Certain series attached to an even number of elliptic modular forms

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1 Results

Let $n \in \mathbf{Z}_{>0}$, $k := (k_1, \ldots, k_n) \in (\mathbf{Z}_{>0})^n$, $m = (m_1, \ldots, m_n) \in (\mathbf{Z}_{>0})^n$ and $s \in \mathbf{C}$. We put

$$\begin{aligned} Q_k^{(n)}(m,s) &:= \int_0^\infty t^{s+|k|-n-1} dt \\ &\cdot \prod_{j=1}^{n} \int_0^\infty u_j^{k_j-2} e^{-4\pi m_j u_j t} (\sqrt{u_j}\theta(iu_j) - 1) du_j \ ; \end{aligned} \tag{1}$$

here $|k| := \sum_{j=1}^n k_j$ and

$$\theta(z) := \sum_{l=-\infty}^{\infty} e^{\pi i l^2 z}$$

is the Jacobi theta function. The right-hand side of (1) converges absolutely and locally uniformly for $\mathrm{Re}(s) > \frac{n}{2}$. It is easy to see

$$Q_k^{(n)}(m,\sigma) > 0 \quad \text{for} \quad \frac{n}{2} < \sigma \in \mathbf{R}.$$

For $w \in \mathbf{Z}$ let M_w be the space of holomorphic modular forms of weight w for $SL_2(\mathbf{Z})$ and S_w be the space of cusp forms in M_w. Let f_j and g_j be elements of M_{k_j} such that $f_j(z)g_j(z)$ is a cusp form for each $j = 1, \ldots, n$. Let

$$f_j(z) = \sum_{l=0}^{\infty} a_j(l) e^{2\pi i l z} \quad \text{and} \quad g_j(z) = \sum_{l=0}^{\infty} b_j(l) e^{2\pi i l z} \tag{2}$$

be the Fourier expansions. The series we treat here is the following:

$$\begin{aligned}&\mathcal{D}(s; f_1, \ldots, f_n; g_1, \ldots, g_n)\\ := &\sum_{m=(m_1,\ldots,m_n)\in(\mathbf{Z}_{>0})^n} \left(\prod_{j=1}^{n} a_j(m_j)\overline{b_j(m_j)}\right) Q_k^{(n)}(m,s).\end{aligned} \tag{3}$$

The right-hand side of (3) converges absolutely and locally uniformly for

$$\mathrm{Re}(s) > \frac{n}{2}(\max_{1\le j\le n}(k_j)+1).$$

Theorem 1.
(i) *The series* (3) *has a meromorphic continuation to the whole s-plane.*
(ii) *Let* (,) *be the Petersson inner product. Then the function*

$$\sum_{\nu=1}^{n} \sum_{1\le i_1<\ldots<i_\nu\le n} \left(\prod_{\substack{j\ne i_1,\ldots,i_\nu\\ 1\le j\le n}} (f_j, g_j)\right) \cdot \mathcal{D}(s; f_{i_1}, \ldots, f_{i_\nu}; g_{i_1}, \ldots, g_{i_\nu})$$

is invariant under the substitution $s \mapsto n-s$; it has possible simple poles at $s=0$ and $s=n$ with residues $-\prod_{j=1}^{n}(f_j,g_j)$ and $\prod_{j=1}^{n}(f_j,g_j)$ respectively, and is holomorphic elsewhere.

In case where every g_j is the Eisenstein series we have

Corollary. *Suppose $f_j \in S_{k_j}$ $(j=1,\ldots,n)$ with Fourier expansions as in* (2). *For $l \in \mathbf{Z}_{>0}$ put*

$$\sigma_\nu(l) := \sum_{d|l} d^\nu \quad \textit{for} \quad \nu \in \mathbf{C}.$$

Then the series

$$\mathcal{S}(s; f_1, \ldots, f_n) := \sum_{m=(m_1,\ldots,m_n)\in(\mathbf{Z}_{>0})^n} \left(\prod_{j=1}^{n} a_j(m_j)\sigma_{k_j-1}(m_j)\right) Q_k^{(n)}(m,s)$$

has a holomorphic continuation to the whole s-plane and satisfies the functional equation

$$\mathcal{S}(s; f_1, \ldots, f_n) = \mathcal{S}(n-s; f_1, \ldots, f_n).$$

2 A key to the proof: an integral of Rankin-Selberg type

We use the following type of Eisenstein series for the Siegel modular group $\Gamma_n := Sp_{2n}(\mathbf{Z})$ whose properties were studied by Kohnen-Skoruppa [2], Yamazaki [5], and Deitmar-Krieg [1]:

$$E_s^{(n)}(Z) := \sum_{M \in \Delta_{n,n-1} \backslash \Gamma_n} \left(\frac{\det(\mathrm{Im}(M\langle Z \rangle))}{\det(\mathrm{Im}(M\langle Z \rangle^*))} \right)^s . \tag{4}$$

Here $s \in \mathbf{C}$, Z is a variable on H_n, the Siegel upper half space of degree n,

$$\Delta_{n,n-1} := \left\{ \begin{pmatrix} * & * \\ 0^{(1,2n-1)} & * \end{pmatrix} \in \Gamma_n \right\},$$

M runs over a complete set of representatives of $\Delta_{n,n-1} \backslash \Gamma_n$; for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with A, B, C, D being $n \times n$ blocks ,

$$M\langle Z \rangle := (AZ + D)(CZ + D)^{-1}$$

and $M\langle Z \rangle^*$ is the upper left $(n-1) \times (n-1)$ block of $M\langle Z \rangle$. We understand that

$$\det(\mathrm{Im}(M\langle Z \rangle^*)) = 1$$

if $n = 1$. The right-hand side of (4) converges absolutely and locally uniformly for $\mathrm{Re}(s) > n$. Put

$$\xi(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

By [1][5], the Eisenstein series (4) has meromorphic continuation in s to the whole s-plane; the function $\xi(2s)E_s^{(n)}(Z)$ is invariant under the substitution $s \mapsto n - s$ and is holomorphic except for the simple poles at $s = 0$ and $s = n$ with residues $-1/2$ and $1/2$, respectively.

