CAP automorphic representations of low rank groups *

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Abstract

In this talk, I report my recent joint work with K. Konno on non-tempered automorphic representations on low rank groups [KK]. We obtain a fairly complete classification of such automorphic representations for the quasisplit unitary groups in four variables.

1 CAP forms

The term CAP in the title is a short hand for the phrase "Cuspidal but Associated to Parabolic subgroups". This is the name given by Piatetski-Shapiro [PS83] to those cuspidal automorphic representations which apparently contradict the generalized Ramanujan conjecture. More precisely, let G be a connected reductive group defined over a number field F, and G^* be its quasisplit inner form. We write $\mathbb{A} = \mathbb{A}_F$ for the adéle ring of F. An irreducible cuspidal representation $\pi = \bigotimes_v \pi_v$ is a *CAP form* if there exists a residual discrete automorphic representation $\pi^* = \bigotimes_v \pi_v^*$ such that, at all but finite number of v, π_v and π_v^* share the same absolute values of Hecke eigenvalues.

It is a consequence of the result of Jacquet-Shalika [JS81a], [JS81b] and Moeglin-Waldspurger [MW89] that there are no CAP forms on the general linear groups. On the other hand, for a central division algebra D of dimension n^2 over $F^\times$, the trivial representation of $D^\times(\mathbb{A})$ is clearly a CAP form which shares the same local component, at any place v where D is unramified, with the residual representation $\mathbf{1}_{GL(n,\mathbb{A})}$. On the other hand, a quasisplit unitary group $U_{E/F}(3)$ of 3-variables already have non-trivial CAP forms, which can be obtained as θ-lifts of some automorphic characters of $U_{E/F}(1)$ [GR90], [GR91]. But the first and the most well-known example of CAP forms are the analogues of the θ_{10} representation by Howe-Piatetski-Shapiro [Sou88] and the Saito-Kurokawa representations of Sp_4 [PS83]. Also Gan-Gurevich-Jiang obtained very interesting example

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of CAP forms on the split group of type G_2 [GGJ02] (see also the article by Gan in this volume).

In any case, the local components of CAP forms at almost all places are non-trivial Langlands quotients by definition, and hence non-tempered in an apparent way. To put such forms into the framework of Langlands' conjecture, J. Arthur proposed a series of conjectures [Art89]. The conjectural description is through the so-called *A-parameters*, homomorphisms ψ from the direct product of the hypothetical Langlands group $\mathcal{L}_F$ of F with $SL(2,\mathbb{C})$ to the L-group ${}^LG$ of G [Bor79]:

$$\psi : \mathcal{L}_F \times SL(2,\mathbb{C}) \longrightarrow {}^LG,$$

considered modulo $\widehat{G}$-conjugation. We write $\Psi(G)$ for the set of $\widehat{G}$-conjugacy classes of A-parameters for G. By restriction, we obtain the local component

$$\psi_v : \mathcal{L}_{F_v} \times SL(2,\mathbb{C}) \to {}^LG_v$$

of ψ at each place v. Here the local Langlands group $\mathcal{L}_{F_v}$ is defined in [Kot84, §12], and ${}^LG_v$ is the L-group of the scalar extension $G_v = G \otimes_F F_v$. The local conjecture, among other things, associates to each ψ_v a finite set $\Pi_{\psi_v}(G_v)$ of isomorphism classes of irreducible unitarizable representations of $G(F_v)$, called an *A-packet*. At all but finite number of v, $\Pi_{\psi_v}(G_v)$ is expected to contain a unique unramified element π_v^1. Using such elements, we can form the global A-packet associated to ψ

$$\Pi_\psi(G) := \left\{ \bigotimes_v \pi_v \;\middle|\; \begin{array}{ll} \text{(i)} & \pi_v \in \Pi_{\psi_v}(G_v),\ \forall v; \\ \text{(ii)} & \pi_v = \pi_v^1,\ \forall' v \end{array} \right\}.$$

Arthur's conjecture predicts the multiplicity of each element in $\Pi_\psi(G)$ in the discrete spectrum of the right regular representation of $G(\mathbb{A})$ on $L^2(G(F)\mathfrak{A}_G\backslash G(\mathbb{A}))$. Here $\mathfrak{A}_G$ is the maximal $\mathbb{R}$-vector subgroup in the center of the infinite component $G(\mathbb{A}_\infty)$ of $G(\mathbb{A})$.

