ON FUNCTORIALITY OF ZELEVINSKI INVOLUTIONS

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Let F be a p-adic field and G a connected reductive algebraic group defined over F. For simplicity, we assume that G is quasi-split. We denote by W_F the Weil group of F. Let ${}^LG = \hat{G} \rtimes W_F$ be the L-group of G. We denote by $\mathcal{L}^G$ the set of standard Levi subgroups of G. For $M \in \mathcal{L}^G$, we denote by $r(M)$ the semisimple split F-rank of M. Let $\Pi(G)$ be the set of equivalence classes of irreducible admissible representations of $G(F)$ and $\mathbb{C}[\Pi(G)]$ the space of virtual characters of $G(F)$. The parabolic induction defines a homomorphism $i_M^G : \mathbb{C}[\Pi(M)] \longrightarrow \mathbb{C}[\Pi(G)]$ and the (normalized) Jacquet functor defines a homomorphism $r_M^G : \mathbb{C}[\Pi(G)] \longrightarrow \mathbb{C}[\Pi(M)]$. Following S. Kato [11], we define the Zelevinski involution $\mathbf{D}_G$ by

$$\mathbf{D}_G = \sum_{M \in \mathcal{L}^G} (-1)^{r(M)} i_M^G \circ r_M^G.$$

Let $\{M\}$ be the set of associate standard Levi subgroups of M. We say that $\pi \in \Pi(G)$ is of type $\{M_\pi\}$ if $r_{M_\pi}^G(\pi)$ is a non-zero linear combination of supercuspidal representations of $M_\pi(F)$. We put $r_\pi = r(M_\pi)$. For $\pi \in \Pi(G)$, we define

$$\mathbf{d}_G(\pi) = (-1)^{r_\pi} \mathbf{D}_G(\pi).$$

A.-M. Aubert [4, 5] proved that $\mathbf{d}_G(\pi)$ is irreducible. Thus the Zelevinski involution preserves the irreducibility. It seems natural to consider the relation between the Zelevinski involution and the conjectural Langlands functoriality. Nevertheless the Zelevinski involution does not preserve the L-packets. We consider the A-packets conjectured by J. Arthur [3, Conjecture 6.1]. For a Langlands parameter $\phi : W_F \times SU_2(\mathbb{C}) \longrightarrow {}^LG$, we denote by $\Pi_\phi(G)$ the corresponding conjectural L-packet. Although $SU_2(\mathbb{C})$ is isomorphic to $SL_2(\mathbb{C})$, we denote the second factor of this group by $SU_2(\mathbb{C})$ in order to distinguish it from the factor $SL_2(\mathbb{C})$ used to define the Arthur parameters in [3]. Let

$$\psi : W_F \times SU_2(\mathbb{C}) \times SL_2(\mathbb{C}) \longrightarrow {}^LG$$

be an Arthur parameter of G. We put

$$S_\psi = \mathrm{Cent}(\psi, \hat{G}),$$
$$\mathbb{S}_\psi = S_\psi / S_\psi^0 \cdot Z_{\hat{G}}^\Gamma,$$

where S_ψ^0 is the identity component of S_ψ and $Z_{\hat{G}}^\Gamma$ is the subgroup of the center $Z_{\hat{G}}$ of $\hat{G}$ consisting of the elements fixed by $\Gamma = \mathrm{Gal}(\overline{F}/F)$.

Let $\Pi_\psi(G)$ be the conjectural A-packet of ψ and $\Pi_{\phi_\psi}(G)$ the L-packet corresponding to ψ. We fix Whittaker data χ of $G(F)$. This determines the base point $\pi_\chi \in \Pi_{\phi_\psi}(G)$ as in [3, §6]. For $\overline{s} \in \mathbb{S}_\psi$ and $\pi \in \Pi_\psi(G)$, we define $\langle \overline{s}, \pi | \pi_\chi \rangle$ as in [3, Conjecture 6.1]. Then it is conjectured that $\langle \cdot \,, \pi | \pi_\chi \rangle$ is an irreducible character of $\mathbb{S}_\psi$. We say that a virtual character $\theta \in \mathbb{C}[\Pi(G)]$ is stable if θ is stable as a distribution on $G(F)$. Let $\mathbb{C}[\Pi(G)]^{st}$ be the space of stable virtual characters of $G(F)$ and $\mathbb{C}[\Pi_\psi(G)]$ the subspace of $\mathbb{C}[\Pi(G)]$ generated by $\Pi_\psi(G)$. We put $\mathbb{C}[\Pi_\psi(G)]^{st} = \mathbb{C}[\Pi(G)]^{st} \cap \mathbb{C}[\Pi_\psi(G)]$. As F is a p-adic field, the following hypothesis is believed.

