On the fibered category of smooth maps and actions of diffeological groupoids

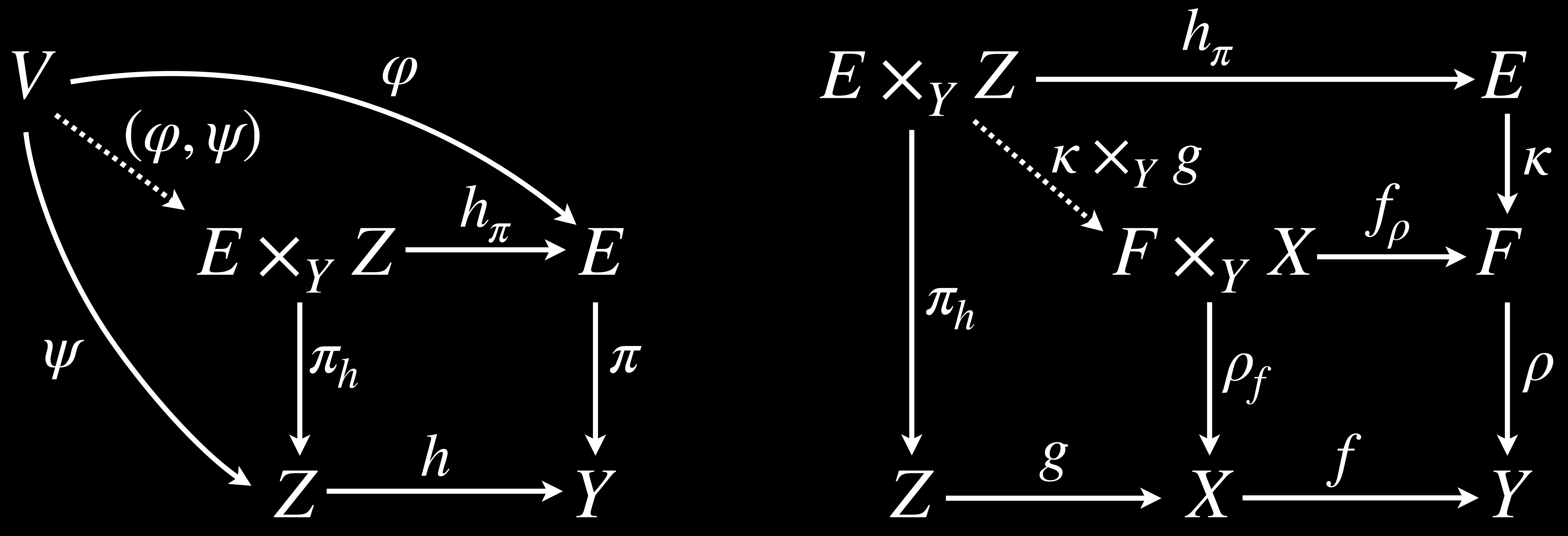
Atsushi Yamaguchi

Contents of this slide

§1. Fibered category of morphisms

Suppose that $Z \overset{\pi_h}{\leftarrow} E \times_Y Z \overset{h_\pi}{\rightarrow} E$ is a limit of a diagram $Z \overset{h}{\rightarrow} Y \overset{\pi}{\leftarrow} E$ in $\mathscr{E}$.
For morphisms $\varphi : V \to F$ and $\psi : V \to Z$ of $\mathscr{E}$ which satisfy $\pi\varphi = h\psi$, we denote by $(\varphi, \psi) : V \to F \times_Y Z$ the unique morphism that makes the following left diagram commute.

Suppose moreover that $X \overset{\rho_f}{\leftarrow} F \times_Y X \overset{f_\rho}{\rightarrow} F$ is a limit of a diagram $X \overset{f}{\rightarrow} Y \overset{\rho}{\leftarrow} F$.
If morphisms $\kappa : E \to F$ and $g : Z \to X$ of $\mathscr{E}$ satisfy $\rho\kappa h_\pi = fg\pi_h$, we denote $(\kappa h_\pi, g\pi_h)$ by $\kappa \times_Y g$.



Let Δ^1 be a category given by $\mathrm{Ob}\Delta^1 = \{0,1\}$ and $\mathrm{Mor}\,\Delta^1 = \{id_0, id_1, 0 \to 1\}$.

For a category $\mathscr{E}$, let us denote by $\mathscr{E}^{(2)}$ the category of functors from Δ^1 to $\mathscr{E}$ and natural transformations. Then, an object of $\mathscr{E}^{(2)}$ is identified with a morphism $\boldsymbol{E} = (E \overset{\pi}{\to} X)$ of $\mathscr{E}$ and a morphism from $\boldsymbol{E} = (E \overset{\pi}{\to} X)$ to $\boldsymbol{F} = (F \overset{\rho}{\to} Y)$ of $\mathscr{E}^{(2)}$ is identified with a pair $\langle \varphi, f \rangle$ of morphisms $\varphi : E \to F$ and $f : X \to Y$ of $\mathscr{E}$ which make the following diagram commute.

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} \\ X & \xrightarrow{f} & Y \end{array}$$

Define a functor $p : \mathscr{E}^{(2)} \to \mathscr{E}$ by $p(\boldsymbol{E}) = X$ if $\boldsymbol{E} = (E \overset{\pi}{\to} X)$ and $p(\langle \varphi, f \rangle) = f$.

For an object X of $\mathscr{E}$, we denote by $\mathscr{E}^{(2)}_X$ a subcategory of $\mathscr{E}^{(2)}$ given by

$$\mathrm{Ob}\mathscr{E}^{(2)}_X = \{\boldsymbol{E} \in \mathrm{Ob}\mathscr{E}^{(2)} \mid p(\boldsymbol{E}) = X\} \text{ and } \mathrm{Mor}\mathscr{E}^{(2)}_X = \{\boldsymbol{\varphi} \in \mathrm{Mor}\mathscr{E}^{(2)} \mid p(\boldsymbol{\varphi}) = id_X\}.$$

Proposition 1.1

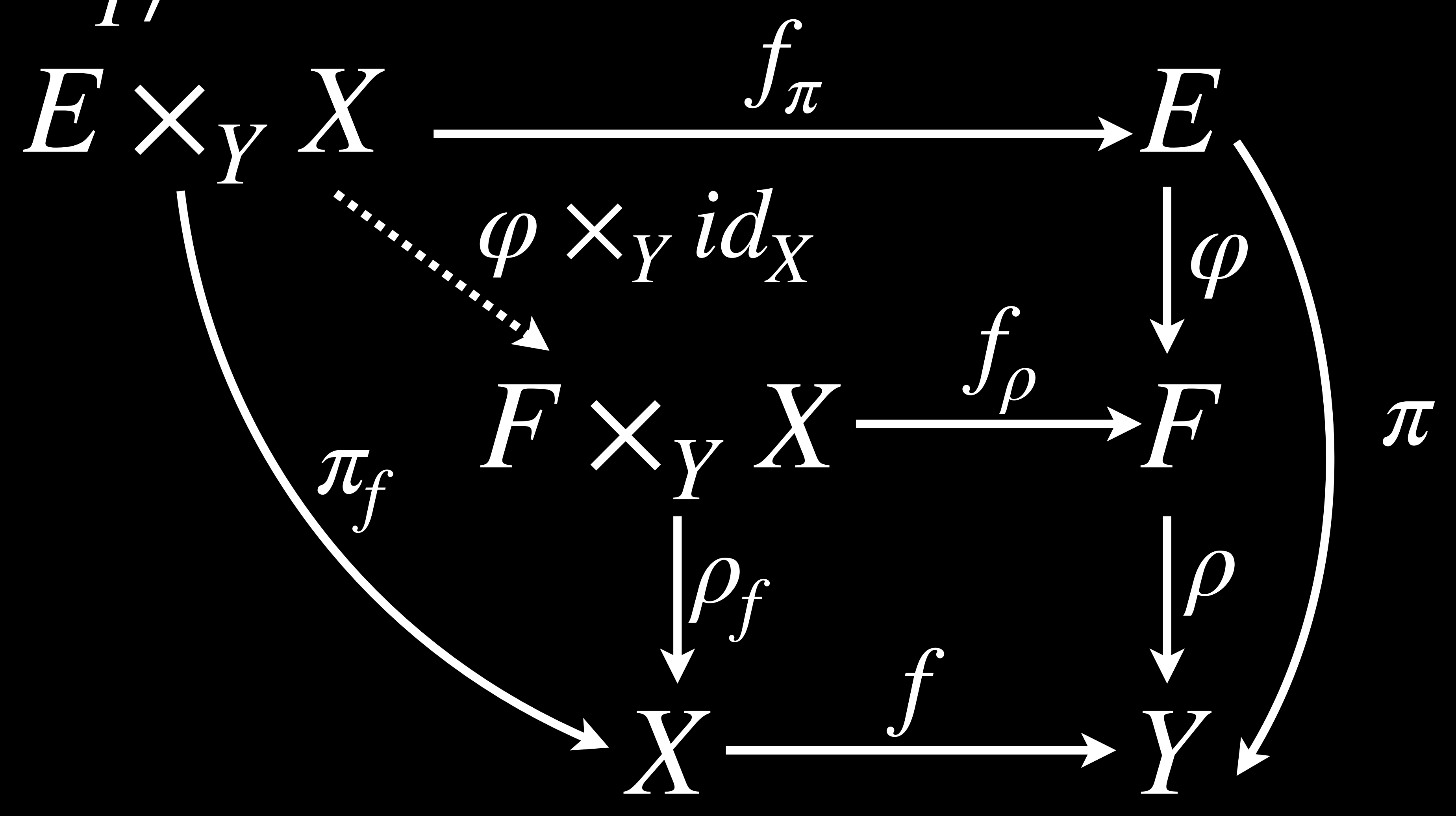
If $\mathscr{E}$ is a category with finite limits, $p:\mathscr{E}^{(2)}\to\mathscr{E}$ is a fibered category.

In fact, for a morphism $f:X\to Y$ of $\mathscr{E}$, the inverse image functor $f^*:\mathscr{E}^{(2)}_Y\to\mathscr{E}^{(2)}_X$ is given as follows.

