v} of unitary representations of $G(F_v)$ associated to ψ_v; this is the so-called local Arthur packet. This finite set is indexed by the irreducible characters of $\mathcal{S}_{\psi_v}$:

$$A_{\psi_v} = \{\pi_{\eta_v} : \eta_v \in \widehat{\mathcal{S}_{\psi_v}}\}.$$

Moreover, it should satisfy the following conditions:

- for almost all v where $\psi_v|_{L_{F_v}}$ is unramified, $\pi_{\mathbf{1_v}}$ is the irreducible unramified representation with Satake parameter
$$s_{\psi_v} := \psi_v \left(\mathrm{Frob}_v \times \begin{pmatrix} q_v^{1/2} & \\ & q_v^{-1/2} \end{pmatrix} \right).$$
- a particular linear combination of the characters of the π_{η_v}'s is a stable distribution.
- certain identities involving transfer to endoscopic groups hold.

Here we have not described the last two conditions precisely as they will not be relevant for us in this talk.

If $\eta = \bigotimes_v \eta_v$ is an irreducible character of $\mathcal{S}_{\psi,\mathbb{A}}$, then we may set

$$\pi_\eta = \bigotimes_v \pi_{\eta_v}.$$

This is possible because for almost all v, $\eta_v = \mathbf{1_v}$ and $\pi_{\mathbf{1_v}}$ is required to be unramified by the above. We can now state the last statement of Arthur's conjecture:

(C) The $G(\mathbb{A})$-submodule $L^2[\psi]$ has a decomposition given by:

$$L^2[\psi] = \bigoplus_{\eta \in \widehat{\mathcal{S}_{\psi,\mathbb{A}}}} m_\eta \cdot \pi_\eta$$

where

$$m_\eta = \langle \epsilon, \Delta^*(\eta) \rangle_{\mathcal{S}_\psi}$$

is the multiplicity of ϵ in the representation $\Delta^*(\eta)$ of $\mathcal{S}_\psi$.

This concludes our discussion of Arthur's conjecture.

3. **The Example of G_2**

Now we examine the special case when $G = G_2$ so that $\widehat{G} = G_2(\mathbb{C})$. We shall write down some Arthur parameters for G_2 and see what Arthur's conjecture says for them. Essentially, the only fact we need to know about G_2 is the following:

Lemma 3.1. *$G_2(\mathbb{C})$ contains a subgroup isomorphic to $SO_3(\mathbb{C}) \times S_3$, where S_3 is the symmetric group on 3 letters. Moreover, the centralizer of $SO_3(\mathbb{C})$ is precisely S_3.*

The map $SL_2(\mathbb{C}) \rightarrow SO_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C})$ corresponds via the Jacobson-Morozzov theorem to the subregular unipotent orbit in $G_2(\mathbb{C})$. With this lemma in hand, we can now write down our first family of Arthur parameters.

3.1. **Cubic unipotent parameters.**

Let E be an étale cubic F-algebra. Then E corresponds to a conjugacy class of maps

$$\rho_E : L_F \longrightarrow Gal(\overline{F}/F) \longrightarrow S_3.$$

Using ρ_E and the natural projection map from $SL_2(\mathbb{C})$ to $SO_3(\mathbb{C})$, we set:

$$\psi_E : L_F \times SL_2(\mathbb{C}) \longrightarrow S_3 \times SO_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C}).$$

The maps ψ_E are the cubic unipotent Arthur parameters.

For simplicity, we shall only consider the case when $E = E_0$ is the split algebra $F \times F \times F$. In this case, ρ_{E_0} is the trivial map, and so we have:

$$\begin{cases} \mathcal{S}_{\psi_{E_0}} = \mathcal{S}_{\psi_{E_0},v} = S_3 \\ \mathcal{S}_{\psi_{E_0},\mathbb{A}} = S_3(\mathbb{A}). \end{cases}$$

The map $\mathcal{S}_\psi \rightarrow \mathcal{S}_{\psi,\mathbb{A}}$ is simply the natural embedding $S_3(F) \hookrightarrow S_3(\mathbb{A})$.

