

# Principal series Whittaker functions on $SL(3, \mathbf{R})$

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This is an extract from a preprint with the same title. The full proofs are contained in that. Here we write only the major results. *The numbering of the statement are the same as the original full paper. Some statements in the original are skipped.*

## Introduction

The study of Whittaker models of algebraic groups over local fields has already some history. The Jacquet integral is named after the investigation of H.Jacquet [7]. Multiplicity free theorem by J.Shalika for quasi-split groups, was later enhanced for the case of the real field by N.Wallach. For reductive groups over the real field, this theme was investigated by M.Hashizume [5], B.Kostant, D. Vogan, H.Matsumoto, and the joint work of R.Goodman and N.Wallach [4].

More specifically  $GL(n, \mathbf{R})$ , explicit expressions for class 1 Whittaker functions are obtained, firstly for  $n = 3$  by D.Bump [2]. The main contributor for the case of general  $n$  seems to be E.Stade. Other related results will be find in the references of the papers of him ([9],[10]).

Let us explain the outline of this paper. The purpose of the master thesis [1] refered above is to investigate the Whittaker functions belonging to the non-spherical principal series representations of  $SL(3, \mathbf{R})$ . The minimal  $K$ -type of such representations is 3-dimensional. So we have to consider vector-valued functions. The main results are, firstly, to obtain the holonomic system of the  $A$ -radial part of such Whittaker functions with minimal  $K$ -type explicitly (§4), and secondly to have 6 formal solutions (§5, Theorem (5.5)), which are considered as examples of confluent hypergeometric series of two variables. We also have integral expressions of these 6 solutions (§5, Theorem (5.6)). In the subsequent section, the Jacquet integral (so to say, the primary Whittaker function) is written as a sum of these 6 *secondary* Whittaker functions (§6-8).

## 1 Preliminaries. Basic terminology

### 1.1 Whittaker model

Given an irreducible admissible representation  $(\pi, H)$  of  $G = SL(3, \mathbf{R})$ , we consider its model or realization in the space of Whittaker functions. This means, for a non-

degenerate unitary character  $\psi$  of a maximal unipotent subgroup  $N = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in G \right\}$  of  $G$  defined by

$$\psi\left(\begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{pmatrix}\right) = \exp\{2\pi\sqrt{-1}(c_1x_{12} + c_2x_{23})\}$$

with  $c_1, c_2 \in \mathbf{R}$  being non-zero, we consider a smooth induction  $C^\infty\text{-Ind}_N^G(\psi)$  to  $G$ , and the space of intertwining operators of smooth  $G$ -modules

$$\text{Hom}_G(H_\infty, C^\infty\text{-Ind}_N^G(\psi))$$

with  $H_\infty$  the subspace consisting of  $C^\infty$ -vectors in  $H$ . Or more algebraically speaking, we might consider the corresponding space in the context of  $(\mathfrak{g}, K)$ -modules (with  $\mathfrak{g} = \text{Lie}(G), K = SO(3)$ ):

$$\text{Hom}_{(\mathfrak{g}, K)}(H_\infty, C^\infty\text{-Ind}_N^G(\psi)).$$

## 1.2 Principal series representations

Let  $P_0$  be a minimal parabolic subgroup of  $G$  given by the upper triangular matrices in  $G$ , and  $P_0 = MAN$  be a Langlands decomposition of  $P_0$  with  $M = K \cap \{\text{diagonals in } G\}$ ,  $A = \exp\mathfrak{a}$ , with

$$\mathfrak{a} = \{\text{diag}(t_1, t_2, t_3) | t_i \in \mathbf{R}, t_1 + t_2 + t_3 = 0\}.$$

In order to define a principal series representation with respect to the minimal parabolic subgroup  $P_0$  of  $G$ , we firstly fix a character  $\sigma$  of the finite abelian group  $M$  of type  $(2, 2)$  and a linear form  $\nu \in \mathfrak{a}^* \otimes_{\mathbf{R}} \mathbf{C} = \text{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{C})$ . For such data, we can define a representation  $\sigma \otimes e^\nu$  of  $MA$ , and extend this to  $P_0$  by the identification  $P_0/N \cong MA$ . Then we set

$$\pi_{\sigma, \nu} = L^2\text{-Ind}_{P_0}^G(\sigma \otimes e^{\nu+\rho} \otimes 1_N).$$

Here  $\nu(\text{diag}(t_1, t_2, t_3)) = \sum_{i=1}^3 \nu_i t_i$  with  $\nu_i \in \mathbf{C}$  and  $\rho$  is the half-sum of positive roots of  $(\mathfrak{g}, \mathfrak{a})$  for  $P_0$ , given as follows. For  $i < j$  ( $1 \leq i, j \leq 3$ ), we put  $\eta_{ij}(a) = a_i/a_j$  for  $a = \text{diag}(a_1, a_2, a_3)$  ( $a_1 a_2 a_3 = 1$ ). Then we have  $a^{2\rho} = \prod_{i < j} a_i/a_j = a_1^2/a_3^2 = a_1^4 a_2^2$  by definition. Hence  $a^\rho = a_1^2 a_2$ .

Here the characters  $\sigma_j$  of  $M$  are identified as follows. The group  $M$  consisting of 4 elements is a finite abelian group of  $(2, 2)$  type, and its elements except for the unity is given by the matrices

$$m_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, m_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, m_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since  $M$  is commutative, all the irreducible unitary representations of it is 1-dimensional. For any  $\sigma \in \widehat{M}$ , we have  $\sigma^2 = 1$ . Therefore the set  $\widehat{M}$  consisting of

4 characters  $\{\sigma_j : j = 0, 1, 2, 3\}$ , where each  $\sigma_j$ , except for the trivial character  $\sigma_0$ , is specified by the following table of values at the elements  $m_i$ .

	$m_1$	$m_2$	$m_3$
$\sigma_1$	1	-1	-1
$\sigma_2$	-1	1	-1
$\sigma_3$	-1	-1	1

**Proposition (1.1)** (i) *If  $\sigma$  is the trivial character of  $M$ , the representation  $\pi_{\sigma,\nu}$  is spherical or class 1, i.e., it has a (unique)  $K$ -invariant vector in the representation space  $H_{\sigma,\nu}$ .*

(ii) *If  $\sigma$  is not trivial, then the minimal  $K$ -type of the restriction  $\pi_{\sigma,\nu}|_K$  to  $K$  is a 3-dimensional representation of  $K = SO(3)$ , which is isomorphic to the unique standard one  $(\tau_2, V_2)$ . The multiplicity of this minimal  $K$ -type is one:*

$$\dim_{\mathbf{C}} \operatorname{Hom}_K(\tau_2, H_{\sigma,\nu}) = 1,$$

*namely there is a unique non-zero  $K$ -homomorphism*

$$\iota : (\tau_2, V_2) \rightarrow (\pi_{\sigma,\nu}|_K, H_{\sigma,\nu})$$

*up to constant multiple.*

## 2 Representations of $K = SO(3)$

### 2.1 The spinor covering

To describe the finite dimensional irreducible representations of  $SO(3)$ , the simplest way seems to utilize the double covering  $s : SU(2) = Spin(3) \rightarrow SO(3)$ , which is realized as follows.

The Hamilton quaternion algebra  $\mathbf{H}$  is realized in  $M_2(\mathbf{C})$  by

$$\mathbf{H} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in M_2(\mathbf{C}) \mid a, b \in \mathbf{C} \right\}.$$

Then  $SU(2)$  is the subgroup of the multiplicative group consisting of quaternions with reduced norm 1, i.e.,

$$SU(2) = \{x \in \mathbf{H} \mid \det x = 1\}.$$

Let  $\mathbf{P} = \{x \in \mathbf{H} \mid \operatorname{tr} x = 0\}$  be the 3-dimensional real Euclidean space consisting of pure quaternions. Then for each  $x \in SU(2)$ , the map

$$p \in \mathbf{P} \mapsto x \cdot p \cdot x^{-1} \in \mathbf{P}$$

preserve the Euclid norm  $p \mapsto \det p$  and the orientation, hence we have a homomorphism

$$s : SU(2) \rightarrow SO(\mathbf{P}, \det) = SO(3),$$

which is surjective, since the range is a connected group. The kernel of this homomorphism is given by  $\{\pm 1_2\}$ .

By the derivation of  $s ds : \mathfrak{su}(2) \rightarrow \mathfrak{so}(3)$ , the standard generators:

$$u_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

are mapped to  $2K_1, 2K_2, 2K_3$  with

$$K_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, K_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{k},$$

respectively. Here  $\mathfrak{k}$  is the Lie algebra of  $K$ .