We say an A-parameter ψ is of *CAP type* if

(i) ψ is *elliptic*. This is the condition for $\Pi_\psi(G)$ to contain an element which occurs in the discrete spectrum.

(ii) $\psi|_{SL(2,\mathbb{C})}$ is non-trivial.

According to the conjecture, the CAP automorphic representations of $G(\mathbb{A})$ is contained in some of the global A-packets associated to such A-parameters. In this talk, we shall classify the CAP forms by such parameters along the line of Arthur's conjecture, in the case of the quasisplit unitary group $U_{E/F}(4)$ of four variables. Although our description of such forms tells nothing about the character relations conjectured in [Art89], it is quite explicit and fairly complete. We hope to apply this to certain analysis of the cohomology of the Shimura variety attached to $GU_{E/F}(4)$.

2 Parameter consideration

Global case Take a quadratic extension E/F of number fields and write σ for the generator of the Galois group of this extension. Let $G = G_n := U_{E/F}(n)$ be the quasisplit unitary groups in n variables associated to E/F. Later we shall mainly be concerned with the case $n = 4$. The L-group ${}^L G$ is the semi-direct product of $\widehat{G} = GL(n, \mathbb{C})$ by the absolute Weil group W_F of F, where W_F acts through $W_F/W_E \simeq \mathrm{Gal}(E/F)$ by

$$\rho_G(\sigma)g = \mathrm{Ad}(I_n)^t g^{-1}, \quad I_n := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{n-1} & & & \end{pmatrix}.$$

Thus an A-parameter ψ for G is determined by its restriction to $\mathcal{L}_E \times SL(2, \mathbb{C})$, which is just a completely reducible representation:

$$\psi|_{\mathcal{L}_E \times SL(2,\mathbb{C})} = \bigoplus_{i=1}^{r} \varphi_{\Pi_i} \otimes \rho_{d_i}.$$

Here Π_i is an irreducible cuspidal representation of $GL(m_i, \mathbb{A}_E)$ enjoying the following properties:

- $\sigma(\Pi_i) := \Pi_i \circ \sigma$ is isomorphic to the contragredient $\Pi_i^\vee$.
- Its central character ω_{Π_i} restricted to $\mathbb{A}^\times$ equals $\omega_{E/F}^{n-d_i-m_i+1}$, where $\omega_{E/F}$ is the quadratic character associated to E/F by the classfield theory.
- Some condition on the order of its twisted Asai L-functions at $s = 1$.

ρ_d is the d-dimensional irreducible representation of $SL(2, \mathbb{C})$. We note that ψ is elliptic if and only if its irreducible components $\varphi_{\Pi_i} \otimes \rho_{d_i}$ are distinct to each other. The S-group

$$\mathcal{S}_\psi(G) := \pi_0(\mathrm{Cent}(\psi, \widehat{G})/Z(\widehat{G}))$$

is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{r-1}$, where $\pi_0(\bullet)$ stands for the group of connected components. This plays a central role in the conjectural multiplicity formula.