What does Arthur's conjecture say for the parameter ψ_{E_0}? Well, statement B predicts that for each place v, the corresponding local Arthur packet has 3 members indexed by the irreducible characters of S_3. So we have:

$$A_{\psi_{E_0}} = \{\pi_{\mathbf{1_v}}, \pi_{r_v}, \pi_{\epsilon_v}\}$$

where ϵ_v is the sign character of S_3 and r_v is the 2-dimensional one. Further, for S a finite set of places of F, let

$$\eta_S = (\otimes_{v\in S} r_v) \bigotimes (\otimes_{v\notin S} \mathbf{1_v}).$$

Then statement C predicts that the representation

$$\pi_S := \pi_{\eta_S} = (\otimes_{v\in S} \pi_{r_v}) \bigotimes (\otimes_{v\notin S} \pi_{\mathbf{1_v}})$$

occurs in $L^2[\psi_{E_0}]$ with multiplicity equal to the multiplicity of the trivial representation in $r \otimes r \otimes .... \otimes r$ ($\#S$ times). A quick computation gives:

$$m_{disc}(\pi_S) \geq \frac{1}{6} \cdot (2^{\#S} + (-1)^{\#S} 2),$$

which is one of the main claims of Theorem 1.1. Thus Arthur's conjecture predicts the existence of a family of representations $\{\pi_S\}$ whose discrete multiplicities are unbounded as $\#S \rightarrow \infty$.

3.2. **Some Tempered Parameters.** Now we consider some tempered Arthur parameters, i.e. those for which ψ is trivial on $SL_2(\mathbb{C})$. Let us start with a cuspidal representation τ of PGL_2 such that

$$\tau_v = \begin{cases} \text{Steinberg representation for } v \in S_\tau; \\ \text{an unramified representation for } v \notin S_\tau \end{cases}$$

for some finite set S_τ of finite places of F. Conjecturally, τ corresponds to a map $\phi_\tau : L_F \longrightarrow SL_2(\mathbb{C})$. Because of our assumptions, the map ϕ_τ is surjective; in fact, for $v \in S_\tau$, the local parameter ϕ_{τ_v} is already surjective, since it corresponds to the Steinberg representation.

Now we construct an Arthur parameter for G_2 using ϕ_τ as follows:

$$\psi_\tau : L_F \longrightarrow SL_2(\mathbb{C}) \to SO_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C}).$$

Then we have:

$$\begin{cases} \mathcal{S}_{\psi_\tau} = \mathcal{S}_{\psi_\tau,v} = S_3 \quad \text{for all } v \in S_\tau. \\ \mathcal{S}_{\psi_\tau,v} = \{1\} \quad \text{for all } v \notin S_\tau. \end{cases}$$

In particular, statement B in Arthur's conjecture predicts that the local packets have the following form:

$$A_{\psi_\tau,v} = \begin{cases} \{\pi'_{\mathbf{1_v}}, \pi'_{r_v}, \pi'_{\epsilon_v}\} \quad \text{if } v \in S_\tau; \\ \{\pi'_{\mathbf{1_v}}\} \quad \text{if } v \notin S_\tau. \end{cases}$$

Moreover, the representations in the local packets should be tempered.

In fact, the parameter ψ_τ is an example of Langlands parameter considered by Lusztig. Hence, in this case, the local packet $A_{\psi_\tau,v}$ has already been defined, and it does consist of 3 discrete series representations (see [GrS]).

Finally, if we set

$$\pi_\tau = \left(\otimes_{v\in S_\tau} \pi'_{r_v}\right) \bigotimes \left(\otimes_{v\notin S_\tau} \pi'_{\mathbf{1_v}}\right),$$

then statement C in Arthur's conjecture implies that

$$m_{disc}(\pi_\tau) \geq \frac{1}{6} \cdot (2^{\# S_\tau} + (-1)^{\# S_\tau} 2).$$

In fact, since the representation π'_τ is tempered, it cannot occur in the residual spectrum, and so we have

$$m_{cusp}(\pi_\tau) \geq \frac{1}{6} \cdot (2^{\# S_\tau} + (-1)^{\# S_\tau} 2).$$

Now one can find cuspidal representations τ of PGL_2 of the above type and with S_τ as big as one wishes (using the trace formula for example). Hence, Arthur's conjecture predicts that one can find a family of tempered representations of $G_2(\mathbb{A})$ whose cuspidal multiplicities are unbounded.

4. **Construction of Unipotent Cusp Forms**

In this section, we explain how one constructs the unipotent representation π_S and demonstrates Theorem 1.1.