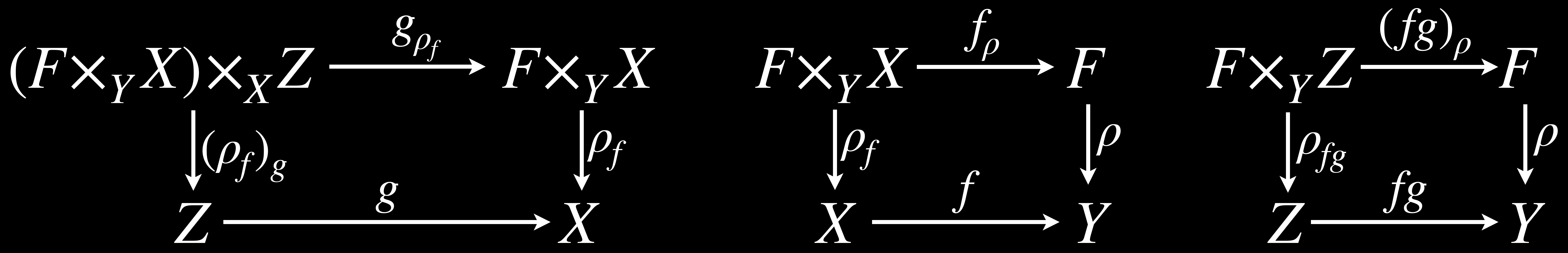
For an object $\boldsymbol{F}=(F\overset{\rho}{\to}Y)$ of $\mathscr{E}^{(2)}_Y$, let $X\overset{\rho_f}{\leftarrow}F\times_Y X\overset{f_\rho}{\to}F$ be a limit of a diagram $X\overset{f}{\to}Y\overset{\rho}{\leftarrow}F$ and set $f^*(\boldsymbol{F})=\left(F\times_Y X\overset{\rho_f}{\to}X\right)$.

The cartesian morphism $\alpha_f(\boldsymbol{F}):f^*(\boldsymbol{F})\to\boldsymbol{F}$ is given by $\alpha_f(\boldsymbol{F})=\langle f_\rho,f\rangle$.

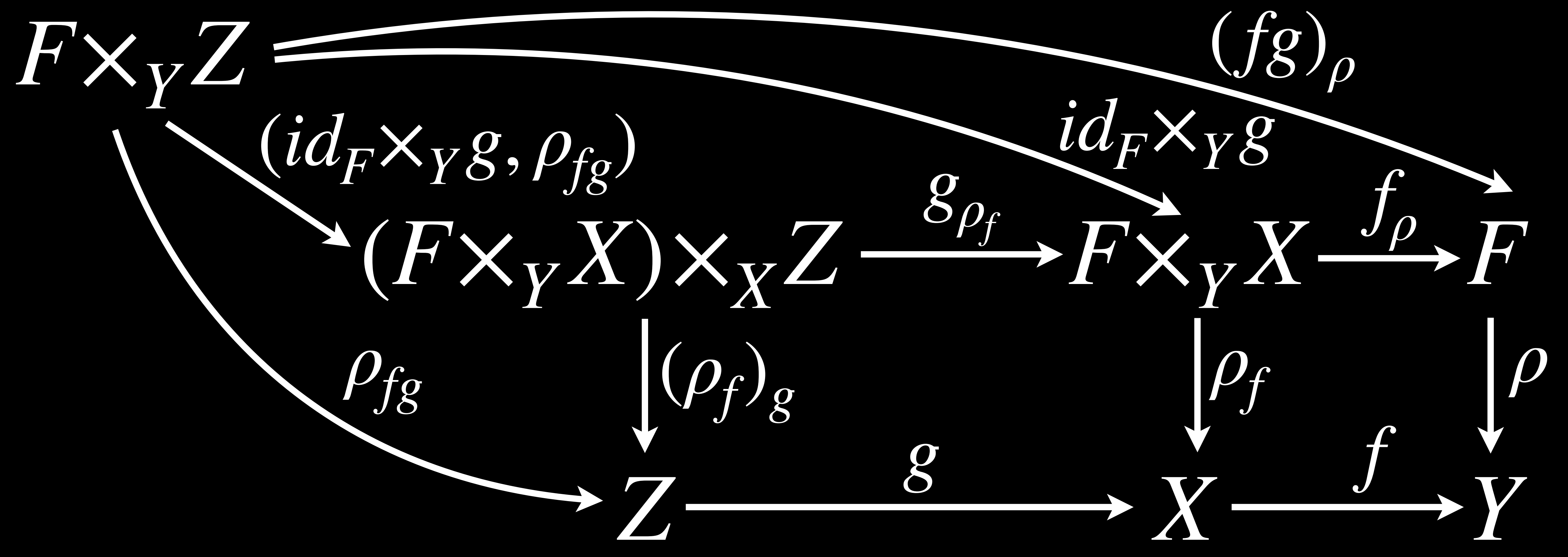
For a morphism $\boldsymbol{\varphi}:\boldsymbol{E}\to\boldsymbol{F}$ of $\mathscr{E}^{(2)}_Y$, we set $f^*(\boldsymbol{\varphi})=\langle\varphi\times_Y id_X, id_X\rangle$ if $\boldsymbol{E}=(E\overset{\pi}{\to}Y)$, $\boldsymbol{F}=(F\overset{\rho}{\to}Y)$ and $\boldsymbol{\varphi}=\langle\varphi, id_Y\rangle$.

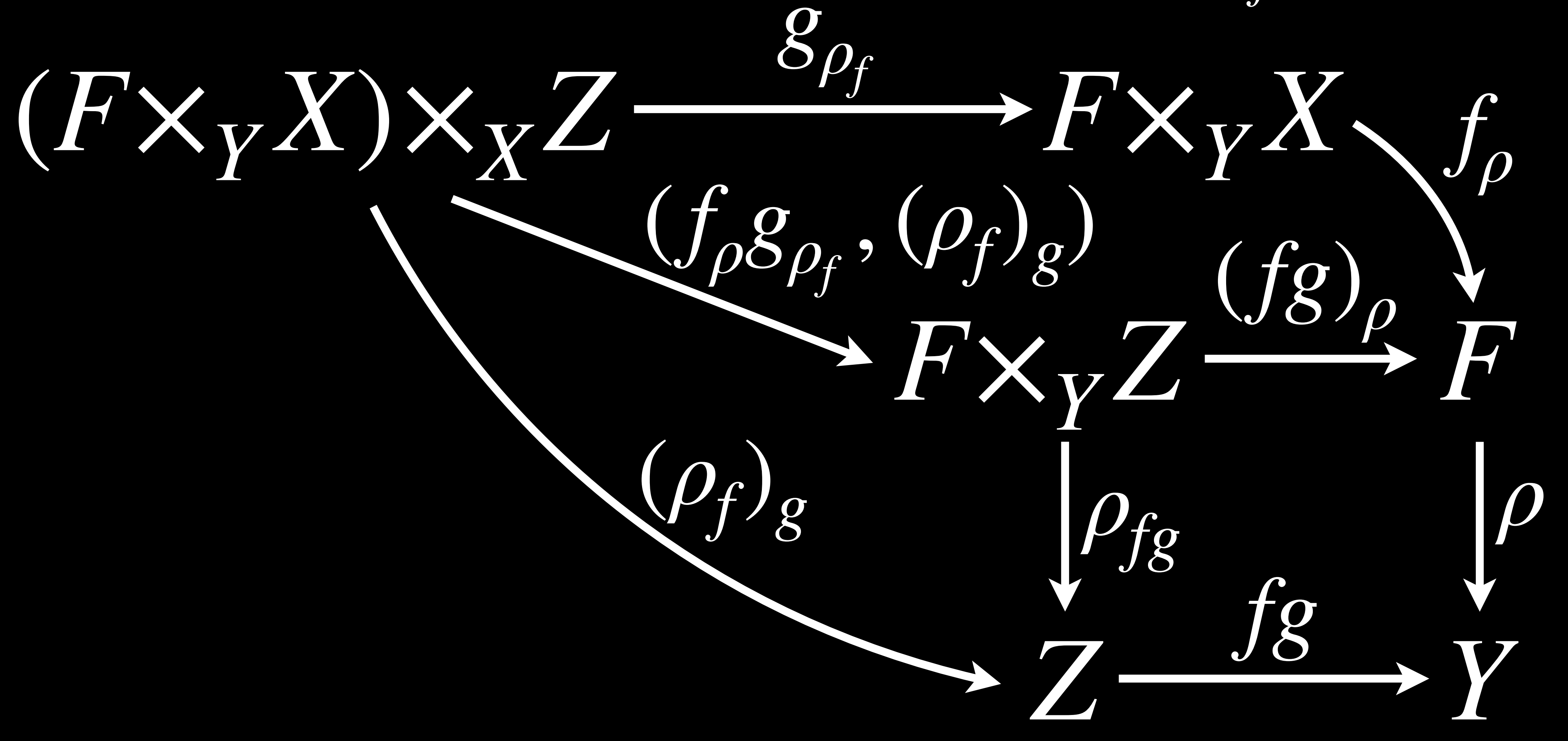


For morphisms $f:X\to Y$, $g:Z\to X$ of $\mathscr{E}$ and an object $\boldsymbol{F}=(F\xrightarrow{\rho}Y)$ of $\mathscr{E}^{(2)}_Y$, consider the following cartesian squares.



Since the outer rectangle of the following left diagram is cartesian, $(id_F\times_Y g,\rho_{fg}):F\times_Y Z\to(F\times_Y X)\times_X Z$ is an isomorphism whose inverse is $(f_\rho g_{\rho_f},(\rho_f)_g)$.





Define $c_{f,g}(\boldsymbol{F}):(fg)^*(\boldsymbol{F})\to g^*(f^*(\boldsymbol{F}))$ by $c_{f,g}(\boldsymbol{F})=\langle(id_F\times_Y g,\rho_{fg}),id_Z\rangle$ which is an isomorphism of $\mathscr{E}^{(2)}_Z$. Hence $p:\mathscr{E}^{(2)}\to\mathscr{E}$ is a fibered category.

From now on, we assume that $\mathscr{E}$ is a category with finite limits.

For morphisms $f:X\to Y$, $g:X\to Z$, $k:V\to X$ of $\mathscr{E}$ and object $\boldsymbol{E}$, $\boldsymbol{F}$ of $\mathscr{E}^{(2)}_Y$, $\mathscr{E}^{(2)}_Z$, respectively, define a map $k^\sharp_{\boldsymbol{E},\boldsymbol{F}}:\mathscr{E}^{(2)}_X(f^*(\boldsymbol{E}),g^*(\boldsymbol{F}))\to\mathscr{E}^{(2)}_V((fk)^*(\boldsymbol{E}),(gk)^*(\boldsymbol{F}))$ to be to be the following composition.

$$\mathscr{E}^{(2)}_X(f^*(\boldsymbol{E}),g^*(\boldsymbol{F}))\xrightarrow{k^*}\mathscr{E}^{(2)}_V(k^*(f^*(\boldsymbol{E})),k^*(g^*(\boldsymbol{F})))\xrightarrow{c_{f,k}(\boldsymbol{E})^*c_{g,k}(\boldsymbol{F})_*^{-1}}\mathscr{E}^{(2)}_V((fk)^*(\boldsymbol{E}),(gk)^*(\boldsymbol{F}))$$

Proposition 1.2

Let $f:X\to Y$, $g:X\to Z$, $h:X\to W$, $k:V\to X$, $j:U\to V$ be morphisms of $\mathscr{E}$ and $\boldsymbol{E}$, $\boldsymbol{F}$, $\boldsymbol{D}$ objects of $\mathscr{E}^{(2)}_Y$, $\mathscr{E}^{(2)}_Z$, $\mathscr{E}^{(2)}_W$, respectively.