Local case Similar description for the A-packets of the unitary group $G = G_n$ associated to a quadratic extension E/F of local fields is also valid. For each A-parameter ψ, we have the associated non-tempered Langlands parameter

$$\phi_\psi : \mathcal{L}_F \ni w \longmapsto \psi\Big(w, \begin{pmatrix} |w|_F^{1/2} & 0 \\ 0 & |w|_F^{-1/2} \end{pmatrix}\Big) \in {}^L G.$$

Here the "absolute value" $|\ |_F$ on $\mathcal{L}_F$ is the composite $|\ |_F : \mathcal{L}_F \twoheadrightarrow W_F^{\mathrm{ab}} \overset{\mathrm{rec}}{\underset{\sim}{\to}} F^\times \xrightarrow{|\ |_F} \mathbb{R}_+^\times$. (rec denotes the reciprocity map in the local classfield theory.) In Arthur's conjecture, it was imposed that the L-packet $\Pi_{\phi_\psi}(G)$ associated to ϕ_ψ should be contained in $\Pi_\psi(G)$. We also have the S-group $\mathcal{S}_\psi(G)$ as in the global case. We postulate the following:

Assumption 2.1. *There exists a bijection $\Pi_\psi(G) \ni \pi \longmapsto (\bar{s} \mapsto \langle \bar{s}, \pi \rangle_\psi) \in \Pi(\mathcal{S}_\psi(G))$. Here $\Pi(\mathcal{S}_\psi(G))$ is the set of isomorphism classes of irreducible representations of $\mathcal{S}_\psi(G)$.*

Now for $n = 4$, the possibilities of $\{(d_i, m_i)\}_i$ for elliptic A-parameters with non-trivial $SL(2, \mathbb{C})$-component are given as follows.

(1) Stable cases. $\{(4, 1)\}$, $\{(2, 2)\}$.

(2) Endoscopic cases.

  (a) $\{(3, 1), (1, 1)\}$;
  (b) $\{(2, 1), (1, 2)\}$;
  (c) $\{(2, 1), (2, 1)\}$;
  (d) $\{(2, 1), (1, 1), (1, 1)\}$.

In the cases (1), (2.a), it follows from Assumption 2.1 that $\Pi_{\phi_\psi}(G) = \Pi_\psi(G)$, and we know from [Kon98] that all the contribution of the corresponding global A-packets belong to the residual spectrum. On the other hand, $\Pi_\psi(G) \setminus \Pi_{\phi_\psi}(G)$ is expected to be non-empty in the rest cases. We shall use the local θ-correspondence to construct the missing members.

3 Local θ-correspondence

Local Howe duality First let us recall the local θ-correspondence. We consider an m-dimensional (non-degenerate) hermitian space $(V, (\ ,\ ))$ and n-dimensional skew-hermitian space $(W, \langle\ ,\ \rangle)$ over E. We write $G(V)$ and $G(W)$ for the unitary groups of V and W, respectively. If we define the symplectic space $(\mathbb{W}, \langle\!\langle\ ,\ \rangle\!\rangle)$ by

$$\mathbb{W} := V \otimes_E W, \quad \langle\!\langle v \otimes w, v' \otimes w' \rangle\!\rangle := \frac{1}{2}\mathrm{Tr}_{E/F}[(v, v')\sigma(\langle w, w' \rangle)],$$

Then $(G(V), G(W))$ form a so-called *dual reductive pair* in the symplectic group $Sp(\mathbb{W})$ of this symplectic space:

$$\iota_{V,W} : G(V) \times G(W) \ni (g, g') \longmapsto g \otimes g' \in Sp(\mathbb{W}).$$

Fixing a non-trivial character ψ_F of F, we have the metaplectic group of $\mathbb{W}$ which is a central extension

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow Mp_{\psi_F}(\mathbb{W}) \longrightarrow Sp(\mathbb{W}) \longrightarrow 1.$$

This admits a unique Weil representation ω_{ψ_F} on which $\mathbb{C}^1$ acts by the multiplication [RR93]. For each pair $\underline{\xi} = (\xi, \xi')$ of characters of $E^\times$ satisfying $\xi|_{F^\times} = \omega_{E/F}^n$, $\xi'|_{F^\times} = \omega_{E/F}^m$, we have the corresponding lifting $\widetilde{\iota}_{V,W,\underline{\xi}} : G(V) \times G(W) \to Mp_{\psi_F}(\mathbb{W})$ of $\iota_{V,W}$:

$$\begin{array}{ccc} G(V) \times G(W) & \xrightarrow{\widetilde{\iota}_{V,W,\underline{\xi}}} & Mp_{\psi_F}(\mathbb{W}) \\ \| & & \downarrow \\ G(V) \times G(W) & \xrightarrow{\iota_{V,W}} & Sp(\mathbb{W}) \end{array}$$

