PRINCIPAL SERIES WHITTAKER FUNCTIONS ON SYMPLECTIC GROUPS

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§1. Class one Whittaker functions

(1.1) Definitions and notation Let G be a semisimple Lie group with finite center and $\mathfrak{g}$ its Lie algebra. Fix a maximal compact subgroup K of G and put $\mathfrak{k} = \mathrm{Lie}(K)$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ and θ the corresponding Cartan involution. For a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ and $\alpha \in \mathfrak{a}^*$, put $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$ and $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ the restricted root system. Denoted by Δ^+ the positive system in Δ and Π the set of simple roots. Then we have an Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ with $\mathfrak{n} = \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$. Let $G = NAK$ be the Iwasawa decomposition corresponding to that of $\mathfrak{g}$. We denote by W the Weyl group of the root system Δ.

Let $P_0 = MAN$ be the minimal parabolic subgroup of G with $M = Z_K(A)$. For a linear form $\nu \in \mathfrak{a}^*_{\mathbf{C}} = \mathfrak{a}^* \otimes_{\mathbf{R}} \mathbf{C}$, define a character e^ν on A by $e^\nu(a) = \exp(\nu(\log a))$ $(a \in A)$. We call the induced representation

$$\pi_\nu = L^2\text{-}\mathrm{Ind}_{P_0}^G(1_M \otimes e^{\nu+\rho} \otimes 1_N)$$

the *class one principal series representation* of G. Here $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} m_\alpha \alpha$ $(m_\alpha = \dim \mathfrak{g}_\alpha)$.

Let $U(\mathfrak{g}_{\mathbf{C}})$ and $U(\mathfrak{a}_{\mathbf{C}})$ be the universal enveloping algebras of $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{a}_{\mathbf{C}}$, the complexifications of $\mathfrak{g}$ and $\mathfrak{a}$ respectively. Set

$$U(\mathfrak{g}_{\mathbf{C}})^K = \{X \in U(\mathfrak{g}_{\mathbf{C}}) \mid \mathrm{Ad}(k)X = X \text{ for all } k \in K\}.$$

Let p be the projection $U(\mathfrak{g}_{\mathbf{C}}) \to U(\mathfrak{a}_{\mathbf{C}})$ along the decomposition $U(\mathfrak{g}_{\mathbf{C}}) = U(\mathfrak{a}_{\mathbf{C}}) \oplus (\mathfrak{n}U(\mathfrak{g}_{\mathbf{C}}) + U(\mathfrak{g}_{\mathbf{C}})\mathfrak{k})$. Define the automorphism γ of $U(\mathfrak{a}_{\mathbf{C}})$ by $\gamma(H) = H + \rho(H)$ for $H \in \mathfrak{a}_{\mathbf{C}}$. For $\nu \in \mathfrak{a}^*_{\mathbf{C}}$, define the algebra homomorphism $\chi_\nu : U(\mathfrak{g}_{\mathbf{C}})^K \to \mathbf{C}$ by

$$\chi_\nu(z) = \nu(\gamma \circ p(z))$$

for $z \in U(\mathfrak{g}_{\mathbf{C}})^K$. Note that χ_ν is trivial on $U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k}$ and the restriction of χ_ν to the center $Z(\mathfrak{g}_{\mathbf{C}})$ of $U(\mathfrak{g}_{\mathbf{C}})$ coincides with the infinitesimal character of the class one principal series representation π_ν. Let η be a unitary character of N. Since $\mathfrak{n} = [\mathfrak{n}, \mathfrak{n}] \oplus \sum_{\alpha\in\Pi} \mathfrak{g}_\alpha$, η is determined by the restriction $\eta_\alpha := \eta|_{\mathfrak{g}_\alpha}$ $(\alpha \in \Pi)$. The length $|\eta_\alpha|$ of η_α is defined as $|\eta_\alpha|^2 = \sum_{1\le i\le m_\alpha} \eta(X_{\alpha,i})$, where the root vector $X_{\alpha,j}$ is chosen as $B(X_{\alpha,i}, \theta X_{\alpha,j}) = -\delta_{i,j}$ $(1 \le i, j \le m_\alpha)$. Here $B(\ ,\ )$ is the Killing form on $\mathfrak{g}$. In this article we assume that η is nondegenerate, that is, $\eta_\alpha \ne 0$ for all $\alpha \in \Pi$.

Definition 1.1 Under the above notation, a smooth function $w = w_{\nu,\eta}$ on G is called *class one Whittaker function* if

(i) $w(ngk) = \eta(n)w(g)$, for all $n \in N$, $g \in G$ and $k \in K$,

(ii) $Zw = \chi_\nu(Z)w$, for all $Z \in U(\mathfrak{g}_{\mathbf{C}})^K$.

We denote by $\mathrm{Wh}(\nu,\eta)$ the space of class one Whittaker functions and $\mathrm{Wh}(\nu,\eta)^{\mathrm{mod}}$ the subspace consisting of moderate growth functions.

Remark. Because of the Iwasawa decomposition, $w \in \mathrm{Wh}(\nu,\eta)$ is determined by its restriction $w|_A$ to A. We call $w|_A$ the *radial part* of w.

(1.2) M and W-Whittaker functions

Theorem 1.2 *The dimension of the space* $\mathrm{Wh}(\nu,\eta)$ *is the order of the Weyl group W and the the dimension of* $\mathrm{Wh}(\nu,\eta)^{\mathrm{mod}}$ *is at most one. Moreover the unique (up to constant) element in* $\mathrm{Wh}(\nu,\eta)^{\mathrm{mod}}$ *is given by Jacquet integral*:

$$W(\nu,\eta;g) = \int_N a(s_0^{-1}ng)^{\nu+\rho}\eta(n)^{-1}dn.$$

Here s_0 is the longest element in W and $g = n(g)a(g)k(g)$ the Iwasawa decomposition of $g \in G$.

