ON THE LIFTING OF HERMITIAN MODULAR FORMS

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Notation

Let K be an imaginary quadratic field with discriminant $-\mathbf{D} = -\mathbf{D}_K$. We denote by $\mathcal{O} = \mathcal{O}_K$ the ring of integers of K. The non-trivial automorphism of K is denoted by $x \mapsto \bar{x}$. The primitive Dirichlet character corresponding to $K/\mathbb{Q}$ is denoted by $\chi = \chi_{\mathbf{D}}$. We denote by $\mathcal{O}^\sharp = (\sqrt{-\mathbf{D}})^{-1}\mathcal{O}$ the inverse different ideal of $K/\mathbb{Q}$.

The special unitary group $G = \mathrm{SU}(m,m)$ is an algebraic group defined over $\mathbb{Q}$ such that

$$G(R) = \left\{ g \in \mathrm{SL}_{2m}(R \otimes K) \,\middle|\, g \begin{pmatrix} \mathbf{0}_m & -\mathbf{1}_m \\ \mathbf{1}_m & \mathbf{0}_m \end{pmatrix} {}^t\bar{g} = \begin{pmatrix} \mathbf{0}_m & -\mathbf{1}_m \\ \mathbf{1}_m & \mathbf{0}_m \end{pmatrix} \right\}$$

for any $\mathbb{Q}$-algebra R. We put $\Gamma_K^{(m)} = G(\mathbb{Q}) \cap \mathrm{GL}_{2m}(\mathcal{O})$.

The hermitian upper half space $\mathcal{H}_m$ is defined by

$$\mathcal{H}_m = \{ Z \in M_m(\mathbb{C}) \mid \frac{1}{2\sqrt{-1}}(Z - {}^t\bar{Z}) > 0 \}.$$

Then $G(\mathbb{R})$ acts on $\mathcal{H}_m$ by

$$g\langle Z \rangle = (AZ + B)(CZ + D)^{-1}, \quad Z \in \mathcal{H}_m, g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We put

$$\Lambda_m(\mathcal{O}) = \{ h = (h_{ij}) \in \mathrm{M}_m(K) \mid h_{ii} \in \mathbb{Z}, h_{ij} = \bar{h}_{ji} \in \mathcal{O}^\sharp, i \neq j \},$$
$$\Lambda_m(\mathcal{O})^+ = \{ h \in \Lambda_m(\mathcal{O}) \mid h > 0 \}.$$

We set $\mathbf{e}(T) = \exp(2\pi\sqrt{-1}\mathrm{tr}(T))$ if T is a square matrix with entries in $\mathbb{C}$. For each prime p, the unique additive character of $\mathbb{Q}_p$ such that $\mathbf{e}_p(x) = \exp(-2\pi\sqrt{-1}x)$ for $x \in \mathbb{Z}[p^{-1}]$ is denoted $\mathbf{e}_p$. Note that $\mathbf{e}_p$ is of order 0. We put $\mathbf{e}\ (x) = \mathbf{e}(x_\infty)\prod_{p<\infty} \mathbf{e}_p(x_p)$ for an adele $x = (x_v)_v \in \mathbb{A}$.

Let $\underline{\chi} = \otimes_v \underline{\chi}_v$ be the Hecke character of $\mathbb{A}^\times/\mathbb{Q}^\times$ determined by χ. Then $\underline{\chi}_v$ is the character corresponding to $\mathbb{Q}_v(\sqrt{-\mathbf{D}})/\mathbb{Q}$ and given by

$$\underline{\chi}_v(t) = \left(\frac{-\mathbf{D}, t}{\mathbb{Q}_v}\right).$$

Let $Q_{\mathbf{D}}$ be the set of all primes which divides $\mathbf{D}$. For each prime $q \in Q_{\mathbf{D}}$, we put $\mathbf{D}_q = q^{\mathrm{ord}_q \mathbf{D}}$. We define a primitive Dirichlet character χ_q by

$$\chi_q(n) = \begin{cases} \chi(n') & \text{if } (n, q) = 1 \\ 0 & \text{if } q | n, \end{cases}$$

where n' is an integer such that

$$n' \equiv \begin{cases} n & \mod \mathbf{D}_q, \\ 1 & \mod \mathbf{D}_q^{-1}\mathbf{D} \end{cases}$$