The composite $\omega_{V,W,\underline{\xi}} := \omega_\psi \circ \widetilde{\iota}_{V,W,\underline{\xi}}$ is the *Weil representation* of the dual reductive pair $(G(V), G(W))$ associated to $\underline{\xi}$. It is the product of the Weil representations $\omega_{W,\underline{\xi}}$ of $G(V)$ and $\omega_{V,\xi'}$ of $G(W)$.

We write $\mathscr{R}(G(V),\omega_{W,\xi})$ for the set of isomorphism classes of irreducible admissible representations of $G(V)$ which appear as quotients of $\omega_{W,\xi}$. For $\pi_V\in\mathscr{R}(G(V),\omega_{W,\xi})$, the maximal π_V-isotypic quotient of $\omega_{V,W,\underline{\xi}}$ is of the form $\pi_V\otimes\Theta_{\underline{\xi}}(\pi_V,W)$ for some smooth representation $\Theta_{\underline{\xi}}(\pi_V,W)$ of $G(W)$. Similarly we have $\mathscr{R}(G(W),\omega_{V,\xi'})$ and $\Theta_{\underline{\xi}}(\pi_W,V)$ for each $\pi_W\in\mathscr{R}(G(W),\omega_{V,\xi'})$. The local Howe duality conjecture, which was proved by R. Howe himself if F is archimedean [How89] and by Waldspurger if F is a non-archimedean local field of odd residual characteristic [Wal90], asserts the following:

(i) $\Theta_{\underline{\xi}}(\pi_V,W)$ (resp. $\Theta_{\underline{\xi}}(\pi_W,V)$) is an admissible representation of finite length of $G(W)$ (resp. $G(V)$), so that it admits an irreducible quotient.

(ii) Moreover its irreducible quotient $\theta_{\underline{\xi}}(\pi_V,W)$ (resp. $\theta_{\underline{\xi}}(\pi_W,V)$) is unique.

(iii) $\pi_V\mapsto\theta_{\underline{\xi}}(\pi_V,W)$, $\pi_W\mapsto\theta_{\underline{\xi}}(\pi_W,V)$ are bijections between $\mathscr{R}(G(V),\omega_{W,\xi})$ and $\mathscr{R}(G(W),\omega_{V,\xi'})$ converse to each other.

Adams' conjecture A link between the local θ-correspondence and A-packets is given by the following conjecture of J. Adams [Ada89]. Suppose $n\geq m$. Then we have an L-embedding $i_{V,W,\underline{\xi}}:{}^LG(V)\to{}^LG(W)$ given by

$$i_{V,W,\underline{\xi}}(g\rtimes w):=\begin{cases}\xi'\xi^{-1}(w)\begin{pmatrix}g & \\ & \mathbf{1}_{n-m}\end{pmatrix}\times w & \text{if } w\in W_E,\\ \begin{pmatrix} & g\\ J_{n-m}^{n-m-1} & \end{pmatrix}\rtimes w_\sigma & \text{if } w=w_\sigma,\end{cases}$$

where w_σ is a fixed element in $W_F\setminus W_E$ and

$$J_n:=\begin{pmatrix}1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & (-1)^{n-1}\end{pmatrix}$$

Let $\mathrm{T}:SL(2,\mathbb{C})\to\mathrm{Cent}(i_{V,W,\underline{\xi}},\widehat{G}(W))$ be the homomorphism which corresponds to a regular unipotent element in $\mathrm{Cent}(i_{V,W,\underline{\xi}},\widehat{G}(W))\simeq GL(n-m,\mathbb{C})$ (the *tail representation* of $SL(2,\mathbb{C})$). Using this, we define the θ-lifting of A-parameters by

$$\theta_{V,W,\underline{\xi}}:\Psi(G(V))\ni\psi\longmapsto(i_{V,W,\underline{\xi}}\circ\psi^\vee)\cdot T\in\Psi(G(W)).$$