Hashizume ([3]) gave a basis of $\mathrm{Wh}(\nu,\eta)$ and express the Jacquet integral as a linear combination of the basis functions. Let $\langle\ ,\ \rangle$ be the inner product on $\mathfrak{a}_{\mathbf{C}}^*$ induced by the Killing form $B(\ ,\ )$. We denote by L the set of linear functions on $\mathfrak{a}_{\mathbf{C}}$ of the form $\sum_{\alpha\in\Pi} n_\alpha\alpha$ with $n_\alpha \in \mathbf{Z}_{\geq 0}$.

For each $\lambda \in L$, we can define the rational function a_λ on $\mathfrak{a}_{\mathbf{C}}^*$ as follows. Put $a_0(\nu) = 1$ and determine a_λ for $\lambda \in L\backslash\{0\}$ by

$$(\langle\lambda,\lambda\rangle + 2\langle\lambda,\nu\rangle)a_\lambda(\nu) = 2\textstyle\sum_{\alpha\in\Pi}|\eta_\alpha|^2 a_{\lambda-2\alpha}(\nu), \tag{1.1}$$

inductively. Here we assumed that $\langle\lambda,\lambda\rangle + 2\langle\lambda,\nu\rangle \neq 0$ for all $\lambda \in L\backslash\{0\}$.

Definition 1.3 For $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ and unitary character η of N, define a series $M(\nu,\eta;a)$ on A by

$$M(\nu,\eta;a) = a^{\nu+\rho}\sum_{\lambda\in L} a_\lambda(\nu)a^\lambda \quad (a \in A)$$

and extend it to the function on G by

$$M(\nu,\eta;g) = \eta(n(g))M(\nu,\eta;a(g)).$$

Definition 1.4 We denote by $'\mathfrak{a}_{\mathbf{C}}^*$ the set of elements $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ satisfying the following:

(i) $\langle\lambda,\lambda\rangle + 2\langle\lambda,s\nu\rangle \neq 0$ *for all* $\lambda \in L\backslash\{0\}$ *and* $s \in W$,

(ii) $s\nu - t\nu \notin \{\sum_{\alpha\in\Pi} m_\alpha\alpha \mid m_\alpha \in \mathbf{Z}\}$ *for all* $s \neq t \in W$.

Theorem 1.5 ([3, Theorem 5.4]) *Let* $\nu \in {}'\mathfrak{a}_{\mathbf{C}}^*$. *Then the set* $\{M(s\nu,\eta;g) \mid s \in W\}$ *forms a basis of* $\mathrm{Wh}(\nu,\eta)$.

We call $W(\nu,\eta;g)$ (resp. $M(\nu,\eta;g)$) W-Whittaker function (resp. M-Whittaker function). Let us recall the linear relation between W and M-Whittaker functions.

Proposition 1.6 ([4, cf. Ch IV]) *Let $c(\nu)$ be the Harish Chandra c-function. Then*

$$\begin{aligned} c(\nu) &:= \int_N a(s_0^{-1}n)^{\nu+\rho} dn \\ &= \prod_{\alpha\in\Delta_0^+} 2^{\frac{m_\alpha - m_{2\alpha}}{2}} \Big(\frac{\pi}{\langle\alpha,\alpha\rangle}\Big)^{\frac{m_\alpha+m_{2\alpha}}{2}} \frac{\Gamma(\nu_\alpha)\Gamma(\frac{1}{2}(\nu_\alpha+\frac{m_\alpha}{2}))}{\Gamma(\nu_\alpha+\frac{m_\alpha}{2})\Gamma(\frac{1}{2}(\nu_\alpha+\frac{m_\alpha}{2}+m_{2\alpha}))}. \end{aligned}$$

Here $\Delta_0^+ = \{\alpha\in\Delta^+ \mid \frac{1}{2}\alpha \notin \Delta\}$.

Definition 1.7 For $\eta\in\widehat{N}$, $\nu\in\mathfrak{a}_{\mathbf{C}}^*$ and $s\in W$, we define $\gamma(s;\nu,\eta)$ as follows. If $s=s_\alpha$ ($\alpha\in\Pi$), the simple reflection,

$$\gamma(s;\nu,\eta) = \Big(\frac{|\eta_\alpha|}{2\sqrt{2\langle\alpha,\alpha\rangle}}\Big)^{2\nu_\alpha} \frac{\Gamma(\frac{1}{2}(-\nu_\alpha+\frac{m_\alpha}{2}+1))\Gamma(\frac{1}{2}(-\nu_\alpha+\frac{m_\alpha}{2}+m_{2\alpha}))}{\Gamma(\frac{1}{2}(\nu_\alpha+\frac{m_\alpha}{2}+1))\Gamma(\frac{1}{2}(-\nu_\alpha+\frac{m_\alpha}{2}+m_{2\alpha}))}.$$

For $s\in W$ and $\alpha\in\Pi$ such that $l(s_\alpha s) = l(s)+1$, then

$$\gamma(s_\alpha s;\nu,\eta) = \gamma(s;\nu,\eta)\gamma(s_\alpha; s\nu,\eta).$$

Here $l(s)$ means the length of s.

Theorem 1.8 ([3, Theorem 7.8]) *If $\nu\in{}'\mathfrak{a}_C^*$,*

$$W(\nu,\eta;g) = \sum_{s\in W} \gamma(s_0 s;\nu,\eta)c(s_0 s\nu)M(s\nu,\eta;g).$$

<u>**Problem**</u> : Find explicit formulas of $W(\nu,\eta;g)$ and $M(\nu,\eta;g)$.

