ON THE LIFTING OF HERMITIAN MODULAR FORMS

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Notation

Let $K$ be an imaginary quadratic field with discriminant $-D = -D_K$. We denote by $\mathcal{O} = \mathcal{O}_K$ the ring of integers of $K$. The non-trivial automorphism of $K$ is denoted by $x \mapsto \bar{x}$. The primitive Dirichlet character corresponding to $K/\mathbb{Q}$ is denoted by $\chi = \chi_D$. We denote by $\mathcal{O}^\sharp = (\sqrt{-D})^{-1}\mathcal{O}$ the inverse different ideal of $K/\mathbb{Q}$.

The special unitary group $G = SU(m, m)$ is an algebraic group defined over $\mathbb{Q}$ such that $G(\mathbb{R}) = \{ g \in SL_2m(\mathbb{R} \otimes K) \mid g \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix} \bar{g} = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix} \}$ for any $\mathbb{Q}$-algebra $R$. We put $\Gamma_K^{(m)} = G(\mathbb{Q}) \cap GL_{2m}(\mathcal{O})$.

The hermitian upper half space $\mathcal{H}_m$ is defined by

$$\mathcal{H}_m = \{ Z \in M_m(\mathbb{C}) \mid \frac{1}{2\sqrt{-1}}(Z - \bar{Z}) > 0 \}. $$

Then $G(\mathbb{R})$ acts on $\mathcal{H}_m$ by

$$g(\langle Z \rangle ) = (AZ + B)(CZ + D)^{-1}, \quad Z \in \mathcal{H}_m, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. $$

We put

$$\Lambda_m(\mathcal{O}) = \{ h = (h_{ij}) \in M_m(K) \mid h_{ii} \in \mathbb{Z}, h_{ij} = \bar{h}_{ji} \in \mathcal{O}^\sharp, i \neq j \},$$

$$\Lambda_m(\mathcal{O})^+ = \{ h \in \Lambda_m(\mathcal{O}) \mid h > 0 \}. $$

We set $\mathbf{e}(T) = \exp(2\pi \sqrt{-1}\text{tr}(T))$ if $T$ is a square matrix with entries in $\mathbb{C}$. For each prime $p$, the unique additive character of $\mathbb{Q}_p$ such that $\mathbf{e}_p(x) = \exp(-2\pi \sqrt{-1}x)$ for $x \in \mathbb{Z}[p^{-1}]$ is denoted $\mathbf{e}_p$. Note that $\mathbf{e}_p$ is of order 0. We put $\mathbf{e}(\langle x \rangle ) = \mathbf{e}(x_\infty) \prod_{p < \infty} \mathbf{e}_p(x_p)$ for an adele $x = (x_v)_v \in \mathbb{A}$.  

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Let $\chi = \otimes_v \chi_v$ be the Hecke character of $\mathbb{A}^\times / \mathbb{Q}^\times$ determined by $\chi$. Then $\chi_v$ is the character corresponding to $\mathbb{Q}_v/(\sqrt{-D})/\mathbb{Q}$ and given by

$$\chi_v(t) = \left( -\frac{D}{v}, t \right).$$

Let $Q_D$ be the set of all primes which divides $D$. For each prime $q \in Q_D$, we put $D_q = q^{\operatorname{ord}_q D}$. We define a primitive Dirichlet character $\chi_q$ by

$$\chi_q(n) = \begin{cases} \chi(n') & \text{if } (n, q) = 1 \\ 0 & \text{if } q | n, \end{cases}$$

where $n'$ is an integer such that

$$n' \equiv \begin{cases} n \pmod{D_q}, \\ 1 \pmod{D_q^{-1}D} \end{cases}$$

Then we have $\chi = \prod_{q \mid D} \chi_q$. Note that

$$\chi_q(n) = \left( \frac{\chi(-1)D_q}{Q_q}, n \right) = \prod_{p | n} \left( \frac{\chi(-1)D_q}{Q_p}, n \right)$$

for $q \nmid n$, $n > 0$. One should not confuse $\chi_q$ with $\chi_q$.