Conjecture 3.1 ([Ada89] Conj.A). *The local θ-correspondence should be subordinated to the map of A-packets: $\Pi_\psi(G(V))\mapsto\Pi_{\theta_{V,W,\underline{\xi}}(\psi)}(G(W))$.*

Here we have said subordinated because $\mathscr{R}(G(V),\omega_{W,\xi})$ is not compatible with A-packets, that is, $\Pi_\psi(G(V))\cap\mathscr{R}(G(V),\omega_{W,\xi})$ is often strictly smaller than $\Pi_\psi(G(V))$. But when these two are assured to coincide, we can expect more:

Conjecture 3.2 ([Ada89] Conj.B). *For V, W in the stable range, that is, the Witt index of W is larger than m, we have*

$$\Pi_{\theta_{V,W,\underline{\xi}}(\psi)}(G(W)) = \bigcup_{V;\, \dim_E V = m} \theta_{\underline{\xi}}(\Pi_\psi(G(V)), W).$$

Now we note that our situation is precisely that of Conj. 3.2 with $m = 2$ and $W = V \oplus -V$. Moreover, we find that the A-parameters in the cases (2.b), (2.c), (2.d) in § 2 are exactly those of the form

$$\theta_{V,W,\underline{\xi}}(\psi), \quad \psi \in \Psi(G(V)).$$

ε-dichotomy We explain the construction of the A-packets when F is non-archimedean. We need one more ingredient.

Proposition 3.3 (ε-dichotomy). *Suppose $\dim_E V = 2$ and write W_1 for the hyperbolic skew-hermitian space $(E^2, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))$. Take an L-packet Π of $G_2(F) = G(W)$ and $\tau \in \Pi$ [Rog90, Ch.11].*
(i) $\tau \in \mathscr{R}(G(W), \omega_{V,\xi'})$ if and only if

$$\varepsilon(1/2, \Pi \times \xi\xi'^{-1}, \psi_F)\omega_\Pi(-1)\lambda(E/F, \psi_F)^{-2} = \omega_{E/F}(-\det V).$$

Here the ε-factor on the right hand side is the standard ε-factor for G_2 twisted by $\xi\xi'^{-1}$ defined by the Langlands-Shahidi theory *[Sha90]. ω_Π is the central character of the elements of Π and $\lambda(E/F, \psi_F)$ is Langlands' λ-factor [Lan70].*
(ii) If this is the case, we have $\theta_{\underline{\xi}}(\tau, V) = (\xi^{-1}\xi')_{G(V)}\tau_V^\vee$. Here $(\xi^{-1}\xi')_{G(V)}$ denotes the character of $G(V)$ given by the composite

$$G(V) \overset{\det}{\to} U_{E/F}(1, F) \ni z/\sigma(z) \mapsto \xi^{-1}\xi'(z) \in \mathbb{C}^\times.$$

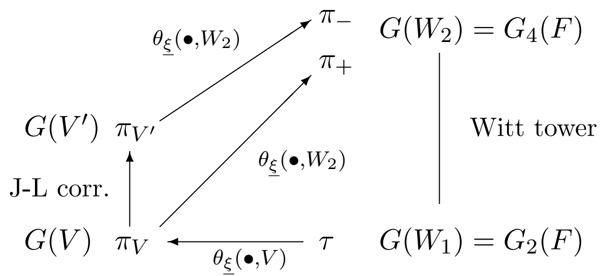
τ_V stands for the Jacquet-Langlands correspondent[1] *of τ.*

This is a special case of the ε-dichotomy of the local θ-correspondence for unitary groups over p-adic fields, which was proved for general unitary groups (at least for supercuspidal representations) in [HKS96]. But since we need to combine this with our description of the residual spectrum [Kon98], we have to use the Langlands-Shahidi ε-factors instead of Piatetski-Shapiro-Rallis's doubling ε-factors adopted by them. By this reason, we deduced this proposition from the analogous result for the unitary similitude groups [Har93] combined with the following description of the base change for G_2.

