Notes on representation theory of internal categories

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Introduction

In [1], J.F.Adams generalized the notion of Hopf algebras which are obtained from generalized homology theories satisfying certain conditions and showed that such a generalized homology theory, say E_*, takes values in the category of comodules over the "generalized Hopf algebra" associated with E_*. This notion introduced by Adams is now called a Hopf algebroid which is a groupoid object in the opposite category of graded commutative rings. We set a categorical foundation of representations of an internal category in [19] by using the notion of fibered category ([6]). Under the framework of [19], a comodule over a Hopf algebroid Γ is a representation of Γ regarded as a groupoid in the opposite category of graded commutative rings.

The aim of this note is to provide various fundamental notions on representations of internal categories under the framework of [19]. Namely, we give definitions and constructions of "restrictions", "trivial representations", "regular representations", "induced representations" and others. In order to develop a theory of representations of internal categories, we study certain properties of fibered categories on representability of presheaves obtained from pairs of inverse image functors in the first section.

Contents

1 Study on fibered categories

The aim of this section is to provide various notions and constructions on fibered categories which are needed to develop a theory of representations of internal category next section.

We begin by reviewing the notion of fibered category following [6] and prove some basic facts which are needed later. In the second subsection, we introduce a notion of "left fibered representable pair" for a fibered category $p : \mathcal{F} \to \mathcal{E}$ which generalizes the notion of fibered product in a category to a fibered category and study its properties. Next, we also introduce a notion of "right fibered representable pair" which is a dual notion of left fibered representable pair and give analogous results.

1.1 Recollections on fibered category

Let $p : \mathcal{F} \to \mathcal{E}$ be a functor. For an object X of $\mathcal{E}$, we denote by $\mathcal{F}_X$ the subcategory of $\mathcal{F}$ consiting of objects M of $\mathcal{F}$ satisfying $p(M) = X$ and morphisms φ satisfying $p(\varphi) = id_X$. For a morphism $f : X \to Y$ in $\mathcal{E}$ and $M \in \mathrm{Ob}\,\mathcal{F}_X$, $N \in \mathrm{Ob}\,\mathcal{F}_Y$, we put $\mathcal{F}_f(M, N) = \{\varphi \in \mathcal{F}(M, N) |\, p(\varphi) = f\}$.

Definition 1.1.1 *([6], p.161 Définition 5.1.) Let $\alpha : M \to N$ be a morphism in $\mathcal{F}$ and set $X = p(M)$, $Y = p(N)$, $f = p(\alpha)$. We call α a cartesian morphism if, for any $M' \in \mathrm{Ob}\,\mathcal{F}_X$, the map $\mathcal{F}_X(M', M) \to \mathcal{F}_f(M', N)$ defined by $\varphi \mapsto \alpha\varphi$ is bijective.*

The following assertion is immediate from the definition.

Proposition 1.1.2 *Let $\alpha_i : M_i \to N_i$ $(i = 1, 2)$ be morphisms in $\mathcal{F}$ such that $p(M_1) = p(M_2)$, $p(N_1) = p(N_2)$, $p(\alpha_1) = p(\alpha_2)$ and $\lambda : N_1 \to N_2$ a morphism in $\mathcal{F}$ such that $p(\lambda) = id_{p(N_1)}$. If α_2 is cartesian, there is unique morphism $\mu : M_1 \to M_2$ such that $p(\mu) = id_{p(M_1)}$ and the following diagram is commutative.*

$$\begin{array}{ccc} M_1 & \xrightarrow{\alpha_1} & N_1 \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \lambda} \\ M_2 & \xrightarrow{\alpha_2} & N_2 \end{array}$$

Corollary 1.1.3 *If $\alpha_i : M_i \to N$ $(i = 1, 2)$ are cartesian morphisms in $\mathcal{F}$ such that $p(M_1) = p(M_2)$ and $p(\alpha_1) = p(\alpha_2)$, there is unique morphism $\mu : M_1 \to M_2$ such that $\alpha_1 = \alpha_2\mu$ and $p(\mu) = id_{p(M_1)}$. Moreover, μ is an isomorphism.*

Definition 1.1.4 *([6], p.162 Définition 5.1.) Let $f : X \to Y$ be a morphism in $\mathcal{E}$ and $N \in \mathrm{Ob}\,\mathcal{F}_Y$. If there exists a cartesian morphism $\alpha : M \to N$ such that $p(\alpha) = f$, M is called an inverse image of N by f. We denote M by $f^*(N)$ and α by $\alpha_f(N) : f^*(N) \to N$. By (1.1.3), $f^*(N)$ is unique up to isomorphism.*

Remark 1.1.5 *For the identity morphism id_X of $X \in \mathrm{Ob}\,\mathcal{E}$ and $N \in \mathrm{Ob}\,\mathcal{F}_X$, the identity morphism id_N of N is obviously cartesian. Hence the inverse image of N by the identity morphism of X always exists and $\alpha_{id_X}(N) : id_X^*(N) \to N$ can be chosen as the identity morphism of N. By the uniqueness of $id_X^*(N)$ up to isomorphism, $\alpha_{id_X}(N) : id_X^*(N) \to N$ is an isomorphism for any choice of $id_X^*(N)$.*

The following assertion is a direct consequece of (1.1.2).

Proposition 1.1.6 *Let $f : X \to Y$ be a morphism in $\mathcal{E}$ and N, N' objects of $\mathcal{F}_Y$. Suppose that there exists a cartesian morphism $\alpha_f(N) : f^*(N) \to N$ for any object N of $\mathcal{F}_Y$. For a morphism $\varphi : N \to N'$ in $\mathcal{F}_Y$, there exists unique morphism $f^*(\varphi) : f^*(N) \to f^*(N')$ that makes the following diagram commute.*

$$\begin{array}{ccc} f^*(N) & \xrightarrow{\alpha_f(N)} & N \\ \downarrow{\scriptstyle f^*(\varphi)} & & \downarrow{\scriptstyle \varphi} \\ f^*(N') & \xrightarrow{\alpha_f(N')} & N' \end{array}$$

Thus we have a functor $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ defined by $N \mapsto f^*(N)$ and $\varphi \mapsto f^*(\varphi)$.

Proof. For the identity morphism id_N of $N \in \mathrm{Ob}\,\mathcal{F}_Y$, we have $f^*(id_N) = id_{f^*(N)}$ by the uniqueness of $f^*(id_N)$. For morphisms $\varphi : N \to N'$ and $\psi : N' \to N''$ in $\mathcal{F}_Y$, we have the following diagram whose trapezoids of the both sides and the outer rectangle are commutative.

$$\begin{tikzcd}
f^*(N) \arrow[rr, "f^*(\psi\varphi)"] \arrow[dr, "f^*(\varphi)"] \arrow[dd, "\alpha_f(N)"] & & f^*(N'') \arrow[dd, "\alpha_f(N'')"] \\
& f^*(N') \arrow[ur, "f^*(\psi)"] \arrow[d, "\alpha_f(N')"] & \\
N \arrow[r, "\varphi"] & N' \arrow[r, "\psi"] & N''
\end{tikzcd}$$

Hence we have $f^*(\psi\varphi) = f^*(\psi)f^*(\varphi)$ by the uniqueness of $f^*(\psi\varphi)$. $\square$

Definition 1.1.7 *([6], p.162 Définition 5.1.) If the assumption of (1.1.6) is satisfied, we say that the functor of the inverse image by f exists.*

Definition 1.1.8 *([6], p.164 Définition 6.1.) If a functor $p : \mathcal{F} \to \mathcal{E}$ satisfies the following condition (i), p is called a prefibered category and if p satisfies both (i) and (ii), p is called a fibered category or p is fibrant.*

(i) For any morphism f in $\mathcal{E}$, the functor of the inverse image by f exists.
(ii) The composition of cartesian morphisms is cartesian.

Definition 1.1.9 *([6], p.170 Définition 7.1.) Let $p : \mathcal{F} \to \mathcal{E}$ be a functor. A map*

$$\kappa : \mathrm{Mor}\,\mathcal{E} \longrightarrow \coprod_{X,Y \in \mathrm{Ob}\,\mathcal{E}} \mathrm{Funct}(\mathcal{F}_Y, \mathcal{F}_X)$$

is called a cleavage if $\kappa(f)$ is an inverse image functor $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ for $(f : X \to Y) \in \mathrm{Mor}\,\mathcal{E}$. A cleavage κ is said to be normalized if $\kappa(id_X) = id_{\mathcal{F}_X}$ for any $X \in \mathrm{Ob}\,\mathcal{E}$. A category $\mathcal{F}$ over $\mathcal{E}$ is called a cloven prefibered category (resp. normalized cloven prefibered category) if a cleavage (resp. normalized cleavage) is given.

$p : \mathcal{F} \to \mathcal{E}$ has a cleavage if and only if p is prefibered. If p is prefibered, p has a normalized cleavage by (1.1.5).

Let $f : X \to Y$, $g : Z \to X$ be morphisms in $\mathcal{E}$ and N an object of $\mathcal{F}_Y$. If $p : \mathcal{F} \to \mathcal{E}$ is a prefibered category, it follows from (1.1.2) that there is unique morphism $c_{f,g}(N) : g^*f^*(N) \to (fg)^*(N)$ in $\mathcal{F}_Z$ such that the following square commutes.

$$\begin{tikzcd}
g^*f^*(N) \arrow[r, "\alpha_g(f^*(N))"] \arrow[d, "c_{f,g}(N)"] & f^*(N) \arrow[d, "\alpha_f(N)"] \\
(fg)^*(N) \arrow[r, "\alpha_{fg}(N)"] & N
\end{tikzcd}$$

Then, we have the following result by (1.1.3).

Proposition 1.1.10 *([6], p.172 Proposition 7.2.) Let $p : \mathcal{F} \to \mathcal{E}$ be a cloven prefibered category. Then, p is a fibered category if and only if $c_{f,g}(N)$ is an isomorphism for any composable morphisms $f : X \to Y$, $g : Z \to X$ in $\mathcal{E}$ and $N \in \mathrm{Ob}\,\mathcal{F}_Y$.*

Proof. Suppose that p is a fibered category. Then, both $\alpha_{fg}(N)$ and $\alpha_f(N)\alpha_g(f^*(N))$ are cartesian morphisms such that $p(\alpha_{fg}(N)) = p(\alpha_f(N)\alpha_g(f^*(N))) = fg$. Hence by (1.1.3), $c_{f,g}(N)$ is an isomorphism.

Conversely, assume that $c_{f,g}(N)$ is an isomorphism for any composable morphisms $f : X \to Y$, $g : Z \to X$ in $\mathcal{E}$ and $N \in \mathrm{Ob}\,\mathcal{F}_Y$. Let $\alpha : M \to N$ and $\beta : L \to M$ be a cartesian morphisms in $\mathcal{F}$. Put $p(M) = X$, $p(N) = Y$, $p(L) = Z$, $p(\alpha) = f$ and $p(\beta) = g$. There is unique morphism $\zeta : L \to (fg)^*(N)$ in $\mathcal{F}_Z$ such that $\alpha_{fg}(N)\zeta = \alpha\beta$. It follows from (1.1.3) that here are isomorphisms $\psi : M \to f^*(N)$ in $\mathcal{F}_X$ and $\xi : L \to g^*(M)$ in $\mathcal{F}_Z$ such that $\alpha = \alpha_f(N)\psi$, $\beta = \alpha_g(M)\xi$. By (1.1.6), the following diagram is commutative.

$$\begin{tikzcd}
& g^*(M) \arrow[r, "g^*(\psi)"] \arrow[dd, "\alpha_g(M)"] & g^*f^*(N) \arrow[r, "c_{f,g}(N)"] \arrow[d, "\alpha_g(f^*(N))"] & (fg)^*(N) \arrow[dd, "\alpha_{fg}(N)"] \\
L \arrow[ur, "\xi", "\cong"'] \arrow[dr, "\beta"'] & & f^*(N) \arrow[dr, "\alpha_f(N)"] & \\
& M \arrow[ur, "\psi", "\cong"'] \arrow[rr, "\alpha"] & & N
\end{tikzcd}$$

Hence we have $\alpha_{fg}(N)c_{f,g}(N)g^*(\psi)\xi = \alpha\beta = \alpha_{fg}(N)\zeta$. Since $c_{f,g}(N)g^*(\psi)\xi, \zeta : L \to (fg)^*(N))$ are morphisms in $\mathcal{F}_Z$, $c_{f,g}(N)g^*(\psi)\xi = \zeta$ holds by the uniqueness of ζ, . Thus ζ is an isomorphism and it follows that $\alpha\beta$ is cartesian. $\square$

Proposition 1.1.11 *For composable morphisms $f : X \to Y$, $g : Z \to X$ in $\mathcal{E}$ and a morphism $\varphi : M \to N$ in $\mathcal{F}_Y$, the following diagram commutes. In other words, $c_{f,g}$ gives a natural transformation $g^*f^* \to (fg)^*$ of functors from $\mathcal{F}_Y$ to $\mathcal{F}_Z$.*

$$\begin{tikzcd}
g^*f^*(M) \arrow[r, "c_{f,g}(M)"] \arrow[d, "g^*f^*(\varphi)"] & (fg)^*(M) \arrow[d, "(fg)^*(\varphi)"] \\
g^*f^*(N) \arrow[r, "c_{f,g}(N)"] & (fg)^*(N)
\end{tikzcd}$$

Proof. It follows from the definition of $c_{f,g}(M)$ and $c_{f,g}(N)$ that the upper and the lower trapezoids of the following diagram are commutative. It also follows from the definition of $f^*(\varphi)$, $g^*f^*(\varphi)$ and $(fg)^*(\varphi)$ that the right trapezoids and the outer and inner rectangle of the following diagram are commutative.

$$\begin{tikzcd}
g^*f^*(M) \arrow[rrr, "\alpha_{g}*(f^*(M))"] \arrow[dr, "c_{f,g}(M)"] \arrow[ddd, "g^*f^*(\varphi)"] & & & f^*(M) \arrow[dl, "\alpha_f(M)"'] \arrow[ddd, "f^*(\varphi)"] \\
& (fg)^*(M) \arrow[r, "\alpha_{fg}(M)"] \arrow[d, "(fg)^*(\varphi)"] & M \arrow[d, "\varphi"] & \\
& (fg)^*(N) \arrow[r, "\alpha_{fg}(N)"] & N & \\
g^*f^*(N) \arrow[rrr, "\alpha_{g}*(f^*(N))"] \arrow[ur, "c_{f,g}(N)"] & & & f^*(N) \arrow[ul, "\alpha_f(N)"']
\end{tikzcd}$$

Hence we have $\alpha_{fg}(N)(fg)^*(\varphi)c_{f,g}(M) = \alpha_{fg}(N)c_{f,g}(N)g^*f^*(\varphi)$. Since $\alpha_{fg}(N)$ is cartesian, the assertion follows. $\square$

Proposition 1.1.12 *([6], p.172 Proposition 7.4.) Let $p : \mathcal{F} \to \mathcal{E}$ be a cloven prefibered category.*

(1) For a morphism $f : X \to Y$ in $\mathcal{E}$ and an object N of $\mathcal{F}_Y$, we have $c_{f,id_X}(N) = \alpha_{id_X}(f^*(N))$ and $c_{id_Y,f}(N) = f^*(\alpha_{id_Y}(N))$.

(2) For a diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in $\mathcal{E}$ and an object M of $\mathcal{F}_W$, the following diagram commutes.

$$\begin{tikzcd}
f^*(g^*h^*)(M) \arrow[r, "f^*(c_{h,g}(M))"] \arrow[d, equal] & f^*(hg)^*(M) \arrow[r, "c_{hg,f}(M)"] & ((hg)f)^*(M) \arrow[d, equal] \\
(f^*g^*)h^*(M) \arrow[r, "c_{g,f}(h^*(M))"] & (gf)^*h^*(M) \arrow[r, "c_{h,gf}(M)"] & (h(gf))^*(M)
\end{tikzcd}$$

Proof. (1) The following diagrams commute by the definition of $c_{f,id_X}(N)$ and $c_{id_Y,f}(N)$.

$$\begin{tikzcd}
id_X^*f^*(N) \arrow[r, "\alpha_{id_X}(f^*(N))"] \arrow[d, "c_{f,id_X}(N)"] & f^*(N) \arrow[d, "\alpha_f(N)"] \\
f^*(N) \arrow[r, "\alpha_f(N)"] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*id_Y^*(N) \arrow[r, "\alpha_f(id_Y^*(N))"] \arrow[d, "c_{id_Y,f}(N)"] & id_Y^*(N) \arrow[d, "\alpha_{id_Y}(N)"] \\
f^*(N) \arrow[r, "\alpha_f(N)"] & N
\end{tikzcd}$$

On the other hand, the following diagrams also commute.

$$\begin{tikzcd}
id_X^*f^*(N) \arrow[r, "\alpha_{id_X}(f^*(N))"] \arrow[d, "\alpha_{id_X}(f^*(N))"] & f^*(N) \arrow[d, "\alpha_f(N)"] \\
f^*(N) \arrow[r, "\alpha_f(N)"] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*id_Y^*(N) \arrow[r, "\alpha_f(id_Y^*(N))"] \arrow[d, "f^*(\alpha_{id_Y}(N))"] & id_Y^*(N) \arrow[d, "\alpha_{id_Y}(N)"] \\
f^*(N) \arrow[r, "\alpha_f(N)"] & N
\end{tikzcd}$$

Hence the assertion follows from the uniqueness of $c_{f,id_X}(N)$ and $c_{id_Y,f}(N)$.

(2) The following diagram is commutative.

$$\begin{tikzcd}
(f^*g^*)h^*(M) \arrow[rr, "c_{g,f}(h^*(M))"] \arrow[d, equal] & & (gf)^*h^*(M) \arrow[d, "\alpha_{gf}(h^*(M))"'] \arrow[ddr, "c_{h,gf}(M)"] & \\
f^*(g^*h^*)(M) \arrow[r, "\alpha_f(g^*h^*(M))"] \arrow[ddr, "f^*(c_{h,g}(M))"'] & g^*h^*(M) \arrow[r, "\alpha_g(h^*(M))"] \arrow[d, "c_{h,g}(M)"] & h^*(M) \arrow[d, "\alpha_h(M)"'] & \\
 & (hg)^*(M) \arrow[r, "\alpha_{hg}(M)"] & M & (h(gf))^*(M) \arrow[l, "\alpha_{hgf}(M)"'] \arrow[d, equal] \\
 & f^*(hg)^*(M) \arrow[u, "\alpha_f((hg)^*(M))"'] \arrow[rr, "c_{hg,f}(M)"] & & ((hg)f)^*(M)
\end{tikzcd}$$

Hence we have $\alpha_{hgf}(M)c_{h,gf}(M)c_{g,f}(h^*(M)) = \alpha_{hgf}(M)c_{hg,f}(M)f^*(c_{h,g}(M))$. Since $\alpha_{hgf}(M)$ is cartesian, $c_{h,gf}(M)c_{g,f}(h^*(M)) = c_{hg,f}(M)f^*(c_{h,g}(M))$ holds. $\square$

Let $p : \mathcal{F} \to \mathcal{E}$, $q : \mathcal{G} \to \mathcal{C}$ be normalized cloven fibered categories and $F : \mathcal{E} \to \mathcal{C}$, $\Phi : \mathcal{F} \to \mathcal{G}$ functors such that $q\Phi = Fp$. For a morphism $f : X \to Y$ in $\mathcal{E}$ and an object M of $\mathcal{F}_Y$, since $\alpha_{F(f)}(\Phi(M)) : F(f)^*(\Phi(M))) \to \Phi(M)$ is a cartesian morphism mapped to $F(f)$ by q and $\Phi(\alpha_f(M)) : \Phi(f^*(M)) \to \Phi(M)$ also mapped to $F(f)$ by q, there exists unique morphism $c_{f,\Phi}(M) : \Phi(f^*(M)) \to F(f)^*(\Phi(M)))$ of $\mathcal{G}_{F(X)}$ that makes the following diagram commute.

$$\begin{tikzcd}
\Phi(f^*(M)) \arrow[r, "\Phi(\alpha_f(M))"] \arrow[d, "c_{f,\Phi}(M)"] & \Phi(M) \\
F(f)^*(\Phi(M)) \arrow[ur, "\alpha_{F(f)}(\Phi(M))"'] &
\end{tikzcd}$$

We note that Φ preserves cartesian morphisms if and only if $c_{f,\Phi}(M)$ is an isomorphism for any morphism $f : X \to Y$ in $\mathcal{E}$ and any object M of $\mathcal{F}_Y$.