(1) For $\varphi\in\mathscr{E}^{(2)}_X(f^*(\boldsymbol{E}),g^*(\boldsymbol{F}))$ and $\psi\in\mathscr{E}^{(2)}_X(g^*(\boldsymbol{F}),h^*(\boldsymbol{D}))$, we have

$$k^\sharp_{\boldsymbol{F},\boldsymbol{D}}(\psi)k^\sharp_{\boldsymbol{E},\boldsymbol{F}}(\varphi)=k^\sharp_{\boldsymbol{E},\boldsymbol{D}}(\psi\varphi).$$

(2) $\mathscr{E}^{(2)}_X(f^*(\boldsymbol{E}),g^*(\boldsymbol{F}))\xrightarrow{k^\sharp_{\boldsymbol{E},\boldsymbol{F}}}\mathscr{E}^{(2)}_V((fk)^*(\boldsymbol{E}),(gk)^*(\boldsymbol{F}))\xrightarrow{j^\sharp_{\boldsymbol{E},\boldsymbol{F}}}\mathscr{E}^{(2)}_U((fkj)^*(\boldsymbol{E}),(gkj)^*(\boldsymbol{F}))$

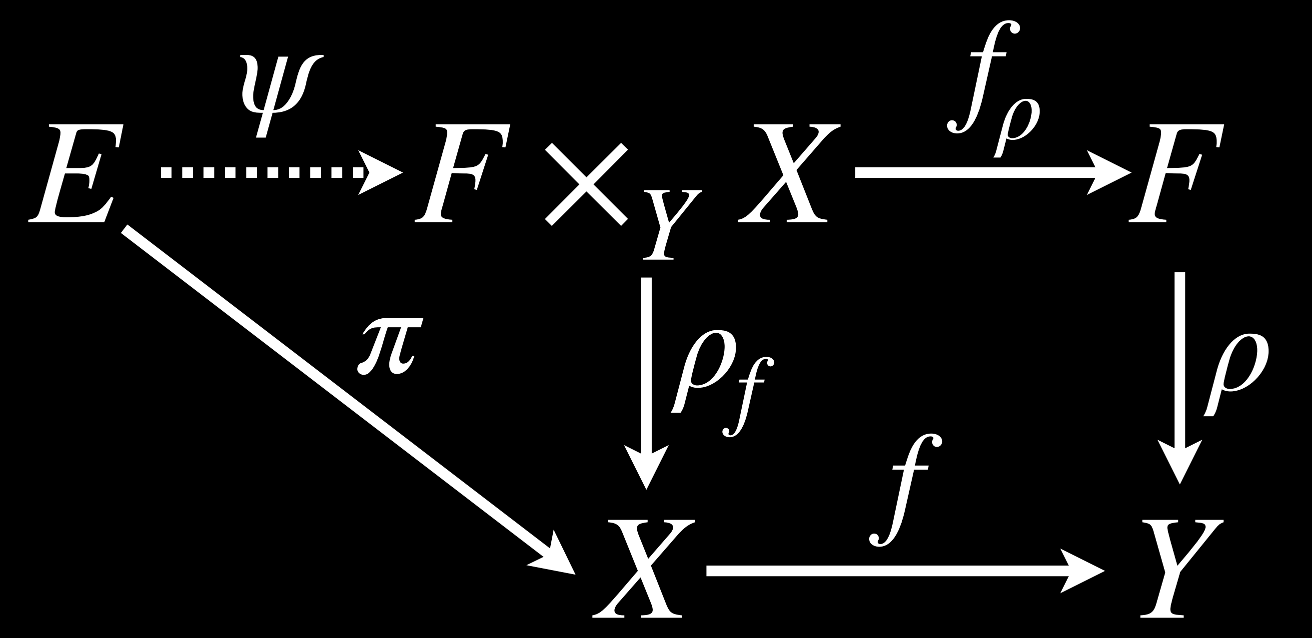
coincides with $(kj)^\sharp_{\boldsymbol{E},\boldsymbol{F}}:\mathscr{E}^{(2)}_X(f^*(\boldsymbol{E}),g^*(\boldsymbol{F}))\to\mathscr{E}^{(2)}_U((fkj)^*(\boldsymbol{E}),(gkj)^*(\boldsymbol{F}))$.

For a morphism $f: X \to Y$ of $\mathscr{E}$, define functor $f_*: \mathscr{E}_X^{(2)} \to \mathscr{E}_Y^{(2)}$ by $f_*(\boldsymbol{E}) = \left(E \xrightarrow{f\pi} Y\right)$ for an object $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ of $\mathscr{E}_X^{(2)}$, $f_*(\boldsymbol{\varphi}) = \langle \varphi, id_Y \rangle$ for a morphism $\boldsymbol{\varphi} = \langle \varphi, id_X \rangle$ of $\mathscr{E}_X^{(2)}$.

For an object $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ of $\mathscr{E}_X^{(2)}$ and an object $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ of $\mathscr{E}_Y^{(2)}$, we have

$$\mathscr{E}_Y^{(2)}(f_*(\boldsymbol{E}), \boldsymbol{F}) = \{\langle \varphi, id_Y \rangle \mid \varphi \in \mathscr{E}(E, F), \rho\varphi = f\pi\} \text{ and}$$

$$\mathscr{E}_Y^{(2)}(\boldsymbol{E}, f^*(\boldsymbol{F})) = \{\langle \psi, id_X \rangle \mid \psi \in \mathscr{E}(E, F \times_Y X), \rho_f \psi = \pi\}.$$



Define $\mathrm{Ad}^f_{\boldsymbol{E},\boldsymbol{F}}: \mathscr{E}_Y^{(2)}(f_*(\boldsymbol{E}), \boldsymbol{F}) \to \mathscr{E}_X^{(2)}(\boldsymbol{E}, f^*(\boldsymbol{F}))$ by $\mathrm{Ad}^f_{\boldsymbol{E},\boldsymbol{F}}(\langle \varphi, id_Y \rangle) = \langle (\varphi, \pi), id_X \rangle$.
Then the inverse of $\mathrm{Ad}^f_{\boldsymbol{E},\boldsymbol{F}}$ is given by $\left(\mathrm{Ad}^f_{\boldsymbol{E},\boldsymbol{F}}\right)^{-1}(\langle \psi, id_X \rangle) = \langle f_\rho \psi, id_Y \rangle$, hence $\mathrm{Ad}^f_{\boldsymbol{E},\boldsymbol{F}}$ is bijective. Thus we have the following result.

Proposition 1.3

$f_*: \mathscr{E}_X^{(2)} \to \mathscr{E}_Y^{(2)}$ is a left adjoint of $f^*: \mathscr{E}_Y^{(2)} \to \mathscr{E}_X^{(2)}$. Hence $p: \mathscr{E}^{(2)} \to \mathscr{E}$ is a bifibered category.

In (1.4) and (1.5) below, let $f\colon X\to Y$, $g\colon X\to Z$, $h\colon V\to Z$, $i\colon V\to W$ be morphisms of $\mathscr{E}$ and $X \overset{h_g}{\leftarrow} X\times_Z V \overset{g_h}{\rightarrow} V$ a limit of a diagram $X \overset{g}{\rightarrow} Z \overset{h}{\leftarrow} V$.

Proposition 1.4

For an object $\boldsymbol{F}=(F \overset{\rho}{\rightarrow} Y)$ of $\mathscr{E}^{(2)}_Y$, consider the following cartesian squares.

$$\begin{array}{ccc} F\times_Y X & \xrightarrow{f_\rho} & F \\ \downarrow{\scriptstyle \rho_f} & & \downarrow{\scriptstyle \rho} \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} F\times_Y (X\times_Z V) & \xrightarrow{(fh_g)_\rho} & F \\ \downarrow{\scriptstyle \rho_{fh_g}} & & \downarrow{\scriptstyle \rho} \\ X\times_Z V & \xrightarrow{fh_g} & Y \end{array} \qquad \begin{array}{ccc} (F\times_Y X)\times_Z V & \xrightarrow{h_{g\rho_f}} & F\times_Y X \\ \downarrow{\scriptstyle (g\rho_f)_h} & & \downarrow{\scriptstyle g\rho_f} \\ V & \xrightarrow{h} & Z \end{array}$$

Then, $\left(id_F\times_Z h_g,\, g_h\rho_{fh_g}\right)\colon F\times_Y(X\times_Z V)\to (F\times_Y X)\times_Z V$ is an isomorphism which satisfies $(g\rho_f)_h\left(id_F\times_Z h_g,\, g_h\rho_{fh_g}\right)=g_h\rho_{fh_g}$. Thus we have an isomorphism of $\mathscr{E}^{(2)}_W$

$$\theta_{f,g,h,i}(\boldsymbol{F})=\left\langle \left(id_F\times_Z h_g,\, g_h\rho_{fh_g}\right), id_W\right\rangle\colon (ig_h)_*((fh_g)^*(\boldsymbol{F}))\to i_*(h^*(g_*(f^*(\boldsymbol{F})))).$$

Proposition 1.5

We assume that the inverse image functors $f^*: \mathscr{E}_Y^{(2)} \to \mathscr{E}_X^{(2)}$, $h^*: \mathscr{E}_Z^{(2)} \to \mathscr{E}_V^{(2)}$ and $(fh_g)^*: \mathscr{E}_{X\times_Z V}^{(2)} \to \mathscr{E}_Y^{(2)}$ have right adjoints $f_!: \mathscr{E}_X^{(2)} \to \mathscr{E}_Y^{(2)}$, $h_!: \mathscr{E}_V^{(2)} \to \mathscr{E}_Z^{(2)}$ and $(fh_g)_!: \mathscr{E}_Y^{(2)} \to \mathscr{E}_{X\times_Z V}^{(2)}$, respectively. Let $\boldsymbol{\varepsilon}^f: f^*f_! \to id_{\mathscr{E}_X^{(2)}}$ and $\boldsymbol{\varepsilon}^h: h^*h_! \to id_{\mathscr{E}_V^{(2)}}$ be the counits of the adjunctions $f^* \dashv f_!$ and $h^* \dashv h_!$, respectively. For an object $\boldsymbol{E}$ of $\mathscr{E}_W^{(2)}$, let $\theta^{f,g,h,i}(\boldsymbol{E}): f_!(g^*(h_!(i^*(\boldsymbol{E})))) \to (fh_g)_!((ig_h)^*(\boldsymbol{E}))$ be the adjoint of the following composition with respect to the adjunction $(fh_g)^* \dashv (fh_g)_!$.