Theorem 1 follows from the following integral representation:

Theorem 2. *For*

$$F_j(z) := \overline{f_j(z)} g_j(z) \mathrm{Im}(z)^{k_j}$$

we have

$$\left(\left(\cdots \left(E_s^{(n)} \begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{pmatrix}, F_1(z_1) \right), \cdots \right), F_n(z_n) \right)$$

$$= \frac{1}{2\xi(2s)} \sum_{\nu=1}^{n} \sum_{1 \le i_1 < \ldots < i_\nu \le n} \left(\prod_{\substack{j \ne i_1, \ldots, i_\nu \\ 1 \le j \le n}} (f_j, g_j) \right)$$

$$\cdot \mathcal{D}(s; f_{i_1}, \ldots, f_{i_\nu}; g_{i_1}, \ldots, g_{i_\nu}).$$

Remark. Define a symmetric positive definite matrix

$$P_Z := \begin{pmatrix} 1_n & {}^tX \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ X & 1_n \end{pmatrix}.$$

Then

$$E_s^{(n)}(Z) = \frac{1}{2\zeta(2s)} \sum_{h \in \mathbf{Z}^{(2n,1)} - \{0\}} ({}^t h P_Z h)^{-s} \quad \text{for} \quad \mathrm{Re}(s) > n.$$

3 Supplementary remarks

(i) Let

$$\varphi_j(z) = \sum_{l=1}^{\infty} c_j(l) e^{2\pi i l z}$$

be holomorphic primitive cusp forms of weight 1 for $\Gamma_0(N_j)$ with odd characters χ_j where $N_j \in \mathbf{Z}_{>0}$ and $j = 1, \ldots, n$. Suppose $n \ge 3$. Then by Kurokawa [3, Theorem 5], the Dirichlet series

$$\sum_{l=1}^{\infty} c_1(l) \cdots c_n(l) l^{-s}$$

has meromorphic continuation in the region $\mathrm{Re}(s) > 0$ but has the line $\mathrm{Re}(s) = 0$ as a natural boundary. (Cf. also [4, Theorem 8].) Thus it is a nontrivial problem to find a series associated with more than two elliptic modular forms which has analytic continuation to the whole s-plane.
(ii) In case $n = 1$ we have

$$\mathcal{D}(s; f_1; g_1) = 2\xi(2s)(4\pi)^{1-k_1-s}\Gamma(s + k_1 - 1) D(s + k_1 - 1, f_1, g_1)$$

for $\mathrm{Re}(s) > (k_1 + 1)/2$, where

$$D(s, f_1, g_1) := \sum_{m=1}^{\infty} a_1(m) \overline{b_1(m)} m^{-s}.$$

Thus in this case Theorem 1 states nothing but the well-known properties of the Rankin series $D(s, f_1, g_1)$.

(iii) In case $n = 2$ we have

$$
\begin{aligned}
&\mathcal{D}(s; f_1, f_2; g_1, g_2) \\
&= 2^{6-2|k|}\pi^{2-|k|}(2\pi)^{-2s}\frac{\Gamma(s)\Gamma(s+|k|-2)\Gamma(s+k_1-1)\Gamma(s+k_2-1)}{\Gamma(2s+|k|-2)} \\
&\cdot \sum_{m_1, m_2 \in \mathbf{Z}_{>0}} a_1(m_1)a_2(m_2)\overline{b_1(m_1)b_2(m_2)}m_1^{1-k_1-s}m_2^{1-k_2} \\
&\cdot \sum_{\lambda_1, \lambda_2 \in \mathbf{Z}_{>0}} \lambda_1^{-2s} F\left(s, s+k_1-1; 2s+|k|-2; 1-\frac{m_2\lambda_2^2}{m_1\lambda_1^2}\right)
\end{aligned}
$$

for $\mathrm{Re}(s) > \max(k_1, k_2) + 1$, where $F = {}_2F_1$ is the hypergeometric function.
(iv) The function $Q_k^{(n)}(m, s)$ has another representation:

$$
\begin{aligned}
Q_k^{(n)}(m,s) \quad &= 2^{3n-|k|+1}\pi^{\frac{n-|k|}{2}-s}\left(\prod_{j=1}^{n} m_j^{\frac{1-k_j}{2}}\right) \cdot \sum_{\lambda_1,\ldots,\lambda_n \in \mathbf{Z}_{>0}} \left(\prod_{j=1}^{n} \lambda_j^{k_j-1}\right) \\
&\cdot \int_0^\infty t^{2s-1+|k|-n} \prod_{j=1}^{n} K_{k_j-1}(4\sqrt{\pi m_j}\lambda_j t)dt
\end{aligned}
$$

for $\mathrm{Re}(s) > n/2$, where K_ν is the modified Bessel function of order ν.

References

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