Parabolic induction and parahoric induction

J.-F. Dat

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1 Introduction

In the same way Eisenstein series theory is a masterpiece of the description of the automorphic spectrum, the so-called parabolic induction and restriction functors are prominent tools in the study of smooth representations of a p-adic group G. Given a parabolic subgroup P of G with Levi component M, we will note i_P^G and r_G^P respectively these functors. These are *a priori* functors between categories of all smooth representations of G and M, but it is well known that these functors restrict to (or respect) the subcategories of admissible, resp. finite length, smooth representations. And actually it is generally believed that only the latter category is relevant for automorphic applications. For example the first interesting question for someone interested in automorphic spectral problems is the study of reducibility (and of composition factors) of representations of G parabolically induced from irreducible ones of M, especially when the latter are local components of some automorphic representation. On this question we will say almost nothing.

But among all automorphic aspects, especially thinking to the links with Galois representations, is the study of congruences between automorphic forms as in the pioneering works of Serre and Ribet. This leads naturally to studying not only *complex* but *finite fields-valued* and even *ring-valued* smooth representations. For example one might be interested in studying stable $\overline{\mathbb{Z}}_l$-lattices in $\overline{\mathbb{Q}}_l$-representations. In this respect, the most prominent work is that of Vignéras for GL_n : she classified the finite coefficients smooth dual *à la* Bernstein-Zelevinski and *à la* Bushnell-Kutzko, she also could thoroughly study lattices as above, and eventually she got a beautiful local Langlands' type correspondance modulo a prime l and compatible with Harris-Taylor-Henniart's one through reduction of lattices. Unfortunately all this was possible only by Gelfand's derivatives theory and Bushnell-Kutzko's types theory which at present only exist for GL_n.

In this note we want to explain a general and systematic approach to the study of ring-valued smooth representations. The proofs may be found in [3]. Our general motivation is a possible further application to finite coefficients local Langland's functoriality.

The first systematic algebraic approach to smooth representation theory was that of Bernstein ; he recognized very soon the interest of working with more general smooth representations than just admissible ones. In this respect, he proved highly non trivial abstract (finiteness and cohomological) properties of parabolic functors and relevant categories. However his results work only for complex coefficients (more generally for coefficients in an algebraically closed field of *banal* characteristic). Our first task has been thus to try and extend his results to general ring coefficients. His approach hinges on a good "spectral" understanding of the parabolic functors, ours hinges rather on a tentative of "geometric" understanding. We use Bruhat-Tits' bulding theory and especially the parahoric groups they have defined after Iwahori's pioneering work. These are compact open subgroups in contrast with parabolic subgroups which are closed non-compact.

2 Problems arising from Bernstein's theory

Let R be a ring such that $p \in R^*$. Let us write $Mod_R(G)$ for the category of all smooth R-valued representations (recall that this merely means that any vector is fixed by an open subgroup). We will sum up Bernstein's theory [2], [1] in the following

Theorem 2.1 *(Bernstein)*

i) There is a categorical decomposition $\mathrm{Mod}_{\mathbb{C}}(G) = \bigoplus_{[M,\pi]} \mathrm{Mod}_{\mathbb{C}}(G)_{M,\pi}$ *where by definition* $\mathrm{Mod}_{\mathbb{C}}(G)_{M,\pi}$ *is the full subcategory of all objects all irreducible subquotients of which have cuspidal support conjugate to some unramified twist of* (M, π) *(and thus the sum runs over conjugacy-unramified-twisting classes of such pairs).*

ii) The category $\mathrm{Mod}_{\mathbb{C}}(G)$ *is noetherian. In particular, for any compact open subgroup* H *of* G*, the Hecke algebra* $\mathcal{H}_{\mathbb{C}}(G, H)$ *of compactly supported bi-*H*-invariant distributions is a noetherian algebra.*

iii) Parabolic induction functors send finitely generated complex *representations on finitely generated representations (the corresponding statement for restriction is also true and easy).*

iv) Parabolic restriction r_G^P *is right adjoint to opposite parabolic induction* $\mathrm{i}_{\overline{P}}^G$ *for* complex *representations (highly non-trivial fact, not to be confused with usual Frobenius reciprocity).*

Bernstein's arguments for the proofs of these statements rest heavily on the following