$$(fh_g)^*(f_!(g^*(h_!(i^*(\boldsymbol{E}))))) \xrightarrow{c_{f,h_g}(f_!(g^*(h_!(i^*(\boldsymbol{E})))))} h_g^*(f^*(f_!(g^*(h_!(i^*(\boldsymbol{E})))))) \xrightarrow{h_g^*(\boldsymbol{\varepsilon}^f_{g^*(h_!(i^*(\boldsymbol{E})))})}$$
$$h_g^*(g^*(h_!(i^*(\boldsymbol{E})))) \xrightarrow{c_{g,h_g}(h_!(i^*(\boldsymbol{E})))^{-1}} (gh_g)^*(h_!(i^*(\boldsymbol{E}))) = (hg_h)^*(h_!(i^*(\boldsymbol{E}))) \xrightarrow{c_{h,g_h}(h_!(i^*(\boldsymbol{E})))}$$
$$g_h^*(h^*(h_!(i^*(\boldsymbol{E})))) \xrightarrow{g_h^*(\boldsymbol{\varepsilon}^h_{i^*(\boldsymbol{E})})} g_h^*(i^*(\boldsymbol{E})) \xrightarrow{c_{i,g_h}(\boldsymbol{E})^{-1}} (ig_h)^*(\boldsymbol{E})$$

Then, $\theta^{f,g,h,i}(\boldsymbol{E})$ is an isomorphism of $\mathscr{E}_Y^{(2)}$.

§2. Fibered category of smooth maps

Let us denote by $\mathscr{D}$ the category of diffeological spaces. Since $\mathscr{D}$ has finite limits, $p : \mathscr{D}^{(2)} \to \mathscr{D}$ is a bifibered category by (1.1) and (1.3). The aim of this section is to show that inverse image functors of $p : \mathscr{D}^{(2)} \to \mathscr{D}$ have right adjoints.

For a diffeological spaces X and Y, we denote by Y^X the set of all smooth maps from X to Y.

Let $f : X \to Y$ be a smooth map and $(E \stackrel{\pi}{\to} X)$ be an object of $\mathscr{D}^{(2)}$. For $y \in Y$, we give $f^{-1}(y)$ the subset diffeology of X and denote by $i_y : f^{-1}(y) \to X$ the inclusion map. We define a subset $E(f;y)$ of $E^{f^{-1}(y)}$ as follows.

$$E(f;y) = \begin{cases} \{\alpha \in E^{f^{-1}(y)} \mid \pi\alpha = i_y\} & f^{-1}(y) \neq \emptyset \\ \emptyset & f^{-1}(y) = \emptyset \end{cases}$$

Let E^f be the disjoint union $\coprod_{y \in Y} E(f;y)$ of $E(f;y)$'s and define a map $f_!(\pi) : E^f \to Y$ by $f_!(\pi)(\alpha) = y$ if $\alpha \in E(f;y)$.

We consider the following left cartesian square and define a map $\varepsilon_\pi^f : E^f \times_Y X \to E$ by $\varepsilon_\pi^f(\alpha, x) = \alpha(x)$ if $\alpha \in E(f; y)$ and $x \in f^{-1}(y)$ for $y \in Y$.

Then, ε_π^f makes the following right diagram commute.

$$\begin{array}{ccc} E^f \times_Y X & \xrightarrow{f_{f_!(\pi)}} & E^f \\ \Big\downarrow {\scriptstyle f_!(\pi)_f} & & \Big\downarrow {\scriptstyle f_!(\pi)} \\ X & \xrightarrow{\quad f \quad} & Y \end{array} \qquad \begin{array}{ccc} E^f \times_Y X & \xrightarrow{\varepsilon_\pi^f} & E \\ \Big\downarrow {\scriptstyle f_!(\pi)_f} & \swarrow {\scriptstyle \pi} & \\ X & & \end{array}$$

Let $\mathscr{D}(E^f)$ be the set of all parameterizations $P : U \to E^f$ of E^f such that compositions $U \xrightarrow{P} E^f \xrightarrow{f_!(\pi)} Y$ and $U \times_Y W \xrightarrow{P \times_Y Q} E^f \times_Y X \xrightarrow{\varepsilon_\pi^f} E$ are smooth for any plot $Q : W \to X$ of X. Here, $U \xleftarrow{\mathrm{pr}_U} U \times_Y W \xrightarrow{\mathrm{pr}_W} W$ is a limit of a diagram $U \xrightarrow{f_!(\pi)P} Y \xleftarrow{fQ} W$ and we give $U \times_Y W$ the subset diffeology of $U \times W$.

Proposition 2.1

$\mathscr{D}(E^f)$ is a diffeology of E^f.

Thus we have a diffeological space E^f with diffeology $\mathscr{D}(E^f)$. Since $f_!(\pi): E^f \to Y$ is smooth by the definition of $\mathscr{D}(E^f)$, we have an object $\left(E^f \xrightarrow{f_!(\pi)} Y\right)$ of $\mathscr{D}^{(2)}_Y$.

Proposition 2.2

$\varepsilon^f_\pi : E^f \times_Y X \to E$ is smooth.

For a morphism $\langle \gamma, id_X \rangle : (E \xrightarrow{\pi} X) \to (F \xrightarrow{\chi} X)$ of $\mathscr{D}^{(2)}_X$, let $\gamma_y : E(f;y) \to F(f;y)$ be a map defined by $\gamma_y(\alpha) = \gamma\alpha$. We denote by $\gamma^f : E^f \to F^f$ the map $\coprod_{y \in Y} \gamma_y$ induced by γ_y's. Then, $f_!(\chi)\gamma^f = f_!(\pi)$ holds.

Proposition 2.3

$\gamma^f : E^f \to F^f$ makes the following diagram commute and it is smooth.

$$\begin{array}{ccc} E^f \times_Y X & \xrightarrow{\varepsilon^f_\pi} & E \\ \Big\downarrow{\scriptstyle \gamma^f \times_Y id_X} & & \Big\downarrow{\scriptstyle \gamma} \\ F^f \times_Y X & \xrightarrow{\varepsilon^f_\chi} & F \end{array}$$

We define a functor $f_! : \mathscr{D}_X^{(2)} \to \mathscr{D}_Y^{(2)}$ by putting $f_!(\boldsymbol{E}) = \left(E^f \xrightarrow{f_!(\pi)} Y\right)$ for an object $\boldsymbol{E}$ of $\mathscr{D}_X^{(2)}$ and $f_!(\boldsymbol{\gamma}) = \langle \gamma^f, id_Y \rangle : f_!(\boldsymbol{E}) \to f_!(\boldsymbol{F})$ for a morphism $\boldsymbol{\gamma} = \langle \gamma, id_X \rangle : \boldsymbol{E} \to \boldsymbol{F}$ of $\mathscr{D}_X^{(2)}$, where $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ and $\boldsymbol{F} = (F \xrightarrow{\chi} X)$. It follows from (2.3) that we have a natural transformation $\boldsymbol{\varepsilon}^f : f^* f_! \to id_{\mathscr{D}_X^{(2)}}$ defined by

$$\boldsymbol{\varepsilon}_{\boldsymbol{E}}^f = \langle \varepsilon_\pi^f, id_X \rangle : \left(E^f \times_Y X \xrightarrow{f_!(\pi)_f} X\right) \to (E \xrightarrow{\pi} X).$$

for an object $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ of $\mathscr{D}_X^{(2)}$.

Let $\boldsymbol{F}=(F\overset{\rho}{\to}Y)$ be an object of $\mathscr{D}^{(2)}_Y$ and consider the following cartesian square.

$$\begin{array}{ccc} F\times_Y X & \xrightarrow{f_\rho} & F \\ \big\downarrow{\scriptstyle \rho_f} & & \big\downarrow{\scriptstyle \rho} \\ X & \xrightarrow{f} & Y \end{array}$$

For $v\in F$, put $y=\rho(v)$ and denote by $c_v: f^{-1}(y)\to F$ the constant map with value v. Note that we have the following equalities.

$$\begin{aligned}(F\times_Y X)^f &= \coprod_{y\in Y}(F\times_Y X)(f;y) = \coprod_{y\in Y}\left\{\alpha\in(F\times_Y X)^{f^{-1}(y)}\,|\,\rho_f\,\alpha=i_y\right\}\\ &= \coprod_{y\in Y}\left\{(\lambda,i_y)\in(F\times_Y X)^{f^{-1}(y)}\,|\,\lambda: f^{-1}(y)\to F \textit{ satisfies } \rho\lambda=fi_y\right\}\\ &= \coprod_{y\in Y}\left\{(\lambda,i_y)\in(F\times_Y X)^{f^{-1}(y)}\,|\,\lambda: f^{-1}(y)\to F \textit{ satisfies } \lambda(f^{-1}(y))\subset\rho^{-1}(y)\right\}\end{aligned}$$

We define a map $\eta^f_\rho: F\to(F\times_Y X)^f$ by $\eta^f_\rho(v)=(c_v,i_y)$ if $\rho(v)=y$. Then, the right diagram is commutative.

$$\begin{array}{ccc} F & \xrightarrow{\eta^f_\rho} & (F\times_Y X)^f \\ & {\scriptstyle\rho}\searrow & \big\downarrow{\scriptstyle f_!(\rho_f)} \\ & & Y \end{array}$$

Lemma 2.4
The following compositions are both identity maps.

$$F\times_Y X \xrightarrow{\eta^f_\rho\times_Y id_X} (F\times_Y X)^f\times_Y X \xrightarrow{\varepsilon^f_{\rho^f}} F\times_Y X\,,\qquad E^f \xrightarrow{\eta^f_{f_!(\pi)}} (E^f\times_Y X)^f \xrightarrow{(\varepsilon^f_\pi)^f} E^f$$

By the definition of the diffeology of $(F\times_Y X)^f$ and (2.4), we have the following.

Proposition 2.5
$\eta^f_\rho : F\to (F\times_Y X)^f$ is smooth.