1. **Fourier coefficients of Eisenstein series on $H_m$**

In this section, we consider Siegel series associated to non-degenerate hermitian matrices. Fix a prime $p$. Put $\xi_p = \chi(p)$, i.e.,

$$\xi_p = \begin{cases} 1 & \text{if } -D \in (\mathbb{Q}_p^\times)^2 \\ -1 & \text{if } \mathbb{Q}_p/(\sqrt{-D})/\mathbb{Q}_p \text{ is unramified quadratic extension} \\ 0 & \text{if } \mathbb{Q}_p/(\sqrt{-D})/\mathbb{Q}_p \text{ is ramified quadratic extension}. \end{cases}$$

For $H \in \Lambda_m(\mathcal{O})$, $\det H \neq 0$, we put

$$\gamma(H) = (-D)^{m/2} \det(H)$$

$$\zeta_p(H) = \chi_p(\gamma(H))^{m-1}.$$ 

The Siegel series for $H$ is defined by

$$b_p(H, s) = \sum_{R \in \operatorname{Herm}(K_p)/\operatorname{Herm}(\mathcal{O}_p)} e_p(\operatorname{tr}(BR)) p^{-\operatorname{ord}_p(\nu(R))s}, \quad \operatorname{Re}(s) \gg 0.$$ 

Here, $\operatorname{Herm}(K_p)$ (resp. $\operatorname{Herm}(\mathcal{O}_p)$) is the additive group of all hermitian matrices with entries in $K_p$ (resp. $\mathcal{O}_p$). The ideal $\nu(R) \subset \mathbb{Z}_p$ is defined...
as follows: Choose a coprime pair \( \{C, D\}, C, D \in M_{2n}(\mathcal{O}_p) \) such that \( C^t D = D^t \bar{C} \), and \( D^{-1} C = R \). Then \( \nu(R) = \det(D)\mathcal{O}_p \cap \mathbb{Z}_p \).

We define a polynomial \( t_p(K/\mathbb{Q}; X) \in \mathbb{Z}[X] \) by

\[
t_p(K/\mathbb{Q}; X) = \prod_{i=1}^{[(m+1)/2]} (1 - p^{2i} X) \prod_{i=1}^{[m/2]} (1 - p^{2i-1} \xi_p X).
\]

There exists a polynomial \( F_p(H; X) \in \mathbb{Z}[X] \) such that

\[
F_p(H; p^{-s}) = b_p(H, s)t_p(K/\mathbb{Q}; p^{-s})^{-1}.
\]

This is proved in [9].

Moreover, \( F_p(H; X) \) satisfies the following functional equation:

\[
F_p(H; p^{-2m}X^{-1}) = \zeta_p(H)(p^{m} X)^{-\ord_p \gamma(H)} F_p(H; X).
\]

This functional equation is a consequence of [7], Proposition 3.1. We will discuss it in the next section.

The functional equation implies that \( \deg F_p(H; X) = \ord_p \gamma(H) \). In particular, if \( p \nmid \gamma(H) \), then \( F_p(H; X) = 1 \). Put

\[
\tilde{F}_p(H; X) = X^{-\ord_p \gamma(H)} F_p(H; p^{-m}X^2).
\]

Then following lemma is a immediate consequence of the functional equation of \( F(H; X) \).

**Lemma 1.** We have

\[
\tilde{F}_p(H; X^{-1}) = \tilde{F}_p(H; X), \quad \text{if } m \text{ is odd.}
\]

\[
\tilde{F}_p(H; \xi_p X^{-1}) = \tilde{F}_p(H; X), \quad \text{if } m \text{ is even and } \xi_p \neq 0.
\]

Let \( k \) be a sufficiently large integer. Put \( n = [m/2] \). The Eisenstein series \( E_{2k+2n}^{(m)}(Z) \) of weight \( 2k + 2n \) on \( \mathcal{H}_m \) is defined by

\[
E_{2k+2n}^{(m)}(Z) = \sum_{\{C, D\}/\sim} \det(CZ + D)^{-2k-2n},
\]

where \( \{C, D\}/\sim \) extends over coprime pairs \( \{C, D\}, C, D \in M_{2n}(\mathcal{O}) \) such that \( C^t D = D^t \bar{C} \) modulo the action of \( GL_m(\mathcal{O}) \). We put

\[
E_{2k+2n}^{(m)}(Z) = A_m^{-1} \prod_{i=1}^{m} L(1 + i - 2k - 2n, \chi^{i-1}) E_{2k+2n}^{(m)}(Z).
\]