Let H be the disconnected linear algebraic group $Spin_8 \rtimes S_3$. For each place v of F, the group $H(F_v)$ has a distinguished representation Π_v known as the minimal representation. To be more precise, Π_v is a particular extension to $H(F_v)$ of the unramified representation of $Spin_8(F_v)$ whose Satake parameter is

$$\iota \begin{pmatrix} q_v^{1/2} & \\ & q_v^{-1/2} \end{pmatrix}$$

where $\iota : SL_2(\mathbb{C}) \longrightarrow PGSO_8(\mathbb{C})$ is the map associated to the subregular unipotent orbit of the dual group $PGSO_8(\mathbb{C})$.

Now H contains the subgroup $S_3 \times G_2$, and one may restrict the representation Π_v to the subgroup $S_3(F_v) \times G_2(F_v)$ to get:

$$\Pi_v = \bigoplus_{\eta_v \in \widehat{S_3(F_v)}} \eta_v \otimes \pi_{\eta_v}.$$

In the beautiful papers [HMS] and [V], Huang-Magaard-Savin (for non-archimedean v) and Vogan (for archimedean v) showed that each π_{η_v} is a non-zero irreducible unitarizable representation and the π_{η_v}'s are mutually distinct. Moreover, the representations π_{η_v} can be completely determined, and $\pi_{\mathbf{1_v}}$ is unramified with Satake parameter $s_{\psi_{E_0,v}}$. In view of these results, it seems natural to take the set of representations π_{η_v} as the elements of the local Arthur packet $A_{\psi_{E_0,v}}$.

Consider now the global situation. If $\Pi = \otimes_v \Pi_v$, then as an abstract representation of $S_3(\mathbb{A}) \times G_2(\mathbb{A})$, we have:

$$\Pi = \bigoplus_{\eta} \eta \otimes \pi_\eta$$

as $\eta = \otimes_v \eta_v$ ranges over the irreducible representations of $S_3(\mathbb{A})$. In particular, for each η, we have an embedding

$$\iota_\eta : \eta \otimes \pi_\eta \hookrightarrow \Pi.$$

Using residues of Eisenstein series, one can construct a $Spin_8(\mathbb{A})$-equivariant embedding

$$\Theta : \Pi \hookrightarrow \mathcal{A}^2(Spin_8)$$

of Π into the space of square-integrable automorphic forms of $Spin_8$. For each η, we may now define a $G_2(\mathbb{A})$-equivariant map Θ_η as follows:

$$\Theta_\eta : \eta \otimes \pi_\eta \xrightarrow{\iota_\eta} \Pi \xrightarrow{\Theta} \mathcal{A}^2(Spin_8) \xrightarrow{\text{restriction}} \{\text{functions on } G_2(F)\backslash G_2(\mathbb{A})\}.$$

Then the following was proved in [GGJ]:

Theorem 4.1. *(i) The image of Θ_η is contained in $\mathcal{A}^2(G_2)$.*

(ii) The restriction of Θ_η to the subspace $\eta^{S_3(F)} \otimes \pi_\eta$ is injective.

The proof of the theorem is not difficult; it involves showing the non-vanishing of certain Fourier coefficients. Also, it is easy to see that the restriction of Θ_η to $(\eta^{S_3(F)})^\perp \otimes \pi_\eta$ is identically zero. In any case, the theorem immediately implies that

$$m_{disc}(\pi_S) \geq \frac{1}{6} \cdot (2^{\#S} + (-1)^{\#S} 2).$$

In fact, in [G], we show that equality holds when F is totally real.

To complete the proof of Theorem 1.1, one may appeal to the determination of the residual spectrum of G_2 by H. Kim [K] and S. Zampera [Z]. Their results show that L^2_{res} has the multiplicity one property, and further that $m_{res}(\pi_S) = 1$. This concludes the proof of Theorem 1.1.

5. Potential Construction of some Tempered Cusp Forms

Finally, we would like to explain how we expect to show that the tempered representation π_τ discussed in Section 3 has cuspidal multiplicity at least that predicted by Arthur's conjecture.