Lemma 3.4. *Let $\widetilde{\pi} = \omega \otimes \pi'$ be an irreducible admissible representation of the unitary similitude group $GU_{E/F}(2) \simeq (E^\times \times GL(2, F))/\Delta F^\times$, and write $\Pi(\widetilde{\pi})$ for the associated L-packet of $G_2(F)$ consisting of the irreducible components of $\widetilde{\pi}|_{G_2(F)}$. Then the standard base change of $\Pi(\widetilde{\pi})$ to $GL(2, E)$ [Rog90, 11.4] is given by $\omega(\det)\pi'_E$, where π'_E is the base change lift of π' to $GL(2, E)$ [Lan80].*

[1]In fact, the Jacquet-Langlands correspondence for unitary groups in two variables is defined only for L-packets and not for each member of the packets [LL79]. We know that $\tau \mapsto \tau_V$ certainly defines a bijection between Π and its Jacquet-Langlands correspondent. But we do not specify the bijection explicitly here. See Rem. 3.6 also.

Now we construct the A-packets. Our construction is summarized in the following picture.



Each A-parameter of our concern is of the form

$$\psi|_{\mathcal{L}_E \times SL(2,\mathbb{C})} = \psi_1|_{\mathcal{L}_E \times SL(2,\mathbb{C})} \oplus (\xi'\xi^{-1} \otimes \rho_2),$$

where ψ_1 is some A-parameter for G_2. Take $\tau \in \Pi_{\psi_1}(G_2)$ and let $(V, (\ ,\ ))$ be the 2-dimensional hermitian space such that the condition of Prop. 3.3 (i) holds. If we write $\pi_V := \theta_{\underline{\xi}}(\tau, V) \simeq (\xi\xi'^{-1})_{G(V)}\tau_V^\vee$, then the result of [Kud86] tells us $\pi_+ := \theta_{\underline{\xi}}(\pi_V, W_2)$, $(\tau \in \Pi_{\psi_1}(G_2))$ form the local residual L-packet $\Pi_{\phi_\psi}(G_4)$. We now suppose that there exists a Jacquet-Langlands corresondent $\pi_{V'} \simeq (\xi\xi'^{-1})_{G(V')}\tau_{V'}^\vee$ of π_V on the unitary group $G(V')$ of the other (isometry class of) 2-dimensional hermitian space. Then Prop. 3.3 (i) tells us that $\pi_{V'} \notin \mathscr{R}(G(V'), \omega_{W_1,\xi})$. Yet its local θ-lifting $\pi_- := \theta_{\underline{\xi}}(\pi_{V'}, W_2)$ to the larger group $G_4(F)$ still exists. This is the so-called *early lift* or the *first occurrence*. Following Conj. 3.2, we define

$$\Pi_\psi(G_4) := \{\pi_\pm \,|\, \tau \in \Pi_\psi(G_2)\}.$$

This gives sufficiently many members of the packet as predicted by Assumption 2.1.

Example 3.5. *(i) Suppose $\Pi_{\psi_1}(G_2)$ is an L-packet consisting of supercuspidal elements. For $\tau \in \Pi_{\psi_1}(G_2)$, π_+ is the Langlands quotient $J_{P_1}^{G_4}(\xi'\xi^{-1}|\ |_E^{1/2} \otimes \tau)$, where P_1 is a parabolic subgroup with the Levi factor $\mathrm{R}_{E/F}\mathbb{G}_m \times G_2$. On the other hand the early lift π_- of the supercuspidal τ is again supercuspidal. Thus $\Pi_\psi(G_4)$ consists of non-tempered members and supercuspidal elements.*
(ii) On the contrary, we take $\xi = \xi'$ and consider $\Pi_{\psi_1}(G_2)$ consists of either the Steinberg representation δ_{G_2} or the trivial representation $\mathbf{1}_{G_2}$.

