RESTRICTION OF HERMITIAN MAASS LIFTS AND THE GROSS-PRASAD CONJECTURE
(JOINT WITH T. IKEDA)

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This note is a report on a joint work with Tamotsu Ikeda [12].

After the discovery of the integral representation of triple product L-functions by Garrett [5], Harris and Kudla [10] determined the transcendental parts of the central critical values of triple product L-functions. The transcendental parts behaves differently according to whether the weights are "balanced" or not. In the "balanced" case, the critical values of triple product L-functions have also been studied by Garrett [5], Orloff [18], Satoh [20], Garrett and Harris [6], Gross and Kudla [7], Böcherer and Schulze-Pillot [4], and so on. By contrast, in the "imbalanced" case, there are no results on the critical values of triple product L-functions except [10] to our knowledge. We express certain period integrals of Maass lifts which appear in the Gross-Prasad conjecture [8], [9], as the algebraic parts of the central critical values in the "imbalanced" case.

1. The Gross-Prasad conjecture

In [8], [9], Gross and Prasad suggested that the central values of certain L-functions control a global obstruction of blanching rules for automorphic representations of special orthogonal groups. Let V be a non-degenerate quadratic space of dimension n over a number field k and $H = \mathrm{SO}(V)$ the special orthogonal group of V. Take a non-degenerate quadratic subspace V' of V of dimension $n-1$ and regard $H' = \mathrm{SO}(V')$ as a subgroup of H. Let $\tau \simeq \otimes_v \tau_v$ (resp. $\tau' \simeq \otimes_v \tau'_v$) be an irreducible cuspidal automorphic representation of $H(\mathbb{A}_k)$ (resp. $H'(\mathbb{A}_k)$).

Conjecture 1.1 (Gross-Prasad). *Assume that τ and τ' are both tempered. Then the period integral*

$$\langle G|_{H'}, F\rangle = \int_{H'(k)\backslash H'(\mathbb{A}_k)} G(h)\overline{F(h)}\, dh$$

does not vanish for some $G \in \tau$ and some $F \in \tau'$ if and only if

(i) $\mathrm{Hom}_{H'(k_v)}(\tau_v, \tau'_v) \neq 0$ *for all places v of k,*

(ii) $L(1/2, \tau \times \tau') \neq 0$.

Remark that a meromorphic continuation of the L-function $L(s, \tau \times \tau')$ has not been established in general, however, it could be described in terms of L-functions of general linear groups by the functoriality. We also note that the conjecture is supported by the results of Waldspurger [22] for $n = 3$, Harris and Kudla [10], [11] for $n = 4$, Böcherer, Furusawa, and Schulze-Pillot [3] for $n = 5$.

Gross and Prasad restricted their conjecture to the tempered cases. According to the Arthur conjecture [2], non-tempered cuspidal automorphic representations exist, and if τ or τ' is non-tempered, then the L-function $L(s, \tau \times \tau')$ could have a pole at $s = 1/2$. Hence a modification to the condition (ii) would be inevitable if one consider the Gross-Prasad conjecture in general (see [3] for $n = 5$). Our result provides an example for $n = 6$ when τ, τ' are both non-tempered. Remark that the triple product L-function considered in this note is only of degree 8 and is a part of the L-function $L(s, \tau \times \tau')$ of degree 24.

2. Saito-Kurokawa lifts

First, we review the notion of Saito-Kurokawa lifts [16], [17], [1], [23]. Let k be a positive even integer. Let

$$F(Z) = \sum_{B>0} A(B) e^{2\pi\sqrt{-1}\,\mathrm{tr}(BZ)} \in S_k(\mathrm{Sp}_2(\mathbb{Z})), \quad Z \in \mathfrak{h}_2$$

be a Siegel modular form of degree 2. Here $\mathfrak{h}_2$ is the Siegel upper half plane given by

$$\mathfrak{h}_2 = \left\{ Z = {}^tZ \in \mathrm{M}_2(\mathbb{C}) \mid \mathrm{Im}(Z) > 0 \right\}.$$

We say that F satisfies the Maass relation if there exists a function $\beta_F^* : \mathbb{N} \to \mathbb{C}$ such that

$$A\left(\begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \right) = \sum_{d|(n,r,m)} d^{k-1} \beta_F^* \left(\frac{4nm - r^2}{d^2} \right).$$

We denote by $S_k^{\mathrm{Maass}}(\mathrm{Sp}_2(\mathbb{Z}))$ the space of Siegel cusp forms which satisfy the Maass relation.