<u>**Known results**</u> (G is real semisimple) :

(1) G is real rank 1 : W (resp. M)-Whittaker functions can be written by modified K (resp. I)-Bessel functions.

(2) $G = SL(n,\mathbf{R})$: In case of $n=3$, Tahtajan-Vinogradov ([14]) and Bump ([1]) obtained explicit formulas of W and M-Whittaker functions. For general n, Stade ([11]) found a recursive integral formula of W-Whittaker function and I ([7]) proved a similar recursive formula of M-Whittaker function conjectured in [13]. When $n=4$, Stade ([12]) also gave a explicit formula of $a_\lambda(\nu)$ by solving the recurrence relation (1.1) and his formula included (terminating) generalized hypergeometric series ${}_4F_3(1)$ (cf. [7]).

(3) $G = Sp(2,\mathbf{R}), SO_o(2,q) (q\geq 3)$: As for W-Whittaker function on $Sp(2,\mathbf{R})$, Niwa ([9]) obtained the formula (3.5) in section (3.1). In the similar way to Proskurin's evaluation of Jacquet integral for $G = Sp(2,\mathbf{C})$ ([10]), I ([5]) found the integral expression (3.7). The explicit formula (3.4) of M-Whittaker function is also obtained in [5]. These results can be extended to $SO_o(2,q)$ in [6] ($\mathfrak{so}(2,3)\cong\mathfrak{sp}(2,\mathbf{R})$, $\mathfrak{so}(2,4)\cong\mathfrak{su}(2,2)$).

Extending the work of Niwa, we consider the problem in case of $G = Sp_n(\mathbf{R})$ and $SO_{n,n}$ in this article.

(1.3) Structure theory for $Sp_n(\mathbf{R})$ and $SO_{n,n}$ We give precise description of the notation in the above subsections. Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be algebraic groups over $\mathbf{Q}$ defined as

$$\mathbf{G}_1 = \mathbf{SO}_{n,n} = \left\{ g \in \mathbf{SL}_{2n} \,\middle|\, {}^t g \begin{pmatrix} & J_n \\ J_n & \end{pmatrix} g = \begin{pmatrix} & J_n \\ J_n & \end{pmatrix} \right\},$$

and

$$\mathbf{G}_2 = \mathbf{Sp}_n = \left\{ g \in \mathbf{SL}_{2n} \,\middle|\, {}^t g \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix} g = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix} \right\}.$$

Here $J_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$ ($n \times n$ matrix). Hereafter we use the notation in sections (1.1) and (1.2) with subscript $_1$ for $G_1 := \mathbf{G}_1(\mathbf{R}) = SO_{n,n}$ and $_2$ for $G_2 := \mathbf{G}_2(\mathbf{R}) = Sp_n(\mathbf{R})$.

< Iwasawa decompositions >

$$\begin{aligned}
\mathfrak{a}_1 &= \{\mathrm{diag}(a_1, \ldots, a_n, -a_n, \ldots, -a_1) \mid a_i \in \mathbf{R}\}, \\
\mathfrak{a}_2 &= \{\mathrm{diag}(t_1, \ldots, t_n, -t_n, \ldots, -t_1) \mid t_i \in \mathbf{R}\}, \\
A_1 &= \{\mathrm{diag}(a_1, \ldots, a_n, a_n^{-1}, \ldots, a_1^{-1}) \mid a_i > 0\}, \\
A_2 &= \{\mathrm{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \mid t_i > 0\}, \\
N_i &= \left\{ \begin{pmatrix} n_0 & * \\ 0 & J_n {}^t n_0^{-1} J_n \end{pmatrix} \in G_i \,\middle|\, n_0 = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}.
\end{aligned}$$

< principal series >

$$\begin{aligned}
&\nu = (\nu_1, \ldots, \nu_n) \in \mathfrak{a}_{i,\mathbf{C}}^* \quad (i = 1, 2), \\
&\rho_1 = \rho_1^{(n)} = (n-1, n-2, \ldots, 1, 0), \quad \rho_2 = \rho_2^{(n)} = (n, n-1, \ldots, 2, 1).
\end{aligned}$$

< Weyl groups > $W_1 = \mathfrak{S}_n \ltimes (\mathbf{Z}/2\mathbf{Z})^{n-1}$, $W_2 = \mathfrak{S}_n \ltimes (\mathbf{Z}/2\mathbf{Z})^n$.

< unitary characters >

$$\begin{aligned}
\eta_1(u) &= \exp\big(2\pi\sqrt{-1}(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n} + u_{n-1,n+1})\big), \\
\eta_2(u) &= \exp\big(2\pi\sqrt{-1}(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n} + u_{n,n+1})\big),
\end{aligned}$$

for $u = (u_{k,l}) \in N_i$.

< $c_i(\nu)$ and $\gamma_i(s; \nu, \eta)$ >

$$c_1(\nu) = \pi^{\frac{n(n-1)}{2}} \prod_{1 \le i < j \le n} \frac{\Gamma(\frac{\nu_i - \nu_j}{2})\Gamma(\frac{\nu_i + \nu_j}{2})}{\Gamma(\frac{\nu_i - \nu_j + 1}{2})\Gamma(\frac{\nu_i + \nu_j + 1}{2})},$$

$$c_2(\nu) = \frac{\pi^{\frac{n^2}{2}}}{2^{\frac{n}{2}}} \prod_{1 \le i \le n} \frac{\Gamma(\frac{\nu_i}{2})}{\Gamma(\frac{\nu_i + 1}{2})} \prod_{1 \le i < j \le n} \frac{\Gamma(\frac{\nu_i - \nu_j}{2})\Gamma(\frac{\nu_i + \nu_j}{2})}{\Gamma(\frac{\nu_i - \nu_j + 1}{2})\Gamma(\frac{\nu_i + \nu_j + 1}{2})},$$