The following result is easily verified from the definitions of η^f_ρ, η^f_π and $(F\times_Y X)^f$.

Proposition 2.6
For a morphism $\langle\gamma, id_Y\rangle : (E\xrightarrow{\pi} Y)\to(F\xrightarrow{\rho} Y)$ of $\mathscr{D}^{(2)}_Y$, the right diagram is commutative.

$$\begin{array}{ccc} E & \xrightarrow{\eta^f_\pi} & (E\times_Y X)^f \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle(\gamma\times_Y id_X)^f} \\ F & \xrightarrow{\eta^f_\rho} & (F\times_Y X)^f \end{array}$$

It follows from (2.5) and (2.6) that we have a natural transformation $\boldsymbol{\eta}^f : id_{\mathscr{D}_Y^{(2)}} \to f_! f^*$ defined by $\boldsymbol{\eta}^f_{\boldsymbol{F}} = \langle \eta^f_\rho, id_Y \rangle : (F \xrightarrow{\rho} Y) \to \left((F \times_Y X)^f \xrightarrow{f_!(\rho_f)} Y \right)$ for an object $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ of $\mathscr{D}_Y^{(2)}$.

For an object $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ of $\mathscr{D}_Y^{(2)}$ and an object $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ of $\mathscr{D}_X^{(2)}$, since

$$f^*(\boldsymbol{F}) \xrightarrow{f^*(\boldsymbol{\eta}^f_{\boldsymbol{F}})} f^* f_! f^*(\boldsymbol{F}) \xrightarrow{\boldsymbol{\varepsilon}^f_{f^*(\boldsymbol{F})}} f^*(\boldsymbol{F}), \quad f_!(\boldsymbol{E}) \xrightarrow{\boldsymbol{\eta}^f_{f_!(\boldsymbol{E})}} f_! f^* f_!(\boldsymbol{E}) \xrightarrow{f_!(\boldsymbol{\varepsilon}^f_{\boldsymbol{E}})} f_!(\boldsymbol{E})$$

are both identity morphisms by (2.4), we have the following result.

Theorem 2.7

$f_!$ is a right adjoint of f^*.

§3. Representations of groupoids

Let $\mathscr{E}$ be a category with finite limits.

Definition 3.1

A groupoid $G=(G_0, G_1; \sigma, \tau, \varepsilon, \mu, \iota)$ in $\mathscr{E}$ consists of the following data.

(1) Two objects G_0, G_1 of $\mathscr{E}$.

(2) Five morphisms $\sigma, \tau : G_1 \to G_0$, $\varepsilon : G_0 \to G_1$, $\mu : G_1 \times_{G_0} G_1 \to G_1$, $\iota : G_1 \to G_1$ of $\mathscr{E}$, where $G_1 \xleftarrow{\mathrm{pr}_1} G_1 \times_{G_0} G_1 \xrightarrow{\mathrm{pr}_2} G_1$ is a limit of a diagram $G_1 \xrightarrow{\tau} G_0 \xleftarrow{\sigma} G_1$, such that $\sigma\varepsilon = \tau\varepsilon = id_{G_0}$, $\sigma\iota = \tau$, $\tau\iota = \sigma$ and the following diagrams commute.

$$\begin{array}{ccccc} G_1 & \xleftarrow{\mathrm{pr}_1} & G_1\times_{G_0}G_1 & \xrightarrow{\mathrm{pr}_2} & G_1 \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\tau} \\ G_0 & \xleftarrow{\sigma} & G_1 & \xrightarrow{\tau} & G_0 \end{array}
\qquad
\begin{array}{ccc} G_1\times_{G_0}G_1\times_{G_0}G_1 & \xrightarrow{\mu\times_{G_0} id_{G_1}} & G_1\times_{G_0}G_1 \\ \downarrow{\scriptstyle id_{G_1}\times_{G_0}\mu} & & \downarrow{\scriptstyle\mu} \\ G_1\times_{G_0}G_1 & \xrightarrow{\mu} & G_1 \end{array}$$

$$\begin{array}{ccccc} & \nearrow^{(id_{G_1}, \varepsilon\tau)} & G_1\times_{G_0}G_1 & \nwarrow^{(\varepsilon\sigma, id_{G_1})} & \\ & & \downarrow{\scriptstyle\mu} & & \\ G_1 & \xrightarrow{id_{G_1}} & G_1 & \xleftarrow{id_{G_1}} & G_1 \end{array}
\qquad
\begin{array}{ccccc} G_1 & \xrightarrow{(id_{G_1}, \iota)} & G_1\times_{G_0}G_1 & \xleftarrow{(\iota, id_{G_1})} & G_1 \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\tau} \\ G_0 & \xrightarrow{\varepsilon} & G_1 & \xleftarrow{\varepsilon} & G_0 \end{array}$$

Here $G_1 \times_{G_0} G_1 \times_{G_0} G_1$ denotes a limit of $G_1 \xrightarrow{\tau} G_0 \xleftarrow{\sigma} G_1 \xrightarrow{\tau} G_0 \xleftarrow{\sigma} G_1$.

$$\begin{array}{ccccccccc} & & & & G_1 \times_{G_0} G_1 \times_{G_0} G_1 & & & & \\ & \swarrow^{\mathrm{pr}_1} & & & \downarrow \mathrm{pr}_2 & & & \searrow^{\mathrm{pr}_3} & \\ G_1 & \xrightarrow{\tau} & G_0 & \xleftarrow{\sigma} & G_1 & \xrightarrow{\tau} & G_0 & \xleftarrow{\sigma} & G_1 \end{array}$$

Definition 3.2

Let $\boldsymbol{G} = (G_0, G_1; \sigma, \tau, \varepsilon, \mu, \iota)$ and $\boldsymbol{H} = (H_0, H_1; \sigma', \tau', \varepsilon', \mu', \iota')$ be groupoids in $\mathscr{E}$. If morphisms $f_0 : G_0 \to G_0$ and $f_1 : G_1 \to G_1$ of $\mathscr{E}$ make the following diagrams commute, we call a pair (f_0, f_1) a morphism of groupoids from $\boldsymbol{G}$ to $\boldsymbol{H}$.

$$\begin{array}{ccccc} G_0 & \xleftarrow{\sigma} & G_1 & \xrightarrow{\tau} & G_0 \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_0 \\ H_0 & \xleftarrow{\sigma'} & H_1 & \xrightarrow{\tau'} & H_0 \end{array} \qquad \begin{array}{ccc} G_1 \times_{G_0} G_1 & \xrightarrow{\mu} & G_1 \\ \downarrow f_1 \times f_1 & & \downarrow f_1 \\ H_1 \times_{H_0} H_1 & \xrightarrow{\mu'} & H_1 \end{array} \qquad \begin{array}{ccc} G_1 & \xrightarrow{\varepsilon} & G_0 \\ \downarrow f_1 & & \downarrow f_0 \\ H_1 & \xrightarrow{\varepsilon'} & H_0 \end{array}$$

Here we denote $(f_1 \mathrm{pr}_1, f_1 \mathrm{pr}_2)$ by $f_1 \times f_1$.

We remark that the commutativity of the above diagrams implies $f_1 \iota = \iota' f_1$.

Notation 3.3

Let $f\colon X\to Y$ be a morphism of $\mathscr{E}$ and $\boldsymbol{E}=(E\xrightarrow{\pi}X)$, $\boldsymbol{F}=(F\xrightarrow{\rho}Y)$ objects of $\mathscr{E}_X^{(2)}$, $\mathscr{E}_Y^{(2)}$, respectively.

(1) For $\boldsymbol{\varphi}=\langle\varphi, id_X\rangle\in\mathscr{E}_X^{(2)}(\boldsymbol{E},f^*(\boldsymbol{F}))$, we denote by $\hat{\boldsymbol{\varphi}}=\langle\hat{\varphi}, id_Y\rangle$ the image of $\boldsymbol{\varphi}$ by the bijection $\left(\mathrm{Ad}^f_{\boldsymbol{E},\boldsymbol{F}}\right)^{-1}:\mathscr{E}_X^{(2)}(\boldsymbol{E},f^*(\boldsymbol{F}))\to\mathscr{E}_Y^{(2)}(f_*(\boldsymbol{E}),\boldsymbol{F})$. Hence $\hat{\varphi}:E\to F$ is a composition $E\xrightarrow{\varphi}F\times_Y X\xrightarrow{f_\rho}F$, where $X\xleftarrow{\rho_f}F\times_Y X\xrightarrow{f_\rho}F$ is a limit of a diagram $X\xrightarrow{f}Y\xleftarrow{\rho}F$.

(2) Suppose that $f^*:\mathscr{E}_Y^{(2)}\to\mathscr{E}_X^{(2)}$ has a right adjoint $f_!:\mathscr{E}_X^{(2)}\to\mathscr{E}_Y^{(2)}$.
For $\boldsymbol{\psi}=\langle\psi, id_X\rangle\in\mathscr{E}_X^{(2)}(f^*(\boldsymbol{E}),\boldsymbol{F})$, we denote by $\check{\boldsymbol{\psi}}=\langle\check{\psi}, id_Y\rangle:\boldsymbol{E}\to f_!(\boldsymbol{F})$ the image of $\boldsymbol{\psi}$ by the bijection $\mathscr{E}_X^{(2)}(f^*(\boldsymbol{E}),\boldsymbol{F})\to\mathscr{E}_Y^{(2)}(\boldsymbol{E},f_!(\boldsymbol{F}))$. If $\boldsymbol{\eta}^f:id_{\mathscr{E}_Y^{(2)}}\to f_!f^*$ is the unit of this adjunction, we have $\check{\boldsymbol{\psi}}=f_!(\boldsymbol{\psi})\boldsymbol{\eta}^f_{\boldsymbol{E}}$.

Definition 3.4

Let $\boldsymbol{G}=(G_0,G_1;\sigma,\tau,\varepsilon,\mu,\iota)$ be a groupoid in $\mathscr{E}$ and $\boldsymbol{E}=(E\xrightarrow{\pi}G_0)$ an object of $\mathscr{E}^{(2)}_{G_0}$. If a morphism $\xi:\sigma^*(\boldsymbol{E})\to\tau^*(\boldsymbol{E})$ of $\mathscr{E}^{(2)}_{G_1}$ satisfies the following conditions (A) and (U), we call a pair $(\boldsymbol{E},\xi)$ a representation of $\boldsymbol{G}$ on $\boldsymbol{E}$.