Proposition 1.1.13 *For a morphism $\varphi : M \to N$ of $\mathcal{F}_Y$, the following digram is commutative.*

$$\begin{tikzcd}
\Phi(f^*(M)) \arrow[r, "\Phi(f^*(\varphi))"] \arrow[d, "c_{f,\Phi}(M)"] & \Phi(f^*(N)) \arrow[d, "c_{f,\Phi}(N)"] \\
F(f)^*(\Phi(M)) \arrow[r, "F(f)^*(\Phi(\varphi))"] & F(f)^*(\Phi(N))
\end{tikzcd}$$

Proof. It follows from (1.1.6) that the lower middle rectangle and the outer trapezoid of the following diagram are commutative. The triangles of the both sides are also commutative by the definition of $c_{f,\Phi}(M)$ and $c_{f,\Phi}(N)$.

$$\begin{tikzcd}
F(f)^*(\Phi(M)) \arrow[rrr, "F(f)^*(\Phi(\varphi))"] \arrow[ddr, "\alpha_{F(f)}(\Phi(M))"'] & & & F(f)^*(\Phi(N)) \arrow[ddl, "\alpha_{F(f)}(\Phi(N))"] \\
 & \Phi(f^*(M)) \arrow[r, "\Phi(f^*(\varphi))"] \arrow[d, "\Phi(\alpha_f(M))"] \arrow[ul, "c_{f,\Phi}(M)"'] & \Phi(f^*(N)) \arrow[d, "\Phi(\alpha_f(N))"'] \arrow[ur, "c_{f,\Phi}(N)"] & \\
 & \Phi(M) \arrow[r, "\Phi(\varphi)"] & \Phi(N) &
\end{tikzcd}$$

Hence we have

$$\alpha_{F(f)}(\Phi(M))c_{f,\Phi}(N)\Phi(f^*(\varphi)) = \alpha_{F(f)}(\Phi(M))F(f)^*(\Phi(\varphi))c_{f,\Phi}(M).$$

Since both $c_{f,\Phi}(N)\Phi(f^*(\varphi))$ and $F(f)^*(\Phi(\varphi))c_{f,\Phi}(M)$ are morphisms in $\mathcal{G}_{F(X)}$ and $\alpha_{F(f)}(\Phi(M))$ is a cartesian morphism, the above equality implies the result. $\square$

Proposition 1.1.14 *For morphisms $f : X \to Y$, $k : V \to X$ in $\mathcal{E}$ and $M \in \operatorname{Ob}\mathcal{F}_Y$, the following diagram is commutative.*

$$\begin{tikzcd}
\Phi(k^*(f^*(M))) \arrow[r, "c_{k,\Phi}(f^*(M))"] \arrow[d, "\Phi(c_{f,k}(M))"] & F(k)^*(\Phi(f^*(M))) \arrow[r, "F(k)^*(c_{f,\Phi}(M))"] & F(k)^*(F(f)^*(\Phi(M))) \arrow[d, "c_{F(f),F(k)}(\Phi(M))"] \\
\Phi((fk)^*(M)) \arrow[rr, "c_{fk,\Phi}(M)"] & & F(fk)^*(\Phi(M))
\end{tikzcd}$$

Proof. The inner triangles are all commutative by (1.1.6) and definitions of $c_{f,k}(M)$, $c_{k,\Phi}(f^*(M))$, $c_{f,\Phi}(M)$, $c_{F(f),F(k)}(\Phi(M))$, $c_{fk,\Phi}(M)$.

$$\begin{tikzcd}
\Phi(k^*(f^*(M))) \arrow[rr, "c_{k,\Phi}(f^*(M))"] \arrow[dr, "\Phi(\alpha_k(f^*(M)))"] \arrow[ddd, "\Phi(c_{f,k}(M))"'] & & F(k)^*(\Phi(f^*(M))) \arrow[dl, "\alpha_{F(k)}(\Phi(f^*(M)))"] \arrow[dr, "F(k)^*(c_{f,\Phi}(M))"] & \\
& \Phi(f^*(M)) \arrow[r, "c_{f,\Phi}(M)"] \arrow[dr, "\Phi(\alpha_f(M))"] & F(f)^*(\Phi(M)) \arrow[d, "\alpha_{F(f)^*}(\Phi(M))"] & F(k)^*(F(f)^*(\Phi(M))) \arrow[l, "\alpha_{F(k)}(F(f)^*(\Phi(M)))"'] \arrow[ddl, "c_{F(f),F(k)}(\Phi(M))"] \\
& & \Phi(M) & \\
\Phi((fk)^*(M)) \arrow[uurr, "\Phi(\alpha_{fk}(M))"] \arrow[rr, "c_{fk,\Phi}(M)"] & & F(fk)^*(\Phi(M)) \arrow[u, "\alpha_{F(fk)}(\Phi(M))"'] &
\end{tikzcd}$$

Thus we have the following equality.

$$\alpha_{F(fk)}(\Phi(M))c_{F(f),F(k)}(\Phi(M))F(k)^*(c_{f,\Phi}(M))c_{k,\Phi}(f^*(M)) = \alpha_{F(fk)}(\Phi(M))c_{fk,\Phi}(M)\Phi(c_{f,k}(M))$$

Since both $c_{F(f),F(k)}(\Phi(M))F(k)^*(c_{f,\Phi}(M))c_{k,\Phi}(f^*(M))$ and $c_{fk,\Phi}(M)\Phi(c_{f,k}(M))$ are morphisms in $\mathcal{G}_{F(V)}$ and $\alpha_{F(fk)}(\Phi(M))$ is a cartesian morphism, the assertion follows from the above equality. $\square$

Let $p : \mathcal{F} \to \mathcal{E}$ be a cloven fibered category. For morphisms $f : X \to Y$ and $g : X \to Z$ in $\mathcal{E}$, we define a functor $F_{f,g} : \mathcal{F}_Y^{op} \times \mathcal{F}_Z \to \mathcal{S}et$ by $F_{f,g}(M,N) = \mathcal{F}_X(f^*(M), g^*(N))$ for $M \in \operatorname{Ob}\mathcal{F}_Y$, $N \in \operatorname{Ob}\mathcal{F}_Z$ and $F_{f,g}(\varphi,\psi) = f^*(\varphi)^*g^*(\psi)_*$ for $\varphi \in \operatorname{Mor}\mathcal{F}_Y$, $\psi \in \operatorname{Mor}\mathcal{F}_Z$. For a morphism $k : V \to X$ in $\mathcal{E}$, $M \in \operatorname{Ob}\mathcal{F}_Y$ and $N \in \operatorname{Ob}\mathcal{F}_Z$, let $k^\sharp_{M,N} : F_{f,g}(M,N) \to F_{fk,gk}(M,N)$ be the following composition.

$$\begin{aligned}
F_{f,g}(M,N) = \mathcal{F}_X(f^*(M), g^*(N)) &\xrightarrow{k^*} \mathcal{F}_V(k^*(f^*(M)), k^*(g^*(N))) \xrightarrow{(c_{f,k}(M)^{-1})^*} \mathcal{F}_V((fk)^*(M), k^*(g^*(N))) \\
&\xrightarrow{c_{g,k}(N)_*} \mathcal{F}_V((fk)^*(M), (gk)^*(N)) = F_{fk,gk}(M,N)
\end{aligned}$$

Let $\varphi : M \to L$ and $\psi : P \to N$ be morphisms in $\mathcal{F}_Y$ and $\mathcal{F}_Z$, respectively. Since the following diagram is commutative by (1.1.11), $k^\sharp_{M,N}$ is natural in M, N and we have a natural transformation $k^\sharp : F_{f,g} \to F_{fk,gk}$.

$$\begin{tikzcd}
\mathcal{F}_X(f^*(L), g^*(P)) \arrow[r, "k^*"] \arrow[d, "f^*(\varphi)^*g^*(\psi)_*"] & \mathcal{F}_V(k^*(f^*(L)), k^*(g^*(P))) \arrow[r, "c_{g,k}(P)_*(c_{f,k}(L)^{-1})^*"] \arrow[d, "k^*(f^*(\varphi))^*k^*(g^*(\psi))_*"] & \mathcal{F}_V((fk)^*(L), (gk)^*(P)) \arrow[d, "(fk)^*(\varphi)^*(gk)^*(\psi)_*"] \\
\mathcal{F}_X(f^*(M), g^*(N)) \arrow[r, "k^*"] & \mathcal{F}_V(k^*(f^*(M)), k^*(g^*(N))) \arrow[r, "c_{g,k}(N)_*(c_{f,k}(M)^{-1})^*"] & \mathcal{F}_V((fk)^*(M), (gk)^*(N))
\end{tikzcd}$$

Proposition 1.1.15 *Let $f : X \to Y$, $g : X \to Z$, $h : X \to W$, $k : V \to X$ be morphisms in $\mathcal{E}$.*

(1) Let L, M, N be objects of $\mathcal{F}_Y$, $\mathcal{F}_Z$, $\mathcal{F}_W$, respectively. For morphisms $\zeta : f^*(L) \to g^*(M)$ and $\xi : g^*(M) \to h^*(N)$ in $\mathcal{F}_X$, we have $k^\sharp_{L,N}(\xi\zeta) = k^\sharp_{M,N}(\xi)k^\sharp_{L,M}(\zeta)$.

(2) For objects M and N of $\mathcal{F}_Y$, a composition

$$\mathcal{F}_Y(M,N) \xrightarrow{f^*} \mathcal{F}_X(f^*(M), f^*(N)) \xrightarrow{k^\sharp_{M,N}} \mathcal{F}_V((fk)^*(M), (fk)^*(N))$$

coincides with $(fk)^* : \mathcal{F}_Y(M,N) \to \mathcal{F}_V((fk)^*(M), (fk)^*(N))$. In particular, $k^\sharp_{M,M} : \mathcal{F}_X(f^*(M), f^*(M)) \to \mathcal{F}_V((fk)^*(M), (fk)^*(M))$ maps the identity morphism of $f^*(M)$ to the identity morphism of $(fk)^*(M)$.

Proof. (1) The assertion follows from

$$\begin{aligned}
k^\sharp_{M,N}(\xi)k^\sharp_{L,M}(\zeta) &= c_{h,k}(N)k^*(\xi)c_{g,k}(M)^{-1}c_{g,k}(M)k^*(\zeta)c_{f,k}(L)^{-1} = c_{h,k}(N)f^*(\xi)f^*(\zeta)c_{f,k}(L)^{-1} \\
&= c_{h,k}(N)f^*(\xi\zeta)c_{f,k}(L)^{-1} = k^\sharp_{L,N}(\xi\zeta).
\end{aligned}$$

(2) The assertion follows from the definition of $k^\sharp$ and (1.1.11). $\square$

Proposition 1.1.16 *For morphisms $f : X \to Y$, $g : X \to Z$, $k : V \to X$ and $j : W \to V$ in $\mathcal{E}$, the following diagram is commutative for any $M \in \operatorname{Ob}\mathcal{F}_Y$ and $N \in \operatorname{Ob}\mathcal{F}_Z$. Hence we have $(kj)^\sharp = j^\sharp k^\sharp$.*

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M),g^*(N)) \arrow[rr, "(kj)^\sharp_{M,N}"] \arrow[dr, "k^\sharp_{M,N}"'] & & \mathcal{F}_W((fkj)^*(M),(gkj)^*(N)) \\
& \mathcal{F}_V((fk)^*(M),(gk)^*(N)) \arrow[ur, "j^\sharp_{M,N}"'] &
\end{tikzcd}$$

Proof. For $M \in \operatorname{Ob}\mathcal{F}_Y$, $N \in \operatorname{Ob}\mathcal{F}_Z$ and $\xi \in \mathcal{F}_X(f^*(M),g^*(N))$, it follows from (1.1.11) and (1.1.12) that

$$\begin{aligned}
j^\sharp_{M,N}k^\sharp_{M,N}(\xi) &= c_{gk,j}(N)j^*(c_{g,k}(N)k^*(\xi)c_{f,k}(M)^{-1})c_{fk,j}(M)^{-1} \\
&= c_{gk,j}(N)j^*(c_{g,k}(N))j^*(k^*(\xi))j^*(c_{f,k}(M)^{-1})c_{fk,j}(M)^{-1} \\
&= c_{gk,j}(N)j^*(c_{g,k}(N))c_{k,j}(g^*(N))^{-1}(kj)^*(\xi)c_{k,j}(f^*(M))j^*(c_{f,k}(M)^{-1})c_{fk,j}(M)^{-1} \\
&= c_{gk,j}(N)j^*(c_{g,k}(N))c_{k,j}(g^*(N))^{-1}(kj)^*(\xi)(c_{fk,j}(M)j^*(c_{f,k}(M))c_{k,j}(f^*(M))^{-1})^{-1} \\
&= c_{g,kj}(N)(kj)^*(\xi)c_{f,kj}(M)^{-1} = (kj)^\sharp_{M,N}(\xi).
\end{aligned}$$

Hence we have $j^\sharp_{M,N}k^\sharp_{M,N} = (kj)^\sharp_{M,N}$ for any $M, N \in \operatorname{Ob}\mathcal{F}_Y$. □

Let $p : \mathcal{F} \to \mathcal{E}$, $q : \mathcal{G} \to \mathcal{C}$ be normalized cloven fibered categories and $F : \mathcal{E} \to \mathcal{C}$, $\Phi : \mathcal{F} \to \mathcal{G}$ functors such that $q\Phi = Fp$ and Φ preserves cartesian morphisms. For morphisms $f : X \to Y$, $g : X \to Z$ in $\mathcal{E}$ and objects M, N of $\mathcal{F}_Y$, $\mathcal{F}_Z$ respectively, we denote by $\Phi^{f,g}_{M,N}$ a composition

$$\begin{aligned}
\mathcal{F}_X(f^*(M),g^*(N)) &\xrightarrow{\Phi} \mathcal{G}_{F(X)}(\Phi(f^*(M)),\Phi(g^*(N))) \xrightarrow{(c_{f,\Phi}(M)^{-1})^*} \mathcal{G}_{F(X)}(F(f)^*(\Phi(M)),\Phi(g^*(N))) \\
&\xrightarrow{c_{g,\Phi}(N)_*} \mathcal{G}_{F(X)}(F(f)^*(\Phi(M)),F(g)^*(\Phi(N))).
\end{aligned}$$

Proposition 1.1.17 *Assume that Φ preserves cartesian morphisms. Let $f : X \to Y$, $g : X \to Z$, $h : X \to W$ be morphisms in $\mathcal{E}$ and objects M, N, L of $\mathcal{F}_Y$, $\mathcal{F}_Z$, $\mathcal{F}_W$, respectively. For $\varphi \in \mathcal{F}_X(f^*(M),g^*(N))$ and $\psi \in \mathcal{F}_X(g^*(N),h^*(L))$, $\Phi^{f,h}_{M,L}(\psi\varphi) = \Phi^{g,h}_{N,L}(\psi)\Phi^{f,g}_{M,N}(\varphi)$ holds.*

Proof. The assertion follows from

$$\begin{aligned}
\Phi^{g,h}_{N,L}(\psi)\Phi^{f,g}_{M,N}(\varphi) &= c_{h,\Phi}(L)\Phi(\psi)c_{g,\Phi}(N)^{-1}c_{g,\Phi}(N)\Phi(\varphi)c_{f,\Phi}(M)^{-1} = c_{h,\Phi}(L)\Phi(\psi)\Phi(\varphi)c_{f,\Phi}(M)^{-1} \\
&= c_{h,\Phi}(L)\Phi(\psi\varphi)c_{f,\Phi}(M)^{-1} = \Phi^{f,h}_{M,L}(\psi\varphi)
\end{aligned}$$

□

Proposition 1.1.18 *Let $p : \mathcal{F} \to \mathcal{E}$, $q : \mathcal{G} \to \mathcal{C}$, $r : \mathcal{H} \to \mathcal{D}$ be normalized cloven fibered categories and $F : \mathcal{E} \to \mathcal{C}$, $G : \mathcal{C} \to \mathcal{D}$, $\Phi : \mathcal{F} \to \mathcal{G}$, $\Psi : \mathcal{G} \to \mathcal{H}$ functors which satisfy $q\Phi = Fp$, $r\Psi = Gq$.*

(1) For a morphism in $f : X \to Y$ $\mathcal{E}$ and an object M of $\mathcal{F}_Y$, the following diagram is commutative.

$$\begin{tikzcd}
\Psi(\Phi(f^*(M))) \arrow[r, "\Psi(c_{f,\Phi}(M))"] \arrow[dr, "c_{f,\Psi\Phi}(M)"'] & \Psi(F(f)^*(\Phi(M))) \arrow[d, "c_{F(f),\Psi}(\Phi(M))"] \\
& G(F(f))^*(\Psi(\Phi(M)))
\end{tikzcd}$$

(2) Let $f : X \to Y$, $g : X \to Z$ be morphisms in $\mathcal{E}$ and objects M, N of $\mathcal{F}_Y$, $\mathcal{F}_Z$, respectively. If Φ and Ψ preserves cartesian morphisms, the following diagram is commutative.

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M),g^*(N)) \arrow[r, "\Phi^{f,g}_{M,N}"] \arrow[dr, "(\Psi\Phi)^{f,g}_{M,N}"'] & \mathcal{G}_{F(X)}(F(f)^*(\Phi(M)),F(g)^*(\Phi(N))) \arrow[d, "\Psi^{F(f),F(g)}_{\Phi(M),\Phi(N)}"] \\
& \mathcal{H}_{G(F(X))}(G(F(f))^*(\Psi(\Phi(M))),G(F(g))^*(\Psi(\Phi(N))))
\end{tikzcd}$$

Proof. (1) The outer triangle, the lower left and right triangles of the following diagram is commutative.

$$\begin{tikzcd}
\Psi(\Phi(f^*(M))) \arrow[rr, "c_{f,\Psi\Phi}(M)"] \arrow[dr, "\Psi(c_{f,\Phi}(M))"] \arrow[ddr, "\Psi(\Phi(\alpha_f(M)))"'] & & G(F(f))^*(\Psi(\Phi(M))) \arrow[ddl, "\alpha_{G(F(f))}(\Psi(\Phi(M)))"] \\
& \Psi(F(f)^*(\Phi(M))) \arrow[ur, "c_{F(f),\Psi}(\Phi(M))"] \arrow[d, "\Psi(\alpha_{F(f)}(\Phi(M)))"] & \\
& \Psi(\Phi(M)) &
\end{tikzcd}$$

Hence we have

$$\begin{aligned}
\alpha_{G(F(f))}(\Psi(\Phi(M)))c_{F(f),\Psi}(\Phi(M))\Psi(c_{f,\Phi}(M)) = \Psi(\alpha_{F(f)}(\Phi(M)))\Psi(c_{f,\Phi}(M)) &= \Psi(\Phi(\alpha_f(M))) \\
&= \alpha_{G(F(f))}(\Psi(\Phi(M)))c_{f,\Psi\Phi}(M).
\end{aligned}$$

Since $\alpha_{G(F(f))}(\Psi(\Phi(M)))$ is a morphism, it follows $c_{F(f),\Psi}(\Phi(M))\Psi(c_{f,\Phi}(M)) = c_{f,\Psi\Phi}(M)$.

(2) For $\varphi \in \mathcal{F}_X(f^*(M), g^*(N))$, since we have

$$\begin{aligned}
\Psi^{F(f),F(g)}_{\Phi(M),\Phi(N)}(\Phi^{f,g}_{M,N}(\varphi)) &= c_{F(g),\Psi}(\Phi(N))\Psi(c_{g,\Phi}(N)\Phi(\varphi)c_{f,\Phi}(M)^{-1})c_{F(f),\Psi}(\Phi(M))^{-1} \\
&= c_{F(g),\Psi}(\Phi(N))\Psi(c_{g,\Phi}(N))\Psi(\Phi(\varphi))\Psi(c_{f,\Phi}(M)^{-1})c_{F(f),\Psi}(\Phi(M))^{-1} \\
(\Psi\Phi)^{f,g}_{M,N}(\varphi) &= c_{g,\Psi\Phi}(N)\Psi(\Phi(\varphi))c_{f,\Psi\Phi}(M)^{-1},
\end{aligned}$$

the assertion follows from (1). □

Proposition 1.1.19 *Assume that Φ preserves cartesian morphisms. For morphisms $f : X \to Y$, $g : X \to Z$, $k : V \to X$ in $\mathcal{E}$ and $M \in \operatorname{Ob}\mathcal{F}_Y$, $N \in \operatorname{Ob}\mathcal{F}_Z$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(N)) \arrow[r, "k^\sharp_{M,N}"] \arrow[d, "\Phi^{f,g}_{M,N}"] & \mathcal{F}_V((fk)^*(M), (gk)^*(N)) \arrow[d, "\Phi^{fk,gk}_{M,N}"] \\
\mathcal{G}_{F(X)}(F(f)^*(\Phi(M)), F(g)^*(\Phi(N))) \arrow[r, "F(k)^\sharp_{\Phi(M),\Phi(N)}"] & \mathcal{G}_{F(V)}(F(fk)^*(\Phi(M)), F(gk)^*(\Phi(N)))
\end{tikzcd}$$

Proof. The following diagram is commutative by (1.1.14), (1.1.6) and the definition of $c_{k,\Phi}(f^*(M))$.

$$\begin{tikzcd}
\Phi((fk)^*(M)) \arrow[dd, "c_{fk,\Phi}(M)"] & \Phi(k^*(f^*(M))) \arrow[l, "\Phi(c_{f,k}(M))"'] \arrow[d, "c_{k,\Phi}(f^*(M))"] \arrow[dr, "\Phi(\alpha_k(f^*(M)))"] & \\
& F(k)^*(\Phi(f^*(M))) \arrow[r, "\alpha_{F(k)}(\Phi(f^*(M)))"] \arrow[d, "F(k)^*(c_{f,\Phi}(M))"] & \Phi(f^*(M)) \arrow[d, "c_{f,\Phi}(M)"] \\
F(fk)^*(\Phi(M)) & F(k)^*(F(f)^*(\Phi(M))) \arrow[l, "c_{F(f),F(k)}(\Phi(M))"'] \arrow[r, "\alpha_{F(k)}(F(f)^*(\Phi(M)))"] & F(f)^*(\Phi(M))
\end{tikzcd}$$

Hence we have the following equality.

$$\Phi(\alpha_k(f^*(M))c_{f,k}(M)^{-1})c_{fk,\Phi}(M)^{-1} = c_{f,\Phi}(M)^{-1}\alpha_{F(k)}(F(f)^*(\Phi(M)))c_{F(f),F(k)}(\Phi(M))^{-1} \quad \cdots (*)$$

