Standard L-functions attached to vector valued Siegel modular forms

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In this report, we study the analytic continuation of standard L-functions attached to vector valued Siegel modular forms. In Section 1, we define vector valued Siegel modular forms and standard L-functions. In Section 2, we describe the results in special cases and tools to prove. In Section 3, we describe one of the tools the differential operator generalized by Ibukiyama, and construct the operator explicity in the cases. In Section 4, we consider in general case.

§1. Vector valued Siegel modular forms and standard L- functions

Let n be a positive integer. Let

$$\mathbf{H}_n := \{Z \in M(n, \mathbf{C}) \mid Z = {}^tZ, \quad \mathrm{Im}(Z) > 0\}$$

be the Siegel upper half space of degree n, and

$$\Gamma_n := Sp(n, \mathbf{Z}) := \{\gamma \in GL(2n, \mathbf{Z}) \mid {}^t\gamma J \gamma = J\}$$

the Siegel modular group of degree n, where $J := \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$. Let (ρ, V_ρ) be an irreducible rational representation of $GL(n, \mathbf{C})$ on a finite-dimensional complex vector space V_ρ such that the highest weight of ρ is $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{Z}^n$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Furthermore, we fix an inner product $\langle \cdot, \cdot \rangle$ on V_ρ such that

$$\langle \rho(g)v, w\rangle = \langle v, \rho({}^t\overline{g})w\rangle \quad \text{for } g \in GL(n, \mathbf{C}),\ v,\ w \in V_\rho.$$

A C^∞-function $f : \mathbf{H}_n \to V_\rho$ is called a V_ρ-valued C^∞-modular form of type ρ if it satisfies

$$\rho(CZ + D)\, f(Z) = f((AZ + B)(CZ + D)^{-1}) \quad \text{for all } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n.$$

The space of all such functions is denoted by M_ρ^∞. The space of V_ρ-valued Siegel modular forms of type ρ is defined by

$$M_\rho := \{f \in M_\rho^\infty \mid f \text{ is holomorphic on } \mathbf{H}_n \text{ (and its cusps)}\},$$

and the space of cuspforms by

$$S_\rho := \left\{ f \in M_\rho \;\middle|\; \lim_{\lambda\to\infty} f(\begin{pmatrix} Z & 0 \\ 0 & i\lambda \end{pmatrix}) = 0 \quad \text{for all } Z \in \mathbf{H}_{n-1} \right\}.$$

Let $\mathcal{H}^n$ be the Hecke algebra for $(\Gamma_n, G^+Sp(n,\mathbf{Q}))$ over $\mathbf{C}$, where

$$G^+Sp(n,\mathbf{Q}) := \left\{ g \in GL(2n,\mathbf{Q}) \;\middle|\; {}^t gJg = rJ \text{ with some } r > 0 \right\}.$$

Then $\mathcal{H}^n$ has the following structure

$$\mathcal{H}^n = \bigotimes_{p:\text{prime}}{}' \; \mathcal{H}_p^n, \quad \mathcal{H}_p^n \simeq \mathbf{C}[X_0^{\pm1}, \dots, X_n^{\pm1}]^W.$$

Here $\mathcal{H}_p^n$ is the Hecke algebra for $(\Gamma_n, G^+Sp(n,\mathbf{Q}) \cap GL(2n,\mathbf{Z}[1/p]))$ over $\mathbf{C}$, and W is the group generated by w_1, ..., w_n and permutations in X_1, ..., X_n, where w_1, ..., w_n are automorphisms on $\mathbf{C}[X_0^{\pm1}, \dots, X_n^{\pm1}]$ defined by

$$w_j(X_i) := \begin{cases} X_0 X_j & \text{if } i = 0, \\ X_i & \text{if } i \neq j, \\ X_i^{-1} & \text{if } i = j. \end{cases}$$