## 2.2 Representations of $SU(2)$

The set of equivalence classes of the finite dimensional continuous representations of  $SU(2)$  is exhausted by the symmetric tensor products  $\tau_l$  ( $l = 0, 1, \dots$ ) of the standard representation. These are realized as follows.

Let  $V_l$  be the subspace consisting of homogeneous polynomials of two variables  $x, y$  in the polynomial ring  $\mathbf{C}[x, y]$ . For  $g \in SU(2)$  with  $g^{-1} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ , and  $f(x, y) \in V_l$  we set

$$\tau_l(g)f(x, y) := f(ax + by, -\bar{b}x + \bar{a}y).$$

Passing to the Lie algebra  $Lie(SU(2)) = \mathfrak{su}(2)$ , the derivation of  $\tau_l$ , denoted by the same symbol, is described as follows by using the standard basis  $\{v_k = x^k y^{l-k} \ (0 \leq k \leq l)\}$  and the standard generators

$$u_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

Namely we have

$$\tau_l(u_1)v_k = \sqrt{-1}(l - 2k)v_k, \quad \tau_l(X_+)v_k = (l - k)v_{k+1}, \quad \tau_l(X_-)v_k = -k \cdot v_{k-1}.$$

Here we put  $X_+ = \frac{1}{2}(u_2 + \sqrt{-1}u_3)$ ,  $X_- = \frac{1}{2}(u_2 - \sqrt{-1}u_3)$ .

The condition that  $\tau_l$  defines a representation of  $SO(3)$  by passing to the quotient with respect  $s : SU(2) \rightarrow SO(3)$  is that  $\tau_l(-1_2) = (-1)^l = +1$ , i.e.,  $l$  is even. Therefore the dimension of  $V_l$ ,  $l + 1$  is odd in this case.

The representation  $\tau_2$  of  $SU(2)$  is equivalent to the spinor homomorphism. Hence passing to the quotient,  $\tau_2$  is equivalent to the tautological representation  $SO(3) \rightarrow GL(3, \mathbf{C})$ .

## 2.3 Irreducible components of $\tau_2 \otimes \tau_4$ and $\tau_2 \otimes Ad_{\mathfrak{p}}$

For our later use, we want to specify the standard basis of the unique irreducible constituent  $\tau_2$  in the tensor product  $\tau_2 \otimes \tau_4$ .

**Lemma (2.1)** *Let  $\{v_i \ (i = 0, 1, 2)\}$  and  $\{w_j \ (0 \leq j \leq 4)\}$  be the standard basis of  $(\tau_2, V_2)$  and  $(\tau_4, V_4)$ , respectively. Then the elements*

$$\begin{aligned} v'_0 &= v_0 \otimes w_2 - 2v_1 \otimes w_1 + v_2 \otimes w_0, \\ v'_1 &= v_0 \otimes w_3 - 2v_1 \otimes w_2 + v_2 \otimes w_1, \\ v'_2 &= v_0 \otimes w_4 - 2v_1 \otimes w_3 + v_2 \otimes w_2 \end{aligned}$$

define a set of standard basis in  $\tau_2 \subset \tau_2 \otimes \tau_4$ , which is unique up to a common scalar multiple.

## 2.4 The $K$ -module isomorphism between $\mathfrak{p}_{\mathbb{C}}$ and $V_4$

We denote by  $\mathfrak{p}_{\mathbb{C}}$  the complexification of the orthogonal complement  $\mathfrak{p}$  of  $\mathfrak{k}$  with respect to the Killing form, on which the group  $K$  acts via the adjoint action  $Ad_{\mathfrak{p}}$ . We denote by  $E_{ij}$  the matrix unit with 1 at  $(i, j)$ -th entry and 0 at other entries. Then  $E_{ii}$  and  $E_{ij} + E_{ji}$  are considered as elements in  $\mathfrak{p}$ . We set  $H_{ij} = E_{ii} - E_{jj}$  for  $i \neq j$ .

**Lemma (2.2)** *Via the unique isomorphism  $V_4$  and  $\mathfrak{p}_{\mathbb{C}}$  as  $K$ -modules we have the identification*

$$\begin{aligned} w_0 &= -2\{H_{23} - \sqrt{-1}(E_{23} + E_{32})\}, \\ w_1 &= \sqrt{-1}\{(E_{12} + E_{21}) - \sqrt{-1}(E_{13} + E_{31})\}, \\ w_2 &= \frac{2}{3}(H_{12} + H_{13}), \\ w_3 &= \sqrt{-1}\{(E_{12} + E_{21}) + \sqrt{-1}(E_{13} + E_{31})\}, \\ w_4 &= -2\{H_{23} + \sqrt{-1}(E_{23} + E_{32})\}. \end{aligned}$$

# 3 Principal series $(\mathfrak{g}, K)$ -modules

## 3.1 The case of the class one principal series

### 3.1.1 The Capelli elements

A set of generators for the center  $Z(\mathfrak{g})$  of the universal enveloping algebra  $U(\mathfrak{g})$  of  $\mathfrak{g} = \mathfrak{sl}_3$  is obtained as Capelli elements, because  $\mathfrak{sl}_3$  is of type  $A_2$ .

Let

$$E'_{ii} = E_{ii} - \frac{1}{3}\left(\sum_{a=1}^3 E_{aa}\right), \quad E'_{ij} = E_{ij} \quad (i \neq j).$$

Then  $E'_{ij} \in \mathfrak{g}$ . Define a matrix  $\mathcal{C}$  of size 3 with entries in  $\mathfrak{g}$  by

$$\mathcal{C} = \begin{pmatrix} E'_{11} & E'_{12} & E'_{13} \\ E'_{21} & E'_{22} & E'_{23} \\ E'_{31} & E'_{32} & E'_{33} \end{pmatrix} - \text{diag}(-1, 0, 1).$$

Then for

$$\mathcal{A} = (A_{ij})_{1 \leq i, j \leq 3} = x \cdot 1_3 - \mathcal{C} \in M_3(\mathfrak{g}[x]) \subset M_3(U(\mathfrak{g})[x]),$$

we define its *vertical* determinant by

$$\det \downarrow (\mathcal{A}) = \sum_{\sigma \in \mathfrak{S}_3} \text{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} A_{3\sigma(3)}.$$

Then it is written in the form  $x^3 + Cp_2x - Cp_3 \in U(\mathfrak{g})[x]$  with some elements  $Cp_2$  and  $Cp_3$  in  $Z(\mathfrak{g})$ .

**Proposition (3.1)** *The set  $\{Cp_2, Cp_3\}$  is a system of independent generators of  $Z(\mathfrak{g})$ . Here are explicit formulae of  $Cp_2$  and  $Cp_3$ :*

$$Cp_2 = (E'_{11} - 1)E'_{22} + E'_{22}(E'_{33} + 1) + (E'_{11} - 1)(E'_{33} + 1) \\ - E_{23}E_{32} - E_{13}E_{31} - E_{12}E_{21},$$

$$Cp_3 = (E'_{11} - 1)E'_{22}(E'_{33} + 1) + E_{12}E_{23}E_{31} + E_{13}E_{21}E_{32} \\ - (E'_{11} - 1)E_{23}E_{32} - E_{13}E'_{22}E_{31} - E_{12}E_{21}(E'_{33} + 1).$$

### Eigenvalues of $Cp_2, Cp_3$

We compute the value  $Cp_2f_0(e)$  and  $Cp_3f_0(e)$ . Let  $S_2(a, b, c) = ab + bc + ca$  and  $S_3(a, b, c) = abc$  be the elementary symmetric functions of three variables of degree 2 and 3, respectively. Then we have the following.

**Proposition (3.2)** *The infinitesimal character of  $\pi_{\sigma, \nu}$  is given by*

$$Cp_2f_0 = S_2\left(\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(2\nu_2 - \nu_1), \frac{1}{3}(\nu_1 + \nu_2)\right)f_0$$

and

$$Cp_3f_0 = S_3\left(\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(2\nu_2 - \nu_1), \frac{1}{3}(\nu_1 + \nu_2)\right)f_0.$$

## 3.2 $(\mathfrak{g}, K)$ -module structure of non-spherical principal series at the minimal $K$ -type

### 3.2.1 Construction of $K$ -equivariant differential operators

**Lemma (3.3)** *Let  $\{f_i (i = 0, 1, 2)\}$  be the set of the standard basis of the minimal  $K$ -type  $\tau \subset \pi_{\sigma, \nu}$  of a non-spherical principal series representation  $\pi_{\sigma, \nu} = \pi$ . Define another three  $C^\infty$ -elements  $\{\varphi_i (i = 0, 1, 2)\}$  by the formulae:*

$$\begin{aligned} \varphi_0 &= \frac{2}{3}\pi(2E_{11} - E_{22} - E_{33})f_0 \\ &\quad - 2\sqrt{-1}\pi(E_{12} + E_{21} - \sqrt{-1}(E_{13} + E_{31}))f_1 \\ &\quad - 2\pi(E_{12} + E_{21} - \sqrt{-1}(E_{23} + E_{32}))f_2, \\ \varphi_1 &= \sqrt{-1}\pi(E_{12} + E_{21} + \sqrt{-1}(E_{13} + E_{31}))f_0 \\ &\quad - \frac{4}{3}\pi(2E_{11} - E_{22} - E_{33})f_1 \\ &\quad + \sqrt{-1}\pi(E_{12} + E_{21} - \sqrt{-1}(E_{13} + E_{31}))f_2, \\ \varphi_2 &= -2\pi(E_{22} - E_{33} + \sqrt{-1}(E_{23} + E_{32}))f_0 \\ &\quad - 2\sqrt{-1}\pi(E_{12} + E_{21}) + \sqrt{-1}(E_{13} + E_{31}))f_1 \\ &\quad + \frac{2}{3}\pi(2E_{11} - E_{22} - E_{33})f_2. \end{aligned}$$

Then  $(\varphi_0, \varphi_1, \varphi_2)$  is a constant multiple of  $(f_0, f_1, f_2)$ .