Hypothesis 1. *The map*

$$\pi \in \Pi_\psi(G) \longrightarrow \langle \cdot \,, \pi | \pi_\chi \rangle \in \Pi(\mathbb{S}_\psi)$$

is injective, where $\Pi(\mathbb{S}_\psi)$ *is the set of irreducible characters of* $\mathbb{S}_\psi$*, and*

$$\dim \mathbb{C}[\Pi_\psi(G)]^{st} = 1.$$

In this article, we assume the Arthur conjecture [3, Conjecture 6.1] and this hypothesis.

Now we turn to the Zelevinski involution. We identify $SU_2(\mathbb{C})$ with $SL_2(\mathbb{C})$ and define $d(\psi)$ by

$$d(\psi)(w \times t \times u) = \psi(w \times u \times t), \qquad w \times t \times u \in W_F \times SU_2(\mathbb{C}) \times SL_2(\mathbb{C}).$$

Then $d(\psi)$ is an Arthur parameter of G constructed from ψ by interchanging the role of $SU_2(\mathbb{C})$ and $SL_2(\mathbb{C})$.

Conjecture 2.

$$\mathbf{d}_G(\Pi_\psi(G)) = \Pi_{d(\psi)}(G).$$

Since $S_\psi = S_{d(\psi)}$, we may identify $\mathbb{S}_\psi$ with $\mathbb{S}_{d(\psi)}$. We denote the base point in $\Pi_{\phi_{d(\psi)}}(G)$ by $\pi_{d,\chi}$.

Conjecture 3. *There exists a one-dimensional character* μ *of* $\mathbb{S}_\psi$ *which satisfies*

$$\langle \overline{s}, \mathbf{d}_G(\pi) | \pi_{d,\chi} \rangle = \mu(\overline{s}) \langle \overline{s}, \pi | \pi_\chi \rangle,$$

for all $\overline{s} \in \mathbb{S}_\psi$*.*

If $\mathbb{S}_\psi = \{1\}$, then $\Pi_\psi(G) = \{\pi_\chi\}$ and $\Pi_{d(\psi)}(G) = \{\pi_{d,\chi}\}$. The following conjecture is a special case of Conjecture 2 .

Conjecture 4. *If* ψ *satisfies* $\mathbb{S}_\psi = \{1\}$*, then*

$$\mathbf{d}_G(\pi_\chi) = \pi_{d,\chi}.$$

In general, nevertheless, $\mathbf{d}_G(\pi_\chi)$ may not be equivalent to $\pi_{d,\chi}$. If $G = SL_2$ and if ψ corresponds to an induced representation of G which is a direct sum of two irreducible tempered representations, then $\mathbf{d}_G$ interchanges these two representations. Thus $\mathbf{d}_G(\pi_\chi) \neq \pi_{d,\chi}$.

In the case that $G = GL_n$, Conjecture 2 follows from the results of C. Moeglin and J.-L. Waldspurger [20]. Recently, K. Konno and T. Konno have checked that Conjecture 2 is compatible with their candidates for the A-packets of $G = U(2,2)$.