Then we have $\chi = \prod_{q|\mathbf{D}} \chi_q$. Note that

$$\chi_q(n) = \left(\frac{\chi_q(-1)\mathbf{D}_q, n}{\mathbb{Q}_q}\right) = \prod_{p|n} \left(\frac{\chi_q(-1)\mathbf{D}_q, n}{\mathbb{Q}_p}\right)$$

for $q \nmid n$, $n > 0$. One should not confuse χ_q with $\underline{\chi}_q$.

1. Fourier coefficients of Eisenstein series on $\mathcal{H}_m$

In this section, we consider Siegel series associated to non-degenerate hermitian matrices. Fix a prime p. Put $\xi_p = \chi(p)$, i.e.,

$$\xi_p = \begin{cases} 1 & \text{if } -\mathbf{D} \in (\mathbb{Q}_p^\times)^2 \\ -1 & \text{if } \mathbb{Q}_p(\sqrt{-\mathbf{D}})/\mathbb{Q}_p \text{ is unramified quadratic extension} \\ 0 & \text{if } \mathbb{Q}_p(\sqrt{-\mathbf{D}})/\mathbb{Q}_p \text{ is ramified quadratic extension.} \end{cases}$$

For $H \in \Lambda_m(\mathcal{O})$, $\det H \neq 0$, we put

$$\gamma(H) = (-\mathbf{D})^{[m/2]} \det(H)$$
$$\zeta_p(H) = \underline{\chi}_p(\gamma(H))^{m-1}.$$

The Siegel series for H is defined by

$$b_p(H, s) = \sum_{R \in \mathrm{Her}_m(K_p)/\mathrm{Her}_m(\mathcal{O}_p)} \mathbf{e}_p(\mathrm{tr}(BR)) p^{-\mathrm{ord}_p(\nu(R))s}, \quad \mathrm{Re}(s) \gg 0.$$

Here, $\mathrm{Her}_m(K_p)$ (resp. $\mathrm{Her}_m(\mathcal{O}_p)$) is the additive group of all hermitian matrices with entries in K_p (resp. $\mathcal{O}_p$). The ideal $\nu(R) \subset \mathbb{Z}_p$ is defined as follows: Choose a coprime pair $\{C, D\}$, C, $D \in \mathrm{M}_{2n}(\mathcal{O}_p)$ such that $C\,{}^t\bar{D} = D\,{}^t\bar{C}$, and $D^{-1}C = R$. Then $\nu(R) = \det(D)\mathcal{O}_p \cap \mathbb{Z}_p$.

We define a polynomial $t_p(K/\mathbb{Q}; X) \in \mathbb{Z}[X]$ by

$$t_p(K/\mathbb{Q}; X) = \prod_{i=1}^{[(m+1)/2]} (1 - p^{2i}X) \prod_{i=1}^{[m/2]} (1 - p^{2i-1}\xi_p X).$$

There exists a polynomial $F_p(H; X) \in \mathbb{Z}[X]$ such that

$$F_p(H; p^{-s}) = b_p(H, s) t_p(K/\mathbb{Q}; p^{-s})^{-1}.$$

This is proved in [9].

Moreover, $F_p(H; X)$ satisfies the following functional equation:

$$F_p(H; p^{-2m}X^{-1}) = \zeta_p(H)(p^m X)^{-\mathrm{ord}_p \gamma(H)} F_p(H; X).$$

This functional equation is a consequence of [7], Proposition 3.1. We will discuss it in the next section.