Consider the cartesian morphism $\alpha_{F(gk)}(\Phi(N)) : F(gk)^*(\Phi(N)) \to \Phi(N)$. For $\varphi \in \mathcal{F}_X(f^*(M), g^*(N))$, we have

$$\begin{aligned}
\alpha_{F(gk)}(\Phi(N))\Phi^{fk,gk}_{M,N}(k^\sharp_{M,N}(\varphi)) &= \alpha_{F(gk)}(\Phi(N))c_{gk,\Phi}(N)\Phi(k^\sharp_{M,N}(\varphi))c_{fk,\Phi}(M)^{-1} \\
&= \Phi(\alpha_{gk}(N))\Phi(k^\sharp_{M,N}(\varphi))c_{fk,\Phi}(M)^{-1} \\
&= \Phi(\alpha_{gk}(N)c_{g,k}(N)k^*(\varphi)c_{f,k}(M)^{-1})c_{fk,\Phi}(M)^{-1} \\
&= \Phi(\alpha_g(N)\alpha_k(g^*(N))k^*(\varphi)c_{f,k}(M)^{-1})c_{fk,\Phi}(M)^{-1} \\
&= \Phi(\alpha_g(N))\Phi(\varphi\alpha_k(f^*(M))c_{f,k}(M)^{-1})c_{fk,\Phi}(M)^{-1} \\
&= \alpha_{F(g)}(\Phi(N))c_{g,\Phi}(N)\Phi(\varphi)\Phi(\alpha_k(f^*(M))c_{f,k}(M)^{-1})c_{fk,\Phi}(M)^{-1} \\
\alpha_{F(gk)}(\Phi(N))F(k)^\sharp_{\Phi(M),\Phi(N)}(\Phi^{f,g}_{M,N}(\varphi)) &= \alpha_{F(gk)}(\Phi(N))F(k)^\sharp_{\Phi(M),\Phi(N)}(c_{g,\Phi}(N)\Phi(\varphi)c_{f,\Phi}(M)^{-1}) \\
&= \alpha_{F(gk)}(\Phi(N))c_{F(g),F(k)}(\Phi(N))F(k)^*(c_{g,\Phi}(N)\Phi(\varphi)c_{f,\Phi}(M)^{-1})c_{F(f),F(k)}(\Phi(M))^{-1} \\
&= \alpha_{F(g)}(\Phi(N))\alpha_{F(k)}(F(g)^*(\Phi(N)))F(k)^*(c_{g,\Phi}(N)\Phi(\varphi)c_{f,\Phi}(M)^{-1})c_{F(f),F(k)}(\Phi(M))^{-1} \\
&= \alpha_{F(g)}(\Phi(N))c_{g,\Phi}(N)\Phi(\varphi)c_{f,\Phi}(M)^{-1}\alpha_{F(k)}(F(f)^*(\Phi(M)))c_{F(f),F(k)}(\Phi(M))^{-1}.
\end{aligned}$$

Then, $(*)$ implies $\alpha_{F(gk)}(\Phi(N))(\Phi^{fk,gk}_{M,N}k^{\sharp}_{M,N}(\varphi)) = \alpha_{F(gk)}(\Phi(N))F(k)^{\sharp}_{\Phi(M),\Phi(N)}(\Phi^{f,g}_{M,N}(\varphi))$. Therefore we have $\Phi^{fk,gk}_{M,N}k^{\sharp}_{M,N}(\varphi) = F(k)^{\sharp}_{\Phi(M),\Phi(N)}\Phi^{f,g}_{M,N}(\varphi)$. $\square$

For a cloven fibered category $p : \mathcal{F} \to \mathcal{E}$, we define a category $\widetilde{\mathcal{F}}$ as follows. Put

$$\operatorname{Ob}\widetilde{\mathcal{F}} = \{(X, M) \,|\, X \in \operatorname{Ob}\mathcal{E},\ M \in \operatorname{Ob}\mathcal{F}_X\}.$$

For $(X, M), (Y, N) \in \operatorname{Ob}\widetilde{\mathcal{F}}$, we put

$$\widetilde{\mathcal{F}}((X, M), (Y, N)) = \{(f, \boldsymbol{\varphi}) \,|\, f \in \mathcal{E}(X, Y),\ \boldsymbol{\varphi} \in \mathcal{F}_X(M, f^*(N))\}.$$

For $(f, \boldsymbol{\varphi}) \in \widetilde{\mathcal{F}}((X, M), (Y, N))$ and $(g, \boldsymbol{\psi}) \in \widetilde{\mathcal{F}}((Y, N), (Z, L))$, define the composition of $(f, \boldsymbol{\varphi})$ and $(g, \boldsymbol{\psi})$ by

$$(g, \boldsymbol{\psi})(f, \boldsymbol{\varphi}) = (gf, c_{g,f}(L)f^*(\boldsymbol{\psi})\boldsymbol{\varphi}).$$

The identity morphism of (X, M) is defined by $id_{(X,M)} = (id_X, \alpha_{id_X}(M)^{-1})$. For $(f, \boldsymbol{\varphi}) \in \widetilde{\mathcal{F}}((X, M), (Y, N))$, $(g, \boldsymbol{\psi}) \in \widetilde{\mathcal{F}}((Y, N), (Z, L))$ and $(h, \boldsymbol{\xi}) \in \widetilde{\mathcal{F}}((Z, L), (W, T))$, it can be verified from (1.1.12) that

$$\begin{aligned}
(f, \boldsymbol{\varphi})(id_X, \alpha_{id_X}(M)^{-1}) &= (f\,id_X, c_{f,id_X}(N)id_X^*(\boldsymbol{\varphi})\alpha_{id_X}(M)^{-1}) = (f, c_{f,id_X}(N)\alpha_{id_X}(f^*(N))^{-1}\boldsymbol{\varphi}) = (f, \boldsymbol{\varphi})\\
(id_Y, \alpha_{id_Y}(N)^{-1})(f, \boldsymbol{\varphi}) &= (id_Y f, c_{id_Y,f}(N)f^*(\alpha_{id_Y}(N)^{-1})\boldsymbol{\varphi}) = (f, \boldsymbol{\varphi})\\
(h, \boldsymbol{\xi})((g, \boldsymbol{\psi})(f, \boldsymbol{\varphi})) &= (h, \boldsymbol{\xi})(gf, c_{g,f}(L)f^*(\boldsymbol{\psi})\boldsymbol{\varphi}) = (hgf, c_{h,gf}(T)(gf)^*(\boldsymbol{\xi})c_{g,f}(L)f^*(\boldsymbol{\psi})\boldsymbol{\varphi})\\
&= (hgf, c_{hg,f}(T)f^*(c_{h,g}(T))f^*(g^*(\boldsymbol{\xi}))f^*(\boldsymbol{\psi})\boldsymbol{\varphi}) = (hgf, c_{hg,f}(T)f^*(c_{h,g}(T)g^*(\boldsymbol{\xi})\boldsymbol{\psi})\boldsymbol{\varphi})\\
&= (hg, c_{h,g}(T)g^*(\boldsymbol{\xi})\boldsymbol{\psi})(f, \boldsymbol{\varphi}) = ((h, \boldsymbol{\xi})(g, \boldsymbol{\psi}))(f, \boldsymbol{\varphi})
\end{aligned}$$

$$\begin{tikzcd}
f^*(g^*(L)) \arrow[rr, "c_{g,f}(L)"] \arrow[d, "f^*(g^*(\xi))"] & & (gf)^*(L) \arrow[r, equal] & (gf)^*(L) \arrow[d, "(gf)^*(\xi)"] \\
f^*(g^*(h^*(T))) \arrow[r, equal] \arrow[d, "f^*(c_{h,g}(T))"] & (f^*g^*)(h^*(T)) \arrow[rr, "c_{g,f}(h^*(T))"] & & (gf)^*(h^*(T)) \arrow[d, "c_{h,gf}(T)"] \\
f^*((hg)^*(T)) \arrow[rr, "c_{hg,f}(T)"] & & ((hg)f)^*(T) \arrow[r, equal] & (h(gf))^*(T)
\end{tikzcd}$$

Therefore $\tilde{\mathcal{F}}$ is a category. We define a functors $\tilde{p} : \widetilde{\mathcal{F}} \to \mathcal{E}$ and $\Theta : \widetilde{\mathcal{F}} \to \mathcal{F}$ by $\tilde{p}(X, M) = X$, $\tilde{p}(f, \boldsymbol{\varphi}) = f$ and $\Theta(X, M) = M$, $\Theta(f, \boldsymbol{\varphi}) = \alpha_f(N)\boldsymbol{\varphi}$ for $(X, M) \in \operatorname{Ob}\widetilde{\mathcal{F}}$ and $(f, \boldsymbol{\varphi}) \in \widetilde{\mathcal{F}}((X, M), (Y, N))$. It is clear that $\tilde{p}$ is a functor and that $p\Theta = \tilde{p}$. Since

$$\begin{aligned}
\Theta(id_X, \alpha_{id_X}(M)^{-1}) &= \alpha_{id_X}(M)\alpha_{id_X}(M)^{-1} = id_M\\
\Theta((g, \boldsymbol{\psi})(f, \boldsymbol{\varphi})) &= \Theta(gf, c_{g,f}(L)f^*(\boldsymbol{\psi})\boldsymbol{\varphi}) = \alpha_{gf}(L)c_{g,f}(L)f^*(\boldsymbol{\psi})\boldsymbol{\varphi} = \alpha_g(L)\alpha_f(g^*(L))f^*(\boldsymbol{\psi})\boldsymbol{\varphi}\\
&= \alpha_g(L)\boldsymbol{\psi}\alpha_f(N)\boldsymbol{\varphi} = \Theta(g, \boldsymbol{\psi})\Theta(f, \boldsymbol{\varphi}),
\end{aligned}$$

Θ is also a functor.

Proposition 1.1.20 *Θ is an isomorphism of categories.*

Proof. Define a functor $\Theta^{-1} : \mathcal{F} \to \widetilde{\mathcal{F}}$ by $\Theta^{-1}(M) = (p(M), M)$ and $\Theta^{-1}(\varphi) = (p(\varphi), \bar{\varphi})$ for $M \in \operatorname{Ob}\mathcal{F}$ and $\varphi \in \mathcal{F}(M, N)$, where $\bar{\varphi} \in \mathcal{F}_{p(M)}(M, p(\varphi)^*(N))$ is unique morphism that is mapped to φ by the bijection $\alpha_{p(\varphi)}(N)_* : \mathcal{F}_{p(M)}(M, p(\varphi)^*(N)) \to \mathcal{F}_{p(\varphi)}(M, N)$. It is clear that Θ^{-1} is the inverse of Θ. $\square$

Suppose that $X \xleftarrow{\pi_f} E \times_Y X \xrightarrow{f_\pi} E$ is a limit of a diagram $X \xrightarrow{f} Y \xleftarrow{\pi} E$ in $\mathcal{E}$. For morphisms $\varphi : V \to E$ and $\psi : V \to X$ in $\mathcal{E}$ which satisfy $\pi\varphi = f\psi$, we denote by $(\varphi, \psi) : V \to E \times_Y X$ the unique morphism that satisfy $f_\pi(\varphi, \psi) = \varphi$ and $\pi_f(\varphi, \psi) = \psi$. Moreover, if $W \xleftarrow{\mathrm{pr}_W} F \times_Y X \xrightarrow{\mathrm{pr}_F} E$ is a limit of a diagram $W \xrightarrow{fg} Y \xleftarrow{\rho\pi} E$ for morphisms $\rho : F \to E$ and $g : W \to X$ in $\mathcal{E}$, we denote $(\rho\mathrm{pr}_F, g\mathrm{pr}_W)$ by $\rho \times_Y g$.

$$\begin{tikzcd}
V \arrow[dr, dotted, "(\varphi,\psi)"] \arrow[drr, bend left, "\varphi"] \arrow[ddr, bend right, "\psi"'] & & \\
& E \times_Y X \arrow[r, "f_\pi"] \arrow[d, "\pi_f"] & E \arrow[d, "\pi"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
F \times_Y W \arrow[rr, "\mathrm{pr}_F"] \arrow[dd, "\mathrm{pr}_W"'] \arrow[dr, dotted, "\rho \times_Y g"] & & F \arrow[d, "\rho"] \\
& E \times_Y X \arrow[r, "f_\pi"] \arrow[d, "\pi_f"] & E \arrow[d, "\pi"] \\
W \arrow[r, "g"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

We need to introduce the notion of "cartesian section" in order to define the notion of trivial representation.

Definition 1.1.21 *([6], p.164 Définition 5.5.) Let $p : \mathcal{F} \to \mathcal{E}$ be a functor. We call a functor $s : \mathcal{E} \to \mathcal{F}$ a cartesian section if $ps = id_{\mathcal{E}}$ and $s(f)$ is cartesian for any $f \in \operatorname{Mor}\mathcal{E}$. The subcategory of $\operatorname{Funct}(\mathcal{E}, \mathcal{F})$ consisting of cartesian sections and morphisms $\varphi : s \to s'$ satisfying $p(\varphi_X) = id_X$ for any $X \in \operatorname{Ob}\mathcal{E}$ is denoted by $\underleftarrow{\operatorname{Lim}}(\mathcal{F}/\mathcal{E})$.*

Proposition 1.1.22 *([4], Lemme 5.7) If $\mathcal{E}$ has a terminal object 1, then the functor $e : \underleftarrow{\operatorname{Lim}}(\mathcal{F}/\mathcal{E}) \to \mathcal{F}_1$ given by $e(s) = s(1)$ and $e(\varphi) = \varphi_1$ is fully faithful. Moreover, if $p : \mathcal{F} \to \mathcal{E}$ is a fibered category, e is an equivalence of categories.*

Remark 1.1.23 *For a cartesian section $s : \mathcal{E} \to \mathcal{F}$ of a fibered category $p : \mathcal{F} \to \mathcal{E}$ and a morphism $f : X \to Y$ in $\mathcal{E}$ and , let us denote by $s_f : s(X) \to f^*(s(Y))$ the unique morphism in $\mathcal{F}_X$ satisfying $\alpha_f(s(Y))s_f = s(f)$. We note that if $s = s_T$ for $T \in \operatorname{Ob}\mathcal{F}_1$, $s_f = c_{o_Y,f}(T)^{-1}$ by the definition of $s_T(f)$ above. Since both $s(f)$ and $\alpha_f(s(Y))$ are cartesian morphisms, s_f is necessarily an isomorphism. Hence, for morphisms $f : X \to Y$ and $g : X \to Z$ in $\mathcal{E}$, we define $s_{f,g} : f^*(s(Y)) \to g^*(s(Z))$ by $s_{f,g} = s_g s_f^{-1}$.*

1.2 Bifibered category

We briefly review the notion of bifibered category following section 10 of [6].

Definition 1.2.1 *Let $p : \mathcal{F} \to \mathcal{E}$ be a functor and $\alpha : M \to N$ a morphism in $\mathcal{F}$. Set $X = p(M)$, $Y = p(N)$, $f = p(\alpha)$. We call α a cocartesian morphism if, for any $N' \in \operatorname{Ob}\mathcal{F}_Y$, the map $\mathcal{F}_X(N, N') \to \mathcal{F}_f(M, N')$ defined by $\varphi \mapsto \varphi\alpha$ is bijective.*

The following assertion is the dual of (1.1.2).

Proposition 1.2.2 *If $\alpha_i : M \to N_i$ $(i = 1, 2)$ are cocartesian morphisms in $\mathcal{F}$ such that $p(N_1) = p(N_2)$ and $p(\alpha_1) = p(\alpha_2)$, there is a unique morphism $\psi : N_1 \to N_2$ such that $\alpha_1 = \alpha_2\psi$ and $p(\psi) = id_{p(N_1)}$. Moreover, ψ is an isomorphism.*

Definition 1.2.3 *Let $f : X \to Y$ be a morphism in $\mathcal{E}$ and $M \in \operatorname{Ob}\mathcal{F}_X$. If there exists a cocartesian morphism $\alpha : M \to N$ such that $p(\alpha) = f$, N is called a direct image of M by f. We denote M by $f_*(N)$ and α by $\alpha^f(M) : M \to f_*(M)$. By (1.2.2), $f_*(N)$ is unique up to isomorphism.*

Proposition 1.2.4 *Let $\alpha : M \to N$, $\alpha' : M' \to N'$ be morphisms in $\mathcal{F}$ such that $p(M) = p(M')$, $p(N) = p(N')$, $p(\alpha) = p(\alpha')(= f)$ and $\lambda : M \to M'$ a morphism in $\mathcal{F}$ such that $p(\lambda) = id_{p(M)}$. If α' is cocartesian, there is a unique morphism $\mu : N \to N'$ such that $p(\mu) = id_{p(N)}$ and $\alpha'\mu = \lambda\alpha$.*

Corollary 1.2.5 *Let $f : X \to Y$ be a morphism in $\mathcal{E}$. If, for any $M \in \operatorname{Ob}\mathcal{F}_X$, there exists a cocartesian morphism $\alpha^f(M) : M \to f_*(M)$, $M \mapsto f_*(M)$ defines a functor $f_* : \mathcal{F}_X \to \mathcal{F}_Y$.*

Definition 1.2.6 *If the assumption of (1.2.5) is satisfied, we say that the functor of the direct image by f exists.*

Definition 1.2.7 *If a functor $p : \mathcal{F} \to \mathcal{E}$ sadisfies the following condition (i), p is called a precofibered category and if p satisfies both (i) and (ii), p is called a cofibered category or p is cofibrant.*

(i) For any morphism f in $\mathcal{E}$, the functor of the direct image by f exists.
(ii) The composition of cocartesian morphisms is cocartesian.

In other words, $p : \mathcal{F} \to \mathcal{E}$ is a precofibered (resp. cofibered) category if and only if $p : \mathcal{F}^{op} \to \mathcal{E}^{op}$ is a prefibered (resp. fibered) category.

Let $p : \mathcal{F} \to \mathcal{E}$ be a functor. A map $\kappa : \operatorname{Mor}\mathcal{E} \to \coprod_{X,Y \in \operatorname{Ob}\mathcal{E}} \operatorname{Funct}(\mathcal{F}_X, \mathcal{F}_Y)$ is called a cocleavage if $\kappa(f)$ is a direct image functor $f_* : \mathcal{F}_X \to \mathcal{F}_Y$ for $(f : X \to Y) \in \operatorname{Mor}\mathcal{E}$. A cocleavage κ is said to be normalized if $\kappa(id_X) = id_{\mathcal{F}_X}$ for any $X \in \operatorname{Ob}\mathcal{E}$. A category $\mathcal{F}$ over $\mathcal{E}$ is called a cloven precofibered category (resp. normalized cloven precofibered category) if a cocleavage (resp. normalized cocleavage) is given.

$p : \mathcal{F} \to \mathcal{E}$ has a cocleavage if and only if p is precofibered. If p is precofibered, p has a normalized cocleavage.

Let $f : X \to Y$, $g : Z \to X$ be morphisms in $\mathcal{E}$ and M an object of $\mathcal{F}_Z$. If $p : \mathcal{F} \to \mathcal{E}$ is a precofibered category, there is a unique morphism $c^{f,g}(M) : (fg)_*(M) \to f_*g_*(M)$ such that the following square commutes and $p(c_{f,g}(M)) = id_Z$.

$$\begin{tikzcd}
M \arrow[r, "\alpha^{fg}(M)"] \arrow[d, "\alpha^{g}(M)"'] & (fg)_*(M) \arrow[d, "c^{f,g}(M)"] \\
g_*(M) \arrow[r, "\alpha^{f}(g_*(M))"] & f_*g_*(M)
\end{tikzcd}$$

The following is the dual of (1.1.9).

Proposition 1.2.8 *Let $p : \mathcal{F} \to \mathcal{E}$ be a cloven precofibered category. Then, p is a cofibered category if and only if $c^{f,g}(M)$ is an isomorphism for any $Z \xrightarrow{g} X \xrightarrow{f} Y$ and $M \in \operatorname{Ob}\mathcal{F}_Z$.*

Proposition 1.2.9 *Let $p : \mathcal{F} \to \mathcal{E}$ be a functor and $f : X \to Y$ a morphism in $\mathcal{E}$.*

(1) Suppose that the functor of the inverse image by f exists. Then, the inverse image $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ by f has a left adjoint if and only if the functor of the direct image by f exists.

(2) Suppose that the functor of the direct image by f exists. Then, the direct image $f_* : \mathcal{F}_X \to \mathcal{F}_Y$ by f has a right adjoint if and only if the functor of the inverse image by f exists.

Proof. (1) Suppose that the functor of the inverse image by f exists and that it has a left adjoint $f_* : \mathcal{F}_X \to \mathcal{F}_Y$. We denote by $\eta : id_{\mathcal{F}_X} \to f^*f_*$ the unit of the adjunction $f_* \dashv f^*$. For $M \in \operatorname{Ob}\mathcal{F}_X$, set $\alpha^f(M) = \alpha_f(f_*(M))\eta_M : M \to f_*(M)$. By the assumption, the following composition is bijective for any $M \in \operatorname{Ob}\mathcal{F}_X$, $N \in \operatorname{Ob}\mathcal{F}_Y$.

$$\mathcal{F}_Y(f_*(M), N) \xrightarrow{f^*} \mathcal{F}_X(f^*f_*(M), f^*(N)) \xrightarrow{\eta_M^*} \mathcal{F}_X(M, f^*(N)) \xrightarrow{\alpha_f(N)_*} \mathcal{F}_f(M, N)$$

We note that, since $\alpha_f(N)f^*(\varphi) = \varphi\alpha_f(f_*(M))$ for $\varphi \in \mathcal{F}_Y(f_*(M), N)$, the above composition coincides with the map $\alpha^f(M)^* : \mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_f(M, N)$ induced by $\alpha^f(M)$. This shows that the functor of the direct image by f exists.