Fact 2.2 *Let* π *be a* complex *irreducible smooth representation of* G*, the following assumptions are equivalent*

i) π *is cuspidal (meaning that its matrix coefficients are compact-modulo-center).*

ii) π *never appears as a subquotient of a parabolically induced representation* $\mathrm{i}_P^G(\sigma)$*.*

iii) π *is a projective object in* $\mathrm{Mod}_{\mathbb{C}}(G)$ *("modulo center").*

Replacing $\mathbb{C}$ by a general algebraically closed field, the three above assumptions may be distinct as soon as the characteristic divides the order of some compact subgroup of G. As a consequence, point i) of the theorem is definitely not true over this kind of fields and no substitute is even conjectured in general. However, points ii), iii) and iv) are expected to hold true in general, even on (noetherian) rings of coefficients.

3 Buildings and parahoric subgroups

3.1 Assume $G = \mathfrak{G}(F)$ for some reductive algebraic group $\mathfrak{G}$ over the p-adic field F. Bruhat and Tits have attached to the pair $(\mathfrak{G}, F)$ an euclidean "extended" building $\mathcal{I}_G$. This is a metric space isomorphic to a product of a euclidean space and a polysimplicial complex with isometric polysimplicial action of G.

Example : In the case of SL_n, the euclidean part is trivial and the polysimplicial part is just simplicial of dimension $n-1$. The set of vertices is in bijection with the homothetic classes of lattices in F^n, while d-simplices correspond to collections of lattices $(\omega_i)_{i=0,\cdots,d-1}$ such that $\omega_0 \subset \omega_1 \subset \cdots \subset \omega_{d-1} \subset \varpi_F^{-1}\omega_0$. This together with obvious incidence relations give the data of a combinatorial polysimplex, and $\mathcal{I}_{SL_n}$ is the standard geometric realisation of this combinatorial polysimplex. One can then identify $\mathcal{I}_{SL_n}$ with the spaces of homothetic classes of norms on F^n. When $n = 2$ we get a homogeneous tree, each vertex belonging to $q+1$ segments.

In the case of a torus T, the simplicial part is trivial and the euclidean part is just $X_*^F(T) \otimes \mathbb{R}$ (rational cocharacters).

When $x \in \mathcal{I}_G$, we note G_x its fixator in G. It is a compact open subgroup, and it is well known that any compact open subgroup is contained in such a fixator. This group G_x has a pro-p-radical noted G_x^+. In general G_x/G_x^+ is isomorphic to the group of rational points of some reductive group over the residue field k_F of F.

Example : For SL_n, the stabilizer of some vertex is always $GL_n(F)$-conjugated to $SL_n(O_F)$ where O_F is the ring of integers of F. The reduction map to k_F sets up a bijection between parabolic subgroups of $SL_n(k_F)$ and fixators of points in the simplicial star of the vertex (*i.e.* the union of all facets whose closure contains the vertex).

3.2 Let M be a F-Levi subgroup of G. Bruhat and Tits have also shown the existence of a (non-unique) isometric and M-equivariant embedding $\mathcal{I}_M \hookrightarrow \mathcal{I}_G$. We will fix such an embedding and consider $\mathcal{I}_M$ as a subset of $\mathcal{I}_G$. Taking up the foregoing notations with M in place of G, it is obvious that $M_x = G_x \cap M$ and it is also true that $M_x^+ = G_x^+ \cap M$. This allows us to use the following general notation : if H is a subgroup of G, we will note $H_x := H \cap G_x$ and $H_x^+ := H \cap G_x^+$.

Example : If T the diagonal torus of SL_n and $\mu \in X_*(T)$ is a rational cocharacter, we can attach to μ the class of the lattice $\sum_{i=1}^n \mu(\varpi_F)_{ii} O_F e_i$ where e_i is the standard basis of F^n. This extends to an embedding of $X_*(T) \otimes \mathbb{R} \hookrightarrow \mathcal{I}_{SL_n}$, and the simplicial structure which is drawn on $X_*(T) \otimes \mathbb{R}$ by the ambient building is that attached to the hyperplane arrangement of $X_*(T) \otimes \mathbb{R}$ given by equations $\{\alpha(x) = k\}_{\alpha,x}$ for all roots α and $k \in \mathbb{Z}$.