The parameter

$$\psi_\tau : L_F \longrightarrow SO_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C})$$

actually factors as:

$$\psi_\tau : L_F \longrightarrow SO_3(\mathbb{C}) \hookrightarrow SL_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C}).$$

Hence, instead of lifting the cuspidal representation τ of PGL_2 directly to G_2, one may first lift it to a cuspidal representation of PGL_3. This is precisely the Gelbart-Jacquet lift, and we denote this cuspidal representation of PGL_3 by $GJ(\tau)$. Note that

$$GJ(\tau)_v = \begin{cases} \text{the Steinberg representation } St_v \text{ if } v \in S_\tau; \\ \text{a specific unramified representation if } v \notin S_\tau. \end{cases}$$

Now it turns out that $PGL_3 \times G_2$ is a dual pair in the split (adjoint) exceptional group of type E_6. This suggests that we may use exceptional theta correspondence to lift $GJ(\tau)$ from PGL_3 to G_2: hopefully we will get the representation π_τ. For this to work out, one should first verify that under local theta correspondence, the Steinberg representation St_v of $PGL_3(F_v)$ lifts to the representation π'_τ of $G_2(F_v)$. However, it was shown in [GS] that the theta lift of St_v is equal to $\pi'_{\mathbf{1}} \oplus \pi'_\epsilon$. So this doesn't work out as planned.

Thankfully, a homomorphism $L_F \longrightarrow SL_3(\mathbb{C})$ is not just a Langlands parameter for PGL_3; it is also a parameter for any inner form of PGL_3. Such an inner form is of the form $PD^\times$ where D is a degree 3 division algebra. Over a p-adic field F_v, there are two such division algebras: D_v and its opposite D_v^{opp}. Being opposite algebras, their groups of invertible elements define isomorphic algebraic groups. Thus, locally, PGL_3 has precisely one inner form $PD^\times$.

Now under the local Jacquet-Langlands correspondence, the Steinberg representation St_v corresponds to the trivial representation $\mathbf{1_v}$ of $PD^\times(F_v)$. Moreover, $PD^\times \times G_2$ is a dual pair in an inner form of E_6. It was shown in [S] that the local theta lift of $\mathbf{1_v}$ is indeed equal to π'_τ.

Hence we are led to the following strategy for embedding π_τ into L^2_{cusp}. Choose a global division algebra D of degree 3 which is ramified precisely at the set S_τ. Then one lifts τ from PGL_2 to G_2 as follows:

$$\begin{array}{ccccccc} PGL_2 & \xrightarrow{\text{Gelbart-Jacquet}} & PGL_3 & \xrightarrow{\text{Jacquet-Langlands}} & PD^\times & \xrightarrow{\text{theta lift}} & G_2 \\ \tau & \longrightarrow & GJ(\tau) & \longrightarrow & JL_D(GJ(\tau)) & \longrightarrow & \Theta(JL_D(GJ(\tau))). \end{array}$$

As an abstract representation, $\Theta(JL_D(GJ(\tau)))$ is indeed isomorphic to π_τ (if it is non-zero).

How does the multiplicity $\frac{1}{6} \cdot (2^{\#S_\tau} + (-1)^{\#S_\tau} 2)$ arise in this case? The answer lies in the following lemma:

Lemma 5.1. *The number of global division algebras of degree 3 ramified precisely at a set S is equal to*

$$\frac{1}{3} \cdot (2^{\#S} + (-1)^{\#S} 2).$$

In particular, the number of inner forms of PGL_3 which are ramified at the set S is half of the above number.

Note that the various inner forms of the lemma are non-isomorphic as algebraic groups, but their groups of adelic points are abstractly isomorphic. Thus the reason for the high multiplicity here is the failure of Hasse principle for the inner forms of PGL_3!

In order for the above strategy to work, it remains to show:

- the non-vanishing of the theta lift $\Theta(JL_D(GJ(\tau)))$;
- the various $\Theta(JL_D(GJ(\tau)))$'s generate linearly independent copies of π_τ in L^2_{cusp}.

At the moment, we are still trying to resolve these questions.

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ON THE LIFTING OF HERMITIAN MODULAR FORMS

TAMOTSU IKEDA

Notation

Let K be an imaginary quadratic field with discriminant $-\mathbf{D} = -\mathbf{D}_K$. We denote by $\mathcal{O} = \mathcal{O}_K$ the ring of integers of K. The non-trivial automorphism of K is denoted by $x \mapsto \bar{x}$. The primitive Dirichlet character corresponding to $K/\mathbb{Q}$ is denoted by $\chi = \chi_{\mathbf{D}}$. We denote by $\mathcal{O}^\sharp = (\sqrt{-\mathbf{D}})^{-1}\mathcal{O}$ the inverse different ideal of $K/\mathbb{Q}$.