$$c_1(s_0 s\nu)\gamma_1(s_0 s;\nu,\eta_1) = \pi^{\frac{n(n-1)}{2}+\langle\nu,\rho_1\rangle}\,\frac{s\left[\pi^{\langle\nu,\rho_1\rangle}\prod_{1\le i<j\le n}\Gamma(\frac{-\nu_i+\nu_j}{2})\Gamma(\frac{-\nu_i-\nu_j}{2})\right]}{\prod_{1\le i<j\le n}\Gamma(\frac{\nu_i-\nu_j+1}{2})\Gamma(\frac{\nu_i+\nu_j+1}{2})},$$

$$c_2(s_0 s\nu)\gamma_2(s_0 s;\nu,\eta_2) = 2^{-\frac{n}{2}}\pi^{\frac{n^2}{2}+\langle\nu,\rho_2\rangle-\frac12\sum_{i=1}^n\nu_i}$$

$$\cdot\frac{s\left[\pi^{\langle\nu,\rho_2\rangle-\frac12\sum_{i=1}^n\nu_i}\prod_{1\le i\le n}\Gamma(-\frac{\nu_i}{2})\prod_{1\le i<j\le n}\Gamma(\frac{-\nu_i+\nu_j}{2})\Gamma(\frac{-\nu_i-\nu_j}{2})\right]}{\prod_{1\le i\le n}\Gamma(\frac{\nu_i+1}{2})\prod_{1\le i<j\le n}\Gamma(\frac{\nu_i-\nu_j+1}{2})\Gamma(\frac{\nu_i+\nu_j+1}{2})}.$$

§2. Symplectic orthogonal theta lifts and main theorem

(2.1) Weil representation and theta lift Let k be a local field and ψ a nontrivial character of k. For a finite dimensional k-vector space Z equipped with symplectic form $\langle\ ,\ \rangle$, put

$$Sp(Z,k) = \{g\in GL(Z,k)\mid \langle z_1g, z_2g\rangle = \langle z_1,z_2\rangle,\quad \forall z_1,z_2\in Z\}.$$

Let $Z = Z^+ + Z^-$ be a polarization, that is, $Z^\pm$ are maximal isotropic subspace of Z. Let ω_ψ be the Weil representation of $\widetilde{Sp}(Z,k)$ on $\mathscr{S}(Z^+)$, the space of Schwartz-Bruhat functions on Z^+. When k is a global field and ψ a nontrivial character on $k\backslash\mathbf{A}$, we can also define Weil representation ω_ψ of $\widetilde{Sp}(Z,\mathbf{A})$ on $\mathscr{S}(Z^+_{\mathbf{A}})$.

Let k be a global field and X a $2n$-dimensional k-vector space of column vectors with symmetric form $(\ ,\ )$ given by $(x,y) = {}^t x\begin{pmatrix} & J_n\\ J_n & \end{pmatrix}y$. Then $\mathbf{G}_1(k) = SO_{n,n}(k)$ acts on X from the left and preserves $(\ ,\ )$. Also let Y be a $2n$-dimensional k-vector space of row vectors with symplectic form $\langle\ ,\ \rangle$ given by $\langle x,y\rangle = x\begin{pmatrix} & J_n\\ -J_n & \end{pmatrix}{}^t y$. Then $\mathbf{G}_2(k) = Sp_n(k)$ acts on Y from the right and preserves $\langle\ ,\ \rangle$. The space $Z := X\otimes Y$ has a symplectic form $(\ ,\ )\otimes\langle\ ,\ \rangle$ and we have a homomorphism $SO_{n,n}(\mathbf{A})\times Sp_n(\mathbf{A})\to Sp(Z,\mathbf{A})$. Let $\{e_1,\ldots,e_n,e_{-n},\ldots,e_{-1}\}$ be the standard basis of X. Then $X^+ = \mathrm{Span}\{e_1,\ldots,e_n\}$ and $X^- = \mathrm{Span}\{e_{-n},\ldots,e_{-1}\}$ give a polarization of X. Also take the standard basis of Y by $\{\varepsilon_1,\ldots,\varepsilon_n,\varepsilon_{-n},\ldots,\varepsilon_{-1}\}$ and put $Y^+ = \mathrm{Span}\{\varepsilon_1,\ldots,\varepsilon_n\}$, $Y^- = \mathrm{Span}\{\varepsilon_{-n},\ldots,\varepsilon_{-1}\}$. We choose a polarization of Z by $Z^\pm = X\otimes Y^\pm$ and denote $\sum_{i=1}^n x_i\otimes\varepsilon_i\in Z^+$ by $(x_1,\ldots,x_n)$.

For ω_ψ and $\phi\in\mathscr{S}(Z^+_{\mathbf{A}})$, define the theta series θ^ϕ_ψ on $\mathbf{G}_1(\mathbf{A})\times\mathbf{G}_2(\mathbf{A})$ by

$$\theta^\phi_\psi(g_1,g_2) = \sum_{z\in Z^+_k}\omega_\psi(g_1,g_2)\phi(z).$$

Let σ be an irreducible cuspidal automorphic representation of $\mathbf{G}_1(\mathbf{A})$. For a cusp form $f\in\sigma$, put

$$F^\phi_f(g_2) = \int_{\mathbf{G}_1(k)\backslash\mathbf{G}_1(\mathbf{A})}\theta^\phi_\psi(g_1,g_2)f(g_1)dg_1.$$

It is known that F^ϕ_f defines a cusp form on $\mathbf{G}_2(\mathbf{A})$ and the space $\Theta_\psi(\sigma) = \langle F^\phi_f\mid f\in\sigma,\phi\in\mathscr{S}(Z^+_{\mathbf{A}})\rangle$ is called the *theta lift* of σ with respect to ψ.