Conversely, assume that the functor of the direct image by f exists. For $M \in \operatorname{Ob}\mathcal{F}_X$, let us denote by $\alpha^f(M) : M \to f_*(M)$ a cocartesian morphism. Then, we have bijections $\alpha^f(M)^* : \mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_f(M, N)$ and $\alpha_f(M)_* : \mathcal{F}_X(M, f^*(N)) \to \mathcal{F}_f(M, N)$ given by $\psi \mapsto \psi\alpha^f(M)$ and $\varphi \mapsto \alpha_f(M)\varphi$, which are natural in $M \in \operatorname{Ob}\mathcal{F}_X$ and $N \in \operatorname{Ob}\mathcal{F}_Y$. Thus we have a natural bijection $\mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_X(M, f^*(N))$.

(2) Suppose that the functor of the direct image by f exists and that it has a right adjoint $f^* : \mathcal{F}_Y \to \mathcal{F}_X$. We denote by $\varepsilon : f_*f^* \to id_{\mathcal{F}_Y}$ the counit of the adjunction $f_* \dashv f^*$. For $N \in \operatorname{Ob}\mathcal{F}_Y$, set $\alpha_f(N) = \varepsilon_N\alpha^f(f^*(N)) : f^*(N) \to N$. By the assumption, the following composition is bijective for any $M \in \operatorname{Ob}\mathcal{F}_X$, $N \in \operatorname{Ob}\mathcal{F}_Y$.

$$\mathcal{F}_X(M, f^*(N)) \xrightarrow{f_*} \mathcal{F}_Y(f_*(M), f_*f^*(N)) \xrightarrow{\varepsilon_{N*}} \mathcal{F}_Y(f_*(M), N) \xrightarrow{\alpha^f(M)^*} \mathcal{F}_f(M, N)$$

We note that, since $f_*(\varphi)\alpha^f(M) = \alpha^f(f^*(N))\varphi$ for $\varphi \in \mathcal{F}_X(M, f^*(N))$, the above composition coincides with the map $\alpha_f(N)_* : \mathcal{F}_X(M, f^*(N)) \to \mathcal{F}_f(M, N)$ induced by $\alpha_f(N)$. This shows that the functor of the inverse image by f exists.

Conversely, assume that the functor of the inverse image by f exists. For $N \in \operatorname{Ob}\mathcal{F}_Y$, let us denote by $\alpha_f(N) : f^*(N) \to N$ a cartesian morphism. Then, we have bijections $\alpha_f(N)_* : \mathcal{F}_X(M, f^*(N)) \to \mathcal{F}_f(M, N)$ and $\alpha^f(M)^* : \mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_f(M, N)$ given by $\varphi \mapsto \alpha_f(N)\varphi$ and $\psi \mapsto \psi\alpha^f(M)\varphi$, which are natural in $M \in \operatorname{Ob}\mathcal{F}_X$ and $N \in \operatorname{Ob}\mathcal{F}_Y$. Thus we have a natural bijection $\mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_X(M, f^*(N))$. □

Remark 1.2.10 *Let $p : \mathcal{F} \to \mathcal{E}$ be a functor and $f : X \to Y$ a morphism in $\mathcal{E}$ such that the functors of the inverse and direct images by f exist. For $M \in \operatorname{Ob}\mathcal{F}_X$ and $N \in \mathcal{F}_Y$, since there exist a cartesian morphism $\alpha_f(N) : f^*(N) \to N$ and a cocartesian morphism $\alpha^f(M) : M \to f_*(M)$, there is a bijection $ad_f(M, N) : \mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_X(M, f^*(N))$ which satisfies $\alpha_f(N)ad_f(M, N)(\varphi) = \varphi\alpha^f(M)$ for any $\varphi \in \mathcal{F}_Y(f_*(M), N)$. Hence the unit $\eta : id_{\mathcal{F}_X} \to f^*f_*$ of the adjunction $f_* \dashv f^*$ is the unique natural transformation satisfying $\alpha_f(f_*(M))\eta_M = \alpha^f(M)$ for any $M \in \operatorname{Ob}\mathcal{F}_X$. Dually, the counit $\varepsilon : f_*f^* \to id_{\mathcal{F}_Y}$ is the unique natural transformation satisfying $\varepsilon_N\alpha^f(f^*(N)) = \alpha_f(N)$ for any $N \in \operatorname{Ob}\mathcal{F}_Y$.*

Proposition 1.2.11 *([6], p.182 Proposition 10.1.) Let $p : \mathcal{E} \to \mathcal{F}$ be a prefibered and precofibered category. Then, it is a fibered category if and only if it is a cofibered category.*

Proof. For a morphism $f : X \to Y$ in $\mathcal{E}$, we denote by $\eta^f : id_{\mathcal{F}_X} \to f^*f_*$ the unit of the adjunction $f_* \dashv f^*$. Let $f : X \to Y$, $g : Z \to X$ be morphisms in $\mathcal{E}$. For $M \in \operatorname{Ob}\mathcal{F}_Z$ and $N \in \operatorname{Ob}\mathcal{F}_Y$, we claim that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}_X(f^*f_*g_*(M), f^*(N)) \arrow[d, "\eta^{f*}_{g_*(M)}"] & \mathcal{F}_Y(f_*g_*(M), N) \arrow[l, "f^*"'] \arrow[r, "c^{f,g}(M)^*"] & \mathcal{F}_Y((fg)_*(M), N) \arrow[d, "(fg)^*"] \\
\mathcal{F}_X(g_*(M), f^*(N)) \arrow[d, "g^*"] & & \mathcal{F}_Z((fg)^*(fg)_*(M), (fg)^*(N)) \arrow[d, "\eta^{fg*}_M"] \\
\mathcal{F}_Z(g^*g_*(M), g^*f^*(N)) \arrow[r, "\eta^{g*}_M"] & \mathcal{F}_Z(M, g^*f^*(N)) \arrow[r, "c_{f,g}(M)_*"] & \mathcal{F}_Z(M, (fg)^*(N))
\end{tikzcd}$$

Let $\psi : f_*g_*(M) \to N$ be a morphism in $\mathcal{F}_Y$. Then we have

$$\begin{aligned}
\alpha_{fg}(N)\eta^{fg*}_M(fg)^*c^{f,g}(M)^*(\psi) &= \alpha_{fg}(N)(fg)^*(\psi)(fg)^*(c^{f,g}(M))\eta^{fg}_M = \psi\alpha_{fg}(f_*g_*(M))(fg)^*(c^{f,g}(M))\eta^{fg}_M \\
&= \psi c^{f,g}(M)\alpha_{fg}((fg)_*(M))\eta^{fg}_M = \psi c^{f,g}(M)\alpha^{fg}(M) = \psi\alpha^f(g_*(M))\alpha^g(M) \\
&= \psi\alpha_f(f_*g_*(M))\eta^f_{g_*(M)}\alpha_g(g_*(M))\eta^g_M = \alpha_f(N)f^*(\psi)\alpha_g(f^*f_*g_*(M))g^*(\eta^f_{g_*(M)})\eta^g_M \\
&= \alpha_f(N)\alpha_g(f^*(N))g^*f^*(\psi)g^*(\eta^f_{g_*(M)})\eta^g_M = \alpha_{fg}(N)c_{f,g}(N)g^*f^*(\psi)g^*(\eta^f_{g_*(M)})\eta^g_M \\
&= \alpha_{fg}(N)c_{f,g}(N)_*\eta^{g*}_M g^*\eta^{f*}_{g_*(M)}(\psi).
\end{aligned}$$

Since $\alpha_{fg}(N) : (fg)^*(N) \to N$ is cartesian and both $\eta^{fg*}_M(fg)^*c^{f,g}(M)^*(\psi)$ and $c_{f,g}(N)_*\eta^{g*}_M g^*\eta^{f*}_{g_*(M)}(\psi)$ are morphisms in $\mathcal{F}_Y$, we see that the above diagram commutes. Note that the compositions $\eta^{f*}_M f^* : \mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_X(M, f^*(N))$, $\eta^{g*}_M g^* : \mathcal{F}_X(g_*(M), N) \to \mathcal{F}_Z(M, g^*(N))$ and $\eta^{fg*}_M(fg)^* : \mathcal{F}_Y((fg)_*(M), N) \to \mathcal{F}_Z(M, (fg)^*(N))$ are bijective. Hence, by the commutativity of the above diagram, $c_{f,g}(N)_*$ is bijective if and only if $c^{f,g}(M)^*$ is so. Then the assertion follows from (1.1.9) and (1.2.8). □

Definition 1.2.12 *We call a functor $p : \mathcal{F} \to \mathcal{E}$ a bifibered category if it is a fibered and cofibered category.*

Let $p : \mathcal{F} \to \mathcal{E}$ be a cloven fibered category. Suppose that morphisms $f, g : X \to Y$ and $h : Y \to Z$ in $\mathcal{E}$ satisfy $hf = hg$ and that functors $f^*, g^* : \mathcal{F}_Y \to \mathcal{F}_X$ and $h^* : \mathcal{F}_Z \to \mathcal{F}_Y$ have left adjoints $f_*, g_* : \mathcal{F}_X \to \mathcal{F}_Y$ and $h_* : \mathcal{F}_Y \to \mathcal{F}_Z$, respectively. We denote by $ad_f(M,N) : \mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_X(M, f^*(N))$, $ad_g(M,N) : \mathcal{F}_Y(g_*(M), N) \to \mathcal{F}_X(M, g^*(N))$, $ad_h(N,L) : \mathcal{F}_Z(h_*(N), L) \to \mathcal{F}_Y(N, h^*(L))$ the natural bijections for $M \in \operatorname{Ob}\mathcal{F}_X$, $N \in \operatorname{Ob}\mathcal{F}_Y$, $L \in \operatorname{Ob}\mathcal{F}_Z$. Let $\Phi_{M,L}$ be the following composition.

$$\begin{aligned}
\mathcal{F}_Z(h_*(f_*(M)), L) &\xrightarrow{ad_h(f_*(M),L)} \mathcal{F}_Y(f_*(M), h^*(L)) \xrightarrow{ad_f(M,h^*(L))} \mathcal{F}_X(M, f^*(h^*(L))) \xrightarrow{c_{h,f}(L)_*} \\
&\mathcal{F}_X(M, (hf)^*(L)) = \mathcal{F}_X(M, (hg)^*(L)) \xrightarrow{c_{h,g}(L)^{-1}} \mathcal{F}_X(M, g^*(h^*(L))) \xrightarrow{ad_g(M,h^*(L))^{-1}} \\
&\mathcal{F}_Y(g_*(M), h^*(L)) \xrightarrow{ad_h(g_*(M),L)^{-1}} \mathcal{F}_Z(h_*(g_*(M)), L)
\end{aligned}$$

Then, $\Phi_{M,L}$ is a natural bijection. We put $\xi_M = \Phi_{M,h_*(f_*(M))}(id_{h_*(f_*(M))}) : h_*(g_*(M)) \to h_*(f_*(M))$. Then, ξ_M gives a natural equivalence $\xi : h_*g_* \to h_*f_*$. For $\varphi \in \mathcal{F}_Z(h_*(f_*(M)), L)$, the following diagram commutes by the naturality of $\Phi_{M,L}$.

$$\begin{tikzcd}
\mathcal{F}_Z(h_*(f_*(M)), h_*(f_*(M))) \arrow[r, "\varphi_*"] \arrow[d, "\Phi_{M,h_*(f_*(M))}"] & \mathcal{F}_Z(h_*(f_*(M)), L) \arrow[d, "\Phi_{M,L}"] \\
\mathcal{F}_Z(h_*(g_*(M)), h_*(f_*(M))) \arrow[r, "\varphi_*"] & \mathcal{F}_Z(h_*(g_*(M)), L)
\end{tikzcd}$$

Thus we have $\Phi_{M,L}(\varphi) = \varphi\xi_M = \xi^*_M(\varphi)$, in other words, the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}_Z(h_*(f_*(M)), L) \arrow[r, "ad_h(f_*(M),L)"] \arrow[d, "\xi^*_M"] & \mathcal{F}_Y(f_*(M), h^*(L)) \arrow[r, "ad_f(M,h^*(L))"] & \mathcal{F}_X(M, f^*(h^*(L))) \arrow[d, "c_{h,f}(L)_*"] \\
\mathcal{F}_Z(h_*(g_*(M)), L) \arrow[d, "ad_h(g_*(M),L)"] & & \mathcal{F}_X(M, (hf)^*(L)) \arrow[d, equal] \\
\mathcal{F}_Y(g_*(M), h^*(L)) \arrow[r, "ad_g(M,h^*(L))"] & \mathcal{F}_X(M, g^*(h^*(L))) \arrow[r, "c_{h,g}(L)_*"] & \mathcal{F}_X(M, (hg)^*(L))
\end{tikzcd}$$

Proposition 1.2.13 *Let $p : \mathcal{F} \to \mathcal{E}$ be a cloven bifibered category. Suppose that a pair of morphisms $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ of $\mathcal{E}$ has a coequalizer $h : Y \to Z$. Let $\varphi, \psi : M \to N$ be morphisms in $\mathcal{F}$ satisfying $p(\varphi) = f$ and $p(\psi) = g$. Let $\tilde{\varphi} : M \to f^*(N)$ and $\tilde{\psi} : M \to g^*(N)$ be unique morphisms in $\mathcal{F}_X$ that satisfy $\alpha_f(N)\tilde{\varphi} = \varphi$ and $\alpha_g(N)\tilde{\psi} = \psi$. We put ${}^t\tilde{\varphi} = ad_f(M,N)^{-1}(\tilde{\varphi}) : f_*(M) \to N$ and ${}^t\tilde{\psi} = ad_g(M,N)^{-1}(\tilde{\psi}) : g_*(M) \to N$. Suppose that there exists a coequalizer $\pi : h_*(N) \to L$ of morphisms $h_*({}^t\tilde{\varphi})\xi_M : h_*(g_*(M)) \to h_*(N)$ and $h_*({}^t\tilde{\psi}) : h_*(g_*(M)) \to h_*(N)$ of $\mathcal{F}_Z$. Then a composition $N \xrightarrow{ad_h(N,L)(\pi)} h^*(L) \xrightarrow{\alpha_h(L)} L$ is a coequalizer of $M \overset{\varphi}{\underset{\psi}{\rightrightarrows}} N$.*

Proof. Since $\pi h_*({}^t\tilde{\psi}) = \pi h_*({}^t\tilde{\varphi})\xi_M = \xi_M^*(\pi h_*({}^t\tilde{\varphi})) = \Phi_{M,L}(\pi h_*({}^t\tilde{\varphi}))$, we have the following equality.

$$c_{h,g}(L)ad_g(M,h^*(L))(ad_h(g_*(M),L)(\pi h_*({}^t\tilde{\psi}))) = c_{h,f}(L)ad_f(M,h^*(L))(ad_h(f_*(M),L)(\pi h_*({}^t\tilde{\varphi}))) \cdots (i)$$

We put $\pi^a = ad_h(N,L)(\pi) : N \to h^*(L)$. Then, by the naturality of ad_f, ad_g, ad_h we have

$$\text{(the left hand side of } (i)) = c_{h,g}(L)ad_g(M,h^*(L))(\pi^{a\,t}\tilde{\psi}) = c_{h,g}(L)g^*(\pi^a)ad_f(M,N)({}^t\tilde{\psi}) = c_{h,g}(L)g^*(\pi^a)\tilde{\psi}$$
$$\text{(the right hand side of } (i)) = c_{h,f}(L)ad_f(M,h^*(L))(\pi^{a\,t}\tilde{\varphi}) = c_{h,f}(L)f^*(\pi^a)ad_f(M,N)({}^t\tilde{\varphi}) = c_{h,f}(L)f^*(\pi^a)\tilde{\varphi}$$

and since the following diagrams commutes, it follows $\alpha_h(L)\pi^a\varphi = \alpha_h(L)\pi^a\psi$.

$$\begin{tikzcd}
(hg)^*(L) \arrow[d, "\alpha_{hg}(L)"] & g^*(h^*(L)) \arrow[l, "c_{h,g}(L)"'] \arrow[d, "\alpha_g(h^*(L))"] & g^*(N) \arrow[l, "g^*(\pi^a)"'] \arrow[d, "\alpha_g(N)"] & M \arrow[l, "\tilde{\psi}"'] \arrow[r, "\tilde{\varphi}"] \arrow[d, equal] & f^*(N) \arrow[r, "f^*(\pi^a)"] \arrow[d, "\alpha_f(N)"] & f^*(h^*(L)) \arrow[r, "c_{h,f}(L)"] \arrow[d, "\alpha_f(h^*(L))"] & (hf)^*(L) \arrow[d, "\alpha_{hf}(L)"] \\
L & h^*(L) \arrow[l, "\alpha_h(L)"'] & N \arrow[l, "\pi^a"'] & M \arrow[l, "\psi"'] \arrow[r, "\varphi"] & N \arrow[r, "\pi^a"] & h^*(L) \arrow[r, "\alpha_h(L)"] & L
\end{tikzcd}$$

Let $\rho : N \to P$ be a morphism in $\mathcal{F}$ which satisfies $\rho\varphi = \rho\psi$. Then $p(\rho)f = p(\rho)g$ and there exists unique morphism $k : Z \to p(P)$ that satisfies $kh = p(\rho)$. Let $\tilde{\rho} : N \to p(\rho)^*(P) = (kh)^*(P)$ the unique morphism in $\mathcal{F}_Y$ that satisfies $\alpha_{kh}(P)\tilde{\rho} = \rho$. Then, $\alpha_{kh}(P)\tilde{\rho}\alpha_f(N)\tilde{\varphi} = \alpha_{kh}(P)\tilde{\rho}\alpha_g(N)\tilde{\psi}$ and this implies the following.

$$\alpha_{khf}(P)c_{kh,f}(P)f^*(\tilde{\rho})\tilde{\varphi} = \alpha_{kh}(P)\alpha_f((kh)^*(P))f^*(\tilde{\rho})\tilde{\varphi} = \alpha_{kh}(P)\alpha_g((kh)^*(P))g^*(\tilde{\rho})\tilde{\psi} = \alpha_{khg}(P)c_{kh,g}(P)g^*(\tilde{\rho})\tilde{\psi}$$

Since $hf = hg$ and $\alpha_{khf}(P)$ is a cartesian morphism, we have $c_{kh,f}(P)f^*(\tilde{\rho})\tilde{\varphi} = c_{kh,g}(P)g^*(\tilde{\rho})\tilde{\psi}$. On the other hand, it follows from (1.1.12) that there are the following equalities.

$$\begin{aligned}
c_{h,f}(k^*(P))^{-1}c_{k,hf}(P)^{-1}c_{kh,f}(P)f^*(\tilde{\rho})\tilde{\varphi} &= (c_{k,hf}(P)c_{h,f}(k^*(P)))^{-1}c_{kh,f}(P)f^*(\tilde{\rho})\tilde{\varphi} \\
&= (c_{kh,f}(P)f^*(c_{k,h}(P)))^{-1}c_{kh,f}(P)f^*(\tilde{\rho})\tilde{\varphi} \\
&= f^*(c_{k,h}(P)^{-1})f^*(\tilde{\rho})\tilde{\varphi} = f^*(c_{k,h}(P)^{-1}\tilde{\rho})\tilde{\varphi} \\
c_{h,g}(k^*(P))^{-1}c_{k,hg}(P)^{-1}c_{kh,g}(P)g^*(\tilde{\rho})\tilde{\psi} &= (c_{k,hg}(P)c_{h,g}(k^*(P)))^{-1}c_{kh,g}(P)g^*(\tilde{\rho})\tilde{\psi} \\
&= (c_{kh,g}(P)g^*(c_{k,h}(P)))^{-1}c_{kh,g}(P)g^*(\tilde{\rho})\tilde{\psi} \\
&= g^*(c_{k,h}(P)^{-1})g^*(\tilde{\rho})\tilde{\psi} = g^*(c_{k,h}(P)^{-1}\tilde{\rho})\tilde{\psi}
\end{aligned}$$

Put $\check{\rho} = c_{k,h}(P)^{-1}\tilde{\rho} : N \to h^*(k^*(P))$ and ${}^t\check{\rho} = ad_h(N,k^*(P))^{-1}(\check{\rho}) : h_*(N) \to k^*(P)$. Then, the above equalities imply the following.

$$c_{h,f}(k^*(P))f^*(\check{\rho})\tilde{\varphi} = c_{h,g}(k^*(P))g^*(\check{\rho})\tilde{\psi} \cdots (ii)$$

Since the following diagrams commute by the naturality of ad_f and ad_g, we have

$$f^*(\check{\rho})\tilde{\varphi} = ad_f(M,h^*(k^*(P)))(\check{\rho}\,{}^t\tilde{\varphi}), \qquad g^*(\check{\rho})\tilde{\psi} = ad_g(M,h^*(k^*(P)))(\check{\rho}\,{}^t\tilde{\psi}) \cdots (iii).$$

$$\begin{tikzcd}
\mathcal{F}_Y(f_*(M),N) \arrow[r, "ad_f(M,N)"] \arrow[d, "\check{\rho}_*"] & \mathcal{F}_X(M,f^*(N)) \arrow[d, "f^*(\check{\rho})_*"] \\
\mathcal{F}_Y(f_*(M),h^*(k^*(P))) \arrow[r, "ad_f(M,h^*(k^*(P)))"] & \mathcal{F}_X(M,f^*(h^*(k^*(P))))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}_Y(g_*(M), N) \arrow[r, "ad_g(M,N)"] \arrow[d, "\check{\rho}_*"] & \mathcal{F}_X(M, g^*(N)) \arrow[d, "g^*(\check{\rho})_*"] \\
\mathcal{F}_Y(g_*(M), (kh)^*(P)) \arrow[r, "ad_g(M,(kh)^*(P))"] \arrow[d, "c_{k,h}(P)_*^{-1}"] & \mathcal{F}_X(M, g^*((kh)^*(P))) \arrow[d, "g^*(c_{k,h}(P)^{-1})_*"] \\
\mathcal{F}_Y(g_*(M), h^*(k^*(P))) \arrow[r, "ad_g(M,h^*(k^*(P)))"] & \mathcal{F}_X(M, g^*(h^*(k^*(P))))
\end{tikzcd}$$