### 3.2.2 Computation of eigenvalues

The previous lemma tells that there exist a scalar  $\lambda(\sigma, \nu)$  depending on  $\sigma$  and  $\nu$  such that  $\varphi_i = \lambda(\sigma, \nu)f_i$  ( $i = 0, 1, 2$ ). We determine this eigenvalue  $\lambda(\sigma, \nu)$  by using explicit models of the principal series  $\pi_{\sigma, \nu}$ .

To do this, we have to find functions in

$$L^2\text{-Ind}_M^K(\sigma_i) = L^2_{M, \sigma_i}(K) = \{f \in L^2(K) \mid f(mk) = \sigma(m)f(k) \text{ for all } m \in M, k \in K\}$$

corresponding to the standard basis in the minimal  $K$ -type for each  $i$ .

In the larger space  $L^2(K)$ , the  $\tau_2$ -isotypic component is generated by the 9 matrix elements  $s_{ij}(k)$  ( $1 \leq i, j \leq 3$ ) of the tautological representation

$$k \in K \mapsto S(k) = (s_{ab}(k))_{1 \leq a, b \leq 3} \in SO(3).$$

It is directly confirmed that  $s_{ib}(k)$  ( $b = 0, 1, 2$ ) belong to the subspace  $L^2_{M, \sigma_i}(K)$  for each  $i$ .

Diagonalizing the action of  $u_1$ , we find that  $s_{i1}$  corresponds to  $v_1$  for each  $i$ . And finally we find that the standard basis is given by

$$v_0 = \sqrt{-1}(s_{i2} - \sqrt{-1}s_{i3}), \quad v_1 = s_{i1}, \quad \text{and } v_2 = \sqrt{-1}(s_{i2} + \sqrt{-1}s_{i3}).$$

We need the values of these standard functions  $f_a(k) = v_a$  ( $a = 0, 1, 2$ ) at the identity  $e \in K$ .

**Lemma (3.4)** *The values of the standard functions at  $e \in K$  is given as follows.*

1. If  $\sigma = \sigma_1$ ,  $(f_0(e), f_1(e), f_2(e)) = (0, 1, 0)$ .
2. If  $\sigma = \sigma_2$ ,  $(f_0(e), f_1(e), f_2(e)) = (\sqrt{-1}, 0, \sqrt{-1})$ .
3. If  $\sigma = \sigma_3$ ,  $(f_0(e), f_1(e), f_2(e)) = (1, 0, -1)$ .

Now we can proceed to the computation of the value  $\lambda(\sigma_i, \nu)$ .

**Lemma (3.5)**

$$\lambda(\sigma_1, \nu) = -\frac{4}{3}(2\nu_1 - \nu_2), \quad \lambda(\sigma_2, \nu) = \frac{4}{3}(\nu_1 - 2\nu_2), \quad \lambda(\sigma_3, \nu) = \frac{4}{3}(\nu_1 + \nu_2).$$

Summing up the lemmata in this section, we have the following.

**Proposition (3.6)** *Let  $\{f_i$  ( $i = 0, 1, 2$ )* be the set of the standard basis of the minimal  $K$ -type  $\tau \subset \pi_{\sigma, \nu}$  of a non-spherical principal series representation  $\pi_{\sigma, \nu} = \pi$ . Define another three  $C^\infty$ -elements  $\{\varphi_i$  ( $i = 0, 1, 2$ ) by the formulae:

$$\begin{aligned} \varphi_0 &= \frac{2}{3}\pi(H_{12} + H_{13})f_0 \\ &\quad - 2\sqrt{-1}\pi(E_{12} + E_{21} - \sqrt{-1}(E_{13} + E_{31}))f_1 \\ &\quad - 2\pi(H_{23} - \sqrt{-1}(2E_{23} + \frac{1}{2}u_1))f_2, \\ \varphi_1 &= \sqrt{-1}\pi(E_{12} + E_{21} + \sqrt{-1}(E_{13} + E_{31}))f_0 \\ &\quad - \frac{4}{3}\pi(H_{12} + H_{13})f_1 \\ &\quad + \sqrt{-1}\pi(E_{12} + E_{21} - \sqrt{-1}(E_{13} + E_{31}))f_2, \\ \varphi_2 &= -2\pi(H_{23} + \sqrt{-1}(2E_{23} + \frac{1}{2}u_1))f_0 \\ &\quad - 2\sqrt{-1}\pi(E_{12} + E_{21}) + \sqrt{-1}(E_{13} + E_{31}))f_1 \\ &\quad + \frac{2}{3}\pi(H_{12} + H_{13})f_2. \end{aligned}$$

Then we have

$$(\varphi_0, \varphi_1, \varphi_2) = \lambda(\sigma_i, \nu)(f_0, f_1, f_2)$$

with eigenvalue  $\lambda(\sigma_i, \nu)$  given by

$$\lambda(\sigma_1, \nu) = -\frac{4}{3}(2\nu_1 - \nu_2), \quad \lambda(\sigma_2, \nu) = \frac{4}{3}(\nu_1 - 2\nu_2), \quad \lambda(\sigma_3, \nu) = \frac{4}{3}(\nu_1 + \nu_2).$$

In the next section, we consider the Whittaker realization of the equation of the above proposition. Then we need the following Iwasawa decomposition of standard elements of  $\mathfrak{g}$ .

**Lemma (3.7)** *We have the following decomposition of standard generators of  $\mathfrak{g}$  with respect to the Iwasawa decomposition  $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$ . For  $H_{ij} \in \mathfrak{a}$  we have*

$$H_{ij} = 0 + H_{ij} + 0.$$

Since  $E_{ij} + E_{ji} = 2E_{ij} - (E_{ij} - E_{ji})$ , we have

$$E_{12} + E_{21} = 2E_{12} + 0 + K_3, \quad E_{13} + E_{31} = 2E_{13} + 0 + (-K_2), \quad E_{23} + E_{32} = 2E_{23} + 0 + K_1.$$

## 4 The holonomic system for the $A$ -radial part of the principal series Whittaker functions

### 4.1 The case of the class one principal series

Let  $I$  be a non-zero Whittaker functional from the class one principal series  $\pi_{\sigma_0, \nu}$  to  $C^\infty\text{-Ind}_N^G(\psi)$ . Let  $F$  be the restriction of the image  $I(f_0)$  of the  $K$ -fixed vector  $f_0$  to  $A$ . We write here the holonomic system for  $F$  with respect to the variables  $y_1 = \eta_{12}(a) = a_1/a_2$ ,  $y_2 = \eta_{23}(a) = a_2/a_3 = a_1/a_2^2$ .

**Proposition (4.1)** *Put  $F(y_1, y_2) = y_1 y_2 G(y_1, y_2)$  (note  $a^\rho = y_1 y_2$ ). Then  $G(y_1, y_2)$  satisfies the partial differential equations:*

$$\Delta_2 G = \frac{1}{3}(\nu_1^2 + \nu_2^2 - \nu_1 \nu_2)G$$

and

$$\{\partial_1(\partial_1 - \partial_2)\partial_2 + 4\pi^2 c_2^2 y_2^2 \partial_1 - 4\pi^2 c_1^2 y_1^2 \partial_2\}G = -\frac{1}{27}(2\nu_1 - \nu_2)(2\nu_2 - \nu_1)(\nu_1 + \nu_2)G.$$

Here  $\partial_i$  is the Euler operator  $y_i \frac{\partial}{\partial y_i}$  for  $i = 1, 2$ . and we write

$$\Delta_2 = (\partial_1^2 + \partial_2^2 - \partial_1 \partial_2) - 4\pi^2(c_1^2 y_1^2 + c_2^2 y_2^2).$$

*Remark* From these equations for the monodromy exponents  $\alpha_1, \alpha_2$  at the origin  $y_1 = 0$ ,  $y_2 = 0$ , we have an equality of sets of complex numbers:

$$\{\alpha_1, -\alpha_1 + \alpha_2, -\alpha_2\} = \left\{ \frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(2\nu_2 - \nu_1), -\frac{1}{3}(\nu_1 + \nu_2) \right\}.$$



## 4.2 The holonomic system for the $A$ -radial part of non-spherical Whittaker functions

Let  $I$  be a non-zero Whittaker functional from the principal series  $\pi_{\sigma_i, \nu}$ . For the set  $\{f_i | (i = 0, 1, 2)\}$  of standard functions, we put  $F_i = I(f_i)$ .