The special unitary group $G = \mathrm{SU}(m, m)$ is an algebraic group defined over $\mathbb{Q}$ such that

$$G(R) = \left\{ g \in \mathrm{SL}_{2m}(R \otimes K) \,\middle|\, g \begin{pmatrix} \mathbf{0}_m & -\mathbf{1}_m \\ \mathbf{1}_m & \mathbf{0}_m \end{pmatrix} {}^t\bar{g} = \begin{pmatrix} \mathbf{0}_m & -\mathbf{1}_m \\ \mathbf{1}_m & \mathbf{0}_m \end{pmatrix} \right\}$$

for any $\mathbb{Q}$-algebra R. We put $\Gamma_K^{(m)} = G(\mathbb{Q}) \cap \mathrm{GL}_{2m}(\mathcal{O})$.

The hermitian upper half space $\mathcal{H}_m$ is defined by

$$\mathcal{H}_m = \{ Z \in M_m(\mathbb{C}) \mid \frac{1}{2\sqrt{-1}}(Z - {}^t\bar{Z}) > 0 \}.$$

Then $G(\mathbb{R})$ acts on $\mathcal{H}_m$ by

$$g\langle Z \rangle = (AZ + B)(CZ + D)^{-1}, \quad Z \in \mathcal{H}_m, g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We put

$$\Lambda_m(\mathcal{O}) = \{ h = (h_{ij}) \in \mathrm{M}_m(K) \mid h_{ii} \in \mathbb{Z}, h_{ij} = \bar{h}_{ji} \in \mathcal{O}^\sharp, i \neq j \},$$
$$\Lambda_m(\mathcal{O})^+ = \{ h \in \Lambda_m(\mathcal{O}) \mid h > 0 \}.$$

We set $\mathbf{e}(T) = \exp(2\pi\sqrt{-1}\mathrm{tr}(T))$ if T is a square matrix with entries in $\mathbb{C}$. For each prime p, the unique additive character of $\mathbb{Q}_p$ such that $\mathbf{e}_p(x) = \exp(-2\pi\sqrt{-1}x)$ for $x \in \mathbb{Z}[p^{-1}]$ is denoted $\mathbf{e}_p$. Note that $\mathbf{e}_p$ is of order 0. We put $\mathbf{e}_{\mathbb{A}}(x) = \mathbf{e}(x_\infty)\prod_{p<\infty} \mathbf{e}_p(x_p)$ for an adele $x = (x_v)_v \in \mathbb{A}$.

Let $\underline{\chi} = \otimes_v \underline{\chi}_v$ be the Hecke character of $\mathbb{A}^\times/\mathbb{Q}^\times$ determined by χ. Then $\underline{\chi}_v$ is the character corresponding to $\mathbb{Q}_v(\sqrt{-\mathbf{D}})/\mathbb{Q}$ and given by

$$\underline{\chi}_v(t) = \left(\frac{-\mathbf{D}, t}{\mathbb{Q}_v}\right).$$

Let $Q_{\mathbf{D}}$ be the set of all primes which divides $\mathbf{D}$. For each prime $q \in Q_{\mathbf{D}}$, we put $\mathbf{D}_q = q^{\mathrm{ord}_q \mathbf{D}}$. We define a primitive Dirichlet character χ_q by

$$\chi_q(n) = \begin{cases} \chi(n') & \text{if } (n,q) = 1 \\ 0 & \text{if } q|n, \end{cases}$$

where n' is an integer such that

$$n' \equiv \begin{cases} n & \mod \mathbf{D}_q, \\ 1 & \mod \mathbf{D}_q^{-1}\mathbf{D} \end{cases}$$

Then we have $\chi = \prod_{q|\mathbf{D}} \chi_q$. Note that

$$\chi_q(n) = \left(\frac{\chi_q(-1)\mathbf{D}_q, n}{\mathbb{Q}_q}\right) = \prod_{p|n} \left(\frac{\chi_q(-1)\mathbf{D}_q, n}{\mathbb{Q}_p}\right)$$

for $q \nmid n$, $n > 0$. One should not confuse χ_q with $\underline{\chi}_q$.