The functional equation implies that $\deg F_p(H; X) = \mathrm{ord}_p \gamma(H)$. In particular, if $p \nmid \gamma(H)$, then $F_p(H; X) = 1$. Put

$$\tilde{F}_p(H; X) = X^{-\mathrm{ord}_p \gamma(H)} F_p(H; p^{-m}X^2).$$

Then following lemma is a immediate consequence of the functional equation of $F(H; X)$.

Lemma 1. *We have*

$$\tilde{F}_p(H; X^{-1}) = \tilde{F}_p(H; X), \qquad \text{if } m \text{ is odd.}$$

$$\tilde{F}_p(H; \xi_p X^{-1}) = \tilde{F}_p(H; X), \qquad \text{if } m \text{ is even and } \xi_p \neq 0.$$

Let k be a sufficiently large integer. Put $n = [m/2]$. The Eisenstein series $E^{(m)}_{2k+2n}(Z)$ of weight $2k + 2n$ on $\mathcal{H}_m$ is defined by

$$E^{(m)}_{2k+2n}(Z) = \sum_{\{C,D\}/\sim} \det(CZ + D)^{-2k-2n},$$

where $\{C, D\}/\sim$ extends over coprime pairs $\{C, D\}$, C, $D \in \mathrm{M}_{2n}(\mathcal{O})$ such that $C\,{}^t\bar{D} = D\,{}^t\bar{C}$ modulo the action of $GL_m(\mathcal{O})$. We put

$$\mathcal{E}^{(m)}_{2k+2n}(Z) = A_m^{-1} \prod_{i=1}^{m} L(1 + i - 2k - 2n, \chi^{i-1}) E^{(m)}_{2k+2n}(Z).$$

Here

$$A_m = \begin{cases} 2^{-4n^2-4n}\mathbf{D}^{2n^2+n} & \text{if } m = 2n + 1, \\ (-1)^n 2^{-4n^2+4n}\mathbf{D}^{2n^2-n} & \text{if } m = 2n. \end{cases}$$

Then the H-th Fourier coefficient of $\mathcal{E}^{(2n+1)}_{2k+2n}(Z)$ is equal to

$$\begin{aligned}|\gamma(H)|^{2k-1}\prod_{p|\gamma(H)}F_p(H;p^{-2k-2n}) =&|\gamma(H)|^{k-(1/2)}\prod_{p|\gamma(H)}\tilde{F}_p(H;p^{-k+(1/2)})\\ =&|\gamma(H)|^{k-(1/2)}\prod_{p|\gamma(H)}\tilde{F}_p(H;p^{k-(1/2)})\end{aligned}$$

for any $H\in\Lambda_{2n+1}(\mathcal{O})^+$ and any sufficiently large integer k.

The H-th Fourier coefficient of $\mathcal{E}^{(2n)}_{2k+2n}(Z)$ is equal to

$$|\gamma(H)|^{2k}\prod_{p|\gamma(H)}F_p(H;p^{-2k-2n}) = |\gamma(H)|^{k}\prod_{p|\gamma(H)}\tilde{F}_p(H;p^{-k})$$

for any $H\in\Lambda_{2n}(\mathcal{O})^+$ and any sufficiently large integer k.

2. **Main theorems**

We first consider the case when $m=2n$ is even.

Let $f(\tau)=\sum_{N=1}^{\infty}a(N)q^N\in S_{2k+1}(\Gamma_0(\mathbf{D}),\chi)$ be a primitive form, whose L-function is given by

$$\begin{aligned}L(f,s)=&\sum_{N=1}^{\infty}a(N)N^{-s}\\ =&\prod_{p\ \mathbf{D}}(1-a(p)p^{-s}+\chi(p)p^{2k-2s})^{-1}\prod_{q|\mathbf{D}}(1-a(q)q^{-s})^{-1}.\end{aligned}$$

For each prime $p\nmid\mathbf{D}$, we define the Satake parameter $\{\alpha_p,\beta_p\}=\{\alpha_p,\chi(p)\alpha_p^{-1}\}$ by

$$(1-a(p)X+\chi(p)p^{2k}X^2)=(1-p^k\alpha_pX)(1-p^k\beta_pX).$$

For $q\mid\mathbf{D}$, we put $\alpha_q=q^{-k}a(q)$.