(A) $\mu^\sharp_{\boldsymbol{E},\boldsymbol{E}}(\xi):(\sigma\mathrm{pr}_1)^*(\boldsymbol{E})=(\sigma\mu)^*(\boldsymbol{E})\to(\tau\mu)^*(\boldsymbol{E})=(\tau\mathrm{pr}_2)^*(\boldsymbol{E})$ coincides with a composition $(\sigma\mathrm{pr}_1)^*(\boldsymbol{E})\xrightarrow{(\mathrm{pr}_1)^\sharp_{\boldsymbol{E},\boldsymbol{E}}(\xi)}(\tau\mathrm{pr}_1)^*(\boldsymbol{E})=(\sigma\mathrm{pr}_2)^*(\boldsymbol{E})\xrightarrow{(\mathrm{pr}_2)^\sharp_{\boldsymbol{E},\boldsymbol{E}}(\xi)}(\tau\mathrm{pr}_2)^*(\boldsymbol{E})$.

(U) $\varepsilon^\sharp_{\boldsymbol{E},\boldsymbol{E}}(\xi):\boldsymbol{E}=(\sigma\varepsilon)^*(\boldsymbol{E})\to(\tau\varepsilon)^*(\boldsymbol{E})=\boldsymbol{E}$ coincides with the identity morphism of $\boldsymbol{E}$.

Let $(\boldsymbol{E},\xi)$ and $(\boldsymbol{F},\zeta)$ be representations of $\boldsymbol{G}$ on $\boldsymbol{E}$ and $\boldsymbol{F}$, respectively.

If a morphism $\varphi:\boldsymbol{E}\to\boldsymbol{F}$ of $\mathscr{E}^{(2)}_{G_0}$ makes the right diagram commute, we call φ a morphism of representations of $\boldsymbol{G}$.

$$\begin{array}{ccc} \sigma^*(\boldsymbol{E}) & \xrightarrow{\sigma^*(\varphi)} & \sigma^*(\boldsymbol{F}) \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \zeta} \\ \tau^*(\boldsymbol{E}) & \xrightarrow{\tau^*(\varphi)} & \tau^*(\boldsymbol{F}) \end{array}$$

We denote by $\mathrm{Rep}(\boldsymbol{G})$ the category consists of objects $(\boldsymbol{E},\xi)$ and morphisms of representations of $\boldsymbol{G}$.

For a groupoid $\boldsymbol{G}=(G_0, G_1; \sigma, \tau, \varepsilon, \mu, \iota)$ in $\mathscr{E}$ and an object $\boldsymbol{E}=(E \xrightarrow{\pi} G_0)$ of $\mathscr{E}^{(2)}_{G_0}$, let $G_1 \xleftarrow{\pi_\sigma} E \times^{\sigma}_{G_0} G_1 \xrightarrow{\sigma_\pi} E$ be a limit of $G_1 \xrightarrow{\sigma} G_0 \xleftarrow{\pi} E$ and $G_1 \xleftarrow{\pi_\tau} E \times^{\tau}_{G_0} G_1 \xrightarrow{\tau_\pi} E$ a limit of $G_1 \xrightarrow{\tau} G_0 \xleftarrow{\pi} E$. Then we have $\sigma^*(\boldsymbol{E})=(E \times^{\sigma}_{G_0} G_1 \xrightarrow{\pi_\sigma} G_1)$ and $\tau^*(\boldsymbol{E})=(E \times^{\tau}_{G_0} G_1 \xrightarrow{\pi_\tau} G_1)$.

Proposition 3.5

For a morphism $\boldsymbol{\xi}=\langle \xi, id_{G_1}\rangle : \sigma^*(\boldsymbol{E}) \to \tau^*(\boldsymbol{E})$ of $\mathscr{E}^{(2)}_{G_1}$, consider the morphism $\hat{\boldsymbol{\xi}}=\langle \hat{\xi}, id_{G_0}\rangle : \tau_*(\sigma^*(\boldsymbol{E})) \to \boldsymbol{E}$ of $\mathscr{E}^{(2)}_{G_0}$. Then $\boldsymbol{\xi}$ satisfies the conditions (A) and (U) of (3.3) if and only if $\hat{\xi} : E \times^{\sigma}_{G_0} G_1 \to E$ makes the following diagrams commute.

$$\begin{array}{ccccc}
E\times^{\sigma}_{G_0}(G_1\times_{G_0}G_1) & \xrightarrow{id_E\times_{G_0}\mu} & E\times^{\sigma}_{G_0}G_1 & & \\
\cong\Big\downarrow ((\mathrm{pr}_E,\, \mathrm{pr}_1\times_{G_0} id_E),\, id_E\times_{G_0}\mathrm{pr}_2) & & & \searrow^{\hat{\xi}} & \\
 & & & & E \\
 & & & \nearrow_{\hat{\xi}} & \\
(E\times^{\sigma}_{G_0}G_1)\times_{G_0}G_1 & \xrightarrow[\hat{\xi}\times_{G_0} id_{G_1}]{} & E\times^{\sigma}_{G_0}G_1 & &
\end{array}
\qquad
\begin{array}{ccc}
E & \xrightarrow{id_E} & E \\
\Big\downarrow (id_E,\, \varepsilon) & \nearrow & \\
E\times^{\sigma}_{G_0}G_1 & \xrightarrow{\hat{\xi}} & E
\end{array}$$

Thus we may call $\hat{\boldsymbol{\xi}}$ a $\boldsymbol{G}$-action on $\boldsymbol{E}$.

Proposition 3.6

Let $(\boldsymbol{E},\boldsymbol{\xi})$ and $(\boldsymbol{F},\boldsymbol{\zeta})$ be representations of $\boldsymbol{G}$ on $\boldsymbol{E}$ and $\boldsymbol{F}$, respectively and $\boldsymbol{\varphi}:\boldsymbol{E}\to\boldsymbol{F}$ a morphism of $\mathscr{E}^{(2)}_{G_0}$. We put $\boldsymbol{E}=(E\overset{\pi}{\to}G_0)$, $\boldsymbol{F}=(F\overset{\rho}{\to}G_0)$, $\boldsymbol{\xi}=\langle\xi,id_{G_1}\rangle$, $\boldsymbol{\zeta}=\langle\zeta,id_{G_1}\rangle$ and $\boldsymbol{\varphi}=\langle\varphi,id_{G_0}\rangle$. Consider morphisms $\hat{\boldsymbol{\xi}}=\langle\hat{\xi},id_{G_0}\rangle$ and $\hat{\boldsymbol{\zeta}}=\langle\hat{\zeta},id_{G_0}\rangle$ of $\mathscr{E}^{(2)}_{G_0}$. Then, $\boldsymbol{\varphi}$ is a morphism of representations of $\boldsymbol{G}$ if and only if the following diagram commutes.

$$\begin{array}{ccc} E\times^{\sigma}_{G_0}G_1 & \xrightarrow{\hat{\xi}} & E \\ \Big\downarrow{\scriptstyle \varphi\times_{G_0}id_{G_1}} & & \Big\downarrow{\scriptstyle \varphi} \\ F\times^{\sigma}_{G_0}G_1 & \xrightarrow{\hat{\zeta}} & F \end{array}$$

Proposition 3.7

Let $\boldsymbol{G}=(G_0, G_1; \sigma, \tau, \varepsilon, \mu, \iota)$ and $\boldsymbol{H}=(H_0, H_1; \sigma', \tau', \varepsilon', \mu', \iota')$ be groupoids in $\mathscr{E}$ and $\boldsymbol{f}=(f_0, f_1):\boldsymbol{H}\to\boldsymbol{G}$ a morphism of groupoids. For an object $(\boldsymbol{E}, \boldsymbol{\xi})$ of $\mathrm{Rep}(\boldsymbol{G})$, define a morphism $\boldsymbol{\xi}_{\boldsymbol{f}}:\sigma'^*(f_0^*(\boldsymbol{E}))\to\tau'^*(f_0^*(\boldsymbol{E}))$ to be the following composition.

$$\sigma'^*(f_0^*(\boldsymbol{E})) \xrightarrow{c_{f_0,\sigma'}(\boldsymbol{E})} (f_0\sigma')^*(\boldsymbol{E})=(\sigma f_1)^*(\boldsymbol{E}) \xrightarrow{(f_1)^\sharp_{\boldsymbol{E},\boldsymbol{E}}(\boldsymbol{\xi})} (\tau f_1)^*(\boldsymbol{E})=(f_0\tau')^*(\boldsymbol{E}) \xrightarrow{c_{f_0,\tau'}(\boldsymbol{E})^{-1}} \tau'^*(f_0^*(\boldsymbol{E}))$$

Then $(f_0^*(\boldsymbol{E}), \boldsymbol{\xi}_{\boldsymbol{f}})$ is a representation of $\boldsymbol{H}$ on $f_0^*(\boldsymbol{E})$.

Proposition 3.8

Let $\boldsymbol{f}=(f_0, f_1):\boldsymbol{H}\to\boldsymbol{G}$ be a morphism of groupoids in $\mathscr{E}$ and $\boldsymbol{\varphi}:(\boldsymbol{E}, \boldsymbol{\xi})\to(\boldsymbol{F}, \boldsymbol{\zeta})$ a morphism of $\mathrm{Rep}(\boldsymbol{G})$. Then $f_0^*(\boldsymbol{\varphi}):(f_0^*(\boldsymbol{E}), \boldsymbol{\xi}_{\boldsymbol{f}})\to(f_0^*(\boldsymbol{F}), \boldsymbol{\zeta}_{\boldsymbol{f}})$ is a morphism of $\mathrm{Rep}(\boldsymbol{H})$.

Definition 3.9

We call $(f_0^*(\boldsymbol{E}), \boldsymbol{\xi}_f)$ the restriction of $(\boldsymbol{E}, \boldsymbol{\xi})$ by $\boldsymbol{f}$.

Define a functor $\boldsymbol{f}^* : \mathrm{Rep}(\boldsymbol{G}) \to \mathrm{Rep}(\boldsymbol{H})$ by putting $\boldsymbol{f}^*((\boldsymbol{E}, \boldsymbol{\xi})) = (f_0^*(\boldsymbol{E}), \boldsymbol{\xi}_f)$ and $\boldsymbol{f}^*(\boldsymbol{\varphi}) = f_0^*(\boldsymbol{\varphi})$ for an object $(\boldsymbol{E}, \boldsymbol{\xi})$ and a morphism $\boldsymbol{\varphi}$ of $\mathrm{Rep}(\boldsymbol{G})$.

We call this functor the restriction functor by $\boldsymbol{f}$.

Proposition 3.10

For a representation $(\boldsymbol{E}, \boldsymbol{\xi})$ of $\boldsymbol{G}$ on $\boldsymbol{E}$, we put $\boldsymbol{E} = (E \xrightarrow{\pi} G_0)$ and $\hat{\boldsymbol{\xi}}_f = \langle \hat{\xi}_f, id_{H_0} \rangle$.