1. Fourier coefficients of Eisenstein series on $\mathcal{H}_m$

In this section, we consider Siegel series associated to non-degenerate hermitian matrices. Fix a prime p. Put $\xi_p = \chi(p)$, i.e.,

$$\xi_p = \begin{cases} 1 & \text{if } -\mathbf{D} \in (\mathbb{Q}_p^\times)^2 \\ -1 & \text{if } \mathbb{Q}_p(\sqrt{-\mathbf{D}})/\mathbb{Q}_p \text{ is unramified quadratic extension} \\ 0 & \text{if } \mathbb{Q}_p(\sqrt{-\mathbf{D}})/\mathbb{Q}_p \text{ is ramified quadratic extension.} \end{cases}$$

For $H \in \Lambda_m(\mathcal{O})$, $\det H \neq 0$, we put

$$\gamma(H) = (-\mathbf{D})^{[m/2]} \det(H)$$
$$\zeta_p(H) = \underline{\chi}_p(\gamma(H))^{m-1}.$$

The Siegel series for H is defined by

$$b_p(H,s) = \sum_{R \in \mathrm{Her}_m(K_p)/\mathrm{Her}_m(\mathcal{O}_p)} \mathbf{e}_p(\mathrm{tr}(BR)) p^{-\mathrm{ord}_p(\nu(R))s}, \quad \mathrm{Re}(s) \gg 0.$$

Here, $\mathrm{Her}_m(K_p)$ (resp. $\mathrm{Her}_m(\mathcal{O}_p)$) is the additive group of all hermitian matrices with entries in K_p (resp. $\mathcal{O}_p$). The ideal $\nu(R) \subset \mathbb{Z}_p$ is defined as follows: Choose a coprime pair $\{C, D\}$, $C,\ D \in \mathrm{M}_{2n}(\mathcal{O}_p)$ such that $C\,{}^t\bar{D} = D\,{}^t\bar{C}$, and $D^{-1}C = R$. Then $\nu(R) = \det(D)\mathcal{O}_p \cap \mathbb{Z}_p$.

We define a polynomial $t_p(K/\mathbb{Q}; X) \in \mathbb{Z}[X]$ by

$$t_p(K/\mathbb{Q}; X) = \prod_{i=1}^{[(m+1)/2]} (1 - p^{2i}X) \prod_{i=1}^{[m/2]} (1 - p^{2i-1}\xi_p X).$$

There exists a polynomial $F_p(H; X) \in \mathbb{Z}[X]$ such that

$$F_p(H; p^{-s}) = b_p(H, s)t_p(K/\mathbb{Q}; p^{-s})^{-1}.$$

This is proved in [9].

Moreover, $F_p(H; X)$ satisfies the following functional equation:

$$F_p(H; p^{-2m}X^{-1}) = \zeta_p(H)(p^m X)^{-\mathrm{ord}_p \gamma(H)} F_p(H; X).$$

This functional equation is a consequence of [7], Proposition 3.1. We will discuss it in the next section.

The functional equation implies that $\deg F_p(H; X) = \mathrm{ord}_p \gamma(H)$. In particular, if $p \nmid \gamma(H)$, then $F_p(H; X) = 1$. Put

$$\tilde{F}_p(H; X) = X^{-\mathrm{ord}_p \gamma(H)} F_p(H; p^{-m}X^2).$$

Then following lemma is a immediate consequence of the functional equation of $F(H; X)$.

Lemma 1. *We have*

$$\tilde{F}_p(H; X^{-1}) = \tilde{F}_p(H; X), \qquad \textit{if } m \textit{ is odd.}$$

$$\tilde{F}_p(H; \xi_p X^{-1}) = \tilde{F}_p(H; X), \qquad \textit{if } m \textit{ is even and } \xi_p \neq 0.$$

Let k be a sufficiently large integer. Put $n = [m/2]$. The Eisenstein series $E^{(m)}_{2k+2n}(Z)$ of weight $2k + 2n$ on $\mathcal{H}_m$ is defined by

$$E^{(m)}_{2k+2n}(Z) = \sum_{\{C,D\}/\sim} \det(CZ + D)^{-2k-2n},$$

where $\{C, D\}/\sim$ extends over coprime pairs $\{C, D\}$, $C,\ D \in \mathrm{M}_{2n}(\mathcal{O})$ such that $C\,{}^t\bar{D} = D\,{}^t\bar{C}$ modulo the action of $GL_m(\mathcal{O})$. We put

$$\mathcal{E}^{(m)}_{2k+2n}(Z) = A_m^{-1} \prod_{i=1}^{m} L(1 + i - 2k - 2n, \chi^{i-1}) E^{(m)}_{2k+2n}(Z).$$