(2.2) Whittaker coefficients To describe Whittaker coefficient, we fix unitary characters ψ_1 and ψ_2 of $\mathbf{N}_1(\mathbf{A})$ and $\mathbf{N}_2(\mathbf{A})$ as follows (cf. section (1.3)).

$$\psi_1(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n} + u_{n-1,n+1}),$$
$$\psi_2(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n} + u_{n,n+1})$$

for $u = (u_{k,l}) \in \mathbf{N}_i(\mathbf{A})$. We say an irreducible cuspidal representation σ_i on $\mathbf{G}_i(\mathbf{A})$ *has a nontrivial ψ_i^{-1}-Whittaker coefficient*, if the integral

$$W_f(g_i) = \int_{\mathbf{N}_i(k)\backslash\mathbf{N}_i(\mathbf{A})} f(ng_i)\psi_i^{-1}(n)dn$$

does not vanish for some $f \in \sigma_i$. Ginzburg, Rallis and Soudry ([2]) proved the following:

Proposition 2.1 ([2, Proposition 3.5]) *We assume that the irreducible cuspidal representation σ of $\mathbf{G}_1(\boldsymbol{A})$ has a nontrivial ψ_1^{-1}-Whittaker coefficient. Then the theta lift $\Theta_\psi(\sigma)$ to $\mathbf{G}_2(\boldsymbol{A})$ is nontrivial and has a ψ_2^{-1}-Whittaker coefficient. Moreover, the ψ_2^{-1}-Whittaker coefficient of $F_f^\phi \in \Theta_\psi(\sigma)$ is*

$$W_{F_f^\phi}(g_2) = \int_{E(\mathbf{A})\backslash\mathbf{G}_1(\mathbf{A})} \omega_\psi(g_1, g_2)\phi(u_0)W_f(g_1)dg_1. \tag{2.1}$$

Here E is the stabilizer of $u_0 = (e_1, \ldots, e_{n-1}, e_n + e_{-n}) \in Z^+$.

If we decompose the right hand side of (2.1) to the local factors, the integral

$$\int_{E(\mathbf{R})\backslash\mathbf{G}_1(\mathbf{R})} \omega_\psi(g_1, g_2)\phi(u_0)W(g_1)dg_1.$$

is expected to represent the Whittaker function on $Sp_n(\mathbf{R})$. Here W is the Whittaker function on $SO_{n,n}$. Then, if we take

$$\phi(X) = \exp[-\pi(\mathrm{tr}({}^tXX))],$$

and compute the integral by using the formulas of Weil representation, we can propose the following:

Theorem 2.2 *For $a \in A_1$ and $t \in A_2$, put*

$$\theta(a,t) = \exp\Big[-\pi\Big\{\Big(\frac{t_1^2}{a_1^2} + \frac{a_1^2}{t_2^2}\Big) + \cdots + \Big(\frac{t_{n-1}^2}{a_{n-1}^2} + \frac{a_{n-1}^2}{t_n^2}\Big) + \Big(\frac{t_n^2}{a_n^2} + t_n^2a_n^2\Big)\Big\}\Big].$$

Then, for $\nu \in {}'\mathfrak{a}_{1,\mathbf{C}}^* \cap {}'\mathfrak{a}_{2,\mathbf{C}}^*$,

$$\frac{\pi^{-\frac{1}{2}\sum_{i=1}^n \nu_i}}{(2\pi)^{\frac{n}{2}}}\prod_{i=1}^n \Gamma\Big(\frac{\nu_i+1}{2}\Big) \cdot t^{-\rho_2}W_2(\nu;t) = \int_{(\mathbf{R}_{\geq 0})^n} \theta(a,t)\cdot a^{-\rho_1}W_1(\nu;a)\prod_{i=1}^n \frac{da_i}{a_i}. \tag{2.2}$$

The right hand side of (2.2) represent a Whittaker function, however, to see that it is just the Whittaker function we want to seek, it seems to need further argument. For example, if we use the similar result of [2] from $\mathbf{Sp}_n$ to $\mathbf{SO}_{n+1,n+1}$, we obtain Whittaker function on $SO_{n+1,n+1}$ from one on $Sp_n(\mathbf{R})$ (see (3.11)). Though in this formula, the parameter of principal series is not general ($\nu_{n+1}=0$). Then in case of $n=2$, Niwa proved this theorem by checking the right hand side (=(3.5)) satisfy the system of partial differential equation for $Sp_2(\mathbf{R})$-Whittaker function by using computer. But in case of general n, the explicit form of differential equation is not known. So we first prove the lifting of M-Whittaker functions (which also seems to be interesting result) and by using Theorem 1.7 we establish the lifting of W-Whittaker functions.

(2.3) Lifting of M-Whittaker functions We first write down the recurrence relation (1.1) explicitly.

Proposition 2.3 *Let*

$$M_1(\nu;a)=a^{\nu+\rho_1}\sum_{\mathbf{m}=(m_1,\dots,m_n)\in(\mathbf{Z}_{\geq 0})^n} c_{1,\mathbf{m}}(\nu)\Big(2\pi\frac{a_1}{a_2}\Big)^{2m_1}\cdots\Big(2\pi\frac{a_{n-1}}{a_n}\Big)^{2m_{n-1}}(2\pi a_{n-1}a_n)^{2m_n}$$

be the radial part of M-Whittaker function on $SO_{n,n}$. If $\nu\in{}'\mathfrak{a}^*_{1,\mathbf{C}}$, the coefficients $c_{1,\mathbf{m}}(\nu)$ are determined by the following recurrence relation:

$$\begin{aligned}\Big[4\Big(\sum_{i=1}^{n} m_i^2-\sum_{i=1}^{n-2} m_i m_{i+1}-m_{n-2}m_n\Big)\\ +2\Big(\sum_{i=1}^{n-1} m_i(\nu_i-\nu_{i+1})+m_n(\nu_{n-1}+\nu_n)\Big)\Big]c_{1,\mathbf{m}}(\nu)=\sum_{i=1}^{n} c_{1,\mathbf{m}-\mathbf{e}_i}(\nu),\end{aligned}\tag{2.3}$$

with $\mathbf{e}_i=(0,\dots,1,\dots,0)$.