Consider the following diagram whose left and right rectangles are cartesian.

$$\begin{array}{ccccc}
(E \times_{G_0} H_0) \times^{\sigma'}_{H_0} H_1 & \xrightarrow{\sigma'_{\pi_{f_0}}} & E \times_{G_0} H_0 & \xrightarrow{(f_0)_\pi} & E \\
\downarrow{\scriptstyle (\pi_{f_0})_{\sigma'}} & & \downarrow{\scriptstyle \pi_{f_0}} & & \downarrow{\scriptstyle \pi} \\
H_1 & \xrightarrow{\sigma'} & H_0 & \xrightarrow{f_0} & G_0
\end{array}$$

Then, $\hat{\xi}_f : (E \times_{G_0} H_0) \times^{\sigma'}_{H_0} H_1 \to E \times_{G_0} H_0$ is the following composition.

$$(E \times_{G_0} H_0) \times^{\sigma'}_{H_0} H_1 \xrightarrow{((f_0)_\pi \sigma'_{\pi_{f_0}},\, f_1 (\pi_{f_0})_{\sigma'})} E \times^{\sigma}_{G_0} G_1 \xrightarrow{\hat{\xi}} E$$

§4. Right induced representations of groupoids

We assume that the inverse image functor $f^*:\mathscr{E}^{(2)}_Y\to\mathscr{E}^{(2)}_X$ has a right adjoint $f_!:\mathscr{E}^{(2)}_X\to\mathscr{E}^{(2)}_Y$ for any morphism $f:X\to Y$ of $\mathscr{E}$ for the rest of this section.

We denote by $\boldsymbol{\varepsilon}^f: f^*f_!\to id_{\mathscr{E}^{(2)}_X}$ the counit of the adjunction $f^*\dashv f_!$.

For morphisms $f:X\to Y$, $g:X\to Z$, $k:V\to X$ of $\mathscr{E}$ and object $\boldsymbol{F}$ of $\mathscr{E}^{(2)}_Z$, we denote by

$$\boldsymbol{F}^k: f_!(g^*(\boldsymbol{F}))\to (fk)_!((gk)^*(\boldsymbol{F}))$$

the adjoint of the following composition with respect to the adjunction $(fk)^*\dashv(fk)_!$.

$$(fk)^*(f_!(g^*(\boldsymbol{F})))\xrightarrow{c_{f,k}(f_!(g^*(\boldsymbol{F})))} k^*(f^*(f_!(g^*(\boldsymbol{F}))))\xrightarrow{k^*\left(\boldsymbol{\varepsilon}^f_{g^*(\boldsymbol{F})}\right)} k^*(g^*(\boldsymbol{F}))\xrightarrow{c_{g,k}(\boldsymbol{F})^{-1}}(gk)^*(\boldsymbol{F})$$

Remark 4.1

For each object X of $\mathscr{E}$, we choose the inverse image functor $id_X^*:\mathscr{E}^{(2)}_X\to\mathscr{E}^{(2)}_X$ of the identity morphism $id_X:X\to X$ of X to be the identity functor of $\mathscr{E}^{(2)}_X$. Hence we choose the identity functor of $\mathscr{E}^{(2)}_X$ as a right adjoint of $id_X^*:\mathscr{E}^{(2)}_X\to\mathscr{E}^{(2)}_X$.

The following result (4.2) and (4.3) are analogs of (3.5) and (3.6), respectively.

Proposition 4.2

For a morphism $\xi:\sigma^*(E)\to\tau^*(E)$ of $\mathscr{E}^{(2)}_{G_1}$, consider the morphism $\check{\xi}:E\to\sigma_!(\tau^*(E))$ of $\mathscr{E}^{(2)}_{G_0}$. Then ξ satisfies the conditions (A) and (U) of (3.3) if and only if $\check{\xi}$ makes the following diagrams commute.

$$\begin{array}{ccccc}
E & \xrightarrow{\check{\xi}} & \sigma_!(\tau^*(E)) & \xrightarrow{\sigma_!(\tau^*(\check{\xi}))} & \sigma_!(\tau^*(\sigma_!(\tau^*(E)))) \\
\downarrow{\check{\xi}} & & & & \cong\downarrow{\theta^{\sigma,\tau,\sigma,\tau}(E)} \\
\sigma_!(\tau^*(E)) & \xrightarrow{E^{\mu}} & (\sigma\mu)_!((\tau\mu)^*(E)) & = & (\sigma\mathrm{pr}_1)_!((\tau\mathrm{pr}_2)^*(E))
\end{array}
\qquad
\begin{array}{ccc}
E & \xrightarrow{\check{\xi}} & \sigma_!(\tau^*(E)) \\
\downarrow{id_E} & & \downarrow{E^{\varepsilon}} \\
E & = & (\sigma\varepsilon)_!((\tau\varepsilon)^*(E))
\end{array}$$

Proposition 4.3

Let (E,ξ) and (F,ζ) be representations of G on E and F, respectively. A morphism $\varphi:E\to F$ of $\mathscr{E}^{(2)}_{G_0}$ is a morphism of representations of G if and only if the right diagram commutes.

$$\begin{array}{ccc}
E & \xrightarrow{\check{\xi}} & \sigma_!(\tau^*(E)) \\
\downarrow{\varphi} & & \downarrow{\sigma_!(\tau^*(\varphi))} \\
F & \xrightarrow{\check{\zeta}} & \sigma_!(\tau^*(F))
\end{array}$$

Let $\boldsymbol{G}=(G_0, G_1; \sigma, \tau, \varepsilon, \mu, \iota)$ and $\boldsymbol{H}=(H_0, H_1; \sigma', \tau', \varepsilon', \mu', \iota')$ be groupoids in $\mathscr{E}$ and $\boldsymbol{f}=(f_0, f_1):\boldsymbol{H}\to\boldsymbol{G}$ a morphism of groupoids. We consider the following diagram whose rectangles are all cartesian.

$$
\begin{array}{ccccccc}
G_1\times_{G_0}G_1\times_{G_0}G_1\times_{G_0}H_0 & \xrightarrow{(\mathrm{pr}_2, \mathrm{pr}_3)\times_{G_0} id_{H_0}} & G_1\times_{G_0}G_1\times_{G_0}H_0 & \xrightarrow{\mathrm{pr}_2\times_{G_0} id_{H_0}} & G_1\times_{G_0}H_0 & \xrightarrow{\tau_{f_0}} & H_0 \\
\downarrow{\scriptstyle (f_0)_{\tau\mathrm{pr}_2(\mathrm{pr}_2,\mathrm{pr}_3)}} & & \downarrow{\scriptstyle (f_0)_{\tau\mathrm{pr}_2}} & & \downarrow{\scriptstyle (f_0)_{\tau}} & & \downarrow{\scriptstyle f_0} \\
G_1\times_{G_0}G_1\times_{G_0}G_1 & \xrightarrow{(\mathrm{pr}_2, \mathrm{pr}_3)} & G_1\times_{G_0}G_1 & \xrightarrow{\mathrm{pr}_2} & G_1 & \xrightarrow{\tau} & G_0 \\
 & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \sigma} & & \\
 & & G_1 & \xrightarrow{\tau} & G_0 & &
\end{array}
$$

Put $(f_0)_{\tau\mathrm{pr}_2}=\tilde{\mathrm{pr}}_{12}$ and $\mathrm{pr}_2\times_{G_0} id_{H_0}=\tilde{\mathrm{pr}}_{23}$ for short. Since $\mu\times_{G_0} id_{H_0}=(\mu\tilde{\mathrm{pr}}_{12}, \tau_{f_0}\tilde{\mathrm{pr}}_{23})$, we have $\sigma(f_0)_{\tau}(\mu\times_{G_0} id_{H_0})=\sigma\mu\tilde{\mathrm{pr}}_{12}=\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}$ and $\tau_{f_0}(\mu\times_{G_0} id_{H_0})=\tau_{f_0}\tilde{\mathrm{pr}}_{23}$.

For an object F of $\mathscr{E}^{(2)}_{H_0}$, we define a morphism

$$\check{\mu}_f(F) : (\sigma(f_0)_\tau)_!\big(\tau^*_{f_0}(F)\big) \to \sigma_!\Big(\tau^*\Big((\sigma(f_0)_\tau)_!\big(\tau^*_{f_0}(F)\big)\Big)\Big)$$

of $\mathscr{E}^{(2)}_{G_0}$ to be the following composition by using (1.5).

$$(\sigma(f_0)_\tau)_!\big(\tau^*_{f_0}(F)\big) \xrightarrow{F^{\mu\times_{G_0} id_{H_0}}} (\sigma(f_0)_\tau(\mu\times_{G_0} id_{H_0}))_!\big((\tau_{f_0}(\mu\times_{G_0} id_{H_0}))^*(F)\big)$$

$$= (\sigma \mathrm{pr}_1 \tilde{\mathrm{pr}}_{12})_!\big((\tau_{f_0}\tilde{\mathrm{pr}}_{23})^*(F)\big) \xrightarrow{\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(F)^{-1}} \sigma_!\Big(\tau^*\Big((\sigma(f_0)_\tau)_!\big(\tau^*_{f_0}(F)\big)\Big)\Big)$$

It can be verified that $\check{\mu}_f(F)$ makes the diagrams of (4.2) commute. Thus we have the following result.

Proposition 4.4

Let $\mu^r_f(F) : \sigma^*\Big((\sigma(f_0)_\tau)_!\big(\tau^*_{f_0}(F)\big)\Big) \to \tau^*\Big((\sigma(f_0)_\tau)_!\big(\tau^*_{f_0}(F)\big)\Big)$ be the adjoint of $\check{\mu}_f(F)$ with respect to the adjunction $\sigma^* \dashv \sigma_!$.

Then $\Big((\sigma(f_0)_\tau)_!\big(\tau^*_{f_0}(F)\big), \mu^r_f(F)\Big)$ is a representation of G.