Suppose f is an eigenform, i.e., a non-zero common eigenfunction of the Hecke algebra $\mathcal{H}^n$. For $T \in \mathcal{H}^n$, let $\lambda(T)$ be the eigenvalue on f of T. Then for any prime number p, we determine $(\alpha_0(p), \dots, \alpha_n(p)) \in (\mathbf{C}^\times)^{n+1}$ such that it gives the homomorphism

$$\lambda\colon \mathcal{H}_p^n \simeq \mathbf{C}[X_0^{\pm1}, \dots, X_n^{\pm1}]^W \xrightarrow{X_j \mapsto \alpha_j(p)} \mathbf{C},$$

where $X_j \mapsto \alpha_j(p)$ means substituting $\alpha_j(p)$ into X_j ($j = 0$, ..., n). The numbers $\alpha_0(p)$, ..., $\alpha_n(p)$ are called the Satake p-parameters of f. Then we define the standard L- function attached to f by

$$L(s, f, \underline{\mathrm{St}}) := \prod_{p:\text{prime}} \left\{ (1 - p^{-s}) \prod_{j=1}^n (1 - \alpha_j(p) p^{-s})(1 - \alpha_j(p)^{-1} p^{-s}) \right\}^{-1}.$$

The right-hand side converges absolutely and locally uniformly for $\mathrm{Re}(s)$ sufficiently large.

§2. Problem and results

Problem. (Langlands [6])

The standard L-function $L(s, f, \underline{\mathrm{St}})$ *has meromorphic continuation to the whole s-plane and satisfies a functional equation.*

More precisely, we expect the following:

Conjecture. (Takayanagi [9])

We put

$$\Lambda(s, f, \underline{\mathrm{St}}) := \Gamma_\rho(s)\, L(s, f, \underline{\mathrm{St}}),$$

where

$$\Gamma_\rho(s) := \Gamma_{\mathbf{R}}(s+\varepsilon) \prod_{j=1}^{n} \Gamma_{\mathbf{C}}(s + \lambda_j - j)$$

with

$$\Gamma_{\mathbf{R}}(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right), \quad \Gamma_{\mathbf{C}}(s) := 2(2\pi)^{-s}\Gamma(s),$$

and

$$\varepsilon := \begin{cases} 0 & \text{if } n \text{ even}, \\ 1 & \text{if } n \text{ odd}. \end{cases}$$

Then $\Lambda(s, f, \underline{\mathrm{St}})$ *satisfies the functional equation*

$$\Lambda(s, f, \underline{\mathrm{St}}) = \Lambda(1-s, f, \underline{\mathrm{St}}).$$

We assume that k is a positive even integer and f is a cuspform.

For $\rho = \det^k$, this conjecture was solved by Andrianov and Kalinin [1], and Böcherer [2], and for $\rho = \det^k \otimes \mathrm{sym}^l$ and $\rho = \det^k \otimes \mathrm{alt}^{n-1}$ was solved by Takayanagi [9], [10].

Result.

We proved the conjecture in the following two cases:

Case 1. $\rho = \det^k \otimes \mathrm{alt}^l$ *(the highest weight* $(\underbrace{k+1, \ldots, k+1}_{l}, \underbrace{k, \ldots, k}_{n-l})$*).*

Case 2. *the highest weight of* ρ *is* $(k+2, \underbrace{k+1, \ldots, k+1}_{l-2}, \underbrace{k, \ldots, k}_{n-l+1})$*.*

To prove the above result, we use the non-holomorphic Eisenstein series and the differential operator generalized by Ibukiyama [4].

First, for $Z \in \mathbf{H}_n$ and a complex number s, we define the Eisenstein series $E_k^n(Z,s)$ by

$$E_k^n(Z,s) := \det(\mathrm{Im}(Z))^s \sum_{(C,D)} \det(CZ+D)^{-k} \left|\det(CZ+D)\right|^{-2s},$$

where (C,D) runs over a complete system of representatives of $\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \,\middle|\, C = 0 \right\} \backslash \Gamma_n$. Then $E_k^n(Z,s)$ converges absolutely and locally uniformly for $k + 2\,\mathrm{Re}(s) > n+1$. Furthermore the following properties are known:

(i) The Eisenstein series $E_k^n(Z,s)$ has meromorphic continuation to the whole s-plane and satisfies a functional equation. (Langlands [7], Kalinin [5] and Mizumoto [8])

(ii) Any partial derivative (in the entries of Z and $\overline{Z}$) of the Eisenstein series $E_k^n(Z,s)$ is slowly increasing (locally uniformly in s). (Mizumoto [8])

Next, we introduce the differential operator $\mathcal{D}$ which sends the Eisenstein series to the tensor product of two V_ρ-valued Siegel modular forms. Using Garrett decomposition [3], we compute $(\mathcal{D}E_k^{2n})\Big(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}, s\Big)$. Taking the Petersson inner product of f and $(\mathcal{D}E_k^{2n})\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}, s\right)$ in the variable W, we obtain the integral representation of the standard L-function $L(s, f, \underline{\mathrm{St}})$, i.e.,

$$\left(f, (\mathcal{D}E_k^{2n})\Big(\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}, \overline{s}\Big)\right) = (\Gamma\text{-factor}) \cdot L(2s+k-n, f, \underline{\mathrm{St}}) \cdot (\iota^{-1}(f))(Z).$$

Using the properties (i) and (ii) of the Eisenstein series, we prove the conjecture.