Moreover, the following diagrams commute by the naturality of ad_h, we have

$$\check{\rho}\,{}^t\tilde{\varphi} = ad_h(f_*(M), k^*(P))({}^t\check{\rho}h_*({}^t\tilde{\varphi})), \qquad \check{\rho}\,{}^t\tilde{\psi} = ad_h(g_*(M), k^*(P))({}^t\check{\rho}h_*({}^t\tilde{\psi})) \cdots (iv).$$

$$\begin{tikzcd}
\mathcal{F}_Z(h_*(N), k^*(P)) \arrow[r, "ad_h(N,k^*(P))"] \arrow[d, "h_*({}^t\tilde{\varphi})^*"] & \mathcal{F}_Y(N, h^*(k^*(P))) \arrow[d, "{}^t\tilde{\varphi}^*"] \\
\mathcal{F}_Z(h_*(f_*(M)), k^*(P)) \arrow[r, "ad_h(f_*(M),k^*(P))"] & \mathcal{F}_Y(f_*(M), h^*(k^*(P)))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}_Z(h_*(N), k^*(P)) \arrow[r, "ad_h(N,k^*(P))"] \arrow[d, "h_*({}^t\tilde{\psi})^*"] & \mathcal{F}_Y(N, h^*(k^*(P))) \arrow[d, "{}^t\tilde{\psi}^*"] \\
\mathcal{F}_Z(h_*(g_*(M)), k^*(P)) \arrow[r, "ad_h(g_*(M),k^*(P))"] & \mathcal{F}_Y(g_*(M), h^*(k^*(P)))
\end{tikzcd}$$

Since the following diagram commutes, it follows from (ii), (iii) and (iv) that ${}^t\check{\rho}h_*({}^t\tilde{\varphi})\xi_M = {}^t\check{\rho}h_*({}^t\tilde{\psi})$.

$$\begin{tikzcd}
\mathcal{F}_Z(h_*(f_*(M)), k^*(P)) \arrow[r, "ad_h(f_*(M),k^*(P))"] \arrow[d, "\xi_M^*"] & \mathcal{F}_Y(f_*(M), h^*(k^*(P))) \arrow[r, "ad_f(M,h^*(k^*(P)))"] & \mathcal{F}_X(M, f^*(h^*(k^*(P)))) \arrow[d, "c_{h,f}(k^*(P))_*"] \\
\mathcal{F}_Z(h_*(g_*(M)), k^*(P)) \arrow[d, "ad_h(g_*(M),k^*(P))"] & & \mathcal{F}_X(M, (hf)^*(k^*(P))) \arrow[d, equal] \\
\mathcal{F}_Y(g_*(M), h^*(k^*(P))) \arrow[r, "ad_g(M,h^*(k^*(P)))"] & \mathcal{F}_X(M, g^*(h^*(k^*(P)))) \arrow[r, "c_{h,g}(k^*(P))_*"] & \mathcal{F}_X(M, (hg)^*(k^*(P)))
\end{tikzcd}$$

Hence there exists unique morphism $\bar{\rho} : L \to k^*(P)$ of $\mathcal{F}_Z$ that satisfies $\bar{\rho}\pi = {}^t\check{\rho}$. By the naturality of ad_h, the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}_Z(h_*(N), L) \arrow[r, "ad_h(N,L)"] \arrow[d, "\bar{\rho}_*"] & \mathcal{F}_Y(N, h^*(L)) \arrow[d, "h^*(\bar{\rho})_*"] \\
\mathcal{F}_Z(h_*(N), k^*(P)) \arrow[r, "ad_h(N,k^*(P))"] & \mathcal{F}_Y(N, h^*(k^*(P)))
\end{tikzcd}$$

Thus $h^*(\bar{\rho})\pi^a = ad_h(N, k^*(P))(\bar{\rho}\pi) = ad_h(N, k^*(P))({}^t\check{\rho}) = \check{\rho} = c_{k,h}(P)^{-1}\tilde{\rho}$, which implies $c_{k,h}(P)h^*(\bar{\rho})\pi^a = \tilde{\rho}$. Therefore we have $\alpha_k(P)\bar{\rho}\alpha_h(L)\pi^a = \alpha_k(P)\alpha_h(k^*(P))h^*(\bar{\rho})\pi^a = \alpha_{kh}(P)c_{k,h}(P)h^*(\bar{\rho})\pi^a = \alpha_{kh}(P)\tilde{\rho} = \rho$.

It remains to show that $\alpha_h(L)\pi^a : N \to L$ is an epimorphism in $\mathcal{F}$. Suppose that morphisms $\beta, \gamma : L \to Q$ of $\mathcal{F}$ satisfy $\beta\alpha_h(L)\pi^a = \gamma\alpha_h(L)\pi^a$. Then, we have $p(\beta)h = p(\gamma)h$ which implies $p(\beta) = p(\gamma)$ since h is an epimorphism. We put $q = p(\beta) = p(\gamma) : Z \to p(Q)$. Let $\tilde{\beta}, \tilde{\gamma} : L \to q^*(Q)$ be the unique morphisms in $\mathcal{F}_Z$ that satisfy $\alpha_q(Q)\tilde{\beta} = \beta$ and $\alpha_q(Q)\tilde{\gamma} = \gamma$, respectively. Then,

$$\begin{aligned}
\alpha_{qh}(Q)c_{q,h}(Q)h^*(\tilde{\beta})\pi^a &= \alpha_q(Q)\alpha_h(q^*(Q))h^*(\tilde{\beta})\pi^a = \alpha_q(Q)\tilde{\beta}\alpha_h(L)\pi^a = \alpha_q(Q)\tilde{\gamma}\alpha_h(L)\pi^a \\
&= \alpha_q(Q)\alpha_h(q^*(Q))h^*(\tilde{\gamma})\pi^a = \alpha_{qh}(Q)c_{q,h}(Q)h^*(\tilde{\gamma})\pi^a
\end{aligned}$$

and it follows $h^*(\tilde{\beta})\pi^a = h^*(\tilde{\gamma})\pi^a \in \mathcal{F}_Y(N, h^*(q^*(Q)))$. By the naturality of ad_h,

$$ad_h(N, q^*(Q))^{-1} : \mathcal{F}_Y(N, h^*(q^*(Q))) \to \mathcal{F}_Z(h_*(N), q^*(Q))$$

maps $h^*(\tilde{\beta})\pi^a$ and $h^*(\tilde{\gamma})\pi^a$ to $\tilde{\beta}\pi$ and $\tilde{\gamma}\pi$, respectively and we see $\tilde{\beta}\pi = \tilde{\gamma}\pi$. Since π is an epimorphism, it follows $\tilde{\beta} = \tilde{\gamma}$ which implies $\beta = \gamma$. $\square$

1.3 Left fibered representable pair

Let $p : \mathcal{F} \to \mathcal{E}$ be a normalized cloven fibered category. For morphisms $f : X \to Y$, $g : X \to Z$ in $\mathcal{E}$ and an object M of $\mathcal{F}_Y$, we define a presheaf $F_{f,g,M} : \mathcal{F}_Z \to \mathcal{S}et$ on $\mathcal{F}_Z^{op}$ by $F_{f,g,M}(N) = F_{f,g}(M,N) = \mathcal{F}_X(f^*(M), g^*(N))$ for $N \in \operatorname{Ob}\mathcal{F}_Z$ and $F_{f,g,M}(\psi) = F_{f,g}(id_M, \psi) = g^*(\psi)_*$ for $\psi \in \operatorname{Mor}\mathcal{F}_Z$.

Suppose that $F_{f,g,M}$ is representable. We choose an object $M_{[f,g]}$ of $\mathcal{F}_Z$ such that there exists a natural equivalence $P_{f,g}(M) : F_{f,g,M} \to \hat{h}_{M_{[f,g]}}$, where $\hat{h}_{M_{[f,g]}}$ is the presheaf on $\mathcal{F}_Z^{op}$ represented by $M_{[f,g]}$. If $X = Z$ and g is the identity morphism of Z, we take $f^*(M)$ as $M_{[f,id_X]}$. Hence $P_{f,id_X}(M)_N$ is the identity map of $\mathcal{F}_X(f^*(M), N)$. Let us denote by $\iota_{f,g}(M) : f^*(M) \to g^*(M_{[f,g]})$ the morphism in $\mathcal{F}_X$ which is mapped to the identity morphism of $M_{[f,g]}$ by $P_{f,g}(M)_{M_{[f,g]}} : \mathcal{F}_X(f^*(M), g^*(M_{[f,g]})) \to \mathcal{F}_Z(M_{[f,g]}, M_{[f,g]})$.

Definition 1.3.1 *We say that a pair (f,g) of morphisms $f : X \to Y$ and $g : X \to Z$ in $\mathcal{E}$ is a left fibered representable pair with respect to an object M of $\mathcal{F}_Y$ if the presheaf $F_{f,g,M}$ on $\mathcal{F}_Z^{op}$ is representable. If (f,g) is a left fibered representable pair with respect to all objects of $\mathcal{F}_Y$, we say that (f,g) is a left fibered representable pair.*

Proposition 1.3.2 *The inverse of $P_{f,g}(M)_N : \mathcal{F}_X(f^*(M), g^*(N)) \to \mathcal{F}_Z(M_{[f,g]}, N)$ is given by the map defined by $\varphi \mapsto g^*(\varphi)\iota_{f,g}(M)$.*

Proof. For $\varphi \in \mathcal{F}_Y(M_{[f,g]}, N)$, the following diagram commutes by naturality of $P_{f,g}(M)$.

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(M_{[f,g]})) \arrow[r, "g^*(\varphi)_*"] \arrow[d, "P_{f,g}(M)_{M_{[f,g]}}"] & \mathcal{F}_X(f^*(M), g^*(N)) \arrow[d, "P_{f,g}(M)_N"] \\
\mathcal{F}_Z(M_{[f,g]}, M_{[f,g]}) \arrow[r, "\varphi_*"] & \mathcal{F}_Z(M_{[f,g]}, N)
\end{tikzcd}$$

It follows that $P_{f,g}(M)_N$ maps $g^*(\varphi)\iota_X(M)$ to φ. □

Remark 1.3.3 *If $g^* : \mathcal{F}_Z \to \mathcal{F}_X$ has a left adjoint $g_* : \mathcal{F}_X \to \mathcal{F}_Z$, $F_{f,g,M} : \mathcal{F}_Y \to \mathcal{S}et$ is representable for any object M of $\mathcal{F}_Y$. In fact, $M_{[f,g]}$ is defined to be $g_*f^*(M)$ in this case and (f,g) is a left fibered representable pair for any morphism f in $\mathcal{E}$ whose domain is X. Hence if $p : \mathcal{F} \to \mathcal{E}$ is a bifibered category, a pair (f,g) of morphisms in $\mathcal{E}$ with same domains is always a left fibered representable pair. If we denote by $(\mathrm{ad}_g)_{P,N} : \mathcal{F}_Y(g_*(P), N) \to \mathcal{F}_X(P, g^*(N))$ the bijection which is natural in $P \in \operatorname{Ob}\mathcal{F}_X$ and $N \in \operatorname{Ob}\mathcal{F}_Y$, we have $P_{f,g}(M)_N = (\mathrm{ad}_g)^{-1}_{f^*(M),N} : \mathcal{F}_X(f^*(M), g^*(N)) \to \mathcal{F}_Z(g_*f^*(M), N)$. Let us denote by $\eta_g : id_{\mathcal{F}_X} \to g^*g_*$ and $\varepsilon_g : g_*g^* \to id_{\mathcal{F}_Z}$ the unit and the counit of the adjunction $g_* \dashv g^*$, respectively. Then, $P_{f,g}(M)_N$ maps $\psi \in \mathcal{F}_X(f^*(M), g^*(N))$ to $(\varepsilon_g)_N g_*(\psi)$ and $P_{f,g}(M)_N^{-1}$ maps $\varphi \in \mathcal{F}_Z(g_*f^*(M), N)$ to $g^*(\varphi)(\eta_g)_{f^*(M)}$. It follows from (1.3.2) that we have $\iota_{f,g}(M) = (\eta_g)_{f^*(M)} : f^*(M) \to g^*g_*f^*(M) = g^*(M_{[f,g]})$. We note that if g^* has a left adjoint if and only if (id_X, g) is a left fibered representable pair.*

For a morphism $\varphi : L \to M$ of $\mathcal{F}_Y$, define a natural transformation $F_{f,g,\varphi} : F_{f,g,M} \to F_{f,g,L}$ by

$$(F_{f,g,\varphi})_N = f^*(\varphi)^* : F_{f,g,M}(N) = \mathcal{F}_X(f^*(M), g^*(N)) \to \mathcal{F}_X(f^*(L), g^*(N)) = F_{f,g,L}(N).$$

It is clear that $F_{f,g,\psi\varphi} = F_{f,g,\varphi}F_{f,g,\psi}$ for morphisms $\psi : M \to P$ and $\varphi : L \to M$ of $\mathcal{F}_Y$. If (f,g) is a left fibered representable pair with respect to M and L we define a morphism $\varphi_{[f,g]} : L_{[f,g]} \to M_{[f,g]}$ of $\mathcal{F}_Z$ by

$$\varphi_{[f,g]} = P_{f,g}(L)_{M_{[f,g]}}((F_{f,g,\varphi})_{M_{[f,g]}}(\iota_{f,g}(M))) = P_{f,g}(L)_{M_{[f,g]}}(\iota_{f,g}(M)f^*(\varphi)) \in \hat{h}_{L_{[f,g]}}(M_{[f,g]}).$$

Proposition 1.3.4 *Let $\varphi : L \to M$ be a morphism in $\mathcal{F}_Y$.*

(1) The following diagrams commute for any $N \in \operatorname{Ob}\mathcal{F}_Z$.

$$\begin{tikzcd}
f^*(L) \arrow[r, "f^*(\varphi)"] \arrow[d, "\iota_{f,g}(L)"] & f^*(M) \arrow[d, "\iota_{f,g}(M)"] \\
g^*(L_{[f,g]}) \arrow[r, "g^*(\varphi_{[f,g]})"] & g^*(M_{[f,g]})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(N)) \arrow[r, "f^*(\varphi)^*"] \arrow[d, "P_{f,g}(M)_N"] & \mathcal{F}_X(f^*(L), g^*(N)) \arrow[d, "P_{f,g}(L)_N"] \\
\mathcal{F}_Z(M_{[f,g]}, N) \arrow[r, "\varphi^*_{[f,g]}"] & \mathcal{F}_Z(L_{[f,g]}, N)
\end{tikzcd}$$

(2) For morphisms $\psi : M \to K$ and $\varphi : L \to M$ of $\mathcal{F}_Y$, we have $(\psi\varphi)_{[f,g]} = \psi_{[f,g]}\varphi_{[f,g]}$.

(3) If $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ preserves epimorphisms (f^* has a right adjoint, for example) and $\varphi : L \to M$ is an epimorphism, so is $\varphi_{[f,g]} : L_{[f,g]} \to M_{[f,g]}$.

Proof. (1) We have $P_{f,g}(L)_{M_{[f,g]}}(\iota_{f,g}(M)f^*(\varphi)) = \varphi_{[f,g]}$ by the definition of $\varphi_{[f,g]}$. On the other hand, $P_{f,g}(L)_{M_{[f,g]}}(g^*(\varphi_{[f,g]})\iota_{f,g}(L)) = \varphi_{[f,g]}$ by (1.3.2). Since $P_{f,g}(L)_{M_{[f,g]}}$ is bijective, the left diagram commutes.

For $\psi \in \mathcal{F}_Z(M_{[f,g]}, N)$, it follows from (1.3.2) and commutativity of the left diagram that we have

$$\begin{aligned} f^*(\varphi)^* P_{f,g}(M)_N^{-1}(\psi) &= g^*(\psi)\iota_{f,g}(M)f^*(\varphi) = g^*(\psi)g^*(\varphi_{[f,g]})\iota_{f,g}(L) = g^*(\psi\varphi_{[f,g]})\iota_{f,g}(L) \\ &= P_{f,g}(L)_N^{-1}(\psi\varphi_{[f,g]}) = P_{f,g}(L)_N^{-1}\varphi^*_{[f,g]}(\psi). \end{aligned}$$

Hence the right diagram commutes.

(2) The following diagram commutes by (1).

$$\begin{tikzcd} \mathcal{F}_X(f^*(K), g^*(K_{[f,g]})) \arrow[r, "f^*(\psi)^*"] \arrow[d, "P_{f,g}(K)_{K_{[f,g]}}"] & \mathcal{F}_X(f^*(M), g^*(K_{[f,g]})) \arrow[r, "f^*(\varphi)^*"] \arrow[d, "P_{f,g}(M)_{K_{[f,g]}}"] & \mathcal{F}_X(f^*(L), g^*(K_{[f,g]}))) \arrow[d, "P_{f,g}(L)_{K_{[f,g]}}"] \\ \mathcal{F}_Z(K_{[f,g]}, K_{[f,g]}) \arrow[r, "\psi^*_{[f,g]}"] & \mathcal{F}_Z(M_{[f,g]}, K_{[f,g]}) \arrow[r, "\varphi^*_{[f,g]}"] & \mathcal{F}_Z(L_{[f,g]}, K_{[f,g]}) \end{tikzcd}$$

Hence, by the definition of $(\psi\varphi)_{[f,g]}$ we have

$$\begin{aligned} \psi_{[f,g]}\varphi_{[f,g]} &= \varphi^*_{[f,g]}\psi^*_{[f,g]}(id_{K_{[f,g]}}) = \varphi^*_{[f,g]}\psi^*_{[f,g]}P_{f,g}(K)_{K_{[f,g]}}(\iota_{f,g}(K)) = P_{f,g}(L)_{K_{[f,g]}}f^*(\varphi)^*f^*(\psi)^*(\iota_{f,g}(K)) \\ &= P_{f,g}(L)_{K_{[f,g]}}(\iota_{f,g}(K)f^*(\varphi\psi)) = (\psi\varphi)_{[f,g]}. \end{aligned}$$

(3) is a direct consequence of (1). □

Remark 1.3.5 *If $g^* : \mathcal{F}_Z \to \mathcal{F}_X$ has a left adjoint $g_* : \mathcal{F}_X \to \mathcal{F}_Z$, for a morphism $\varphi : L \to M$ of $\mathcal{F}_Y$, we have $\varphi_{[f,g]} = g_*f^*(\varphi) : L_{[f,g]} = g_*f^*(L) \to g_*f^*(M) = M_{[f,g]}$. In fact, if we denote by $\varepsilon_g : g^*g_* \to id_{\mathcal{F}_X}$ the counit of the adjunction $g_* \dashv g^*$, we have $\varphi_{[f,g]} = P_{f,g}(L)_{M_{[f,g]}}(\iota_{f,g}(M)f^*(\varphi)) = (\mathrm{ad}_g)^{-1}_{f^*(L),M_{[f,g]}}((\eta_g)_{f^*(M)}f^*(\varphi)) = (\varepsilon_g)_{g_*f^*(M)}g_*((\eta_g)_{f^*(M)})g_*f^*(\varphi) = g_*f^*(\varphi)$.*

Lemma 1.3.6 *Let $\xi : f^*(L) \to g^*(M)$ and $\zeta : f^*(N) \to g^*(K)$ be morphisms in $\mathcal{F}_X$ for morphisms $\varphi : L \to N$ and $\psi : M \to K$ of $\mathcal{F}_Y$ and $\mathcal{F}_Z$, respectively. We put $\hat{\xi} = P_{f,g}(L)_M(\xi)$ and $\hat{\zeta} = P_{f,g}(N)_K(\zeta)$. The following left diagram commutes if and only if the right one commutes.*

$$\begin{tikzcd} f^*(L) \arrow[r, "\xi"] \arrow[d, "f^*(\varphi)"] & g^*(M) \arrow[d, "g^*(\psi)"] \\ f^*(N) \arrow[r, "\zeta"] & g^*(K) \end{tikzcd} \qquad \begin{tikzcd} L_{[f,g]} \arrow[r, "\hat{\xi}"] \arrow[d, "\varphi_{[f,g]}"] & M \arrow[d, "\psi"] \\ N_{[f,g]} \arrow[r, "\hat{\zeta}"] & K \end{tikzcd}$$

Proof. The following diagram is commutative by (1.3.4).

$$\begin{tikzcd} \mathcal{F}_X(f^*(L), g^*(M)) \arrow[r, "g^*(\psi)_*"] \arrow[d, "P_{f,g}(L)_M"] & \mathcal{F}_X(f^*(L), g^*(K)) \arrow[d, "P_{f,g}(L)_K"] & \mathcal{F}_X(f^*(N), g^*(K)) \arrow[l, "f^*(\varphi)^*"'] \arrow[d, "P_{f,g}(N)_K"] \\ \mathcal{F}_Z(L_{[f,g]}, M) \arrow[r, "\psi_*"] & \mathcal{F}_Z(L_{[f,g]}, K) & \mathcal{F}_Z(N_{[f,g]}, K) \arrow[l, "\varphi^*_{[f,g]}"'] \end{tikzcd}$$

Since $\hat{\xi} = P_{f,g}(L)_M(\xi)$, $\hat{\zeta} = P_{f,g}(N)_K(\zeta)$ and $P_{f,g}(L)_K$ is bijective, $g^*(\psi)\xi = g^*(\psi)_*(\xi) = f^*(\varphi)^*(\zeta) = \zeta f^*(\varphi)$ if and only if $\psi\hat{\xi} = \psi_*(\hat{\xi}) = \varphi^*_{[f,g]}(\hat{\zeta}) = \hat{\zeta}\varphi_{[f,g]}$. □

For morphisms $f : X \to Y$, $g : X \to Z$, $k : V \to X$ in $\mathcal{E}$ and $M \in \mathrm{Ob}\,\mathcal{F}_Y$, suppose that suppose that (f, g) and (fk, gk) are left fibered representable pairs with respect to M. We define a morphism $M_k : M_{[fk,gk]} \to M_{[f,g]}$ of $\mathcal{F}_Z$ by

$$M_k = P_{fk,gk}(M)_{M_{[f,g]}}(k^\sharp_{M,M_{[f,g]}}(\iota_{f,g}(M))).$$

Proposition 1.3.7 *(1) The following diagrams commute for any $N \in \mathrm{Ob}\,\mathcal{F}_Z$.*

$$\begin{tikzcd} \mathcal{F}_X(f^*(M), g^*(N)) \arrow[r, "k^\sharp_{M,N}"] \arrow[d, "P_{f,g}(M)_N"] & \mathcal{F}_V((fk)^*(M), (gk)^*(N)) \arrow[d, "P_{fk,gk}(M)_N"] \\ \mathcal{F}_Z(M_{[f,g]}, N) \arrow[r, "M_k^*"] & \mathcal{F}_Z(M_{[fk,gk]}, N) \end{tikzcd} \qquad \begin{tikzcd} (fk)^*(M) \arrow[rr, "k^\sharp_{M,M_{[f,g]}}(\iota_{f,g}(M))"] \arrow[dr, "\iota_{fk,gk}(M)"'] & & (gk)^*(M_{[f,g]}) \\ & (gk)^*(M_{[fk,gk]}) \arrow[ur, "(gk)^*(M_k)"'] & \end{tikzcd}$$

(2) For morphisms $f : X \to Y$, $g : X \to Z$, $k : V \to X$, $h : U \to V$ and $M \in \operatorname{Ob}\mathcal{F}_Y$, suppose that (f,g), (fk, gk) and (fkh, gkh) are left fibered representable pairs with respect to M. Then, we have $M_{kh} = M_k M_h$.