Conjecture 3 implies that the Zelevinski involutions behave well under the endoscopic transfers. Thus it turns our attention to the relation between the Zelevinski involutions and the endoscopic transfers. Since $i_M^G(\mathbb{C}[\Pi(M)]^{st}) \subset \mathbb{C}[\Pi(G)]^{st}$ and $r_M^G(\mathbb{C}[\Pi(G)]^{st}) \subset \mathbb{C}[\Pi(M)]^{st}$, we have

$$\mathbf{D}_G(\mathbb{C}[\Pi(G)]^{st}) = \mathbb{C}[\Pi(G)]^{st}.$$

Let $(\mathcal{H}, H, s, \xi)$ be (standard) endoscopic data. For the sake of brevity, we assume that $\mathcal{H} \cong {}^L H$. Unfortunately the existence of the endoscopic transfer is still hypothetical. In this article, to define the endoscopic transfer of virtual characters, *we assume the fundamental lemma for groups* [1, Hypothesis 3.1] *and for Lie algebras* [21, Conjecture 1.3]. Let

$$\mathrm{Tran}_H^G : \mathbb{C}[\Pi(H)]^{st} \longrightarrow \mathbb{C}[\Pi(G)]$$

be the endoscopic transfer from H to G. Let A_0 (resp. $A_{H,0}$) be a maximal split torus of G (resp. H). We put $a(G) = \dim(A_0)$ and $a(H) = \dim(A_{H,0})$. Then we have the following theorem.

Theorem 5. *Assume the fundamental lemma for groups and for Lie algebras. Then we have*

$$\mathbf{D}_G \circ \mathrm{Tran}_H^G = (-1)^{a(G)-a(H)} \mathrm{Tran}_H^G \circ \mathbf{D}_H.$$

By using this theorem, we can reduce Conjecture 2 to Conjecture 4. Moreover, we can show that Conjecture 4 implies the following formula;

$$\langle \overline{s}, \mathbf{d}_G(\pi) | \pi_{d,\chi} \rangle = \langle \overline{s}, \mathbf{d}_G(\pi_\chi) | \pi_{d,\chi} \rangle \langle \overline{s}, \pi | \pi_\chi \rangle,$$

where $\langle \cdot, \mathbf{d}_G(\pi_\chi) | \pi_{d,\chi} \rangle$ is a one-dimensional character of $\mathbb{S}_\psi$. This is Conjecture 3.

To prove Theorem 5, we show some properties of the double cosets of the Weyl groups (a generalization of [7, Proposition 2.7.7]) and an analogue of the geometric lemma [6, Lemma 2.12].

We fix an F-splitting $(B_0, T_0, \{X_\alpha\})$ of G, an F-splitting $(B_{H,0}, T_{H,0}, \{Y_\alpha\})$ of H, a Γ-splitting $(\mathcal{B}, \mathcal{T}, \{\mathcal{X}_{\check{\alpha}}\})$ of $\hat{G}$ and a Γ-splitting $(\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{Y}_{\check{\alpha}}\})$ of $\hat{H}$. Then we may identify $\hat{T}_0$ (resp. $\hat{T}_{H,0}$) with $\mathcal{T}$ (resp. $\mathcal{T}_H$). We may assume that $A_0 \subset T_0$ and that $A_{H,0} \subset T_{H,0}$. We say that a subtorus of A_0 is standard if it is equal to the split component of the center of a standard Levi subgroup of G. We assume that $s \in \mathcal{T}$, $\xi(\mathcal{T}_H) = \mathcal{T}$ and $\xi(\mathcal{B}_H) \subset \mathcal{B}$. Let $i_0 : T_{H,0} \longrightarrow T_0$ be the dual homomorphism of $\xi^{-1} : \mathcal{T} \longrightarrow \mathcal{T}_H$. We may assume that $i_0(A_{H,0})$ is a standard subtorus of A_0. We identify $A_{H,0}$ with the image $i_0(A_{H,0})$ in A_0. Put $M_H = \mathrm{Cent}(A_{H,0}, G)$.

We discuss the properties of the double cosets of the Weyl groups with respect to the endoscopic groups. Let

$$\Omega(G) = \mathrm{Norm}(A_0, G)/\,\mathrm{Cent}(A_0, G),$$
$$\Omega(H) = \mathrm{Norm}(A_{H,0}, H)/\,\mathrm{Cent}(A_{H,0}, H),$$

be the Weyl groups. We denote the set of roots of (G, A_0) (resp. $(H, A_{H,0})$) by $R(G) = R(G, A_0)$ (resp. $R(H) = R(H, A_{H,0})$). For $\omega_H \in \Omega(H)$, there exists a unique $\omega_G \in \Omega(G)$ which satisfies the following three conditions.