Proposition 2.4 *Let*

$$M_2(\nu;t)=t^{\nu+\rho_2}\sum_{\mathbf{k}=(k_1,\dots,k_n)\in(\mathbf{Z}_{\geq 0})^n} c_{2,\mathbf{k}}(\nu)\Big(2\pi\frac{t_1}{t_2}\Big)^{2k_1}\cdots\Big(2\pi\frac{t_{n-1}}{t_n}\Big)^{2k_{n-1}}(2\pi t_n^2)^{2k_n}$$

be the radial part of M-Whittaker function on $Sp_n(\boldsymbol{R})$. If $\nu\in{}'\mathfrak{a}^*_{2,\mathbf{C}}$, the coefficients $c_{2,\mathbf{k}}(\nu)$ are determined by the following recurrence relation:

$$\begin{aligned}\Big[4\Big(\sum_{i=1}^{n-1} k_i^2+2k_n^2-\sum_{i=1}^{n-2} k_i k_{i+1}-2k_{n-1}k_n\Big)\\ +2\Big(\sum_{i=1}^{n-1} k_i(\nu_i-\nu_{i+1})+2k_n\nu_n\Big)\Big]c_{2,\mathbf{k}}(\nu)=\sum_{i=1}^{n-1} c_{2,\mathbf{k}-\mathbf{e}_i}(\nu)+2c_{2,\mathbf{k}-\mathbf{e}_n}(\nu).\end{aligned}\tag{2.4}$$

From the above propositions we can prove the following:

Theorem 2.5 *If $\nu\in{}'\mathfrak{a}^*_{1,\mathbf{C}}\cap{}'\mathfrak{a}^*_{2,\mathbf{C}}$,*

$$c_{2,\mathbf{k}}(\nu)=\frac{1}{\prod_{i=1}^{n}(\frac{\nu_i}{2}+1)_{k_i}}$$

$$\cdot \sum_{\mathbf{m}\in S(\mathbf{k})} \frac{(-1)^{m_1+\cdots+m_{n-1}} 4^{\sum_{i=1}^{n}(m_i-k_i)} \prod_{i=1}^{n-1}(-k_{i+1}-\frac{\nu_{i+1}}{2})_{m_i} \cdot c_{1,\mathbf{m}}(\nu)}{(k_1-m_1)!\dots(k_{n-2}-m_{n-2})!(k_{n-1}-m_{n-1}-m_n)!(k_n-m_n)!}.$$

Here we use the notation

$$S(\mathbf{k}) = \left\{ \mathbf{m}\in \mathbf{Z}^n_{\geq 0} \;\middle|\; \begin{array}{l} 0\leq m_1\leq k_1,\dots,0\leq m_{n-2}\leq k_{n-2}, \\ 0\leq m_{n-1},\ m_{n-1}+m_n\leq k_{n-1},\ 0\leq m_n\leq k_n \end{array} \right\}$$

and $(a)_n = \Gamma(a+n)/\Gamma(a)$.

By using this Theorems 2.5 and 1.7, we compute the right hand side of (2.2), then we can reach the Theorem 2.2 after somewhat complicated but elementary calculus.

§3. Examples of explicit formulas

From now on we adopt the notation $W_1^{(n)}(\nu;a)$ (resp. $W_2^{(n)}(\nu;t)$) for the radial part of W-Whittaker function on $SO_{n,n}$ (resp. $Sp_n(\mathbf{R})$), etc.

(3.1) From $SO_{2,2}$ to $Sp_2(\mathbf{R})$

Proposition 3.1

$$M_1^{(2)}(\nu;a) = a_1^{\nu_1+1}a_2^{\nu_2} \sum_{m_1,m_2\geq 0} \frac{(\pi a_1/a_2)^{2m_1}(\pi a_1a_2)^{2m_2}}{m_1!m_2!(\frac{\nu_1-\nu_2}{2}+1)_{m_1}(\frac{\nu_1+\nu_2}{2}+1)_{m_2}}. \tag{3.1}$$

Proposition 3.2 $W_1^{(2)}(\nu;a)$ *has the following expressions.*

$$c_1^{(2)} a_1 K_{\frac{\nu_1-\nu_2}{2}}\Big(2\pi\frac{a_1}{a_2}\Big)K_{\frac{\nu_1+\nu_2}{2}}(2\pi a_1a_2), \tag{3.2}$$

$$c_1^{(2)} a_1 \int_{(\mathbf{R}_{\geq 0})^2} \exp\Big[-\pi\Big\{\frac{a_1^2}{t^2}+\Big(\frac{t^2}{a_2^2}+a_2^2t^2\Big)+\Big(\frac{t^2}{b^2}+t^2b^2\Big)\Big\}\Big]\cdot b^{\nu_1}\Big(\frac{a_1a_2b}{t^2(1+a_2^2b^2)}\Big)^{\nu_2}\frac{dt}{t}\frac{db}{b}, \tag{3.3}$$

with some constant $c_1^{(2)}$.