(3) The image of the identity morphism of $k^*(M)$ by $P_{k,k}(M)_M$ is $M_k : M_{[k,k]} \to M_{[id_X, id_X]} = M$ if $X = Y$.

(4) A composition $k^*(M) \xrightarrow{\iota_{k,k}(M)} k^*(M_{[k,k]}) \xrightarrow{k^*(M_k)} k^*(M_{[id_X, id_X]}) = k^*(M)$ is the identity morphism of $k^*(M)$ if $X = Y$.

Proof. (1) For $\varphi \in \mathcal{F}_Z(M_{[f,g]}, N)$, it follows from the naturality of $k^\sharp_{M,N}$ and (1.3.2) that we have

$$\begin{aligned}
k^\sharp_{M,N} P_{f,g}(M)_N^{-1}(\varphi) &= k^\sharp_{M,N}(g^*(\varphi)\iota_{f,g}(M)) = k^\sharp_{M,N} g^*(\varphi)_*(\iota_{f,g}(M)) = (gk)^*(\varphi)_* k^\sharp_{M,M_{[f,g]}}(\iota_{f,g}(M)) \\
&= (gk)^*(\varphi)_* P_{fk,gk}(M)^{-1}_{M_{[f,g]}}(M_k) = (gk)^*(\varphi)(gk)^*(M_k)\iota_{fk,gk}(M) = (gk)^*(\varphi M_k)\iota_{fk,gk}(M) \\
&= (gk)^*(M_k^*(\varphi))\iota_{fk,gk}(M) = P_{fk,gk}(M)_N^{-1} M_k^*(\varphi).
\end{aligned}$$

The commutativity of the right diagram follows from (1.3.2) and the commutativity of the left diagram for the case $N = M_{[f,g]}$.

(2) The following diagram commutes by (1). Hence the assertion follows from (1.1.16).

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(N)) \arrow[r, "k^\sharp_{M,N}"] \arrow[d, "P_{f,g}(M)_N"] & \mathcal{F}_V((fk)^*(M), (gk)^*(N)) \arrow[r, "h^\sharp_{M,N}"] \arrow[d, "P_{fk,gk}(M)_N"] & \mathcal{F}_U((fkh)^*(M), (gkh)^*(N)) \arrow[d, "P_{fkh,gkh}(M)_N"] \\
\mathcal{F}_Z(M_{[f,g]}, N) \arrow[r, "M_k^*"] & \mathcal{F}_Z(M_{[fk,gk]}, N) \arrow[r, "M_h^*"] & \mathcal{F}_Z(M_{[fkh,gkh]}, N)
\end{tikzcd}$$

(3) Apply (1) for $N = M$, $Z = Y = X$ and $f = g = id_X$.

(4) It follows from (1.3.2) that $P_{k,k}(M)_M : \mathcal{F}_V(k^*(M), k^*(M)) \to \mathcal{F}_X(M_{[k,k]}, M)$ maps $k^*(M_k)\iota_{k,k}(M)$ to $M_k : M_{[k,k]} \to M$. Thus the assertion follows from (3). $\square$

Remark 1.3.8 *Suppose that the inverse image functors $g^* : \mathcal{F}_Z \to \mathcal{F}_X$ and $(gk)^* : \mathcal{F}_Z \to \mathcal{F}_V$ have left adjoints $g_* : \mathcal{F}_X \to \mathcal{F}_Z$ and $(gk)_* : \mathcal{F}_V \to \mathcal{F}_Z$, respectively.*

(1) Since $k^\sharp_{M,M_{[f,g]}}(\iota_{f,g}(M)) = c_{g,k}(M_{[f,g]})k^*((\eta_g)_{f^*(M)})c_{f,k}(M)^{-1}$ by (1.3.3) and

$$P_{fk,gk}(M)_{M_{[f,g]}} = (\mathrm{ad}_{gk})^{-1}_{(fk)^*(M),M_{[f,g]}} : \mathcal{F}_V((fk)^*(M), (gk)^*(M_{[f,g]})) \to \mathcal{F}_Z(M_{[fk,gk]}, M_{[f,g]})$$

maps $\varphi \in \mathcal{F}_V((fk)^*(M), (gk)^*(M_{[f,g]}))$ to $(\varepsilon_{gk})_{M_{[f,g]}}(gk)_*(\varphi)$, $M_k : M_{[fk,gk]} \to M_{[f,g]}$ coincides with the following composition.

$$\begin{aligned}
M_{[fk,gk]} = (gk)_*(fk)^*(M) &\xrightarrow{(gk)_*(c_{f,k}(M))^{-1}} (gk)_*k^*f^*(M) \xrightarrow{(gk)_*k^*((\eta_g)_{f^*(M)})} (gk)_*k^*g^*g_*f^*(M) \\
&= (gk)_*k^*g^*(M_{[f,g]}) \xrightarrow{(gk)_*(c_{g,k}(M_{[f,g]}))} (gk)_*(gk)^*(M_{[f,g]}) \xrightarrow{(\varepsilon_{gk})_{M_{[f,g]}}} M_{[f,g]}
\end{aligned}$$

We remark that M_k is the adjoint of the following composition with respect to the adjunction $(gk)_* \dashv (gk)^*$.

$$(fk)^*(M) \xrightarrow{c_{f,k}(M)^{-1}} k^*f^*(M) \xrightarrow{k^*((\eta_g)_{f^*(M)})} k^*g^*g_*f^*(M) = k^*g^*(M_{[f,g]}) \xrightarrow{c_{g,k}(M_{[f,g]})} (gk)^*(M_{[f,g]})$$

(2) The following diagram commutes by (1.3.7) if $X = Y = Z$ and $f = g = id_X$.

$$\begin{tikzcd}
\mathcal{F}_X(M_{[id_X,id_X]}, M) \arrow[r, "M_k^*"] \arrow[d, "(\mathrm{ad}_{id_X})_{id_X^*(M),M}"] & \mathcal{F}_X(k_*(k^*(M)), M) \arrow[d, "(\mathrm{ad}_k)_{k^*(M),M}"] \\
\mathcal{F}_X(id_X^*(M), id_X^*(M)) \arrow[r, "k^\sharp_{M,M}"] & \mathcal{F}_V(k^*(M), k^*(M))
\end{tikzcd}$$

Since id_X^* is the identity functor of $\mathcal{F}_X$, so is id_{X*}. Hence $M_{[k,k]} : k_*k^*(M) = M_{[k,k]} \to M_{[id_X,id_X]} = M$ is identified with the counit $(\varepsilon_k)_M : k_*k^*(M) \to M$ of the adjunction $k_* \dashv k^*$ by the above diagram.

Proposition 1.3.9 *For morphisms $f : X \to Y$, $g : X \to Z$, $k : V \to X$ in $\mathcal{E}$ and a morphism $\varphi : L \to M$ of $\mathcal{F}_Y$, the following diagram commutes.*

$$\begin{tikzcd}
L_{[fk,gk]} \arrow[r, "L_k"] \arrow[d, "\varphi_{[fk,gk]}"] & L_{[f,g]} \arrow[d, "\varphi_{[f,g]}"] \\
M_{[fk,gk]} \arrow[r, "M_k"] & M_{[f,g]}
\end{tikzcd}$$

Proof. The following diagram commutes by the naturality of $k^\sharp$.

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(N)) \arrow[r, "k^\sharp_{M,N}"] \arrow[d, "f^*(\varphi)^*"] & \mathcal{F}_V((fk)^*(M), (gk)^*(N)) \arrow[d, "(fk)^*(\varphi)^*"] \\
\mathcal{F}_X(f^*(L), g^*(N)) \arrow[r, "k^\sharp_{L,N}"] & \mathcal{F}_V((fk)^*(L), (fk)^*(N))
\end{tikzcd}$$

Then, it follows from the commutativity of four diagrams

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(N)) \arrow[r, "P_{f,g}(M)_N"] \arrow[d, "f^*(\varphi)^*"] & \mathcal{F}_Z(M_{[f,g]}, N) \arrow[d, "(\varphi_{[f,g]})^*"] \\
\mathcal{F}_X(f^*(L), g^*(N)) \arrow[r, "P_{f,g}(L)_N"] & \mathcal{F}_Z(L_{[f,g]}, N)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}_V((fk)^*(M), (gk)^*(N)) \arrow[r, "P_{fk,gk}(M)_N"] \arrow[d, "(fk)^*(\varphi)^*"] & \mathcal{F}_Z(M_{[fk,gk]}, N) \arrow[d, "(\varphi_{[fk,gk]})^*"] \\
\mathcal{F}_V((fk)^*(L), (gk)^*(N)) \arrow[r, "P_{fk,gk}(L)_N"] & \mathcal{F}_Z(L_{[fk,gk]}, N)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(N)) \arrow[r, "P_{f,g}(M)_N"] \arrow[d, "k^\sharp_{M,N}"] & \mathcal{F}_Z(M_{[f,g]}, N) \arrow[d, "M_k^*"] \\
\mathcal{F}_V((fk)^*(M), (gk)^*(N)) \arrow[r, "P_{fk,gk}(M)_N"] & \mathcal{F}_Z(M_{[fk,gk]}, N)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}_X(f^*(L), g^*(N)) \arrow[r, "P_{f,g}(L)_N"] \arrow[d, "k^\sharp_{L,N}"] & \mathcal{F}_Z(L_{[f,g]}, N) \arrow[d, "L_k^*"] \\
\mathcal{F}_V((fk)^*(L), (gk)^*(N)) \arrow[r, "P_{fk,gk}(L)_N"] & \mathcal{F}_Z(L_{[fk,gk]}, N)
\end{tikzcd}$$

and the fact that $P_{f,g}(M)_N : \mathcal{F}_X(f^*(M), g^*(N)) \to \mathcal{F}_Z(M_{[f,g]}, N)$ is bijective that the following diagram commutes for any $N \in \operatorname{Ob}\mathcal{F}_1$.

$$\begin{tikzcd}
\mathcal{F}_Z(M_{[f,g]}, N) \arrow[r, "M_k^*"] \arrow[d, "\varphi^*_{[f,g]}"] & \mathcal{F}_Z(M_{[fk,gk]}, N) \arrow[d, "\varphi^*_{[fk,gk]}"] \\
\mathcal{F}_Z(L_{[f,g]}, N) \arrow[r, "L_k^*"] & \mathcal{F}_Z(L_{[fk,gk]}, N)
\end{tikzcd}$$

Thus the assertion follows. □

Remark 1.3.10 *We denote by $\varphi_{[f,g],k} : L_{[fk,gk]} \to M_{[f,g]}$ the composition $M_k\varphi_{[fk,gk]} = \varphi_{[f,g]}L_k$. For morphisms $i : W \to Z$, $j : W \to T$, $h : U \to W$ in $\mathcal{E}$, it follows from (1.3.9) that the following diagram commutes.*

$$\begin{tikzcd}
(M_{[fk,gk]})_{[ih,jh]} \arrow[r, "(M_{[fk,gk]})_h"] \arrow[d, "(M_k)_{[ik,jk]}"] & (M_{[fk,gk]})_{[i,j]} \arrow[d, "(M_k)_{[i,j]}"] \\
(M_{[f,g]})_{[ih,jh]} \arrow[r, "(M_{[f,g]})_h"] & (M_{[f,g]})_{[i,j]}
\end{tikzcd}$$

Namely, we have $(M_k)_{[i,j],h} = (M_{[f,g]})_h(M_k)_{[ih,jh]} = (M_k)_{[i,j]}(M_{[fk,gk]})_h$ which we denote by $(M_k)_h$ for short.

For morphisms $f : X \to Y$, $g : X \to Z$, $h : X \to W$ in $\mathcal{E}$ and $M \in \operatorname{Ob}\mathcal{F}_Y$, we define a morphism $\delta_{f,g,h,M} : M_{[f,h]} \to (M_{[f,g]})_{[g,h]}$ of $\mathcal{F}_W$ to be the image of $\iota_{g,h}(M_{[f,g]})\iota_{f,g}(M) \in \mathcal{F}_X(f^*(M), h^*((M_{[f,g]})_{[g,h]}))$ by

$$P_{f,h}(M)_{(M_{[f,g]})_{[g,h]}} : \mathcal{F}_X(f^*(M), h^*((M_{[f,g]})_{[g,h]})) \to \mathcal{F}_W(M_{[f,h]}, (M_{[f,g]})_{[g,h]}).$$

Proposition 1.3.11 *The following diagram commutes for any $N \in \operatorname{Ob}\mathcal{F}_W$.*

$$\begin{tikzcd}
\mathcal{F}_X(g^*(M_{[f,g]}), h^*(N)) \arrow[r, "\iota_{f,g}(M)^*"] \arrow[d, "P_{g,h}(M_{[f,g]})_N"] & \mathcal{F}_X(f^*(M), h^*(N)) \arrow[d, "P_{f,h}(M)_N"] \\
\mathcal{F}_W((M_{[f,g]})_{[g,h]}, N) \arrow[r, "\delta^*_{f,g,h,M}"] & \mathcal{F}_W(M_{[f,h]}, N)
\end{tikzcd}$$

Proof. For $\varphi \in \mathcal{F}_W((M_{[f,g]})_{[g,h]}, N)$, by the definition of $\delta_{f,g,h,M}$ and the naturality of $P_X(M)$, we have

$$\begin{aligned}\iota_{f,g}(M)^* P_{g,h}(M_{[f,g]})_N^{-1}(\varphi) = h^*(\varphi)\iota_{g,h}(M_{[f,g]})\iota_{f,g}(M) &= h^*(\varphi)_* P_{f,h}(M)^{-1}_{(M_{[f,g]})_{[g,h]}}(\delta_{f,g,h,M}) \\ &= P_{f,h}(M)_N^{-1}\varphi_*(\delta_{f,g,h,M}) = P_{f,h}(M)_N^{-1}\delta^*_{f,g,h,M}(\varphi).\end{aligned}$$

□

We note that $\delta_{f,g,h,M} : M_{[f,h]} \to (M_{[f,g]})_{[g,h]}$ is the unique morphism that makes the diagram of (1.3.11) commute for any $N \in \operatorname{Ob}\mathcal{F}_W$.

Remark 1.3.12 *If $g^* : \mathcal{F}_Z \to \mathcal{F}_X$ and $h^* : \mathcal{F}_W \to \mathcal{F}_X$ have left adjoints $g_* : \mathcal{F}_X \to \mathcal{F}_Z$ and $h_* : \mathcal{F}_X \to \mathcal{F}_W$ respectively, the following diagram is commutative for any $N \in \operatorname{Ob}\mathcal{F}_W$ by the naturality of ad_h.*

$$\begin{tikzcd}
\mathcal{F}_X(g^*g_*f^*(M), h^*(N)) \arrow[r, "(\eta_g)^*_{f^*(M)}"] \arrow[d, "(\mathrm{ad}_h)^{-1}_{g^*g_*f^*(M),N}"] & \mathcal{F}_X(f^*(M), h^*(N)) \arrow[d, "(\mathrm{ad}_h)^{-1}_{f^*(M),N}"] \\
\mathcal{F}_W(h_*g^*g_*f^*(M), N) \arrow[r, "h_*((\eta_g)_{f^*(M)})^*"] & \mathcal{F}_W(h_*f^*(M), N)
\end{tikzcd}$$

It follows that $\delta_{f,g,h,M} = h_*\big((\eta_g)_{f^*(M)}\big)$.

Proposition 1.3.13 *For morphisms $f : X \to Y$, $g : X \to Z$, $h : X \to W$, $k : V \to X$ in $\mathcal{E}$ and a morphism $\varphi : L \to M$ of $\mathcal{F}_Y$, the following diagrams are commutative.*

$$\begin{tikzcd}
L_{[f,h]} \arrow[r, "\delta_{f,g,h,L}"] \arrow[d, "\varphi_{[f,h]}"] & (L_{[f,g]})_{[g,h]} \arrow[d, "(\varphi_{[f,g]})_{[g,h]}"] \\
M_{[f,h]} \arrow[r, "\delta_{f,g,h,M}"] & (M_{[f,g]})_{[g,h]}
\end{tikzcd}
\qquad
\begin{tikzcd}
M_{[fk,hk]} \arrow[r, "\delta_{fk,gk,hk,M}"] \arrow[d, "M_k"] & (M_{[fk,gk]})_{[gk,hk]} \arrow[d, "(M_k)_k"] \\
M_{[f,h]} \arrow[r, "\delta_{f,g,h,M}"] & (M_{[f,g]})_{[g,h]}
\end{tikzcd}$$

Proof. The following diagram is commutative for any $N \in \operatorname{Ob}\mathcal{F}_W$ by (1) of (1.3.4).

$$\begin{tikzcd}
\mathcal{F}_X(g^*((M_{[f,g]}), h^*(N)) \arrow[r, "\iota_{f,g}(M)^*"] \arrow[d, "g^*(\varphi_{[f,g]})^*"] & \mathcal{F}_X(f^*(M), h^*(N)) \arrow[d, "f^*(\varphi)^*"] \\
\mathcal{F}_X(g^*((L_{[f,g]}), h^*(N)) \arrow[r, "\iota_{f,g}(L)^*"] & \mathcal{F}_X(f^*(L), h^*(N))
\end{tikzcd}$$

Hence the following diagram commutes by (1.3.11) and (1) of (1.3.4).

$$\begin{tikzcd}
\mathcal{F}_W((M_{[f,g]})_{[g,h]}, N) \arrow[r, "\delta^*_{f,g,h,M}"] \arrow[d, "(\varphi_{[f,g]})^*_{[g,h]}"] & \mathcal{F}_1(M_{[f,h]}, N) \arrow[d, "\varphi^*_{[f,h]}"] \\
\mathcal{F}_W((L_{[f,g]})_{[g,h]}, N) \arrow[r, "\delta^*_{f,g,h,L}"] & \mathcal{F}_W(L_{[f,h]}, N)
\end{tikzcd}$$

Thus the left diagram is commutative.