Kohnen [13] introduced the plus subspace $S_{k-1/2}^+(\Gamma_0(4))$ given by

$$S_{k-1/2}^+(\Gamma_0(4)) = \{ h(\tau) = \sum_{N>0} c(N) q^N \in S_{k-1/2}(\Gamma_0(4)) \mid \\ c(N) = 0 \text{ if } -N \not\equiv 0, 1 \bmod 4 \}.$$

For $F \in S_k^{\mathrm{Maass}}(\mathrm{Sp}_2(\mathbb{Z}))$, put

$$\Omega^{\mathrm{SK}}(F)(\tau) = \sum_{\substack{N \geq 0 \\ -N \equiv 0,1 \bmod 4}} \beta_F^*(N) q^N.$$

Then $\Omega^{\mathrm{SK}}(F) \in S_{k-1/2}^{+}(\Gamma_0(4))$, and the linear map

$$\Omega^{\mathrm{SK}} : S_k^{\mathrm{Maass}}(\mathrm{Sp}_2(\mathbb{Z})) \longrightarrow S_{k-1/2}^{+}(\Gamma_0(4))$$

is an isomorphism.

3. HERMITIAN MAASS LIFTS

Next, we recall an analogue of Saito-Kurokawa lifts for hermitian modular forms by Kojima [14], Sugano [21], and Krieg [15]. Let $K = \mathbb{Q}(\sqrt{-\mathbf{D}})$ be an imaginary quadratic field with discriminant $-\mathbf{D} < 0$, $\mathcal{O}$ the ring of integers of K, w_K the number of roots of unity contained in K, and χ be the primitive Dirichlet character corresponding to $K/\mathbb{Q}$. Write

$$\chi = \prod_{q \in Q_{\mathbf{D}}} \chi_q,$$

where $Q_{\mathbf{D}}$ is the set of all primes dividing $\mathbf{D}$ and χ_q is a primitive Dirichlet character mod $q^{\mathrm{ord}_q \mathbf{D}}$ for each $q \in Q_{\mathbf{D}}$.

Let k be a positive integer such that $w_K \mid k$. Let

$$G(Z) = \sum_{H \in \Lambda_2(\mathcal{O})^+} A(H) e^{2\pi\sqrt{-1}\,\mathrm{tr}(HZ)} \in S_k(U(2,2)), \quad Z \in \mathcal{H}_2$$

be a hermitian modular form of degree 2. Here $\mathcal{H}_2$ is the hermitian upper half plane given by

$$\mathcal{H}_2 = \left\{ Z \in \mathrm{M}_2(\mathbb{C}) \;\middle|\; \frac{1}{2\sqrt{-1}}(Z - {}^t\bar{Z}) > 0 \right\},$$

and

$$\Lambda_2(\mathcal{O})^+ = \left\{ H = {}^t\bar{H} \in \frac{1}{\sqrt{-\mathbf{D}}} \mathrm{M}_2(\mathcal{O}) \;\middle|\; \mathrm{diag}(H) \in \mathbb{Z}^2,\, H > 0 \right\}.$$

We say that G satisfies the Maass relation if there exists a function $\alpha_G^* : \mathbb{N} \to \mathbb{C}$ such that

$$A(H) = \sum_{d \mid \varepsilon(H)} d^{k-1} \alpha_G^* \left(\frac{\mathbf{D} \det(H)}{d^2} \right),$$

where

$$\varepsilon(H) = \max\{ n \in \mathbb{N} \mid n^{-1} H \in \Lambda_2(\mathcal{O})^+ \}.$$

We denote by $S_k^{\mathrm{Maass}}(U(2,2))$ the space of hermitian cusp forms which satisfy the Maass relation.