**Theorem (4.4)** *Let  $F(a) = {}^t(F_0(a), F_1(a), F_2(a)) = (y_1 y_2)^t(G_0(y), G_1(y), G_2(y))$  be the vector of the  $A$ -radial part of the standard Whittaker functions with minimal  $K$ -type of the principal series representation  $\pi_{\sigma, \nu}$  with non-trivial  $\sigma = \sigma_i$ . Then it satisfies the following partial differential equations:*

(i):

$$\begin{pmatrix} \partial_1 & 4\pi c_1 y_1 & \partial_1 - 2\partial_2 - 4\pi c_2 y_2 \\ -2\pi c_1 y_1 & -2\partial_1 & -2\pi c_1 y_1 \\ \partial_1 - 2\partial_2 + 4\pi c_2 y_2 & 4\pi c_1 y_1 & \partial_1 \end{pmatrix} \begin{pmatrix} G_0(y) \\ G_1(y) \\ G_2(y) \end{pmatrix} = \frac{1}{2} \lambda_i \begin{pmatrix} G_0(y) \\ G_1(y) \\ G_2(y) \end{pmatrix},$$

(ii):

$$\Delta_2 \cdot 1_3 \cdot \begin{pmatrix} G_0(y) \\ G_1(y) \\ G_2(y) \end{pmatrix} - 2\pi c_2 y_2 \begin{pmatrix} G_0(y) \\ 0 \\ -G_2(y) \end{pmatrix} + 2\pi c_1 y_1 \begin{pmatrix} G_1(y) \\ \frac{1}{2}(G_0(y) + G_2(y)) \\ G_1(y) \end{pmatrix} = \frac{1}{3} \mu \begin{pmatrix} G_0(y) \\ G_1(y) \\ G_2(y) \end{pmatrix}.$$

Moreover the eigenvalues  $\lambda_i$  and  $\mu$  depending on the representation  $\pi_{\sigma, \nu}$  are given by

$$\begin{cases} \lambda_1 = -\frac{4}{3}(2\nu_1 - \nu_2) & (\sigma = \sigma_1) \\ \lambda_2 = \frac{4}{3}(\nu_1 - 2\nu_2) & (\sigma = \sigma_2) \\ \lambda_3 = \frac{4}{3}(\nu_1 + \nu_2) & (\sigma = \sigma_3) \end{cases} \quad \text{and} \quad \mu = \nu_1^2 + \nu_2^2 - \nu_1 \nu_2.$$

*Remark* We can write the differential equations (i) and (ii) of the above Theorem as

$$(i): \quad \mathcal{D}_1 G = \lambda_i G \quad (ii): \quad \mathcal{D}_2 G = \mu G,$$

with  $\mathcal{D}_i$  ( $i = 1, 2, 3$ ) by 3 by 3 matrix-valued differential operators. Then we have

$$\mathcal{D}_1 \cdot \mathcal{D}_2 - \mathcal{D}_2 \cdot \mathcal{D}_1 = 0.$$

## 4.3 The equations via the tautological basis

Let  $k \in K \mapsto S(k) = (s_{ij}(k))_{1 \leq i, j \leq 3}$  be the tautological representation of  $K = SO(3)$ . Let  $I \in \text{Hom}_{\mathfrak{g}, K}(\pi_{\sigma_i, \nu}, \text{Ind}_N^G(\psi))$  be a Whittaker functional and define function  $T_{ij}$  on  $A$  by

$$I(s_{ij})|_A = y_1 y_2 T_{ij}(y) \quad (1 \leq i, j, \leq 3).$$

Then

$$\begin{pmatrix} G_0 \\ G_1 \\ G_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{-1} & 1 \\ 1 & 0 & 0 \\ 0 & \sqrt{-1} & -1 \end{pmatrix} \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix}.$$

Then for each  $i$ , the equation (i) of the above theorem is transformed to

$$\begin{pmatrix} -\partial_1 & -2\pi\sqrt{-1}c_1y_1 & 0 \\ -2\pi\sqrt{-1}c_1y_1 & \partial_1 - \partial_2 & -2\pi\sqrt{-1}c_2y_2 \\ 0 & -2\pi\sqrt{-1}c_2y_2 & \partial_2 \end{pmatrix} \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix} = \frac{1}{2}\lambda_i \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix},$$

and the equation (ii) to

$$\left[ \Delta_2 \cdot 1_3 + \begin{pmatrix} 0 & 2\pi\sqrt{-1}c_1y_1 & 0 \\ -2\pi\sqrt{-1}c_1y_1 & 0 & 2\pi\sqrt{-1}c_2y_2 \\ 0 & -2\pi\sqrt{-1}c_2y_2 & 0 \end{pmatrix} \right] \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix} = \frac{1}{3}\mu \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix}.$$

## 5 Power series solutions at the origin

We determine 6 linearly independent formal power series at the origin  $(y_1, y_2) = (0, 0)$  for generic parameter  $\nu$  in this section. These formal solutions converges because the singularity at the origin is a regular singularity. These solutions do not have exponential decay at infinity, different from the unique ‘good’ solution given by Jacquet integral. We refer to these solutions as *secondary* Whittaker functions sometimes.

### 5.1 The case of the class one principal series

This case is more or less discussed in the paper of Bump [2], up to some difference of notations. We omit its explicit formula.

An integral expression of this power series solution was found by Stade ([9, Lemma 3.10], [11, Theorem 2]) as an analogue of an integral formula for Jacquet integral by Vinogradov and Takhadzhyan [12]. The same as non-spherical case discussed later, we let  $\{e_1, e_2, e_3\}$  be a permutation of the three complex numbers  $\{-\frac{1}{3}(2\nu_1 - \nu_2), -\frac{1}{3}(2\nu_2 - \nu_1), \frac{1}{3}(\nu_1 + \nu_2)\} = \{\frac{1}{4}\lambda_1, \frac{1}{4}\lambda_2, \frac{1}{4}\lambda_3\}$

**Theorem (5.2)** For  $\text{Re}(e_2 - e_1) > 2$ ,

$$\begin{aligned} \Phi(y_1, y_2) &= \Gamma\left(\frac{e_2 - e_1}{2} + 1\right)\Gamma\left(\frac{e_3 - e_1}{2} + 1\right)\Gamma\left(\frac{e_2 - e_3}{2} + 1\right)(\pi c_1 y_1)^{\frac{e_3}{2}}(\pi c_2 y_2)^{-\frac{e_3}{2}}(\pi c_1)^{e_1}(\pi c_2)^{-e_2} \\ &\cdot \frac{1}{2\pi\sqrt{-1}} \int_{|u|=1} I_{\frac{e_2 - e_1}{2}}(2\pi c_1 y_1 \sqrt{1 + 1/u}) I_{\frac{e_2 - e_1}{2}}(2\pi c_2 y_2 \sqrt{1 + u}) u^{-\frac{3}{4}e_3} \frac{du}{u}. \end{aligned}$$

### 5.2 The case of the non-spherical principal series

In this case also, the holonomic system obtained in Theorem (4.4) has regular singularities at the origin  $(y_1, y_2) = (0, 0)$  with rank 6, i.e., the order of the Weyl group of  $SL(3, \mathbf{R})$ , for generic values of parameter  $\nu$ . We determine the characteristic indices and the convergent formal power series solutions at  $y = 0$ . Here to abridge the notation, we write the set of variables  $(y_1, y_2)$  as  $y$  collectively.

By inspection we find that it is convenient to introduce scalar functions  $\Phi_i(y_1, y_2)$  ( $i = 0, 1, 2$ ) by

$$F(y) = y_1 y_2 G(y) = y_1 y_2 \left\{ \Phi_0(y) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \Phi_1(y) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \Phi_2(y) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

### 5.3 The holonomic system for $\Phi_i(y)$

Now we can rewrite the holonomic system for  $G_i$  to that for  $\Phi_i$ .