From the above two propositions, we have the followings:

Proposition 3.3

$$M_2^{(2)}(\nu;t) = t_1^{\nu_1+2}t_2^{\nu_2+1} \sum_{m_1,m_2\geq 0} {}_3F_2\left(\begin{array}{c} -m_2,\ -m_1-\frac{\nu_1}{2},\ m_1+\frac{\nu_1}{2}+1 \\ \frac{\nu_1}{2}+1,\ \frac{\nu_2}{2}+1 \end{array} \;\middle|\; 1 \right) \cdot \frac{(\pi t_1/t_2)^{2m_1}(\pi t_2^2)^{2m_2}}{m_1!\,m_2!\,(\frac{\nu_1-\nu_2}{2}+1)_{m_1}(\frac{\nu_1+\nu_2}{2}+1)_{m_2}}. \tag{3.4}$$

Proposition 3.4 $W_2^{(2)}(\nu;t)$ *has following integral expressions.*

$$c_2^{(2)}t_1^2t_2 \int_{(\mathbf{R}_{\geq 0})^2} \exp\Big[-\pi\Big\{\Big(\frac{t_1^2}{a_1^2}+\frac{a_1^2}{t_2^2}\Big)+\Big(\frac{t_2^2}{a_2^2}+t_2^2a_2^2\Big)\Big\}\Big] \cdot K_{\frac{\nu_1-\nu_2}{2}}\Big(2\pi\frac{a_1}{a_2}\Big)K_{\frac{\nu_1+\nu_2}{2}}(2\pi a_1a_2)\frac{da_1da_2}{a_1a_2}, \tag{3.5}$$

$$c_2^{(2)} t_1^2 t_2 \int_{(\mathbf{R}_{\geq 0})^4} \exp\Big[-\pi\Big\{\Big(\frac{t_1^2}{a_1^2}+\frac{a_1^2}{t_2^2}\Big)+\Big(\frac{t_2^2}{a_2^2}+t_2^2a_2^2\Big)+\frac{a_1^2}{u^2}+\Big(\frac{u^2}{a_2^2}+a_2^2u^2\Big) + \Big(\frac{u^2}{b^2}+u^2b^2\Big)\Big\}\Big] \cdot b^{\nu_1}\Big(\frac{a_1a_2b}{u^2(1+a_2^2b^2)}\Big)^{\nu_2}\frac{da_1da_2}{a_1a_2}\frac{du}{u}\frac{db}{b}, \tag{3.6}$$

$$\frac{1}{4}c_2^{(2)} t_1^{2+\frac{\nu_1}{2}} t_2^{1-\frac{3\nu_2}{2}} \int_{(\mathbf{R}_{\geq 0})^2} K_{\frac{\nu_1}{2}}\Big(2\pi\frac{t_1}{t_2}\sqrt{1+x+y}\Big)K_{\frac{\nu_2}{2}}\Big(2\pi t_2^2\sqrt{(1+1/x)(1+1/y)}\Big) \cdot\Big(\frac{x^2y^2}{1+x+y}\Big)^{\frac{\nu_1}{4}}\Big(\frac{x(1+x)}{y(1+y)}\Big)^{\frac{\nu_2}{4}}\frac{dxdy}{xy}, \tag{3.7}$$

with some constant $c_2^{(2)}$.

Remark. As mentioned before, (3.5) is the result of [9] and (3.7) is of [5]. The equivalence of these two expressions can be checked by way of (3.6) and slight change of variables.

(3.2) From $SO_{3,3}$ to $Sp_3(\mathbf{R})$ By virtue of $\mathfrak{so}_{3,3} \cong \mathfrak{sl}_4(\mathbf{R})$, we can find the integral expressions of $W_1^{(3)}(\nu;a)$ by the result of Stade ([11]) for W-Whittaker functions on $SL(n,\mathbf{R})$.

Proposition 3.5 $W_1^{(3)}(\nu;a)$ can be written as follows.

$$c_1^{(3)}a_1^2a_2 \int_{(\mathbf{R}_{\geq 0})^2} K_{\frac{\nu_1+\nu_2}{2}}\Big(2\pi a_2a_3\sqrt{1+u_1^{-2}}\Big)K_{\frac{\nu_1+\nu_2}{2}}\Big(2\pi\frac{a_2}{a_3}\sqrt{1+u_2^2}\Big) \cdot K_{\frac{\nu_1+\nu_2}{2}}\Big(2\pi\frac{a_1}{a_2}\sqrt{(1+u_1^2)(1+u_2^{-2})}\Big)K_{\frac{\nu_1-\nu_2}{2}}\Big(2\pi\frac{a_1}{a_2}\frac{u_1}{u_2}\Big) \cdot\Big(\frac{a_3}{u_1u_2}\Big)^{\nu_3}\frac{du_1du_2}{u_1u_2}, \tag{3.8}$$

$$c_1^{(3)}a_1^2a_2 \int_{(\mathbf{R}_{\geq 0})^6} \exp\Big[-\pi\Big\{\frac{a_1^2}{t_1^2}+\Big(\frac{t_1^2}{a_2^2}+\frac{a_2^2}{t_2^2}\Big)+\Big(\frac{t_2^2}{a_3^2}+a_3^2t_2^2\Big)+\Big(\frac{t_1^2}{b_1^2}+\frac{b_1^2}{t_2^2}\Big) + \Big(\frac{t_2^2}{b_2^2}+t_2^2b_2^2\Big)+\frac{b_1^2}{s^2}+\Big(\frac{s^2}{b_2^2}+b_2^2s^2\Big)+\Big(\frac{s^2}{c^2}+s^2c^2\Big)\Big\}\Big] \cdot c^{\nu_1}\Big(\frac{b_1b_2c}{s^2(1+b_2^2c^2)}\Big)^{\nu_2}\Big(\frac{a_1a_2a_3b_1b_2}{t_1^2t_2^2(1+a_3^2b_2^2)}\Big)^{\nu_3}\frac{dt_1dt_2}{t_1t_2}\frac{db_1db_2}{b_1b_2}\frac{ds}{s}\frac{dc}{c}, \tag{3.9}$$

with some constant $c_1^{(3)}$.