- *δ_{G_2} lifts to $\pi_V = \mathbf{1}_{G(V)}$, where V is anisotropic. $\pi_{V'} = \delta_{G_2}$. $\pi_+ = J_{P_1}^{G_4}(|\ |_E^{1/2} \otimes \delta_{G_2})$ and π_- is an irreducible tempered but not square integrable representation.*

- *$\mathbf{1}_{G_2}$ lifts to $\pi_V = \mathbf{1}_{G(V)}$ but V is hyperbolic this time. $\pi_{V'}$ is again $\mathbf{1}_{G(V')}$ but this should be viewed as the Jacquet-Langlands correspondent of the A-packet $\{\mathbf{1}_{G(V)}\}$. We have $\pi_+ = J_{P_2}^{G_4}(I_{\mathbf{B}}^{GL(2)_E}(\mathbf{1}\otimes\mathbf{1})|\det|_E^{1/2})$, where P_2 is the so-called Siegel parabolic subgroup with the Levi factor $GL(2,E)$. Obviously $\pi_- = J_{P_1}^{G_4}(|\ |_E^{1/2} \otimes \delta_{G_2})$. This last representation is shared by the two packets considered here.*

Real case We end this section by some comments on the case $E/F = \mathbb{C}/\mathbb{R}$. Similar results are obtained by applying the argument of Adams-Barbasch [AB95]. In fact, the local θ-correspondence between unitary groups *of the same size* is described quite explicitly and in full generality in [Pau98]. Their argument also works in the present case. Let me explain some example.

We write $G_{p,q} = U(p,q)$. For a regular integral infinitesimal character $\lambda = (\lambda_1, \lambda_2)$ for $G_{1,1}$, consider the extended L-packet:

$$\Pi_\lambda = \{\delta^+_{1,1}, \delta^-_{1,1}, \delta_{2,0}, \delta_{0,2}\}$$

consisting of the discrete series representation of various $G_{p,q}$ with the infinitesimal character λ. The subscript p,q indicates that $\delta^\bullet_{p,q}$ lives on $G_{p,q}$. We can write $\xi'\xi^{-1}(z) = (z/\bar{z})^n$, $\forall z \in \mathbb{C}$ for some $n \in \mathbb{Z}$. An analogue of Prop. 3.3 in the real case asserts that the local θ-correspondence under the Weil representation $\omega_{V,W,\underline{\xi}}$ gives a bijection

$$\theta_{\underline{\xi}} : \Pi_\lambda \xrightarrow{\sim} \Pi_{n-\lambda},$$

where $n - \lambda = (n - \lambda_2, n - \lambda_1)$.

If λ is sufficiently regular, by which we mean $|\lambda_i - n| > 1$, then it is proved by J.-S. Li [Li90] that $\theta_{\underline{\xi}}(\theta_{\underline{\xi}}(\delta^\pm_{1,1}), W_2)$ is a non-tempered cohomological representation $A_{\mathfrak{q}}(\lambda')$, where the Levi factor of the θ-stable parabolic subalgebra $\mathfrak{q}$ is $\mathfrak{u}(1,1) \oplus \mathfrak{u}(1)^2$. As for the other elements $\delta_{p,q} \in \Pi_{n-\lambda}$, $\theta_{\underline{\xi}}(\delta_{p,q}, W_2)$ is a discrete series representation $A_{\mathfrak{q}}(\lambda')$. This time $\mathfrak{q}$ has the Levi factor $\mathfrak{u}(2) \oplus \mathfrak{u}(1)^2$. The resulting A-packet $\theta_{\underline{\xi}}(\Pi_{n-\lambda})$ is exactly the cohomological A-packet defined by Adams-Johnson [AJ87].

For the complete list of the packets both in the archimedean and non-archimedean case, see our paper [KK].