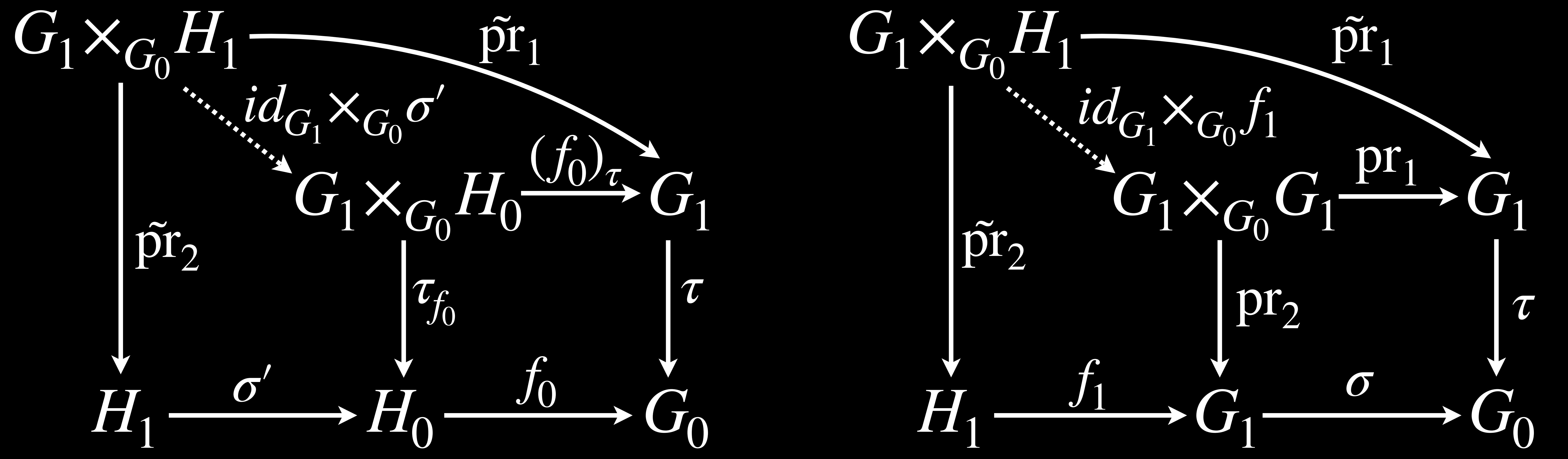
Proposition 4.5

Let $\varphi : E \to F$ be a morphism of $\mathscr{E}^{(2)}_{G_0}$. Then, the following diagram is commutative.

$$\begin{array}{ccc} \sigma^*\Big((\sigma(f_0)_\tau)_!\big(\tau^*_{f_0}(E)\big)\Big) & \xrightarrow{\sigma^*\left((\sigma(f_0)_\tau)_!\left(\tau^*_{f_0}(\varphi)\right)\right)} & \sigma^*\Big((\sigma(f_0)_\tau)_!\big(\tau^*_{f_0}(F)\big)\Big) \\ \Big\downarrow \mu^r_f(E) & & \Big\downarrow \mu^r_f(F) \\ \tau^*\Big((\sigma(f_0)_\tau)_!\big(\tau^*_{f_0}(E)\big)\Big) & \xrightarrow{\tau^*\left((\sigma(f_0)_\tau)_!\left(\tau^*_{f_0}(\varphi)\right)\right)} & \tau^*\Big((\sigma(f_0)_\tau)_!\big(\tau^*_{f_0}(F)\big)\Big) \end{array}$$

Thus $(\sigma(f_0)_\tau)_!\big(\tau^*_{f_0}(\varphi)\big) : \Big((\sigma(f_0)_\tau)_!\big(\tau^*_{f_0}(E)\big), \mu^r_f(E)\Big) \to \Big((\sigma(f_0)_\tau)_!\big(\tau^*_{f_0}(F)\big), \mu^r_f(F)\Big)$ is a morphism of representations of G.

Let $G_1 \xleftarrow{\tilde{\mathrm{pr}}_1} G_1 \times_{G_0} H_1 \xrightarrow{\tilde{\mathrm{pr}}_2} H_1$ be a limit of a diagram $G_1 \xrightarrow{\tau} G_0 \xleftarrow{f_0\sigma'} H_1$. There exist unique morphisms $id_{G_1} \times_{G_0} \sigma' : G_1 \times_{G_0} H_1 \to G_1 \times_{G_0} H_0$ and $id_{G_1} \times_{G_0} f_1 : G_1 \times_{G_0} H_1 \to G_1 \times_{G_0} G_1$ of $\mathscr{E}$ that make the following diagrams commute.



Since $\tau\mu(id_{G_1} \times_{G_0} f_1) = \tau \mathrm{pr}_2(id_{G_1} \times_{G_0} f_1) = \tau f_1 \tilde{\mathrm{pr}}_2 = f_0 \tau' \tilde{\mathrm{pr}}_2$, there exists unique morphism $(\mu(id_{G_1} \times_{G_0} f_1), \tau' \tilde{\mathrm{pr}}_2) : G_1 \times_{G_0} H_1 \to G_1 \times_{G_0} H_0$ that satisfies $(f_0)_\tau(\mu(id_{G_1} \times_{G_0} f_1), \tau' \tilde{\mathrm{pr}}_2) = \mu(id_{G_1} \times_{G_0} f_1)$ and $\tau_{f_0}(\mu(id_{G_1} \times_{G_0} f_1), \tau' \tilde{\mathrm{pr}}_2) = \tau' \tilde{\mathrm{pr}}_2$. It follows $\sigma(f_0)_\tau(\mu(id_{G_1} \times_{G_0} f_1), \tau' \tilde{\mathrm{pr}}_2) = \sigma\mu(id_{G_1} \times_{G_0} f_1) = \sigma \mathrm{pr}_1(id_{G_1} \times_{G_0} f_1) = \sigma \tilde{\mathrm{pr}}_1 = \sigma(f_0)_\tau(id_{G_1} \times_{G_0} \sigma')$.

Since the following diagram is cartesian, (1.5) implies that we have an isomorphism

$$\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(\boldsymbol{F})\colon(\sigma(f_0)_\tau)_!(\tau_{f_0}^*(\sigma'_!(\tau'^*(\boldsymbol{F}))))\to(\sigma(f_0)_\tau(id_{G_1}\times_{G_0}\sigma'))_!((\tau'\tilde{\mathrm{pr}}_2)^*(\boldsymbol{E})).$$

$$\begin{array}{ccc} G_1\times_{G_0}H_1 & \xrightarrow{id_{G_1}\times_{G_0}\sigma'} & G_1\times_{G_0}H_0 \\ \downarrow{\scriptstyle \tilde{\mathrm{pr}}_2} & & \downarrow{\scriptstyle \tau_{f_0}} \\ H_1 & \xrightarrow{\sigma'} & H_0 \end{array}$$

For a representation $(\boldsymbol{F},\zeta)$ of $\boldsymbol{H}$, let us denote by $\check{\zeta}\colon\boldsymbol{F}\to\sigma'_!(\tau'^*(\boldsymbol{F}))$ the adjoint of $\zeta\colon\sigma'^*(\boldsymbol{F})\to\tau'^*(\boldsymbol{F})$. Let $E^f_{(\boldsymbol{F},\zeta)}\colon(\boldsymbol{F},\zeta)^f\to(\sigma(f_0)_\tau)_!(\tau_{f_0}^*(\boldsymbol{F}))$ be an equalizer in $\mathscr{E}^{(2)}_{G_0}$ of

$$(\sigma(f_0)_\tau)_!(\tau_{f_0}^*(\boldsymbol{F}))\xrightarrow{\boldsymbol{F}^{(\mu(id_{G_1}\times_{G_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}}(\sigma(f_0)_\tau(\mu(id_{G_1}\times_{G_0}f_1),\tau'\tilde{\mathrm{pr}}_2))_!(\tau_{f_0}(\mu(id_{G_1}\times_{G_0}f_1),\tau'\tilde{\mathrm{pr}}_2))^*(\boldsymbol{F}))$$
$$=(\sigma(f_0)_\tau(id_{G_1}\times_{G_0}\sigma'))_!((\tau'\tilde{\mathrm{pr}}_2)^*(\boldsymbol{E}))$$

and the following composition.

$$(\sigma(f_0)_\tau)_!(\tau_{f_0}^*(\boldsymbol{F}))\xrightarrow{(\sigma(f_0)_\tau)_!(\tau_{f_0}^*(\check{\zeta}))}(\sigma(f_0)_\tau)_!(\tau_{f_0}^*(\sigma'_!(\tau'^*(\boldsymbol{F}))))$$
$$\xrightarrow{\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(\boldsymbol{F})}(\sigma(f_0)_\tau(id_{G_1}\times_{G_0}\sigma'))_!((\tau'\tilde{\mathrm{pr}}_2)^*(\boldsymbol{E}))$$

Proposition 4.6

For a representation (F,ζ) of H, there exists unique morphism

$$\zeta_f^r : \sigma^*((F,\zeta)^f) \to \tau^*((F,\zeta)^f)$$

of $\mathscr{E}_{G_1}^{(2)}$ such that $\left((F,\zeta)^f, \zeta_f^r\right)$ is a representation of G and that

$$E_{(F,\zeta)}^f : \left((F,\zeta)^f, \zeta_f^r\right) \to \left((\sigma(f_0)_\tau)_!\left(\tau_{f_0}^*(F)\right), \mu_f^r(F)\right)$$

is a morphism of representations of G.

Proposition 4.7

For a morphism $\varphi : (E,\xi) \to (F,\zeta)$ of representations of H, there exists unique morphism $\varphi^f : (E,\xi)^f \to (F,\zeta)^f$ that makes the following diagram commute.

$$\begin{array}{ccc} (E,\xi)^f & \xrightarrow{\varphi^f} & (F,\zeta)^f \\ \downarrow E_{(E,\xi)}^f & & \downarrow E_{(F,\zeta)}^f \\ (\sigma(f_0)_\tau)_!(\tau_{f_0}^*(E)) & \xrightarrow{(\sigma(f_0)_\tau)_!\left(\tau_{f_0}^*(\varphi)\right)} & (\sigma(f_0)_\tau)_!(\tau_{f_0}^*(F)) \end{array}$$

Moreover, $\varphi^f : \left((E,\xi)^f, \xi_f^r\right) \to \left((F,\zeta)^f, \zeta_f^r\right)$ is a morphism of representations of G.

We define a functor $f_!:\mathrm{Rep}(\boldsymbol{H})\to\mathrm{Rep}(\boldsymbol{G})$ by putting $f_!((\boldsymbol{F},\zeta))=\left((\boldsymbol{F},\zeta)^f,\zeta_f^r\right)$ and $f_!(\varphi)=\varphi^f$ for an object $(\boldsymbol{F},\zeta)$ and a morphism φ of $\mathrm{Rep}(\boldsymbol{H})$.

Theorem 4.8

$f_!:\mathrm{Rep}(\boldsymbol{H})\to\mathrm{Rep}(\boldsymbol{G})$ is a right adjoint of $f^*:\mathrm{Rep}(\boldsymbol{G})\to\mathrm{Rep}(\boldsymbol{H})$.