Put

$$A(H)=|\gamma(H)|^k\prod_{p|\gamma(H)}\tilde{F}_p(H,\alpha_p),\quad H\in\Lambda_{2n}(\mathcal{O})^+$$

$$F(Z)=\sum_{H\in\Lambda_{2n}(\mathcal{O})^+}A(H)\mathbf{e}(HZ),\quad Z\in\mathcal{H}_{2n}.$$

Then our first main theorem is as follows:

Theorem 1. *Assume that $m=2n$ is even. Let $f(\tau)$, $A(H)$ and $F(Z)$ be as above. Then we have $F\in S_{2k+2n}(\Gamma_K^{(2n)})$. Moreover, F is a Hecke eigenform. $F=0$ if and only if $f(\tau)$ comes from a Hecke character of K and n is odd.*

Now we consider the case when $m = 2n+1$ is odd.

Let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform, whose L-function is given by

$$\begin{aligned} L(f,s) &= \sum_{N=1}^{\infty} a(N)N^{-s} \\ &= \prod_p (1 - a(p)p^{-s} + p^{2k-1-2s})^{-1} \end{aligned}$$

For each prime p, we define the Satake parameter $\{\alpha_p, \alpha_p^{-1}\}$ by

$$(1 - a(p)X + p^{2k-1}X^2) = (1 - p^{k-(1/2)}\alpha_p X)(1 - p^{k-(1/2)}\alpha_p^{-1}X).$$

Put

$$A(H) = |\gamma(H)|^{k-(1/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H, \alpha_p), \quad H \in \Lambda_{2n+1}(\mathcal{O})^+$$

$$F(Z) = \sum_{H \in \Lambda_{2n+1}(\mathcal{O})^+} A(H)\mathbf{e}(HZ), \quad Z \in \mathcal{H}_{2n+1}.$$

Theorem 2. *Assume that $m = 2n+1$ is odd. Let $f(\tau)$, $A(H)$ and $F(Z)$ be as above. Then we have $F \in S_{2k+2n}(\Gamma_K^{(2n+1)})$. Moreover, F is a non-zero Hecke eigenform.*

References

[1] J. Arthur, *Unipotent automorphic representations: conjectures*, Astérisque **171-172** (1989), 13–71.

[2] S. Böcherer, *Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen. I*, Math. Z. **183** (1983), 21–46.

[3] S. Breulmann and M. Kuss, *On a conjecture of Duke-Imamoglu*, Proc. Amer. Math. Soc **107** (2000),

[4] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Mathematics **55** Birkhäuser Boston, Inc., Boston, Mass. 1985.

[5] T. Ikeda, *On the theory of Jacobi forms and the Fourier-Jacobi coefficients of Eisenstein series*, J. Math. Kyoto Univ. **34** (1994), 615–636.

[6] ———, *On the lifting of elliptic cusp forms to Siegel cusp forms of degree* $2n$, Ann. of Math. **154** (2001), 641–681.

[7] S. Kudla and W. J. Sweet Jr. *Degenerate principal series representations for* $\mathrm{U}(n,n)$. Israel J. Math. bf 98 (1997), 253–306.

[8] G. Shimura *Introduction to the arithmetic theory of automorphic functions*, Publ. Math. Soc. Japan **11** Iwanami Shoten and Princeton University Press, 1971.

[9] ———, *Euler products and Eisenstein series*, CBMS Regional Conference Series in Mathematics **93** the American Mathematical Society, Providence, RI, 1997.