For $N \in \operatorname{Ob}\mathcal{F}_W$ and $\xi \in \mathcal{F}_X(g^*(M_{[f,g]}), h^*(N))$, it follows from (1.3.7) and (1.1.15) that we have

$$k^\sharp_{M_{[f,g]},N}(\xi)(gk)^*(M_k)\iota_{fk,gk}(M) = k^\sharp_{M_{[f,g]},N}(\xi)k^\sharp_{M,M_{[f,g]}}(\iota_{f,g}(M)) = k^\sharp_{M,N}(\xi\iota_{f,g}(M)).$$

This shows that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}_X(g^*(M_{[f,g]}), h^*(N)) \arrow[r, "\iota_{f,g}(M)^*"] \arrow[d, "(gk)^*(M_k)^*k^\sharp_{M_{[f,g]},N}"] & \mathcal{F}_X(f^*(M), h^*(N)) \arrow[d, "k^\sharp_{M,N}"] \\
\mathcal{F}_V((gk)^*(M_{[fk,gk]}), (hk)^*(N)) \arrow[r, "\iota_{fk,gk}(M)^*"] & \mathcal{F}_V((fk)^*(M), (hk)^*(N))
\end{tikzcd}$$

The following diagram commutes by (1) of (1.3.4) and (1.3.7).

$$\begin{CD}
\mathcal{F}_X(g^*(M_{[f,g]}),h^*(N)) @>{k^\sharp_{M_{[f,g]},N}}>> \mathcal{F}_V((gk)^*(M_{[f,g]}),(hk)^*(N)) @>{(gk)^*(M_k)^*}>> \mathcal{F}_V((gk)^*(M_{[fk,gk]}),(hk)^*(N))\\
@VV{P_{g,h}(M_{[f,g]})_N}V @VV{P_{gk,hk}(M_{[f,g]})_N}V @VV{P_{gk,hk}(M_{[fk,gk]})_N}V\\
\mathcal{F}_W((M_{[f,g]})_{[g,h]},N) @>{(M_{[f,g]})^*_k}>> \mathcal{F}_W((M_{[f,g]})_{[gk,hk]},N) @>{(M_k)^*_{[gk,hk]}}>> \mathcal{F}_W((M_{[fk,gk]})_{[gk,hk]},N)
\end{CD}$$

Since $(M_k)_k = (M_{[f,g]})_k(M_k)_{[gk,hk]}$, it follows from (1.3.11) and (1) of (1.3.7) that the following diagram commutes for any $N \in \mathrm{Ob}\,\mathcal{F}_W$.

$$\begin{CD}
\mathcal{F}_W((M_{[f,g]})_{[g,h]},N) @>{\delta^*_{f,g,h,M}}>> \mathcal{F}_W(M_{[f,h]},N)\\
@VV{(M_k)^*_k}V @VV{M^*_k}V\\
\mathcal{F}_W((M_{[fk,gk]})_{[gk,hk]},N) @>{\delta^*_{fk,gk,hk,M}}>> \mathcal{F}_W(M_{[fk,hk]},N)
\end{CD}$$

Thus the right diagram is also commutative. □

Proposition 1.3.14 *For morphisms $f : X \to Y$, $g : X \to Z$, $h : X \to W$, $i : X \to V$ in $\mathcal{E}$ and an object M of $\mathcal{F}_Y$, the following diagrams are commutative.*

$$\begin{CD}
f^*(M) @>{\iota_{f,g}(M)}>> g^*(M_{[f,g]})\\
@VV{\iota_{f,h}(M)}V @VV{\iota_{g,h}(M_{[f,g]})}V\\
h^*(M_{[f,h]}) @>{h^*(\delta_{f,g,h,M})}>> h^*((M_{[f,g]})_{[g,h]})
\end{CD}
\qquad
\begin{CD}
M_{[f,i]} @>{\delta_{f,g,i,M}}>> (M_{[f,g]})_{[g,i]}\\
@VV{\delta_{f,h,i,M}}V @VV{\delta_{g,h,i,M_{[f,g]}}}V\\
(M_{[f,h]})_{[h,i]} @>{(\delta_{f,g,h,M})_{[h,i]}}>> ((M_{[f,g]})_{[g,h]})_{[h,i]}
\end{CD}$$

Proof. It follows from the definition of $\delta_{f,g,h,M}$ and (1.3.2) that

$$\iota_{g,h}(M_{[f,g]})\iota_{f,g}(M) = P_{f,h}(M)^{-1}_{(M_{[f,g]})_{[g,h]}}(\delta_{f,g,h,M}) = h^*(\delta_{f,g,h,M})\iota_{f,h}(M).$$

Hence the following diagram commutes for $N \in \mathrm{Ob}\,\mathcal{F}_V$.

$$\begin{CD}
\mathcal{F}_X(h^*((M_{[f,g]})_{[g,h]}),i^*(N)) @>{h^*(\delta_{f,g,h,M})^*}>> \mathcal{F}_X(h^*(M_{[f,h]}),i^*(N))\\
@VV{\iota_{g,h}(M_{[f,g]})^*}V @VV{\iota_{f,h}(M)^*}V\\
\mathcal{F}_X(g^*(M_{[f,g]}),i^*(N)) @>{\iota_{f,g}(M)^*}>> \mathcal{F}_X(f^*(M),i^*(N))
\end{CD}$$

Therefore the following diagram commutes by (1.3.11) and (1) of (1.3.4).

$$\begin{CD}
\mathcal{F}_V(((M_{[f,g]})_{[g,h]})_{[h,i]},N) @>{(\delta_{f,g,h,M})^*_{[h,i]}}>> \mathcal{F}_V((M_{[f,h]})_{[h,i]},N)\\
@VV{\delta^*_{g,h,i,M_{[f,g]}}}V @VV{\delta^*_{f,h,i,M}}V\\
\mathcal{F}_V((M_{[f,g]})_{[g,i]},N) @>{\delta^*_{f,g,i,M}}>> \mathcal{F}_V(M_{[f,i]},N)
\end{CD}$$

□

Proposition 1.3.15 *For morphisms $f : X \to Y$, $g : X \to Z$ in $\mathcal{E}$ and an object M of $\mathcal{F}_Y$, the following compositions coincide with the identity morphism of $M_{[f,g]}$.*

$$M_{[f,g]} \xrightarrow{\delta_{f,g,g,M}} (M_{[f,g]})_{[g,g]} \xrightarrow{(M_{[f,g]})_g} (M_{[f,g]})_{[id_Z,id_Z]} = M_{[f,g]}$$

$$M_{[f,g]} \xrightarrow{\delta_{f,f,g,M}} (M_{[f,f]})_{[f,g]} \xrightarrow{(M_f)_{[f,g]}} (M_{[id_Y,id_Y]})_{[f,g]} = M_{[f,g]}$$

Proof. The following diagram commutes for any $N \in \mathrm{Ob}\,\mathcal{F}_Z$ by (1) of (1.3.7) and (1.3.11).

$$\begin{CD}
\mathcal{F}_Z(id^*_Z(M_{[f,g]}),id^*_Z(N)) @>{g^\sharp_{M_{[f,g]},N}}>> \mathcal{F}_X(g^*(M_{[f,g]}),g^*(N)) @>{\iota_{f,g}(M)^*}>> \mathcal{F}_X(f^*(M),g^*(N))\\
@VV{P_{id_Z,id_Z}(M_{[f,g]})_N}V @VV{P_{g,g}(M_{[f,g]})_N}V @VV{P_{f,g}(M)_N}V\\
\mathcal{F}_Z((M_{[f,g]})_{[id_Z,id_Z]},N) @>{(M_{[f,g]})^*_g}>> \mathcal{F}_Z((M_{[f,g]})_{[g,g]},N) @>{\delta^*_{f,g,g,M}}>> \mathcal{F}_Z(M_{[f,g]},N)
\end{CD}$$

It follows from (1.3.2) that $\delta^*_{f,g,g,M}(M_{[f,g]})^*_g : \mathcal{F}_Z(M_{[f,g]}, N) = \mathcal{F}_Z((M_{[f,g]})_{[id_Z, id_Z]}, N) \to \mathcal{F}_Z(M_{[f,g]}, N)$ is the identity map of $\mathcal{F}_Z(M_{[f,g]}, N)$.

The following diagram commutes for any $N \in \operatorname{Ob}\mathcal{F}_Z$ by (1) of (1.3.4) and and (1.3.11).

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M_{[id_Y,id_Y]}), g^*(N)) \arrow[r, "f^*(M_f)^*"] \arrow[d, "P_{f,g}(M_{[id_Y,id_Y]})_N"] & \mathcal{F}_X(f^*(M_{[f,f]}), g^*(N)) \arrow[r, "\iota_{f,f}(M)^*"] \arrow[d, "P_{f,g}(M_{[f,f]})_N"] & \mathcal{F}_X(f^*(M), g^*(N)) \arrow[d, "P_{f,g}(M)_N"] \\
\mathcal{F}_Z((M_{[id_Y,id_Y]})_{[f,g]}, N) \arrow[r, "(M_f)^*_{[f,g]}"] & \mathcal{F}_Z((M_{[f,f]})_{[f,g]}, N) \arrow[r, "\delta^*_{f,f,g,M}"] & \mathcal{F}_Z(M_{[f,g]}, N)
\end{tikzcd}$$

Since the composition of the upper horizontal maps of the above diagram coincides with the identity map of $\mathcal{F}_X(f^*(M), g^*(N))$ by (4) of (1.3.7), the composition of the lower horizontal maps of the above diagram is the identity map of $\mathcal{F}_Z(M_{[f,g]}, N)$. □

Let $f : X \to Y$, $g : X \to Z$, $h : X \to W$ be morphisms in $\mathcal{E}$ and L, M, N objects of $\mathcal{F}_Y$, $\mathcal{F}_Z$, $\mathcal{F}_W$, respectively. We define a map

$$\gamma^{f,g,h}_{L,M,N} : \mathcal{F}_Z(L_{[f,g]}, M) \times \mathcal{F}_W(M_{[g,h]}, N) \to \mathcal{F}_W(L_{[f,h]}, N)$$

as follows. For $\varphi \in \mathcal{F}_Z(L_{[f,g]}, M)$ and $\psi \in \mathcal{F}_W(M_{[g,h]}, N)$, let $\gamma^{f,g,h}_{L,M,N}(\varphi, \psi)$ be the following composition.

$$L_{[f,h]} \xrightarrow{\delta_{f,g,h,L}} (L_{[f,g]})_{[g,h]} \xrightarrow{\varphi_{[g,h]}} M_{[g,h]} \xrightarrow{\psi} N$$

Proposition 1.3.16 *The following diagram is commutative.*

$$\begin{tikzcd}
\mathcal{F}_X(f^*(L), g^*(M)) \times \mathcal{F}_X(g^*(M), h^*(N)) \arrow[r, "composition"] \arrow[d, "P_{f,g}(L)_M \times P_{g,h}(M)_N"] & \mathcal{F}_X(f^*(L), h^*(N)) \arrow[d, "P_{f,h}(L)_N"] \\
\mathcal{F}_Z(L_{[f,g]}, M) \times \mathcal{F}_W(M_{[g,h]}, N) \arrow[r, "\gamma^{f,g,h}_{L,M,N}"] & \mathcal{F}_W(L_{[f,h]}, N)
\end{tikzcd}$$

Proof. For $\zeta \in \mathcal{F}_X(f^*(L), g^*(M))$ and $\xi \in \mathcal{F}_X(g^*(M), h^*(N))$, we put $\varphi = P_{f,g}(L)_M(\zeta)$ and $\psi = P_{g,h}(M)_N(\xi)$. Then, we have $\psi\varphi_{[g,h]} = P_{[g,h]}(L_{[f,g]})_N(\xi g^*(\varphi))$ by (1.3.4). It follows from (1.3.11) and (1.3.2) that

$$\psi\varphi_{[g,h]}\delta_{f,g,h,L} = \delta^*_{f,g,h,L}P_{g,h}(L_{[f,g]})_N(\xi g^*(\varphi)) = P_{f,h}(L)_N(\xi g^*(\varphi)\iota_{f,g}(L)) = P_{f,h}(L)_N(\xi\zeta).$$

Thus the result follows. □

We define a poset $\mathcal{P}$ as follows. Set $\operatorname{Ob}\mathcal{P} = \{0, 1, 2, 3, 4, 5\}$ and $\mathcal{P}(i,j)$ is not an empty set if and only if $i = j$ or $i = 0$ or $(i,j) = (1,3), (1,4), (2,4), (2,5)$. We put $\mathcal{P}(i,j) = \{\tau_{ij}\}$ if $\mathcal{P}(i,j)$ is not empty. For a functor $D : \mathcal{P} \to \mathcal{E}$ and an object M of $\mathcal{F}_{D(3)}$, we put $D(\tau_{ij}) = f_{ij}$ and define a morphism

$$\theta_D(M) : M_{[f_{13}f_{01}, f_{25}f_{02}]} \to (M_{[f_{13}, f_{14}]})_{[f_{24}, f_{25}]}$$

of $\mathcal{F}_{D(5)}$ to be the following composition.

$$M_{[f_{13}f_{01}, f_{25}f_{02}]} \xrightarrow{\delta_{f_{13}f_{01}, f_{14}f_{01}, f_{25}f_{02}, M}} (M_{[f_{13}f_{01}, f_{14}f_{01}]})_{[f_{24}f_{02}, f_{25}f_{02}]} \xrightarrow{(M_{f_{01}})_{f_{02}}} (M_{[f_{13}, f_{14}]})_{[f_{24}, f_{25}]}$$

Proposition 1.3.17 *We assume that the inverse image functors $f^*_{14} : \mathcal{F}_{D(4)} \to \mathcal{F}_{D(1)}$, $f^*_{25} : \mathcal{F}_{D(5)} \to \mathcal{F}_{D(2)}$, $(f_{14}f_{01})^* : \mathcal{F}_{D(5)} \to \mathcal{F}_{D(0)}$ and $(f_{25}f_{02})^* : \mathcal{F}_{D(5)} \to \mathcal{F}_{D(0)}$ have left adjoints $(f_{14})_* : \mathcal{F}_{D(1)} \to \mathcal{F}_{D(4)}$, $(f_{25})_* : \mathcal{F}_{D(2)} \to \mathcal{F}_{D(5)}$, $(f_{14}f_{01})_* : \mathcal{F}_{D(0)} \to \mathcal{F}_{D(4)}$ and $(f_{25}f_{02})_* : \mathcal{F}_{D(0)} \to \mathcal{F}_{D(5)}$, respectively. Let $\eta_{f_{14}} : id_{\mathcal{F}_{D(1)}} \to f^*_{14}(f_{14})_*$ and $\eta_{f_{25}} : id_{\mathcal{F}_{D(2)}} \to f^*_{25}(f_{25})_*$ be the units of the adjunctions $f^*_{14} \dashv (f_{14})_*$ and $f^*_{25} \dashv (f_{25})_*$, respectively. For an object M of $\mathcal{F}_{D(1)}$,*

$$\theta_D(M) : M_{[f_{13}f_{01}, f_{25}f_{02}]} = (f_{25}f_{02})_*((f_{13}f_{01})^*(M)) \to (f_{25})_*(f^*_{24}((f_{14})_*(f^*_{13}(M)))) = (M_{[f_{13}, f_{14}]})_{[f_{24}, f_{25}]}$$

coincides with the adjoint of the following composition with respect to the adjunction $(f_{25}f_{02})_* \dashv (f_{25}f_{02})^*$.

$$(f_{13}f_{01})^*(M) \xrightarrow{c_{f_{13}, f_{01}}(M)^{-1}} f^*_{01}(f^*_{13}(M)) \xrightarrow{f^*_{01}\left((\eta_{f_{14}})_{f^*_{13}(M)}\right)} f^*_{01}(f^*_{14}((f_{14})_*(f^*_{13}(M)))) \xrightarrow{c_{f_{14}, f_{01}}((f_{14})_*(f^*_{13}(M)))}$$
$$(f_{14}f_{01})^*((f_{14})_*(f^*_{13}(M))) = (f_{24}f_{02})^*((f_{14})_*(f^*_{13}(M))) \xrightarrow{c_{f_{24}, f_{02}}((f_{14})_*(f^*_{13}(M)))^{-1}} f^*_{02}(f^*_{24}((f_{14})_*(f^*_{13}(M))))$$
$$\xrightarrow{f^*_{02}\left((\eta_{f_{25}})_{f^*_{24}((f_{14})_*(f^*_{13}(M)))}\right)} f^*_{02}(f^*_{25}((f_{25})_*(f^*_{24}((f_{14})_*(f^*_{13}(M)))))) \xrightarrow{c_{f_{25}, f_{02}}((f_{25})_*(f^*_{24}((f_{14})_*(f^*_{13}(M)))))}$$
$$(f_{25}f_{02})^*((f_{25})_*(f^*_{24}((f_{14})_*(f^*_{13}(M)))))$$

Proof. By the definition of $\theta_D(M)$ and (1.3.12), $\theta_D(M)$ is the following composition.

$$
\begin{aligned}
M_{[f_{13}f_{01},f_{25}f_{02}]} = (f_{25}f_{02})_*(f_{13}f_{01})^*(M) &\xrightarrow{(f_{25}f_{02})_*((\eta_{f_{14}f_{01}})_{(f_{13}f_{01})^*(M)})} (f_{25}f_{02})_*(f_{14}f_{01})^*(f_{14}f_{01})_*(f_{13}f_{01})^*(M) \\
&\xrightarrow{(f_{25}f_{02})_*(f_{14}f_{01})^*(M_{f_{01}})} (f_{25}f_{02})_*(f_{14}f_{01})^*(f_{14})_*f_{13}^*(M) = (M_{[f_{13},f_{14}]})_{[f_{14}f_{01},f_{25}f_{02}]} \\
&= (M_{[f_{13},f_{14}]})_{[f_{24}f_{02},f_{25}f_{02}]} \xrightarrow{(M_{[f_{13},f_{14}]})_{f_{02}}} (M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}
\end{aligned}
$$

It follows from (1) of (1.3.8) that the adjoint of $(M_{[f_{13},f_{14}]})_{f_{02}} : (M_{[f_{13},f_{14}]})_{[f_{24}f_{02},f_{25}f_{02}]} \to (M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}$ with respect to the adjunction $(f_{25}f_{02})_* \dashv (f_{25}f_{02})^*$ is the following composition.

$$
\begin{aligned}
(f_{24}f_{02})^*(M_{[f_{13},f_{14}]}) &\xrightarrow{c_{f_{24},f_{02}}(M_{[f_{13},f_{14}]})^{-1}} f_{02}^*f_{24}^*(M_{[f_{13},f_{14}]}) \xrightarrow{f_{02}^*((\eta_{f_{25}})_{f_{24}^*(M_{[f_{13},f_{14}]})})} f_{02}^*f_{25}^*(f_{25})_*f_{24}^*(M_{[f_{13},f_{14}]}) \\
&= f_{02}^*f_{25}^*((M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}) \xrightarrow{c_{f_{25},f_{02}}((M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]})} (f_{25}f_{02})^*((M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]})
\end{aligned}
$$

It also follows from (1) of (1.3.8) that $M_{f_{01}} : M_{[f_{13}f_{01},f_{14}f_{01}]} \to M_{[f_{13},f_{14}]}$ coincides with the following composition.

$$
\begin{aligned}
M_{[f_{13}f_{01},f_{14}f_{01}]} = (f_{14}f_{01})_*(f_{13}f_{01})^*(M) &\xrightarrow{(f_{14}f_{01})_*(c_{f_{13},f_{01}}(M))^{-1}} (f_{14}f_{01})_*f_{01}^*f_{13}^*(M) \xrightarrow{(f_{14}f_{01})_*f_{01}^*((\eta_{f_{14}})_{f_{13}^*(M)})} \\
(f_{14}f_{01})_*f_{01}^*f_{14}^*(f_{14})_*f_{13}^*(M) &= (f_{14}f_{01})_*f_{01}^*f_{14}^*(M_{[f_{13},f_{14}]}) \xrightarrow{(f_{14}f_{01})_*(c_{f_{14},f_{01}}(M_{[f_{13},f_{14}]}))} \\
(f_{14}f_{01})_*(f_{14}f_{01})^*(M_{[f_{13},f_{14}]}) &\xrightarrow{(\varepsilon_{f_{14}f_{01}})_{M_{[f_{13},f_{14}]}}} M_{[f_{13},f_{14}]}
\end{aligned}
$$

Hence if we put $\varphi = c_{f_{14},f_{01}}(M_{[f_{13},f_{14}]})f_{01}^*((\eta_{f_{14}})_{f_{13}^*(M)})c_{f_{13},f_{01}}(M)^{-1} : (f_{13}f_{01})^*(M) \to (f_{14}f_{01})^*(M_{[f_{13},f_{14}]})$, the adjoint of $\theta_D(M)$ with respect to the adjunction $(f_{25}f_{02})_* \dashv (f_{25}f_{02})^*$ is the following composition.

$$
\begin{aligned}
(f_{13}f_{01})^*(M) &\xrightarrow{(\eta_{f_{14}f_{01}})_{(f_{13}f_{01})^*(M)}} (f_{14}f_{01})^*(f_{14}f_{01})_*(f_{13}f_{01})^*(M) \xrightarrow{(f_{14}f_{01})^*(f_{14}f_{01})_*(\varphi)} \\
(f_{14}f_{01})^*(f_{14}f_{01})_*(f_{14}f_{01})^*(M_{[f_{13},f_{14}]}) &\xrightarrow{(f_{14}f_{01})^*((\varepsilon_{f_{14}f_{01}})_{M_{[f_{13},f_{14}]}})} (f_{14}f_{01})^*(M_{[f_{13},f_{14}]}) = (f_{24}f_{02})^*(M_{[f_{13},f_{14}]}) \\
&\xrightarrow{c_{f_{24},f_{02}}(M_{[f_{13},f_{14}]})^{-1}} f_{02}^*f_{24}^*(M_{[f_{13},f_{14}]}) \xrightarrow{f_{02}^*((\eta_{f_{25}})_{f_{24}^*(M_{[f_{13},f_{14}]})})} f_{02}^*f_{25}^*(f_{25})_*f_{24}^*(M_{[f_{13},f_{14}]}) \\
&= f_{02}^*f_{25}^*((M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}) \xrightarrow{c_{f_{25},f_{02}}((M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]})} (f_{25}f_{02})^*((M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]})
\end{aligned}
$$

By the naturality of $\eta_{f_{14}f_{01}}$, the composition of the first three morphisms in the above diagram coincides with $(f_{14}f_{01})^*((\varepsilon_{f_{14}f_{01}})_{M_{[f_{13},f_{14}]}})(\eta_{f_{14}f_{01}})_{(f_{14}f_{01})^*(M_{[f_{13},f_{14}]})}\varphi = \varphi$, which implies the assertion. □