**Proposition (5.3)** *The holonomic system in Theorem (4.4) is equivalent to the following system for  $\Phi_i = \Phi_i(y_1, y_2)$  ( $i = 0, 1, 2$ ).*

- (1) (i)  $[\partial_1 + \frac{1}{4}\lambda_i]\Phi_0 + (2\pi c_1 y_1)\Phi_1 = 0,$   
(ii)  $[\partial_1 - \partial_2 - \frac{1}{4}\lambda_i]\Phi_1 + (2\pi c_1 y_1)\Phi_0 + (2\pi c_2 y_2)\Phi_2 = 0,$   
(iii)  $[\partial_2 - \frac{1}{4}\lambda_i]\Phi_2 - (2\pi c_2 y_2)\Phi_1 = 0,$
- (2) (i)  $[\Delta_2 - \frac{1}{3}\mu]\Phi_0 + (2\pi c_1 y_1)\Phi_1 = 0,$   
(ii)  $[\Delta_2 - \frac{1}{3}\mu]\Phi_1 + (2\pi c_1 y_1)\Phi_0 - (2\pi c_2 y_2)\Phi_2 = 0,$   
(iii)  $[\Delta_2 - \frac{1}{3}\mu]\Phi_2 - (2\pi c_2 y_2)\Phi_1 = 0.$

### 5.4 The characteristic indices at the origin $(y_1, y_2) = (0, 0)$ and the recurrence formulae.

Let

$$\Phi_k(y) = y_1^{-e_1} y_2^{e_2} \sum_{n_1, n_2 \geq 0} c_{k; n_1, n_2} (\pi c_1 y_1)^{n_1} (\pi c_2 y_2)^{n_2}, \quad (k = 0, 1, 2)$$

be a system of formal power series solutions at the origin  $y = 0$ .

Now we can determine the 6 pairs  $(-e_1, e_2)$  of characteristic indices at the origin, and the corresponding initial values conditions for  $F$  or  $\Phi_i$ . the system at the origin and to determine the first coefficients Moreover we have the recurrence relations between the coefficients.

**Lemma (5.4)** *When  $\sigma = \sigma_i$  for  $i = 1, 2$  or  $3$ , we have the following:*

- (1) *The characteristic indices take the six values:*

$$(-e_1, e_2) = (-\frac{1}{4}\lambda_k, \frac{1}{4}\lambda_l) \quad (1 \leq k \neq l \leq 3).$$

- (2) *For each case, the set of first coefficients, or the initial values at the origin are given as follows:*

- (i) *If  $(-e_1, e_2) = (-\frac{1}{4}\lambda_i, \frac{1}{4}\lambda_k)$  ( $k \neq i$ ),*

$$(y_1^{e_1} y_2^{-e_2} G)(0, 0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ i.e., } (y_1^{e_1} y_2^{-e_2} \Phi_0)(0, 0) = 1, \text{ and } (y_1^{e_1} y_2^{-e_2} \Phi_j)(0, 0) =$$

0 for other  $j$ .

- (ii) *If  $(-e_1, e_2) = (-\frac{1}{4}\lambda_k, \frac{1}{4}\lambda_l)$  ( $k \neq i, l \neq i, k \neq l$ ),*

$$(y_1^{e_1} y_2^{-e_2} G)(0, 0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \text{ i.e., } (y_1^{e_1} y_2^{-e_2} \Phi_1)(0, 0) = 1, \text{ and } (y_1^{e_1} y_2^{-e_2} \Phi_j)(0, 0) =$$

0 for other  $j$ .

(iii) If  $(-e_1, e_2) = (-\frac{1}{4}\lambda_k, \frac{1}{4}\lambda_i)$  ( $k \neq i$ ),

$$(y_1^{e_1} y_2^{-e_2} G)(0, 0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \text{ i.e., } (y_1^{e_1} y_2^{-e_2} \Phi_2)(0, 0) = 1, \text{ and } (y_1^{e_1} y_2^{-e_2} \Phi_j)(0, 0) = 0 \text{ for other } j.$$

(3) We have the following recurrence relations for the coefficients:

- (i)  $(n_1 - e_1 + \frac{1}{4}\lambda_i)c_{0;n_1,n_2} + 2c_{1;n_1-1,n_2} = 0;$
- (ii)  $(n_1 - n_2 - e_1 - e_2 - \frac{1}{4}\lambda_i)c_{1;n_1,n_2} + 2c_{0;n_1-1,n_2} + 2c_{2;n_1,n_2-1} = 0;$
- (iii)  $(n_2 + e_2 - \frac{1}{4}\lambda_i)c_{2;n_1,n_2} - 2c_{1;n_1,n_2-1} = 0.$

## 5.5 Power series solutions at the origin

Now we can show the following formulae for the power series solutions.

**Theorem (5.5)** Assume that  $\frac{1}{4}(\lambda_k - \lambda_l) \notin \mathbf{Z}$ . Then we have the following.

(I) When  $\sigma = \sigma_1$  we have the following six independent solutions.

$$\begin{aligned} & {}^t(\Phi_0^{1,I}, \Phi_1^{1,I}, \Phi_2^{1,I}) = y_1^{-\frac{\lambda_1}{4}} y_2^{\frac{\lambda_2}{4}} \\ & \left( \begin{aligned} & \sum_{m_1, m_2 \geq 0} \frac{(\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2})_{m_1 + m_2}}{(\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2})_{m_1} (\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2})_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1} (\pi c_2 y_2)^{2m_2}}{m_1! m_2! (\frac{\lambda_3 - \lambda_1}{8} + \frac{1}{2})_{m_1} (\frac{\lambda_2 - \lambda_3}{8} + 1)_{m_2}} \\ & - \sum_{m_1, m_2 \geq 0} \frac{(\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2})_{m_1 + m_2 + 1}}{(\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2})_{m_1 + 1} (\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2})_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1 + 1} (\pi c_2 y_2)^{2m_2}}{m_1! m_2! (\frac{\lambda_3 - \lambda_1}{8} + \frac{1}{2})_{m_1 + 1} (\frac{\lambda_2 - \lambda_3}{8} + 1)_{m_2}} \\ & - \sum_{m_1, m_2 \geq 0} \frac{(\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2})_{m_1 + m_2 + 1}}{(\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2})_{m_1 + 1} (\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2})_{m_2 + 1}} \cdot \frac{(\pi c_1 y_1)^{2m_1 + 1} (\pi c_2 y_2)^{2m_2 + 1}}{m_1! m_2! (\frac{\lambda_3 - \lambda_1}{8} + \frac{1}{2})_{m_1 + 1} (\frac{\lambda_2 - \lambda_3}{8} + 1)_{m_2}} \end{aligned} \right), \\ & {}^t(\Phi_0^{1,III}, \Phi_1^{1,III}, \Phi_2^{1,III}) = y_1^{-\frac{\lambda_2}{4}} y_2^{\frac{\lambda_3}{4}} \\ & \left( \begin{aligned} & \sum_{m_1, m_2 \geq 0} \frac{(\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_1 + m_2}}{(\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_1} (\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1 + 1} (\pi c_2 y_2)^{2m_2}}{m_1! m_2! (\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_1 + 1} (\frac{\lambda_3 - \lambda_1}{8} + \frac{1}{2})_{m_2}} \\ & - \sum_{m_1, m_2 \geq 0} \frac{(\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_1 + m_2}}{(\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_1} (\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1} (\pi c_2 y_2)^{2m_2}}{m_1! m_2! (\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_1} (\frac{\lambda_3 - \lambda_1}{8} + \frac{1}{2})_{m_2}} \\ & - \sum_{m_1, m_2 \geq 0} \frac{(\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_1 + m_2}}{(\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_1} (\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1} (\pi c_2 y_2)^{2m_2 + 1}}{m_1! m_2! (\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_1} (\frac{\lambda_3 - \lambda_1}{8} + \frac{1}{2})_{m_2 + 1}} \end{aligned} \right), \\ & {}^t(\Phi_0^{1,V}, \Phi_1^{1,V}, \Phi_2^{1,V}) = y_1^{-\frac{\lambda_2}{4}} y_2^{\frac{\lambda_1}{4}} \\ & \left( \begin{aligned} & - \sum_{m_1, m_2 \geq 0} \frac{(\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_1 + m_2 + 1}}{(\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_1 + 1} (\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_2 + 1}} \cdot \frac{(\pi c_1 y_1)^{2m_1 + 1} (\pi c_2 y_2)^{2m_2 + 1}}{m_1! m_2! (\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_1} (\frac{\lambda_1 - \lambda_3}{8} + \frac{1}{2})_{m_2 + 1}} \\ & \cdot \sum_{m_1, m_2 \geq 0} \frac{(\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_1 + m_2 + 1}}{(\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_1} (\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_2 + 1}} \cdot \frac{(\pi c_1 y_1)^{2m_1} (\pi c_2 y_2)^{2m_2 + 1}}{m_1! m_2! (\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_1} (\frac{\lambda_1 - \lambda_3}{8} + \frac{1}{2})_{m_2 + 1}} \\ & \cdot \sum_{m_1, m_2 \geq 0} \frac{(\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_1 + m_2}}{(\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_1} (\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2})_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1} (\pi c_2 y_2)^{2m_2}}{m_1! m_2! (\frac{\lambda_3 - \lambda_2}{8} + 1)_{m_1} (\frac{\lambda_1 - \lambda_3}{8} + \frac{1}{2})_{m_2}} \end{aligned} \right), \end{aligned}$$

and other three solutions  $\Phi_i^{1,II}$ ,  $\Phi_i^{1,IV}$  and  $\Phi_i^{1,VI}$  are given by exchanging the role of  $\lambda_2$  and  $\lambda_3$  in the expression for  $\Phi_i^{1,I}$ ,  $\Phi_i^{1,III}$  and  $\Phi_i^{1,V}$ , respectively.