One can easily check that the S-groups in the cases (2.b), (2.c), (2.d) satisfy $\mathcal{S}_\psi(G_4) \simeq \mathcal{S}_{\psi_1}(G_2) \times \mathbb{Z}/2\mathbb{Z}$. Now we define the bijection in Assumption 2.1 by

- $\langle \bar{s}, \pi_\pm \rangle_\psi := \langle \bar{s}, \tau \rangle_{\psi_1}$ on $\bar{s} \in \mathcal{S}_{\psi_1}(G_2)$;
- $\langle \, , \pi_\pm \rangle_\psi$ on $\mathbb{Z}/2\mathbb{Z}$ equals the sign character if π_- and trivial character otherwise.

For the other cases, only the first one in this definition is enough to give a complete bijection. This finishes our local task.

Remark 3.6. *In the above, we do not mention the definition of the pairing $\langle \, , \, \rangle_{\psi_1}$. There are several choices for this, and we can choose one by fixing a non-trivial character ψ_F of F [LL79]. Also we did not specify the correspondence $\pi_V \mapsto \pi_{V'}$, which is again a subtle problem. In fact, we need to make a choice of (absolute) transfer factor as in [LL79] which again involves a choice of ψ_F (appearing in $\lambda(E/F, \psi_F)$ in the transfer factor). Using this specific transfer, we label the members of endoscopic L-packets of anisotropic unitary group. The correspondence $\pi_V \mapsto \pi_{V'}$ can be described in terms of these data, but we do not go into details here.*

4 Multiplicity formula

We now go back to the global situation where E/F is a quadratic extension of number fields. We note that there always exists a homomorphism $\mathcal{S}_\psi(G_4) \ni \bar{s} \mapsto \bar{s}(v) \in \mathcal{S}_{\psi_v}(G_{4,v})$. We can now state the main result of this talk. Although we treat only the number field case, we believe the result holds also over function fields of one variable over a finite field of odd characteristic.

Theorem 4.1. *Let ψ be an A-parameter of CAP type for $G_4 = U_{E/F}(4)$. As was explained in § 1, we form the global A- packet $\Pi_\psi(G_4) := \bigotimes_v \Pi_{\psi_v}(G_{4,v})$. Then the multiplicity $m(\pi)$ of $\pi = \bigotimes_v \pi_v \in \Pi_\psi(G_4)$ in $L^2(G(F)\backslash G(\mathbb{A}))$ is given by*

$$m(\pi) = \frac{1}{|\mathcal{S}_\psi(G_4)|} \sum_{\bar{s} \in \mathcal{S}_\psi(G_4)} \epsilon_\psi(\bar{s}) \prod_v \langle \bar{s}(v), \pi_v \rangle_{\psi_v},$$

where the sign character ϵ_ψ is defined by

$$\epsilon_\psi = \begin{cases} \mathrm{sgn}_{\mathcal{S}_\psi(G_4)} & \begin{array}{l} \text{if } \psi_1 \text{ is a stable } L\text{-parameter} \\ \text{and } \varepsilon(1/2, \psi_1 \otimes \xi\xi'^{-1}) = -1, \end{array} \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Here $\varepsilon(s, \psi_1 \otimes \xi\xi'^{-1})$ is the Artin root number attached to ψ_1, which equals the standard ε-function for $\Pi_{\psi_1}(G_2) \times \xi\xi'^{-1}$.

The proof divides into two parts. Our local construction together with the global θ-correspondence shows that the multiplicity is no less than the right hand side. Note that we also relies on the multiplicity formula of Labesse-Langlands for unitary groups in two variables [LL79], [Rog90]. Then we prove a characterization of the image of such θ-lifts by poles of certain L-functions, which gives the converse inequality. This also shows that all the CAP forms for $U_{E/F}(4)$ are obtained in the above as the contribution of the A-packets we constructed. In particular the A-packets contains the sufficiently many members at least for global purposes, so that our Assumption 2.1 is justified.

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