Krieg [15] introduced the space $S_{k-1}^*(\Gamma_0(\mathbf{D}),\chi)$ which is an analogue of the Kohnen plus subspace and is given by

$$S_{k-1}^*(\Gamma_0(\mathbf{D}),\chi)=\{g^*(\tau)=\sum_{N>0}a_{g^*}(N)q^N\in S_{k-1}(\Gamma_0(\mathbf{D}),\chi)\mid \\ a_{g^*}(N)=0 \text{ if } \mathbf{a}_{\mathbf{D}}(N)=0\},$$

where

$$\mathbf{a}_{\mathbf{D}}(N)=\prod_{q\in Q_{\mathbf{D}}}(1+\chi_q(-N)).$$

Let

$$g(\tau)=\sum_{N>0}a_g(N)q^N\in S_{k-1}(\Gamma_0(\mathbf{D}),\chi)$$

be a primitive form. For each $Q\subset Q_{\mathbf{D}}$, set

$$\chi_Q=\prod_{q\in Q}\chi_q,\quad \chi'_Q=\prod_{q\in Q_{\mathbf{D}}-Q}\chi_q.$$

Then there exists a primitive form

$$g_Q(\tau)=\sum_{N\geq 0}a_{g_Q}(N)q^N\in S_{k-1}(\Gamma_0(\mathbf{D}),\chi)$$

such that

$$a_{g_Q}(p)=\begin{cases}\chi_Q(p)a_g(p) & \text{if } p\notin Q,\\ \chi'_Q(p)\overline{a_g(p)} & \text{if } p\in Q,\end{cases}$$

for each prime p. Put

$$g^*=\sum_{Q\subset Q_{\mathbf{D}}}\chi_Q(-1)g_Q. \tag{3.1}$$

Then $g^*\in S_{k-1}^*(\Gamma_0(\mathbf{D}),\chi)$. When g runs over primitive forms in $S_{k-1}(\Gamma_0(\mathbf{D}),\chi)$, the forms g^* span $S_{k-1}^*(\Gamma_0(\mathbf{D}),\chi)$.

For $G\in S_k^{\mathrm{Maass}}(U(2,2))$, put

$$\Omega(G)(\tau)=\sum_{N>0}\mathbf{a}_{\mathbf{D}}(N)\alpha_G^*(N)q^N.$$

Then $\Omega(G)\in S_{k-1}^*(\Gamma_0(\mathbf{D}),\chi)$, and the linear map

$$\Omega: S_k^{\mathrm{Maass}}(U(2,2))\longrightarrow S_{k-1}^*(\Gamma_0(\mathbf{D}),\chi)$$

is an isomorphism.

4. STATEMENT OF THE MAIN THEOREM

Let k be a positive integer such that $w_K \mid k$. Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a primitive form and $h(\tau) = \sum_{N>0} c(N)q^N \in S^+_{k-1/2}(\Gamma_0(4))$ a Hecke eigenform which corresponds to f by the Shimura correspondence. Note that h is unique up to scalars. Let $F = (\Omega^{\mathrm{SK}})^{-1}(h) \in S_k^{\mathrm{Maass}}(\mathrm{Sp}_2(\mathbb{Z}))$ be the Saito-Kurokawa lift of f. Define the Petersson norms of f and F by

$$\begin{aligned}\langle f, f\rangle &= \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}_1} |f(\tau)|^2 y^{2k-4} d\tau,\\ \langle F, F\rangle &= \int_{\mathrm{Sp}_2(\mathbb{Z})\backslash\mathfrak{h}_2} |F(Z)|^2 |\det \mathrm{Im}(Z)|^{k-3} dZ,\end{aligned}$$

respectively.

Let $g(\tau) = \sum_{N>0} a_g(N)q^N \in S_{k-1}(\Gamma_0(\mathbf{D}), \chi)$ be a primitive form and $G = \Omega^{-1}(g^*) \in S_k^{\mathrm{Maass}}(U(2,2))$ the hermitian Maass lift of g, where $g^* \in S^*_{k-1}(\Gamma_0(\mathbf{D}), \chi)$ is given by (3.1). Observe that $\mathfrak{h}_2 \subset \mathcal{H}_2$, and by [15], the restriction $G|_{\mathfrak{h}_2}$ belongs to $S_k^{\mathrm{Maass}}(\mathrm{Sp}_2(\mathbb{Z}))$.