(II) When  $\sigma = \sigma_2$ , exchange  $\lambda_1$  and  $\lambda_2$  in the part (I).

(III) When  $\sigma = \sigma_3$ , exchange  $\lambda_1$  and  $\lambda_3$  in the part (I).

## 5.6 Integral representations of the secondary Whittaker functions

In this subsection, we rewrite the power series solutions of the previous subsection by integral expressions.

**Theorem (5.6)** (I) When  $\sigma = \sigma_1$  we have

$$\begin{aligned} & {}^t(\Phi_0^{1,I}, \Phi_1^{1,I}, \Phi_2^{1,I}) = (\pi c_1 y_1)^{\frac{\lambda_3}{8} + \frac{1}{2}} (\pi c_2 y_2)^{-\frac{\lambda_3}{8} + \frac{1}{2}} \\ & \cdot (2\pi\sqrt{-1})^{-1} \Gamma\left(\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda_3 - \lambda_1}{8} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda_2 - \lambda_3}{8} + 1\right) (\pi c_1)^{\frac{\lambda_1}{4}} (\pi c_2)^{-\frac{\lambda_2}{4}} \\ & \cdot \left( \begin{aligned} & \int_{|u|=1} I_{\frac{\lambda_2 - \lambda_1}{8} - \frac{1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_2 - \lambda_1}{8} - \frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u}) u^{-\frac{3}{16}\lambda_3 + \frac{1}{4}} \frac{du}{u} \\ & (-1) \int_{|u|=1} I_{\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_2 - \lambda_1}{8} - \frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u}) u^{-\frac{3}{16}\lambda_3 - \frac{1}{4}} (1+u)^{\frac{1}{2}} \frac{du}{u} \\ & (-1) \int_{|u|=1} I_{\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_2 - \lambda_1}{8} + \frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u}) u^{-\frac{3}{16}\lambda_3 - \frac{1}{4}} \frac{du}{u} \end{aligned} \right) \end{aligned}$$

for  $\operatorname{Re}(\frac{\lambda_2 - \lambda_1}{8}) > \frac{3}{2}$ ,

$$\begin{aligned} & {}^t(\Phi_0^{1,III}, \Phi_1^{1,III}, \Phi_2^{1,III}) = (\pi c_1 y_1)^{\frac{\lambda_1}{8}} (\pi c_2 y_2)^{-\frac{\lambda_1}{8}} \\ & \cdot (2\pi\sqrt{-1})^{-1} \Gamma\left(\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda_3 - \lambda_1}{8} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda_3 - \lambda_2}{8} + 1\right) (\pi c_1)^{\frac{\lambda_2}{4}} (\pi c_2)^{-\frac{\lambda_3}{4}} \\ & \cdot \left( \begin{aligned} & (\pi c_1 y_1) \int_{|u|=1} I_{\frac{\lambda_3 - \lambda_2}{8}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_3 - \lambda_2}{8}}(2\pi c_2 y_2 \sqrt{1+u}) u^{-\frac{3}{16}\lambda_3 - \frac{1}{2}} \frac{du}{u} \\ & (-1) \int_{|u|=1} \left[ \pi c_1 y_1 \sqrt{1+1/u} I_{\frac{\lambda_3 - \lambda_2}{8} - 1}(2\pi c_1 y_1 \sqrt{1+1/u}) + \left(\frac{\lambda_1 - \lambda_3}{8} + \frac{1}{2}\right) \right. \\ & \quad \left. \cdot I_{\frac{\lambda_3 - \lambda_2}{8}}(2\pi c_1 y_1 \sqrt{1+1/u}) \right] I_{\frac{\lambda_3 - \lambda_2}{8}}(2\pi c_2 y_2 \sqrt{1+u}) u^{-\frac{3}{16}\lambda_3 - \frac{1}{2}} \frac{du}{u} \\ & (-1) (\pi c_2 y_2) \int_{|u|=1} I_{\frac{\lambda_3 - \lambda_2}{8}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_3 - \lambda_2}{8}}(2\pi c_2 y_2 \sqrt{1+u}) u^{-\frac{3}{16}\lambda_3 + \frac{1}{2}} \frac{du}{u} \end{aligned} \right) \end{aligned}$$

for  $\operatorname{Re}(\frac{\lambda_3 - \lambda_2}{8}) > 1$ ,

$$\begin{aligned} & {}^t(\Phi_0^{1,V}, \Phi_1^{1,V}, \Phi_2^{1,V}) = (\pi c_1 y_1)^{\frac{\lambda_3}{8} + \frac{1}{2}} (\pi c_2 y_2)^{-\frac{\lambda_3}{8} + \frac{1}{2}} \\ & \cdot (2\pi\sqrt{-1})^{-1} \Gamma\left(\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda_1 - \lambda_3}{8} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda_3 - \lambda_2}{8} + 1\right) (\pi c_1)^{\frac{\lambda_2}{4}} (\pi c_2)^{-\frac{\lambda_1}{4}} \\ & \cdot \left( \begin{aligned} & (-1) \int_{|u|=1} I_{\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u}) u^{-\frac{3}{16}\lambda_3 + \frac{1}{4}} \frac{du}{u} \\ & \int_{|u|=1} I_{\frac{\lambda_1 - \lambda_2}{8} - \frac{1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_1 - \lambda_2}{8} + \frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u}) u^{-\frac{3}{16}\lambda_3 - \frac{1}{4}} (1+u)^{\frac{1}{2}} \frac{du}{u} \\ & \int_{|u|=1} I_{\frac{\lambda_1 - \lambda_2}{8} - \frac{1}{2}}(2\pi c_1 y_1 \sqrt{1+1/u}) I_{\frac{\lambda_1 - \lambda_2}{8} - \frac{1}{2}}(2\pi c_2 y_2 \sqrt{1+u}) u^{-\frac{3}{16}\lambda_3 - \frac{1}{4}} \frac{du}{u} \end{aligned} \right) \end{aligned}$$

for  $\operatorname{Re}(\frac{\lambda_1 - \lambda_2}{8}) > \frac{3}{2}$ .

To have the integral expression for  $\Phi_i^{1,II}, \Phi_i^{1,IV}$  and  $\Phi_i^{1,VI}$ , we have to exchange the role of  $\lambda_2$  and  $\lambda_3$  in the expression for  $\Phi_i^{1,I}, \Phi_i^{1,III}$  and  $\Phi_i^{1,V}$ , respectively.

(II) When  $\sigma = \sigma_2$ , exchange  $\lambda_1$  and  $\lambda_2$  in (I).

(III) When  $\sigma = \sigma_3$ , exchange  $\lambda_1$  and  $\lambda_3$  in (I).

## 6 Evaluation of Jacquet integrals

We give explicit descriptions of Jacquet integrals for non-spherical principal series Whittaker functions here. These are similar to the class one case ([12]).

### 6.1 Jacquet integrals

Let us denote by  $g = n(g)a(g)k(g)$  the Iwasawa decomposition of  $g \in G$ . We define *Jacquet integral*  $J_{ij}$  for  $\sigma_i \in \widehat{M}$  ( $1 \leq i, j \leq 3$ ) as

$$J_{ij}(g) = \int_N \psi(n)^{-1} a(s_0^{-1}ng) s_{ij}(k(s_0^{-1}ng)) dn$$

for  $1 \leq j \leq 3$ . Here

$$s_0 = \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}$$

the longest element in the Weyl group of  $SL(3, \mathbf{R})$  and  $s_{ij}(k)$  is the element of the tautological representation of  $K$  (cf. [4, (7.1)]).