Proposition 3.6 $W_2^{(3)}(\nu;t)$ *is of the form*

$$c_2^{(3)}t_1^3t_2^2t_3 \int_{(\mathbf{R}_{\geq 0})^5} K_{\frac{\nu_1+\nu_2}{2}}\Big(2\pi a_2a_3\sqrt{1+u_1^{-2}}\Big)K_{\frac{\nu_1+\nu_2}{2}}\Big(2\pi\frac{a_2}{a_3}\sqrt{1+u_2^2}\Big) \cdot K_{\frac{\nu_1+\nu_2}{2}}\Big(2\pi\frac{a_1}{a_2}\sqrt{(1+u_1^2)(1+u_2^{-2})}\Big)K_{\frac{\nu_1-\nu_2}{2}}\Big(2\pi\frac{a_1}{a_2}\frac{u_1}{u_2}\Big) \cdot\exp\Big[-\pi\Big\{\Big(\frac{t_1^2}{a_1^2}+\frac{a_1^2}{t_2^2}\Big)+\Big(\frac{t_2^2}{a_2^2}+\frac{a_2^2}{t_3^2}\Big)+\Big(\frac{t_3^2}{a_3^2}+t_3^2a_3^2\Big)\Big\}\Big] \cdot\Big(\frac{a_3}{u_1u_2}\Big)^{\nu_3}\frac{du_1du_2}{u_1u_2}\frac{da_1da_2da_3}{a_1a_2a_3}, \tag{3.10}$$

with some constant $c_2^{(3)}$.

Remark. We also have a formula for $M_2^{(3)}(\nu;t)$ by using the formula in [12], however, our result is not satisfactory form now.

(3.3) Conjecture for general n [2, Proposition 2.7] also computed Whittaker coefficient of theta lift from $\mathbf{Sp}_n$ to $\mathbf{SO}_{n+1,n+1}$. In view of the result, it seems to hold

$$\begin{aligned}
&a^{-\rho_1^{(n+1)}} W_1^{(n+1)}((\nu_1,\dots,\nu_n,0);a)\\
&\quad = c\int_{\mathbf{R}_{\geq 0}^n} \widetilde{\theta}(a,t)\cdot t^{-\rho_2^{(n)}} W_2^{(n)}((\nu_1,\dots,\nu_n);t)\prod_{i=1}^n \frac{dt_i}{t_i},
\end{aligned} \tag{3.11}$$

where

$$\widetilde{\theta}(a,t)=\exp\Big[-\pi\Big\{\frac{a_1^2}{t_1^2}+\Big(\frac{t_1^2}{a_1^2}+\frac{a_2^2}{t_2^2}\Big)+\cdots+\Big(\frac{t_{n-1}^2}{a_{n-1}^2}+\frac{a_n^2}{t_n^2}\Big)+\Big(\frac{t_n^2}{a_{n+1}^2}+a_{n+1}^2t_n^2\Big)\Big\}\Big].$$

It may be impossible to extend $(\nu_1,\dots,\nu_n,0)\to(\nu_1,\dots,\nu_{n+1})$ by adding some terms containing ν_{n+1} to the integrand, however, we can propose the following conjecture from the results for $n=2,3$ ((3.3), (3.9)).

Conjecture 3.7 *Let* $b=\mathrm{diag}(b_1,\dots,b_{n+1},b_{n+1}^{-1},\dots,b_1^{-1})$. *Then* $W_1^{(n+1)}((\nu_1,\dots,\nu_{n+1});b)$ *has the following expressions.*

$$\begin{aligned}
&c\,b^{\rho_1^{(n+1)}}\int_{(\mathbf{R}_{\geq 0})^{2n}} \widetilde{\theta}(b,t)\theta(t,a)\cdot a^{-\rho_1^{(n)}} W_1^{(n)}((\nu_1,\dots,\nu_n);a)\\
&\qquad\cdot\Big(\frac{b_1\cdots b_{n+1}a_1\cdots a_n}{(t_1\cdots t_n)^2(1+b_{n+1}^2a_n^2)}\Big)^{\nu_{n+1}}\prod_{i=1}^n\frac{dt_i}{t_i}\frac{da_i}{a_i},
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&c\,b^{\rho_1^{(n+1)}}\int_{(\mathbf{R}_{\geq 0})^{n}} \prod_{i=1}^{n-1} K_{\nu_{n+1}}\Bigg(2\pi\frac{b_i}{b_{i+1}}\sqrt{\Big(1+\frac{a_{i-1}^2}{b_i^2}\Big)\Big(1+\frac{b_{i+1}^2}{a_i^2}\Big)}\Bigg)\\
&\qquad\cdot K_{\nu_{n+1}}\Bigg(2\pi b_nb_{n+1}\sqrt{\Big(1+\frac{a_{n-1}^2}{b_n^2}\Big)\Big(1+\frac{a_n^2}{b_{n+1}^2}\Big)\Big(1+\frac{1}{a_n^2b_{n+1}^2}\Big)}\Bigg)\\
&\qquad\cdot a^{-\rho_1^{(n)}} W_1^{(n)}((\nu_1,\dots,\nu_n);a)\Big(\frac{a_n^2+b_{n+1}^2}{1+a_n^2b_{n+1}^2}\Big)^{\frac{\nu_{n+1}}{2}}\prod_{i=1}^n\frac{da_i}{a_i},
\end{aligned} \tag{3.13}$$

with some constant c.

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