Here

$$A_m = \begin{cases} 2^{-4n^2-4n}\mathbf{D}^{2n^2+n} & \text{if } m = 2n + 1, \\ (-1)^n 2^{-4n^2+4n}\mathbf{D}^{2n^2-n} & \text{if } m = 2n. \end{cases}$$

Then the H-th Fourier coefficient of $\mathcal{E}^{(2n+1)}_{2k+2n}(Z)$ is equal to

$$|\gamma(H)|^{2k-1} \prod_{p|\gamma(H)} F_p(H; p^{-2k-2n}) = |\gamma(H)|^{k-(1/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H; p^{-k+(1/2)})$$
$$= |\gamma(H)|^{k-(1/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H; p^{k-(1/2)})$$

for any $H \in \Lambda_{2n+1}(\mathcal{O})^+$ and any sufficiently large integer k.

The H-th Fourier coefficient of $\mathcal{E}^{(2n)}_{2k+2n}(Z)$ is equal to

$$|\gamma(H)|^{2k} \prod_{p|\gamma(H)} F_p(H; p^{-2k-2n}) = |\gamma(H)|^{k} \prod_{p|\gamma(H)} \tilde{F}_p(H; p^{-k})$$

for any $H \in \Lambda_{2n}(\mathcal{O})^+$ and any sufficiently large integer k.

2. **Main theorems**

We first consider the case when $m = 2n$ is even.

Let $f(\tau) = \sum_{N=1}^{\infty} a(N) q^N \in S_{2k+1}(\Gamma_0(\mathbf{D}), \chi)$ be a primitive form, whose L-function is given by

$$L(f, s) = \sum_{N=1}^{\infty} a(N) N^{-s}$$
$$= \prod_{p \nmid \mathbf{D}} (1 - a(p)p^{-s} + \chi(p)p^{2k-2s})^{-1} \prod_{q|\mathbf{D}} (1 - a(q)q^{-s})^{-1}.$$

For each prime $p \nmid \mathbf{D}$, we define the Satake parameter $\{\alpha_p, \beta_p\} = \{\alpha_p, \chi(p)\alpha_p^{-1}\}$ by

$$(1 - a(p)X + \chi(p)p^{2k}X^2) = (1 - p^k \alpha_p X)(1 - p^k \beta_p X).$$

For $q \mid \mathbf{D}$, we put $\alpha_q = q^{-k} a(q)$.

Put

$$A(H) = |\gamma(H)|^k \prod_{p|\gamma(H)} \tilde{F}_p(H, \alpha_p), \quad H \in \Lambda_{2n}(\mathcal{O})^+$$
$$F(Z) = \sum_{H \in \Lambda_{2n}(\mathcal{O})^+} A(H) \mathbf{e}(HZ), \quad Z \in \mathcal{H}_{2n}.$$

Then our first main theorem is as follows:

Theorem 1. *Assume that $m = 2n$ is even. Let $f(\tau)$, $A(H)$ and $F(Z)$ be as above. Then we have $F \in S_{2k+2n}(\Gamma_K^{(2n)})$. Moreover, F is a Hecke eigenform. $F = 0$ if and only if $f(\tau)$ comes from a Hecke character of K and n is odd.*

Now we consider the case when $m = 2n+1$ is odd.

Let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform, whose L-function is given by

$$L(f,s) = \sum_{N=1}^{\infty} a(N)N^{-s}$$
$$= \prod_p (1 - a(p)p^{-s} + p^{2k-1-2s})^{-1}$$

For each prime p, we define the Satake parameter $\{\alpha_p, \alpha_p^{-1}\}$ by

$$(1 - a(p)X + p^{2k-1}X^2) = (1 - p^{k-(1/2)}\alpha_p X)(1 - p^{k-(1/2)}\alpha_p^{-1}X).$$

Put

$$A(H) = |\gamma(H)|^{k-(1/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H, \alpha_p), \quad H \in \Lambda_{2n+1}(\mathcal{O})^+$$

$$F(Z) = \sum_{H \in \Lambda_{2n+1}(\mathcal{O})^+} A(H)\mathbf{e}(HZ), \quad Z \in \mathcal{H}_{2n+1}.$$

Theorem 2. *Assume that $m = 2n+1$ is odd. Let $f(\tau)$, $A(H)$ and $F(Z)$ be as above. Then we have $F \in S_{2k+2n}(\Gamma_K^{(2n+1)})$. Moreover, F is a non-zero Hecke eigenform.*

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