Since

$$v_0 = \sqrt{-1}(s_{i2} - \sqrt{-1}s_{i3}), \quad v_1 = s_{i1}, \quad v_2 = \sqrt{-1}(s_{i2} + \sqrt{-1}s_{i3})$$

(§3.2.2) and

$$\Phi_0 = G_1, \quad 2\Phi_1 = G_0 + G_2, \quad 2\Phi_2 = G_0 - G_2,$$

(§5.2) the vector of integrals  ${}^t(J_{i1}, \sqrt{-1}J_{i2}, J_{i3})$  has the same  $K$ -type as  ${}^t(\Phi_0, \Phi_1, \Phi_2)$ .

For an element  $a \in A$ , we use the coordinates  $(y_1, y_2) = (a_1/a_2, a_1a_2^2)$ . In the Iwasawa decomposition of the element  $s_0^{-1}na$  its  $A$ -part  $a(s_0^{-1}na)$  is given by

$$a(s_0^{-1}na) = \left( \frac{y_1^{\frac{1}{3}} y_2^{\frac{2}{3}}}{\sqrt{\Delta_1}}, \left( \frac{y_2}{y_1} \right)^{\frac{1}{3}} \sqrt{\frac{\Delta_1}{\Delta_2}} \right)$$

with

$$\Delta_1 = y_1^2 y_2^2 + y_1^2 n_2^2 + (n_1 n_2 - n_3)^2, \quad \Delta_2 = y_1^2 y_2^2 + y_2^2 n_1^2 + n_3^2.$$

Under the symbol above

$$J_{ij}(y) = y_1^{(2\nu_1 - \nu_2)/3 + 1} y_2^{(\nu_1 + \nu_2)/3 + 1} \cdot \int_{\mathbf{R}^3} \Delta_1^{(\nu_2 - \nu_1 - 1)/2} \Delta_2^{(-\nu_2 - 1)/2} k_{ij} \exp(-2\pi\sqrt{-1}(c_1 n_1 + c_2 n_2)) dn_1 dn_2 dn_3.$$

Here  $(k_{ij})_{1 \leq i, j \leq 3} = k(s_0^{-1}na)$ .

## 6.2 Integral representations of Jacquet integrals

To write down our results, we use the following notation.

**Notation.**

$$K(\alpha, \beta, \gamma, \delta; y) := 4\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}(y_1 y_2)(\pi|c_1|y_1)^{\frac{\lambda_2}{8}}(\pi|c_2|y_2)^{-\frac{\lambda_2}{8}} \\ \cdot \int_0^\infty K_{\frac{\lambda_3-\lambda_1}{8}+\alpha}(2\pi|c_1|y_1\sqrt{1+1/v})K_{\frac{\lambda_3-\lambda_1}{8}+\beta}(2\pi|c_2|y_2\sqrt{1+v})v^{-\frac{3}{16}\lambda_2+\gamma}(1+v)^\delta \frac{dv}{v}$$

with  $K_\nu(z)$  the  $K$ -Bessel function.

### 6.2.1 The case of the class one principal series

In the case of class one, the Jacquet integral  $J_0(y)$  is

**Theorem (6.2)** ([12]) *For  $\operatorname{Re}(\lambda_2 - \lambda_1) > 0$ ,  $\operatorname{Re}(\lambda_3 - \lambda_2) > 0$ ,*

$$J_0(y) = \frac{1}{\Gamma(\frac{\lambda_2-\lambda_1}{8} + \frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8} + \frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8} + \frac{1}{2})} K(0, 0, 0, 0; y).$$

### 6.2.2 The case of the non-spherical principal series

**Theorem (6.3)** *For  $\operatorname{Re}(\lambda_2 - \lambda_1) > 0$ ,  $\operatorname{Re}(\lambda_3 - \lambda_2) > 0$ , the Jacquet integrals  $J_{ij}$  can be written as follows.*

$$\begin{pmatrix} J_{11}(y) \\ J_{12}(y) \\ J_{13}(y) \end{pmatrix} = \frac{(\pi|c_1|)^{\frac{1}{2}}(\pi|c_2|)^{\frac{1}{2}}(\pi|c_1|y_1)^{\frac{1}{2}}(\pi|c_2|y_2)^{\frac{1}{2}}}{\Gamma(\frac{\lambda_2-\lambda_1}{8} + 1)\Gamma(\frac{\lambda_3-\lambda_2}{8} + \frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8} + 1)} \cdot \begin{pmatrix} \varepsilon_1 \varepsilon_2 K(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, 0; y) \\ -\sqrt{-1} \varepsilon_2 K(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{2}; y) \\ -K(\frac{1}{2}, \frac{1}{2}, -\frac{1}{4}, 0; y) \end{pmatrix},$$

$$\begin{pmatrix} J_{21}(y) \\ J_{22}(y) \\ J_{23}(y) \end{pmatrix} = \frac{1}{\Gamma(\frac{\lambda_2-\lambda_1}{8} + 1)\Gamma(\frac{\lambda_3-\lambda_2}{8} + 1)\Gamma(\frac{\lambda_3-\lambda_1}{8} + \frac{1}{2})} \cdot \begin{pmatrix} -\sqrt{-1} \varepsilon_1 K(0, 0, -\frac{1}{2}, 0; y) \\ -K(0, 0, \frac{1}{2}, -1; y) \\ \sqrt{-1} \varepsilon_2 K(0, 0, \frac{1}{2}, 0; y) \end{pmatrix},$$

$$\begin{pmatrix} J_{31}(y) \\ J_{32}(y) \\ J_{33}(y) \end{pmatrix} = \frac{(\pi|c_1|)^{\frac{1}{2}}(\pi|c_2|)^{\frac{1}{2}}(\pi|c_1|y_1)^{\frac{1}{2}}(\pi|c_2|y_2)^{\frac{1}{2}}}{\Gamma(\frac{\lambda_2-\lambda_1}{8} + \frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8} + 1)\Gamma(\frac{\lambda_3-\lambda_1}{8} + 1)} \cdot \begin{pmatrix} -K(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, 0; y) \\ \sqrt{-1} \varepsilon_1 K(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{4}, \frac{1}{2}; y) \\ \varepsilon_1 \varepsilon_2 K(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{4}, 0; y) \end{pmatrix}.$$

Here  $\varepsilon_i$  ( $i = 1, 2$ ) means 1 if  $c_i > 0$  and  $-1$  if  $c_i < 0$ .

## 7 Integral expression of Mellin-Barnes type

As in [9], we consider the Mellin-Barnes integral expression for  $J_{ij}(y)$  to find linear relations between Jacquet integrals  $J_{ij}$  and power series solutions  $\Phi_k^{i,*}$ . We discuss only the non-spherical case.

**Lemma (7.1)** *For  $p, q \in \mathbf{C}$ ,*

$$\begin{aligned} & (\pi|c_1|y_1)^p(\pi|c_2|y_2)^q \int_0^\infty K_\alpha(2\pi|c_1|y_1\sqrt{1+1/v})K_\beta(2\pi|c_2|y_2\sqrt{1+v})v^\gamma(1+v)^\delta \frac{dv}{v} \\ &= \frac{1}{2^4(2\pi\sqrt{-1})^2} \int_{\rho_1-\sqrt{-1}\infty}^{\rho_1+\sqrt{-1}\infty} \int_{\rho_2-\sqrt{-1}\infty}^{\rho_2+\sqrt{-1}\infty} V_0(s_1, s_2)(\pi|c_1|y_1)^{-s_1}(\pi|c_2|y_2)^{-s_2} ds_1 ds_2, \end{aligned}$$

with

$$V_0(s_1, s_2) = \frac{\Gamma\left(\frac{s_1+p+\alpha}{2}\right)\Gamma\left(\frac{s_1+p-\alpha}{2}\right)\Gamma\left(\frac{s_1+p+2\gamma}{2}\right)\Gamma\left(\frac{s_2+q+\beta}{2}\right)\Gamma\left(\frac{s_2+q-\beta}{2}\right)\Gamma\left(\frac{s_2+q-2\gamma-2\delta}{2}\right)}{\Gamma\left(\frac{s_1+s_2+p+q}{2} - \delta\right)}.$$

Here the lines of integration are taken as to the right of all poles of the integrand.