In the above cases, we can construct the differential operator explicitly and compute the integral representation of the standard L-function.

§3. Differential operator

In this section, we describe the differential operator generalized Ibukiyama and in the above cases we construct the operator explicitly.

Let (ρ'_j, V_j) $(j = 1,\, 2)$ be irreducible rational representations of $GL(n, \mathbf{C})$ such that ρ'_1 is equivalent to ρ'_2.

We assume $k \geq n$, and put $\rho_j := \det^k \otimes \rho'_j$.

If a polynomial P

$$P \colon M(n, 2k; \mathbf{C}) \times M(n, 2k; \mathbf{C}) \to V_1 \otimes V_2$$

satisfies

(C1) $P(a_1X_1, a_2X_2) = \rho'(a_1) \otimes \rho'(a_2)\, P(X_1, X_2) \quad$ for all $a_1,\, a_2 \in GL(n, \mathbf{C})$,

(C2) $P(X_1g, X_2g) = P(X_1, X_2) \quad$ for all $g \in O(2k)$

(C3) $P(X_1, X_2)$ is pluri-harmonic for each X_1, X_2,

then there exists a polynomial Q

$$Q \colon \mathrm{sym}(2n, \mathbf{C}) \to V_1 \otimes V_2$$

such that

$$P(X_1, X_2) = Q(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} {}^t\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}).$$

Here $O(2k)$ is the orthogonal group of degree $2k$, and $\mathrm{sym}(2n, \mathbf{C})$ the set of all $\mathbf{C}$-valued symmetric matrices of size $2n$. And for $j = 1,\, 2$, let $X_j = (x^{(j)}_{\mu\nu})$ be variables, then P is called pluri- harmonic for X_j if

$$\sum_{\kappa=1}^{2k} \frac{\partial}{\partial x^{(j)}_{\mu\kappa}} \frac{\partial}{\partial x^{(j)}_{\nu\kappa}} P = 0 \quad \text{for all } \mu,\, \nu.$$

We define the differential operator $\mathcal{D}$ by

$$\mathcal{D} := Q(\partial),$$

where

$$\partial := \left(\frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial z_{ij}} \right)_{1 \leq i,j \leq 2n}, \quad \mathcal{Z} = (z_{ij})_{1 \leq i,j \leq 2n} \in \mathbf{H}_{2n}.$$

Here δ_{ij} is the Kronecker's delta. Then

Theorem. (Ibukiyama)

If f is a C^∞-modular form (resp. a Siegel modular form) of degree $2n$ and type $\det^k$, then

$$(\mathcal{D}f)(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}) \in M^\infty_{\rho_1} \otimes M^\infty_{\rho_2} \quad (\textit{resp. } M_{\rho_1} \otimes M_{\rho_2}).$$

In the above cases, we construct the differential operators explicitly.

First we write (ρ'_j, V_j) $(j = 1, 2)$ explicity. We put

$$W_1 := \mathbf{C}e_1 \oplus \cdots \oplus \mathbf{C}e_n, \quad W_2 := \mathbf{C}e_{n+1} \oplus \cdots \oplus \mathbf{C}e_{2n}.$$

Let l be an even integer. Let $T^l(W_j)$ be the l-th tensor product of W_j, i.e.,

$$T^l(W_j) := \underbrace{W_j \otimes \cdots \otimes W_j}_{l},$$

and ρ'_j the standard representation of $GL(n, \mathbf{C})$ on $T^l(W_j)$. Let c_j be the Young symmetrizer of $(\lambda'_1, \ldots, \lambda'_n)$ on $T^l(W_j)$ such that $\lambda'_1 \geq \ldots \geq \lambda'_n$ and $\lambda'_1 + \cdots + \lambda'_n = l$. In Case 1, $(\lambda'_1, \ldots, \lambda'_n) = (\underbrace{1, \ldots, 1}_{l}, \underbrace{0, \ldots, 0}_{n-l})$, and in Case 2, $(\lambda'_1, \ldots, \lambda'_n) = (2, \underbrace{1, \ldots, 1}_{l-2}, \underbrace{0, \ldots, 0}_{n-l+1})$. We put $V_j := c_j(T^l(W_j))$. Then (ρ', V_j) is an irreducible representation of $GL(n, \mathbf{C})$.