The completed triple product L-function $\Lambda(s, g \times g \times f)$ is given by

$$\Lambda(s, g\times g\times f) = (2\pi)^{-4s+4k-8}\Gamma(s)\Gamma(s-2k+4)\Gamma(s-k+2)^2 L(s, g\times g\times f),$$

and satisfies a functional equation which replaces s with $4k - 6 - s$.

Our main result is as follows.

Theorem 4.1.

$$\frac{\Lambda(2k-3, g\times g\times f)}{\langle f, f\rangle^2} = -2^{4k-6}\mathbf{D}^{-2k+3}c(\mathbf{D})^2 \frac{\langle G|_{\mathfrak{h}_2}, F\rangle^2}{\langle F, F\rangle^2}$$

5. PROOF

Theorem 4.1 follows from the following seesaws.

(5.1)

$$\begin{tikzcd}
\mathrm{O}(4,2) \arrow[d, dash] \arrow[dr, dash] & \widetilde{\mathrm{SL}}_2 \times \widetilde{\mathrm{SL}}_2 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \mathrm{O}(2,2) \arrow[dl, dash] \arrow[d, dash] \\
\mathrm{O}(3,2) \times \mathrm{O}(1) & \mathrm{SL}_2 & \mathrm{O}(2,1) \times \mathrm{O}(1)
\end{tikzcd}$$

(5.2)

$$\begin{tikzcd}
\mathrm{Sp}_6 \arrow[d, dash] \arrow[dr, dash] & \mathrm{O}(2,2)^3 \arrow[dl, dash] \arrow[d, dash] \\
\mathrm{SL}_2^3 & \mathrm{O}(2,2)
\end{tikzcd}$$

To explain these seesaws more precisely, we introduce some notation. In [13], Kohnen defined a linear map

$$\begin{aligned}\mathcal{S}^+_{-\mathbf{D}} : S^+_{k-1/2}(\Gamma_0(4)) &\longrightarrow S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})),\\ \sum_{N>0} c(N)q^N &\longmapsto \sum_{N>0}\sum_{d\mid N} \chi(d)d^{k-2}c\left(\frac{N^2}{d^2}\mathbf{D}\right)q^N.\end{aligned}$$

If $h(\tau) = \sum_{N>0} c(N)q^N \in S^+_{k-1/2}(\Gamma_0(4))$ is a Hecke eigenform and corresponds to $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ by the Shimura correspondence, then

$$\mathcal{S}^+_{-\mathbf{D}}(h) = c(\mathbf{D})f.$$

Let $\mathrm{Tr}^{\mathbf{D}}_1$ denote the trace operator given by

$$\begin{aligned}\mathrm{Tr}^{\mathbf{D}}_1 : S_{2k-2}(\Gamma_0(\mathbf{D})) &\longrightarrow S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})),\\ f &\longmapsto \sum_{\gamma\in\Gamma_0(\mathbf{D})\backslash \mathrm{SL}_2(\mathbb{Z})} f|\gamma.\end{aligned}$$

The seesaw (5.1) accounts for the following identity.

Proposition 5.1.

$$\mathcal{S}^+_{-\mathbf{D}}(\Omega^{\mathrm{SK}}(G|_{\mathfrak{h}_2})) = a_g(\mathbf{D})^2\,\mathrm{Tr}^{\mathbf{D}}_1(g^2).$$

This identity is proved by computing the Fourier coefficients of the both sides explicitly.

The seesaw (5.2) accounts for the following refinement of the main identity by Harris and Kudla [10].

Proposition 5.2.

$$\Lambda(2k-3, g\times g\times f) = -2^{4k-6}\mathbf{D}^{-2k+3}a_g(\mathbf{D})^4\langle \mathrm{Tr}^{\mathbf{D}}_1(g^2), f\rangle^2$$

This identity is proved by computing the local zeta integrals which arise in the integral representation of triple product L-functions by Garrett [5], Piatetski-Shapiro and Rallis [19] at bad primes.

Now Theorem 4.1 follows from Propositions 5.1 and 5.2.

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