**Proposition (7.2)** *Let*

$$\begin{aligned} & M(a_1, a_2, a_3; b_1, b_2, b_3; c; y) \\ &= \frac{1}{(2\pi\sqrt{-1})^2} \int_{\rho_1 - \sqrt{-1}\infty}^{\rho_1 + \sqrt{-1}\infty} \int_{\rho_2 - \sqrt{-1}\infty}^{\rho_2 + \sqrt{-1}\infty} V(s_1, s_2) (\pi|c_1|y_1)^{-s_1} (\pi|c_2|y_2)^{-s_2} ds_1 ds_2, \end{aligned}$$

with

$$V(s_1, s_2) = \frac{\Gamma\left(\frac{s_1+a_1-\lambda_1}{2}\right)\Gamma\left(\frac{s_1+a_2-\lambda_2}{2}\right)\Gamma\left(\frac{s_1+a_3-\lambda_3}{2}\right)\Gamma\left(\frac{s_1+b_1+\lambda_1}{2}\right)\Gamma\left(\frac{s_1+b_2+\lambda_2}{2}\right)\Gamma\left(\frac{s_1+b_3+\lambda_3}{2}\right)}{\Gamma\left(\frac{s_1+s_2+c}{2}\right)}.$$

Here the lines of integration are taken as to the right of all poles of the integrand. Then

$$\begin{aligned} \begin{pmatrix} J_{11}(y) \\ J_{12}(y) \\ J_{13}(y) \end{pmatrix} &= \frac{\pi^{\frac{3}{2}} (\pi|c_1|)^{\frac{\lambda_3}{4}} (\pi|c_2|)^{-\frac{\lambda_1}{4}} y_1 y_2}{4 \Gamma\left(\frac{\lambda_2-\lambda_1}{8} + 1\right) \Gamma\left(\frac{\lambda_3-\lambda_2}{8} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda_3-\lambda_1}{8} + 1\right)} \cdot \begin{pmatrix} \varepsilon_1 \varepsilon_2 M(0, 1, 1; 1, 0, 0; 1; y) \\ -\sqrt{-1} \varepsilon_2 M(1, 0, 0; 1, 0, 0; 0; y) \\ -M(1, 0, 0; 0, 1, 1; 1; y) \end{pmatrix}, \\ \begin{pmatrix} J_{21}(y) \\ J_{22}(y) \\ J_{23}(y) \end{pmatrix} &= \frac{\pi^{\frac{3}{2}} (\pi|c_1|)^{\frac{\lambda_3}{4}} (\pi|c_2|)^{-\frac{\lambda_1}{4}} y_1 y_2}{4 \Gamma\left(\frac{\lambda_2-\lambda_1}{8} + 1\right) \Gamma\left(\frac{\lambda_3-\lambda_2}{8} + 1\right) \Gamma\left(\frac{\lambda_3-\lambda_1}{8} + \frac{1}{2}\right)} \cdot \begin{pmatrix} -\sqrt{-1} \varepsilon_1 M(1, 0, 1; 0, 1, 0; 1; y) \\ -M(0, 1, 0; 0, 1, 0; 0; y) \\ \sqrt{-1} \varepsilon_2 M(0, 1, 0; 1, 0, 1; 1; y) \end{pmatrix}, \\ \begin{pmatrix} J_{31}(y) \\ J_{32}(y) \\ J_{33}(y) \end{pmatrix} &= \frac{\pi^{\frac{3}{2}} (\pi|c_1|)^{\frac{\lambda_3}{4}} (\pi|c_2|)^{-\frac{\lambda_1}{4}} y_1 y_2}{4 \Gamma\left(\frac{\lambda_2-\lambda_1}{8} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda_3-\lambda_2}{8} + 1\right) \Gamma\left(\frac{\lambda_3-\lambda_1}{8} + 1\right)} \cdot \begin{pmatrix} -M(1, 1, 0; 0, 0, 1; 1; y) \\ \sqrt{-1} \varepsilon_1 M(0, 0, 1; 0, 0, 1; 0; y) \\ \varepsilon_1 \varepsilon_2 M(0, 0, 1; 1, 1, 0; 1; y) \end{pmatrix}. \end{aligned}$$

*Proof.* It is obvious from Lemma (7.1).  $\square$

*Remark.* In view of this proposition, we can see the following symmetry for  $J_{ij}$  with respect to the parameter  $(\lambda_1, \lambda_2, \lambda_3)$ . This is natural but is not immediately seen from the formulae for  $J_{ij}$  (Theorem 6.3). We denote

$$\tilde{J}_i(\lambda_1, \lambda_2, \lambda_3) = \left( \frac{\pi^{\frac{3}{2}} (\pi|c_1|)^{\frac{\lambda_3}{4}} (\pi|c_2|)^{-\frac{\lambda_1}{4}} y_1 y_2}{\Gamma\left(\frac{\lambda_2-\lambda_1}{8} + p_i\right) \Gamma\left(\frac{\lambda_3-\lambda_2}{8} + q_i\right) \Gamma\left(\frac{\lambda_3-\lambda_1}{8} + r_i\right)} \right)^{-1} {}^t(J_{i1}(y), J_{i2}(y), J_{i3}(y))$$

with  $(p_i, q_i, r_i) = (1, \frac{1}{2}, 1)$  ( $i = 1$ ),  $(1, 1, \frac{1}{2})$  ( $i = 2$ ),  $(\frac{1}{2}, 1, 1)$  ( $i = 3$ ). Then

$$\tilde{J}_2(\lambda_1, \lambda_2, \lambda_3) = (-\sqrt{-1}) \varepsilon_2 \tilde{J}_1(\lambda_2, \lambda_1, \lambda_3), \quad \tilde{J}_3(\lambda_1, \lambda_2, \lambda_3) = -\varepsilon_1 \varepsilon_2 \tilde{J}_1(\lambda_3, \lambda_2, \lambda_1).$$

## 8 Relation between Jacquet integrals and power series solutions.

We omit the case of the class one principal series here, which is discussed by other people. In the same way of [9] for class one case, we move the lines of Mellin-Barnes



integral expression in Proposition (7.2) to the left and sum up the residues at the poles. Then we obtain the following.

**Theorem (8.2)**

$$\begin{aligned}
{}^t(J_{11}(y), J_{12}(y), J_{13}(y)) &= \frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)} \\
&\cdot \left[ \varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8})t(\Phi_0^{1,I}, \Phi_1^{1,I}, \Phi_2^{1,I}) \right. \\
&+ \varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_3}{8})t(\Phi_0^{1,II}, \Phi_1^{1,II}, \Phi_2^{1,II}) \\
&- \varepsilon_2(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_3}{8})t(\Phi_0^{1,III}, \Phi_1^{1,III}, \Phi_2^{1,III}) \\
&- \varepsilon_2(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8})t(\Phi_0^{1,IV}, \Phi_1^{1,IV}, \Phi_2^{1,IV}) \\
&- (\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_3}{8})t(\Phi_0^{1,V}, \Phi_1^{1,V}, \Phi_2^{1,V}) \\
&\left. - (\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8})t(\Phi_0^{1,VI}, \Phi_1^{1,VI}, \Phi_2^{1,VI}) \right], \\
{}^t(J_{21}(y), J_{22}(y), J_{23}(y)) &= \frac{-\sqrt{-1}\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})} \\
&\cdot \left[ \varepsilon_1(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8})t(\Phi_0^{2,I}, \Phi_1^{2,I}, \Phi_2^{2,I}) \right. \\
&+ \varepsilon_1(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\frac{\lambda_2-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_3}{8})t(\Phi_0^{2,II}, \Phi_1^{2,II}, \Phi_2^{2,II}) \\
&- (\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_3}{8})t(\Phi_0^{2,III}, \Phi_1^{2,III}, \Phi_2^{2,III}) \\
&- (\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8})t(\Phi_0^{2,IV}, \Phi_1^{2,IV}, \Phi_2^{2,IV}) \\
&- \varepsilon_2(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_3}{8})t(\Phi_0^{2,V}, \Phi_1^{2,V}, \Phi_2^{2,V}) \\
&\left. - \varepsilon_2(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8})t(\Phi_0^{2,VI}, \Phi_1^{2,VI}, \Phi_2^{2,VI}) \right], \\
{}^t(J_{31}(y), J_{32}(y), J_{33}(y)) &= \frac{-\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)} \\
&\cdot \left[ (\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_1}{8})t(\Phi_0^{3,I}, \Phi_1^{3,I}, \Phi_2^{3,I}) \right. \\
&+ (\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_2}{8})t(\Phi_0^{3,II}, \Phi_1^{3,II}, \Phi_2^{3,II}) \\
&- \varepsilon_1(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_2}{8})t(\Phi_0^{3,III}, \Phi_1^{3,III}, \Phi_2^{3,III}) \\
&- \varepsilon_1(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\frac{\lambda_2-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_1}{8})t(\Phi_0^{3,IV}, \Phi_1^{3,IV}, \Phi_2^{3,IV}) \\
&- \varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_2}{8})t(\Phi_0^{3,V}, \Phi_1^{3,V}, \Phi_2^{3,V}) \\
&\left. - \varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\frac{\lambda_2-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})\Gamma(\frac{\lambda_2-\lambda_1}{8})t(\Phi_0^{3,VI}, \Phi_1^{3,VI}, \Phi_2^{3,VI}) \right].
\end{aligned}$$

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