On the other hand, let $e_i^{(\alpha)}$ $(i = 1, \ldots, 2n,\ \alpha = 1, \ldots, l)$ be indeterminants. And for a symmetric matrix A of size $2n$ and positive integers α, β $(1 \leq \alpha, \beta \leq l)$, we define

$$\begin{aligned} A^{\alpha\beta} &:= (e_1^{(\alpha)}, \ldots, e_n^{(\alpha)}, 0, \ldots, 0)\, A\, {}^t(e_1^{(\beta)}, \ldots, e_n^{(\beta)}, 0, \ldots, 0), \\ A^{\alpha}_{\beta} &:= (e_1^{(\alpha)}, \ldots, e_n^{(\alpha)}, 0, \ldots, 0)\, A\, {}^t(0, \ldots, 0, e_{n+1}^{(\beta)}, \ldots, e_{2n}^{(\beta)}), \\ A_{\alpha\beta} &:= (0, \ldots, 0, e_{n+1}^{(\alpha)}, \ldots, e_{2n}^{(\alpha)})\, A\, {}^t(0, \ldots, 0, e_{n+1}^{(\beta)}, \ldots, e_{2n}^{(\beta)}). \end{aligned}$$

We consider a product

$$A^{\alpha_1\alpha_2} \ldots A^{\alpha_{2\nu-1}\alpha_{2\nu}} A_{\beta_1\beta_2} \ldots A_{\beta_{2\nu-1}\beta_{2\nu}} A^{\alpha_{2\nu+1}}_{\beta_{2\nu+1}} \ldots A^{\alpha_l}_{\beta_l}$$

with $\{\alpha_1, \ldots, \alpha_l\} = \{\beta_1, \ldots, \beta_l\} = \{1, \ldots, l\}$. Then this product is

$$\sum_{\substack{1 \leq r_j \leq n \\ n+1 \leq s_j \leq 2n}} (\text{coefficient})\, e_{r_1}^{(1)} \ldots e_{r_l}^{(l)} e_{s_1}^{(1)} \ldots e_{s_l}^{(l)}.$$

Now we identify $e_{r_1}^{(1)} \ldots e_{r_l}^{(l)} e_{s_1}^{(1)} \ldots e_{s_l}^{(l)}$ with $e_{r_1} \otimes \ldots \otimes e_{r_l} \otimes e_{s_1} \otimes \ldots \otimes e_{s_l} \in T^l(W_1) \otimes T^l(W_2)$. Then this product belongs to $T^l(W_1) \otimes T^l(W_2)$.

We call a linear combination of such products a "homogeneous polynomial" of A. If $Q\colon \mathrm{sym}(2n, \mathbf{C}) \to V_1 \otimes V_2$ is "homogeneous polynomial", then $Q(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} {}^t\begin{pmatrix} X_1 \\ X_2 \end{pmatrix})$ satisfies (C1), (C2). Therefore if $Q(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} {}^t\begin{pmatrix} X_1 \\ X_2 \end{pmatrix})$

is pluri- harmonic for each X_1, X_2, then we obtain the differential operator $\mathcal{D}$.

We put $S := \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}^t \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$. Then in Case 1,

$$c_1 c_2 S_1^1 \dots S_l^l$$

is pluri-harmonic for each X_1, X_2, and in Case 2,

$$c_1 c_2 (S_1^1 \dots S_l^l - \frac{l}{2(2k-(l-2))} S^{12} S_{12} S_3^3 \dots S_l^l)$$

is pluri-harmonic for each X_1, X_2. Therefore we can compute $(\mathcal{D}E_k^{2n})$ $\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}, s\right)$. And we obtain the integral representation of the standard L-function $L(s, f, \underline{\mathrm{St}})$.

§4. Supplement

In general case, there exist three difficulties in proving the conjecture, i.e.,

(i) to construct the differential operator $\mathcal{D}$ explicitly,

(ii) to compute $(\mathcal{D}E_k^{2n}) \left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}, s\right)$,

(iii) to compute the Petersson inner product $\left(f, (\mathcal{D}E_k^{2n})\left(\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}, \overline{s}\right)\right)$.