3.3 Let P be a parabolic subgroup of G with Levi component M, and let $\overline{P}$ be the opposed parabolic subgroup . It is known that the group G_x^+ has a so-called Iwahori decomposition, meaning that the product map $U_x^+ \times M_x^+ \times \overline{U}_x^+ \longrightarrow G_x^+$ is a bijection, whatever ordering is chosen to make the product. We will briefly account for such decompositions by the simple notation $G_x^+ = U_x^+ M_x^+ \overline{U}_x^+$. Notice that G_x^+ by definition is a normal subgroup of G_x, so that the set $G_{x,P} := P_x G_x^+$ is a group. This group will be called a *parahoric subgroup* of G ; this differs slightly from the Bruhat-Tits definition. It also has a Iwahori decomposition $G_{x,P} = U_x M_x \overline{U}_x^+$.

3.4 Given x, M and P, we would like to construct functors $Mod_R(M_x) \longrightarrow Mod_R(G_{x,P}) \longrightarrow Mod_R(G_x)$ with model the classical construction of parabolic induction $Mod_R(M) \longrightarrow Mod_R(P) \longrightarrow Mod_R(G)$ where the first functor is inflation and the second one is induction. The problem in the parahoric situation is the inflation stage which is impossible since M_x *is not* a quotient of $G_{x,P}$. Next lemma is intended to solve this problem. We need some notations ; for any subgroup H of G we will note $\mathbb{Z}[\frac{1}{p}][H]$ the algebra of all $\mathbb{Z}[\frac{1}{p}]$-values compactly supported distributions. If K is pro-p-subgroup of H, we will note e_K the element of $\mathbb{Z}[\frac{1}{p}][H]$ given by the normalized Haar measure on K.

Lemma 3.5 *There is a central and invertible element $z_{x,P} \in \mathbb{Z}[\frac{1}{p}][G_{x,P}]$ such that $\varepsilon_{x,P} := z_{x,P}^{-1} e_{U_x} e_{\overline{U}_x^+}$ is an idempotent in $\mathbb{Z}[\frac{1}{p}][G_{x,P}]$.*

Notice that by our assumption $p \in R^*$, the algebra $\mathbb{Z}[\frac{1}{p}][G_{x,p}]$ naturaly acts on any smooth R-valued representation of $G_{x,P}$, in particular on the space $\mathcal{C}_R^\infty(G_x)$ of smooth R-valued functions on G_x. Thus we may define $E_{x,P} := \varepsilon_{x,P}.\mathcal{C}_R^\infty(G_x)$. This R-module is endowed with smooth action of G_x on the right and M_x on the left, since M_x normalizes $\varepsilon_{x,P}$. We may thus define functors

$$\begin{array}{rrcl} R_{x,P}: & Mod_R(G_x) & \rightarrow & Mod_R(M_x) \\ & V & \mapsto & E_{x,P} \otimes_{RG_x} V \end{array}$$

and

$$\begin{array}{rrcl} I_{x,P}: & Mod_R(M_x) & \rightarrow & Mod_R(G_x) \\ & W & \mapsto & E_{x,P} \otimes_{RM_x} W \end{array}$$

where tensor products are taken with respect to adequate (right or left) actions. The above lemma implies that $I_{x,P}$ is left adjoint to $R_{x,P}$.

3.6 Given x and M, next question is to what extend these functors rely on the choice of P. As already said, for any parabolic subgroup P containing M, $G_{x,P}$ is a parahoric subgroup of G_x. But the map $P \mapsto G_{x,P}$ is not injective in general : for example if x is inside a maximal simplex, all $G_{x,P}$ are equal to G_x which in this case is a Iwahori subgroup. But when one proves the former lemma, one can also prove that the above functors actually depend only on $G_{x,P}$ and not on P. By the way this justifies the name "parahoric induction/restriction".

But the following question remains open : does parahoric induction really depend on the parahoric subgroup $G_{x,P}$?

Thinking to the parabolic analog, it is well known that even for complex coefficients, the parabolic functors heavily depend on the choice of a parabolic subgroup. In contrast, for a finite group of Lie type, it was shown by Howlett and Lehrer [4] that the parabolic functors don't depend on this choice. Inspired by their work, we can restate our question of dependance in purely algebraic terms :

Question 3.7 *Fix x, M and let P be a parabolic subgroup with Levi component M. Do we have $\varepsilon_{x,P} \in \mathbb{Z}[\frac{1}{p}][G_x]\varepsilon_{x,\overline{P}}\varepsilon_{x,P}$ and $\varepsilon_{x,\overline{P}} \in \mathbb{Z}[\frac{1}{p}][G_x]\varepsilon_{x,P}\varepsilon_{x,\overline{P}}$?*

Next section will justify our interest in answering this question. The only cases we can treat at present are summed up in

Proposition 3.8 *i) If M is a minimal Levi subgroup, then the answer is positive for any parabolic P with Levi component M.*

ii) In general, we have $\varepsilon_{x,P}e_{M_x^+} \in \mathbb{Z}[\frac{1}{p}][G_x]\varepsilon_{x,\overline{P}}\varepsilon_{x,P}e_{M_x^+}$.