1) $\omega_G(A_{H,0}) = A_{H,0}$,
2) $\omega_G|_{A_{H,0}} = \omega_H$,
3) $\omega_G(R^+(M_H)) > 0$.

By identifying ω_H with ω_G, we may regard $\Omega(H)$ as a subgroup of $\Omega(G)$. For $M \in \mathcal{L}^G$, we put

$$\Omega(G)_{M,H} = \{\omega \in \Omega(G) |\, \omega(A_{H,0}) \supset A_M\},$$

where A_M is the split component of the center of M. We also put

$$\tilde{D}_M = \{\omega \in (\Omega(G)_{M,H})^{-1} |\, \omega(R^+(M)) > 0\}.$$

Let $\alpha \in R^+(H)$ and $\omega \in (\tilde{D}_M)^{-1}$. Choose $\tilde{\alpha} \in R^+(G)$ whose restriction to $A_{H,0}$ is α. We say that $\omega\alpha$ *is positive* (and write $\omega\alpha > 0$) if $\omega\tilde{\alpha}$ is contained in $R^+(G)$. It is not hard to show that the positivity of $\omega\alpha$ does not depend on the choice of $\tilde{\alpha}$. We define $D_{M,H}$ by

$$D_{M,H} = \{\omega \in (\tilde{D}_M)^{-1} |\, \omega(R^+(H)) > 0\}.$$

Lemma 6. *(1) The set $D_{M,H}$ is a system of representatives for*

$$\Omega(M)\backslash\Omega(G)_{M,H}/\Omega(H).$$

(2) For $\omega \in D_{M,H}$, put

$$M_\omega = \mathrm{Cent}((\omega \circ i_0)^{-1}(A_M), H),$$

then M_ω is a standard Levi subgroup of H.

For $L \in \mathcal{L}^H$, we put

$$D_{M,H,L} = \{\omega \in D_{M,H} |\, M_\omega = L\}$$

and

$$a_{M,H,L} = \sharp D_{M,H,L}.$$

Then we have the following formula, which is a generalization of [7, Proposition 2.7.7].

Proposition 7.

$$\sum_{M\in\mathcal{L}^G} (-1)^{r(M)} a_{M,H,L} = (-1)^{a(G)-a(H)} \cdot (-1)^{r(L)}.$$

Let ${}^LM_\omega$ be the L-group of M_ω. Then we may regard ${}^LM_\omega$ as a subgroup of ${}^LH$. Since G is quasi-split, we may regard $\Omega(G)$ as a subgroup of $\Omega(G, T_0)$. The choice of the splittings defines an isomorphism $\Omega(G, T_0) \longrightarrow \Omega(\hat{G}, \mathcal{T})$. We choose a representative $\hat{n}_\omega \in \mathrm{Norm}(\mathcal{T}, \hat{G})$ of

$$\omega \in \Omega(G) \subset \Omega(G, T_0) \cong \Omega(\hat{G}, \mathcal{T}).$$

We put $s_\omega = \mathrm{Int}\, \hat{n}_\omega(s)$ and $\xi_\omega = \mathrm{Int}\, \hat{n}_\omega \circ \xi$. Then $({}^LM_\omega, M_\omega, s_\omega, \xi_\omega)$ is endoscopic data of M. We choose absolute transfer factors of these endoscopic data and choose Haar measures of standard Levi subgroups and tori suitably. The following formula is an analogue of the formula of Bernstein–Zelevinski [6, Lemma 2.12].

Proposition 8. *Assume the fundamental lemma for groups and for Lie algebras. Then we have*

$$r_M^G \circ \mathrm{Tran}_H^G = \sum_{\omega \in D_{M,H}} \mathrm{Tran}_{M_\omega}^M \circ r_{M_\omega}^H.$$

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