However, if we cannot construct the differential operator explicitly, the following holds:

Proposition 1.

If $Q(S)$ is a "homogeneous polynomial" of $S := \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}^t \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ and pluri-harmonic for each X_1, X_2, then there exists a "homogeneous polynomial" $\mathcal{P}(X, s)$ of X such that

$$\mathcal{D}(\delta^{-k} |\delta|^{-2s} \varepsilon^s)|_{\mathcal{Z}=\mathcal{Z}_0} = (\delta^{-k} |\delta|^{-2s} \varepsilon^s \cdot \mathcal{P}(\Delta - \mathrm{E}, s))|_{\mathcal{Z}=\mathcal{Z}_0}.$$

Here for $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{2n}$ *and* $\mathcal{Z} \in \mathbf{H}_{2n}$*, we put* $\delta := \det(C\mathcal{Z}+D)$*,* $\varepsilon := \det(\mathrm{Im}(\mathcal{Z}))$*,* $\Delta := (C\mathcal{Z}+D)^{-1}C$*, and* $\mathrm{E} := \dfrac{1}{2i}(\mathrm{Im}(\mathcal{Z}))^{-1}$*. And we put* $\mathcal{Z}_0 := \begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}$*.*

For example, in Case 1, the "homogeneous polynomial" $\mathcal{P}(X,s)$ is

$$\mathcal{P}(X,s) = c_1c_2 \prod_{j=1}^{l}\Big(-k-s+\frac{j-1}{2}\Big)\, X_1^1 \dots X_l^l,$$

and in Case 2,

$$\begin{aligned}\mathcal{P}(X,s) &= c_1c_2 \prod_{j=1}^{l-1}\Big(-k-s+\frac{j-1}{2}\Big) \\ &\quad\times\Big\{\Big(-k-s-\frac{1}{2}+\frac{l}{2(2k-(l-2))}\Big)X_1^1X_2^2\dots X_l^l \\ &\qquad+\frac{ls}{2(2k-(l-2))}X^{12}X_{12}X_3^3\dots X_l^l\Big\}.\end{aligned}$$

Furthermore, using the "homogeneous polynomial" $\mathcal{P}(X,s)$, we obtain the following:

Proposition 2.

Under the assumption of Proposition 1, the Petersson inner product $\Big(f, (\mathcal{D}E_k^{2n})\Big(\begin{pmatrix} -\overline{Z} & 0 \\ 0 & * \end{pmatrix}, \overline{s}\Big)\Big)$ *is equal to*

$$\begin{aligned}&(\Gamma\text{-factor}) \cdot L(2s+k-n, f, \underline{\mathrm{St}}) \\ &\times \frac{1}{\langle v,v\rangle}\Big\langle \int_{\mathbf{S}_n} \langle \rho_2(1_n - \overline{S}S)\,\iota(v), \mathcal{P}(R,\overline{s})\rangle \det(1_n-\overline{S}S)^{s-n-1}\,dS, v\Big\rangle \\ &\times(\iota^{-1}(f))(Z),\end{aligned}$$

where $v \in V_1$,

$$\mathbf{S}_n := \{S \in M(n,\mathbf{C}) \mid S = {}^tS, \quad 1_n - S\overline{S} > 0\},$$

$$R := -\frac{1}{2i}\begin{pmatrix} S & -2i\,1_n \\ -2i\,1_n & 2^2\overline{S}(1_n - S\overline{S})^{-1} \end{pmatrix},$$

and $\iota\colon V_1 \to V_2$ *is the isomorphism defined by* $\iota(e_j) = e_{n+j}$ *for* $j = 1, \dots, n$.

And if

$$\frac{1}{\langle v,v\rangle}\Big\langle \int_{\mathbf{S}_n} \langle \rho_2(1_n-\overline{S}S)\,\iota(v),\mathcal{P}(R,\overline{s})\rangle\ \det(1_n-\overline{S}S)^{s-n-1}\,dS, v\Big\rangle$$

is equal to

$$(\text{constant})\times\prod_{j=1}^{n}\frac{\Gamma(2s+k-n+\lambda_j-j)}{\Gamma(2s+2k+1-2j)},$$

then the conjecture holds.

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