Here

\[
A_m = \begin{cases} 2^{-4n^2-4n} D^{2n^2+n} & \text{if } m = 2n + 1, \\ (-1)^n 2^{-4n^2+4n} D^{2n^2-n} & \text{if } m = 2n. \end{cases}
\]
Then the $H$-th Fourier coefficient of $E^{(2n+1)}_{2k+2n}(Z)$ is equal to
\[ |\gamma(H)|^{2k-1} \prod_{p \mid \gamma(H)} F_p(H; p^{-2k-2n}) = |\gamma(H)|^{k-(1/2)} \prod_{p \mid \gamma(H)} \tilde{F}_p(H; p^{-k+(1/2)}) \]
\[ = |\gamma(H)|^{k-(1/2)} \prod_{p \mid \gamma(H)} \tilde{F}_p(H; p^{k-(1/2)}) \]
for any $H \in \Lambda_{2n+1}(O)^+$ and any sufficiently large integer $k$.

The $H$-th Fourier coefficient of $E^{(2n)}_{2k+2n}(Z)$ is equal to
\[ |\gamma(H)|^{2k} \prod_{p \mid \gamma(H)} F_p(H; p^{-2k-2n}) = |\gamma(H)|^k \prod_{p \mid \gamma(H)} \tilde{F}_p(H; p^{-k}) \]
for any $H \in \Lambda_{2n}(O)^+$ and any sufficiently large integer $k$.

2. Main theorems

We first consider the case when $m = 2n$ is even.

Let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in S_{2k+1}(\Gamma_0(D), \chi)$ be a primitive form, whose $L$-function is given by
\[ L(f, s) = \sum_{N=1}^{\infty} a(N)N^{-s} = \prod_{p \mid D} (1 - a(p)p^{-s} + \chi(p)p^{2k-2s})^{-1} \prod_{q \mid D} (1 - a(q)q^{-s})^{-1}. \]

For each prime $p \nmid D$, we define the Satake parameter $\{\alpha_p, \beta_p\} = \{\alpha_p, \chi(p)\alpha_p^{-1}\}$ by
\[ (1 - a(p)X + \chi(p)p^{2k}X^2) = (1 - p^k\alpha_p X)(1 - p^k\beta_p X). \]

For $q \mid D$, we put $\alpha_q = q^{-k}a(q)$.

Put
\[ A(H) = |\gamma(H)|^k \prod_{p \mid \gamma(H)} \tilde{F}_p(H, \alpha_p), \quad H \in \Lambda_{2n}(O)^+ \]
\[ F(Z) = \sum_{H \in \Lambda_{2n}(O)^+} A(H) e(HZ), \quad Z \in \mathcal{H}_{2n}. \]

Then our first main theorem is as follows:

**Theorem 1.** Assume that $m = 2n$ is even. Let $f(\tau)$, $A(H)$ and $F(Z)$ be as above. Then we have $F \in S_{2k+2n}(\Gamma^{(2n)}_K)$. Moreover, $F$ is a Hecke eigenform. $F = 0$ if and only if $f(\tau)$ comes from a Hecke character of $K$ and $n$ is odd.
Now we consider the case when \( m = 2n + 1 \) is odd.

Let \( f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in S_{2k}(\text{SL}_2(\mathbb{Z})) \) be a normalized Hecke eigenform, whose \( L \)-function is given by
\[
L(f, s) = \sum_{N=1}^{\infty} a(N)N^{-s} = \prod_p \left( 1 - a(p)p^{-s} + p^{2k-2-2s} \right)^{-1}
\]

For each prime \( p \), we define the Satake parameter \( \{\alpha_p, \alpha_p^{-1}\} \) by
\[
(1 - a(p)X + p^{2k-1}X^2) = (1 - p^{k-(1/2)}\alpha_p X)(1 - p^{k-(1/2)}\alpha_p^{-1} X).
\]

Put
\[
A(H) = |\gamma(H)|^{k-(1/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H, \alpha_p), \quad H \in \Lambda_{2n+1}(\mathcal{O})^+
\]
\[
F(Z) = \sum_{H \in \Lambda_{2n+1}(\mathcal{O})^+} A(H) e(HZ), \quad Z \in \mathcal{H}_{2n+1}.
\]

**Theorem 2.** Assume that \( m = 2n + 1 \) is odd. Let \( f(\tau), A(H) \) and \( F(Z) \) be as above. Then we have \( F \in S_{2k+2n}(\Gamma_K^{(2n+1)}) \). Moreover, \( F \) is a non-zero Hecke eigenform.

**References**