The second point is a direct consequence of Howlett and Lehrer's results.

4 Applications of parahoric functors

Theorem 4.1 *Fix a parabolic subgroup P with Levi component M and assume that question 3.7 has a positive answer for any $x \in \mathcal{I}_M$. Then the map*

$$\begin{array}{rcl} \varepsilon_{x,P}.\mathcal{C}_R^{\infty,c}(G) & \rightarrow & \mathcal{C}_R^{\infty,c}(U\backslash G) \\ f & \mapsto & (g \mapsto \int_U f(ug)du) \end{array}$$

is an isomorphism of $M_x \times G$ smooth R-representations, for any $x \in \mathcal{I}_M$.

In order to stress up the scope of the displayed statement in the theorem, let us explain some consequences. First for any x, M, P as above we get an isomorphism of functors on R-representations

$$Res_M^{M_x} \circ r_G^P \simeq R_{x,P} \circ Res_G^{G_x}.$$

Notice that this immediately implies that parabolic restriction respects admissibility, which is generally not known on non-Artinian rings of coefficients. On another hand we get after little further work an isomorphism of functors, still on R-representations,

$$ind_{G_x}^G \circ I_{x,P} \simeq i_P^G \circ ind_{M_x}^M.$$

As an immediate application, this clearly shows that parabolic induction sends finitely generated objects on finitely generated objects.

Next consequence rests on ideas of Bernstein and deserves a special treatment

Corollary 4.2 *Under the same hypothesis as in previous theorem, the functor* i_P^G *is left adjoint to the functor* r_G^P.

As an immediate application, we see that parabolic induction preserves projective objects while parabolic restriction preserves injective ones.

Resting on these results, we can then prove

Proposition 4.3 *Assume now that the answer to 3.7 is positive for any* x, M, P. *Then*

i) For any compact open H*, there is a compact-modulo-center subset* $S_H \subset G$ *supporting all cuspidal bi-*H*-invariant functions on* G*,* regardless of the ring of coefficients.

ii) The category $\mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]}(G)$ *is noetherian.*

Other applications, to shape of reducibility points and to K-theory are given in [3], under the same assumptions as in this proposition.

Recall now that our theorem rests on a basic assumption we cannot grant in full generality. By the proposition in the former section, this assumption is fulfilled when M is minimal, and in this case our theorem gives a real result and the former proposition applies for any relative rank 1 group G. By the same proposition we can also state results on the "level 0 subcategory". We mention first :

Fact 4.4 *(Moy-Prasad-Vigneras [6] +ε) There is a decomposition*

$$\mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]}(G) = \mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]}(G)_0 \bigoplus \mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]}(G)^0$$

where by definition $\mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]}(G)_0$ *is the full subcategory of all objects generated by their* G_x^+*-invariants,* x *running through* $\mathcal{I}_G$ *(called the level* 0 *subcategory). Moreover, the parabolic functors preserve level* 0 *subcategories.*

For level 0 representations, our theorem and its consequences are listed in

Proposition 4.5 *i) For any* x, M, P*, the morphism*

$$\begin{array}{rcl} e_{M_x^+}\varepsilon_{x,P}.\mathcal{C}_R^{\infty,c}(G) & \to & e_{M_x^+}\mathcal{C}_R^{\infty,c}(U\backslash G) \\ f & \mapsto & (g \mapsto \int_U f(ug)du) \end{array}$$

is an isomorphim of $M_x \times G$ *representations.*

ii) On the level 0 *subcategories, the functor* i_P^G *is left adjoint to* r_G^P.

iii) The level 0 *subcategory* $\mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]}(G)_0$ *is noetherian.*

About proofs in [3]: that of the lemma is elementary algebra, that of the theorem rests on a dynamical argument on the building inpired by work of Moy-Prasad [5], that of the corollary rests on “completions” as in Bernstein’s unpublished work [1], that of noetheriannity requires new other arguments.

References

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