Proposition 1.3.18 *The following diagram is commutative.*

$$
\begin{tikzcd}
(f_{13}f_{01})^*(M) \arrow[r, "f_{01}^\sharp(\iota_{f_{13},f_{14}}(M))"] \arrow[d, "\iota_{f_{13}f_{01},f_{25}f_{02}}(M)"] & (f_{14}f_{01})^*(M_{[f_{13},f_{14}]}) \arrow[r, equal] & (f_{24}f_{02})^*(M_{[f_{13},f_{14}]}) \arrow[d, "f_{02}^\sharp(\iota_{f_{24},f_{25}}(M_{[f_{13},f_{14}]}))"] \\
(f_{25}f_{02})^*(M_{[f_{13}f_{01},f_{25}f_{02}]}) \arrow[rr, "(f_{25}f_{02})^*(\theta_D(M))"] & & (f_{25}f_{02})^*((M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]})
\end{tikzcd}
$$

Proof. By the naturality of $P_{f_{13}f_{01},f_{25}f_{02}}(M)$, $\theta_D(M)$ is the image of

$$(f_{25}f_{02})^*((M_{f_{01}})_{f_{02}})\iota_{f_{14}f_{01},f_{25}f_{02}}(M_{[f_{13}f_{01},f_{14}f_{01}]})\iota_{f_{13}f_{01},f_{14}f_{01}}(M) : (f_{13}f_{01})^*(M) \to (f_{25}f_{02})^*((M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]})$$

by $P_{f_{13}f_{01},f_{25}f_{02}}(M)_{(M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}}$. Hence the following equality holds by (1.3.2).

$$(f_{25}f_{02})^*(\theta_D(M))\iota_{f_{13}f_{01},f_{25}f_{02}}(M) = (f_{25}f_{02})^*((M_{f_{01}})_{f_{02}})\iota_{f_{14}f_{01},f_{25}f_{02}}(M_{[f_{13}f_{01},f_{14}f_{01}]})\iota_{f_{13}f_{01},f_{14}f_{01}}(M) \quad \cdots (*)$$

It follows from (1.3.7), (1.1.11) and (1.3.4) that we have

$$
\begin{aligned}
&(f_{25}f_{02})^*((M_{f_{01}})_{f_{02}})\iota_{f_{24}f_{02},f_{25}f_{02}}(M_{[f_{13}f_{01},f_{14}f_{01}]})\\
&=(f_{25}f_{02})^*((M_{f_{01}})_{[f_{24},f_{25}]})(f_{25}f_{02})^*((M_{[f_{13}f_{01},f_{14}f_{01}]})_{f_{02}})\iota_{f_{24}f_{02},f_{25}f_{02}}(M_{[f_{13}f_{01},f_{14}f_{01}]})\\
&=(f_{25}f_{02})^*((M_{f_{01}})_{[f_{24},f_{25}]})f_{02}^{\sharp}(\iota_{f_{24},f_{25}}(M_{[f_{13}f_{01},f_{14}f_{01}]}))\\
&=(f_{25}f_{02})^*((M_{f_{01}})_{[f_{24},f_{25}]})c_{f_{25},f_{02}}((M_{[f_{13}f_{01},f_{14}f_{01}]})_{[f_{24},f_{25}]})f_{02}^*(\iota_{f_{24},f_{25}}(M_{[f_{13}f_{01},f_{14}f_{01}]}))c_{f_{24},f_{02}}(M_{[f_{13}f_{01},f_{14}f_{01}]})^{-1}\\
&=c_{f_{25},f_{02}}((M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]})f_{02}^*(f_{25}^*((M_{f_{01}})_{[f_{24},f_{25}]}))f_{02}^*(\iota_{f_{24},f_{25}}(M_{[f_{13}f_{01},f_{14}f_{01}]}))c_{f_{24},f_{02}}(M_{[f_{13}f_{01},f_{14}f_{01}]})^{-1}\\
&=c_{f_{25},f_{02}}((M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]})f_{02}^*(\iota_{f_{24},f_{25}}(M_{[f_{13},f_{14}]}))f_{02}^*(f_{24}^*(M_{f_{01}}))c_{f_{24},f_{02}}(M_{[f_{13}f_{01},f_{14}f_{01}]})^{-1}\\
&=c_{f_{25},f_{02}}((M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]})f_{02}^*(\iota_{f_{24},f_{25}}(M_{[f_{13},f_{14}]}))c_{f_{24},f_{02}}(M_{[f_{13},f_{14}]})^{-1}(f_{24}f_{02})^*(M_{f_{01}})\\
&=f_{02}^{\sharp}(\iota_{f_{24},f_{25}}(M_{[f_{13},f_{14}]}))(f_{24}f_{02})^*(M_{f_{01}})
\end{aligned}
$$

Therefore we have

$$(*)=f_{02}^{\sharp}(\iota_{f_{24},f_{25}}(M_{[f_{13},f_{14}]}))(f_{24}f_{02})^*(M_{f_{01}})\iota_{f_{13}f_{01},f_{14}f_{01}}(M)=f_{02}^{\sharp}(\iota_{f_{24},f_{25}}(M_{[f_{13},f_{14}]}))f_{01}^{\sharp}(\iota_{f_{13},f_{14}}(M))$$

which implies the assertion. □

Proposition 1.3.19 *For a morphism $\varphi : L \to M$ of $\mathcal{F}_Y$, the following diagram commutes.*

$$
\begin{tikzcd}
L_{[f_{13}f_{01},f_{25}f_{02}]} \arrow[r, "\theta_D(L)"] \arrow[d, "\varphi_{[f_{13}f_{01},f_{25}f_{02}]}"] & (L_{[f_{13},f_{14}]})_{[f_{24},f_{25}]} \arrow[d, "(\varphi_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}"] \\
M_{[f_{13}f_{01},f_{25}f_{02}]} \arrow[r, "\theta_D(M)"] & (M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}
\end{tikzcd}
$$

Proof. The following diagram commutes by (1.3.13), (1.3.9), (1.3.4) and (1.3.7).

$$
\begin{tikzcd}
L_{[f_{13}f_{01},f_{25}f_{02}]} \arrow[r, "\delta_{f_{13}f_{01},f_{14}f_{01},f_{25}f_{02},L}"] \arrow[d, "\varphi_{[f_{13}f_{01},f_{25}f_{02}]}"] & (L_{[f_{13}f_{01},f_{14}f_{01}]})_{[f_{24}f_{02},f_{25}f_{02}]} \arrow[r, "(L_{f_{01}})_{f_{02}}"] \arrow[d, "(\varphi_{[f_{13}f_{01},f_{14}f_{01}]})_{[f_{24}f_{02},f_{25}f_{02}]}"] & (L_{[f_{13},f_{14}]})_{[f_{24},f_{25}]} \arrow[d, "(\varphi_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}"] \\
M_{[f_{13}f_{01},f_{25}f_{02}]} \arrow[r, "\delta_{f_{13}f_{01},f_{14}f_{01},f_{25}f_{02},M}"] & (M_{[f_{13}f_{01},f_{14}f_{01}]})_{[f_{24}f_{02},f_{25}f_{02}]} \arrow[r, "(M_{f_{01}})_{f_{02}}"] & (M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}
\end{tikzcd}
$$

Hence the assertion follows. □

Proposition 1.3.20 *Let $E : \mathcal{P} \to \mathcal{E}$ be a functor which satisfies $E(i) = D(i)$ for $i = 3,4,5$ and $\lambda : D \to E$ a natural transformation which satisfies $\lambda_i = id_{D(i)}$ for $i = 3,4,5$. We put $E(\tau_{ij}) = g_{ij}$, then the following diagram commutes.*

$$
\begin{tikzcd}
M_{[f_{13}f_{01},f_{25}f_{02}]} \arrow[r, "\theta_D(M)"] \arrow[d, "M_{\lambda_0}"] & (M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]} \arrow[d, "(M_{\lambda_1})_{\lambda_2}"] \\
M_{[g_{13}g_{01},g_{25}g_{02}]} \arrow[r, "\theta_E(M)"] & (M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}
\end{tikzcd}
$$

Proof. Since $f_{ij} = g_{ij}\lambda_i$ for $i = 1,2$, we have $f_{13}f_{01} = g_{13}\lambda_1 f_{01} = g_{13}g_{01}\lambda_0$, $f_{14}f_{01} = g_{14}\lambda_1 f_{01} = g_{14}g_{01}\lambda_0$ and $f_{25}f_{02} = g_{25}\lambda_2 f_{02} = g_{25}g_{02}\lambda_0$. It follows from (1.3.7), (1.3.9) and (1.3.13) that

$$
\begin{tikzcd}
M_{[f_{13}f_{01},f_{25}f_{02}]} \arrow[r, "\delta_{f_{13}f_{01},f_{14}f_{01},f_{25}f_{02},M}"] \arrow[d, "M_{\lambda_0}"] & (M_{[f_{13}f_{01},f_{14}f_{01}]})_{[f_{24}f_{02},f_{25}f_{02}]} \arrow[r, "(M_{f_{01}})_{f_{02}}"] \arrow[d, "(M_{\lambda_0})_{\lambda_0}"] & (M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]} \arrow[d, "(M_{\lambda_1})_{\lambda_2}"] \\
M_{[g_{13}g_{01},g_{25}g_{02}]} \arrow[r, "\delta_{g_{13}g_{01},g_{14}g_{01},g_{25}g_{02},M}"] & (M_{[g_{13}g_{01},g_{14}g_{01}]})_{[g_{24}g_{02},g_{25}g_{02}]} \arrow[r, "(M_{g_{01}})_{g_{02}}"] & (M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}
\end{tikzcd}
$$

is commutative. □

For morphisms $f : X \to Y$, $g : X \to Z$, $h : V \to Z$, $i : V \to W$ in $\mathcal{E}$, let $X \xleftarrow{\mathrm{pr}_X} X \times_Z V \xrightarrow{\mathrm{pr}_V} V$ be a limit of a diagram $X \xrightarrow{g} Z \xleftarrow{h} V$. We define a functor $D_{f,g,h,i} : \mathcal{P} \to \mathcal{E}$ by $D_{f,g,h,i}(0) = X \times_Z V$, $D_{f,g,h,i}(1) = X$, $D_{f,g,h,i}(2) = V$, $D_{f,g,h,i}(3) = Y$, $D_{f,g,h,i}(4) = Z$, $D_{f,g,h,i}(5) = W$ and $D_{f,g,h,i}(\tau_{01}) = \mathrm{pr}_X$, $D_{f,g,h,i}(\tau_{02}) = \mathrm{pr}_V$, $D_{f,g,h,i}(\tau_{13}) = f$, $D_{f,g,h,i}(\tau_{14}) = g$, $D_{f,g,h,i}(\tau_{24}) = h$, $D_{f,g,h,i}(\tau_{25}) = i$. For an object M of $\mathcal{F}_Y$, we denote $\theta_{D_{f,g,h,i}}(M)$ by $\theta_{f,g,h,i}(M)$. The following facts are special cases of (1.3.19) and (1.3.20).

Proposition 1.3.21 *Let $f : X \to Y$, $g : X \to Z$, $h : V \to Z$, $i : V \to W$, $j : S \to X$, $k : T \to V$ be morphisms in $\mathcal{E}$ and $\varphi : L \to M$ a morphism in $\mathcal{F}_Y$. The following diagrams are commutative.*

$$\begin{tikzcd} L_{[f\mathrm{pr}_X,\, i\mathrm{pr}_V]} \arrow[r, "\theta_{f,g,h,i}(L)"] \arrow[d, "\varphi_{[f\mathrm{pr}_X,\, i\mathrm{pr}_V]}"] & (L_{[f,g]})_{[h,i]} \arrow[d, "(\varphi_{[f,g]})_{[h,i]}"] \\ M_{[f\mathrm{pr}_X,\, i\mathrm{pr}_V]} \arrow[r, "\theta_{f,g,h,i}(M)"] & (M_{[f,g]})_{[h,i]} \end{tikzcd} \qquad \begin{tikzcd} M_{[fj\mathrm{pr}_S,\, ik\mathrm{pr}_T]} \arrow[r, "\theta_{fj,gj,hk,ik}(M)"] \arrow[d, "M_{j\times_Z k}"] & (M_{[fj,gj]})_{[hk,\, ik]} \arrow[d, "(M_j)_k"] \\ M_{[f\mathrm{pr}_X,\, i\mathrm{pr}_V]} \arrow[r, "\theta_{f,g,h,i}(M)"] & (M_{[f,g]})_{[h,i]} \end{tikzcd}$$

Remark 1.3.22 *If $X \xleftarrow{\mathrm{pr}'_X} X \times'_Z V \xrightarrow{\mathrm{pr}'_V} V$ is another limit of a diagram $X \xrightarrow{g} Z \xleftarrow{h} V$, there exists unique isomorphism $l : X \times'_Z V \to X \times_Z V$ that satisfies $\mathrm{pr}'_X = \mathrm{pr}_X l$ and $\mathrm{pr}'_V = \mathrm{pr}_V l$. We denote by $\theta'_{f,g,h,i}(M) : M_{[f\mathrm{pr}'_X, i\mathrm{pr}'_V]} \to (M_{[f,g]})_{[h,i]}$ the morphism in $\mathcal{F}_W$ obtained from $X \xleftarrow{\mathrm{pr}'_X} X \times'_Z V \xrightarrow{\mathrm{pr}'_V} V$. Then, $M_l : M_{[f\mathrm{pr}'_X,\, i\mathrm{pr}'_V]} \to M_{[f\mathrm{pr}_X,\, i\mathrm{pr}_V]}$ is an isomorphism and (1.3.20) implies $\theta'_{f,g,h,i}(M) = \theta_{f,g,h,i}(M) M_l$.*

Definition 1.3.23 *Let $f : X \to Y$, $g : X \to Z$, $h : V \to Z$, $i : V \to W$ be morphisms in $\mathcal{E}$ and M an object of $\mathcal{F}_Y$. We say that a quadruple (f, g, h, i) is an associative left fibered representable quadruple with respect to M if the following conditions are satisfied.*

(i) A limit $X \xleftarrow{\mathrm{pr}_X} X \times_Z V \xrightarrow{\mathrm{pr}_V} V$ of a diagram $X \xrightarrow{g} Z \xleftarrow{h} V$ exists.
(ii) (f, g) is a left fibered representable pair with respect to M.
(iii) (h, i) is a left fibered representable pair with respect to $M_{[f,g]}$.
(iv) $(f\mathrm{pr}_X, i\mathrm{pr}_V)$ is a left fibered representable pair with respect to M.
(v) $\theta_{f,g,h,i}(M) : M_{[f\mathrm{pr}_X, i\mathrm{pr}_V]} \to (M_{[f,g]})_{[h,i]}$ is an isomorphism.

If (f, g, h, i) is an associative left fibered representable quadruple with respect to any object of $\mathcal{F}_Y$, we say that (f, g, h, i) is an associative left fibered representable quadruple.

Proposition 1.3.24 *Suppose that the following diagram in $\mathcal{E}$ is commutative.*

$$\begin{tikzcd}[column sep=small] & & & Q \arrow[dl, "v"'] \arrow[dr, "w"] & & & \\ & & R \arrow[dl, "r"'] \arrow[dr, "s"] & & S \arrow[dl, "t"'] \arrow[dr, "u"] & & \\ & X \arrow[dl, "f"'] \arrow[dr, "g"] & & V \arrow[dl, "h"'] \arrow[dr, "i"] & & T \arrow[dl, "j"'] \arrow[dr, "k"] & \\ Y & & Z & & W & & U \end{tikzcd}$$

Define functors $D_l : \mathcal{P} \to \mathcal{E}$ for $l = 1, 2, 3, 4$ as follows.

$$\begin{array}{llllll}
D_1(0) = S & D_1(1) = V & D_1(2) = T & D_1(3) = Z & D_1(4) = W & D_1(5) = U \\
D_1(\tau_{01}) = t & D_1(\tau_{02}) = u & D_1(\tau_{13}) = h & D_1(\tau_{14}) = i & D_1(\tau_{24}) = j & D_1(\tau_{25}) = k \\
D_2(0) = Q & D_2(1) = R & D_2(2) = T & D_2(3) = Y & D_2(4) = W & D_2(5) = U \\
D_2(\tau_{01}) = v & D_2(\tau_{02}) = uw & D_2(\tau_{13}) = fr & D_2(\tau_{14}) = is & D_2(\tau_{24}) = j & D_2(\tau_{25}) = k \\
D_3(0) = Q & D_3(1) = X & D_3(2) = S & D_3(3) = Y & D_3(4) = Z & D_3(5) = U \\
D_3(\tau_{01}) = rv & D_3(\tau_{02}) = w & D_3(\tau_{13}) = f & D_3(\tau_{14}) = g & D_3(\tau_{24}) = ht & D_3(\tau_{25}) = ku \\
D_4(0) = R & D_4(1) = X & D_4(2) = V & D_4(3) = Y & D_4(4) = Z & D_4(5) = W \\
D_4(\tau_{01}) = r & D_4(\tau_{02}) = s & D_4(\tau_{13}) = f & D_4(\tau_{14}) = g & D_4(\tau_{24}) = h & D_4(\tau_{25}) = i
\end{array}$$

Then, the following diagram is commutative.

$$\begin{tikzcd} M_{[frv,kuw]} \arrow[r, "\theta_{D_3}(M)"] \arrow[d, "\theta_{D_2}(M)"] & (M_{[f,g]})_{[ht,ku]} \arrow[d, "\theta_{D_1}(M_{[f,g]})"] \\ (M_{[fr,is]})_{[j,k]} \arrow[r, "\theta_{D_4}(M)_{[j,k]}"] & ((M_{[f,g]})_{[h,i]})_{[j,k]} \end{tikzcd}$$

Proof. The following diagrams are commutative by (1.3.14), (1.3.13), (1.3.9), (1.3.4) and (1.3.7).

$$\begin{tikzcd}
M_{[frv,kuw]} \arrow[r, "\delta_{frv,htw,kuw,M}"] \arrow[d, "\delta_{frv,isv,kuw,M}"] & (M_{[frv,grv]})_{[htw,kuw]} \arrow[r, "(M_{[frv,grv]})_w"] \arrow[d, "\delta_{htw,isv,kuw,M_{[frv,grv]}}"'] & (M_{[frv,grv]})_{[ht,ku]} \arrow[d, "\delta_{ht,it,ku,M_{[frv,grv]}}"'] \\
(M_{[frv,isv]})_{[juw,kuw]} \arrow[r, "(\delta_{frv,grv,isv,M})_{[juw,kuw]}"] \arrow[d, "(M_v)_{[juw,kuw]}"] & ((M_{[frv,grv]})_{[grv,isv]})_{[juw,kuw]} \arrow[r, "((M_{[frv,grv]})_w)_w"] \arrow[d, "((M_v)_v)_{[juw,kuw]}"'] & ((M_{[frv,grv]})_{[ht,it]})_{[ju,ku]} \arrow[d, "((M_{rv})_t)_{[ju,ku]}"'] \\
(M_{[fr,is]})_{[juw,kuw]} \arrow[r, "(\delta_{fr,gr,is,M})_{[juw,kuw]}"] \arrow[d, "(M_{[fr,is]})_{uw}"] & ((M_{[fr,gr]})_{[hs,is]})_{[juw,kuw]} \arrow[r, "((M_r)_s)_w"] \arrow[d, "((M_{[fr,gr]})_{[hs,is]})_{uw}"'] & ((M_{[f,g]})_{[h,i]})_{[ju,ku]} \arrow[d, "((M_{[f,g]})_{[h,i]})_u"'] \\
(M_{[fr,is]})_{[j,k]} \arrow[r, "(\delta_{fr,gr,is,M})_{[j,k]}"] & ((M_{[fr,gr]})_{[hs,is]})_{[j,k]} \arrow[r, "((M_r)_s)_{[j,k]}"] & ((M_{[f,g]})_{[h,i]})_{[j,k]}
\end{tikzcd}$$

$$\begin{tikzcd}
(M_{[frv,grv]})_{[ht,ku]} \arrow[r, "(M_{rv})_{[ht,ku]}"] \arrow[d, "\delta_{ht,it,ku,M_{[frv,grv]}}"] & (M_{[f,g]})_{[ht,ku]} \arrow[d, "\delta_{ht,it,ku,M_{[f,g]}}"] \\
((M_{[frv,grv]})_{[ht,it]})_{[ju,ku]} \arrow[r, "((M_{rv})_{[ht,it]})_{[ju,ku]}"] \arrow[d, "((M_{rv})_t)_{[ju,ku]}"] & ((M_{[f,g]})_{[ht,it]})_{[ju,ku]} \arrow[dl, "((M_{[f,g]})_t)_{[ju,ku]}"] \\
((M_{[f,g]})_{[h,i]})_{[ju,ku]} &
\end{tikzcd}$$

Hence the assertion follows from the definition of $\theta_{D_l}(M)$. $\square$

For morphisms $g : X \to Z$, $h : V \to Z$, $i : V \to W$, $j : T \to W$ in $\mathcal{E}$, let $X \xleftarrow{\mathrm{pr}_X} X \times_Z V \xrightarrow{\mathrm{pr}_{2V}} V$ and $V \xleftarrow{\mathrm{pr}_{1V}} V \times_W T \xrightarrow{\mathrm{pr}_T} T$ be limits of diagrams $X \xrightarrow{g} Z \xleftarrow{h} V$ and $V \xrightarrow{i} W \xleftarrow{j} T$, respectively. We also assume that a limit $X \times_Z V \xleftarrow{\mathrm{pr}_{X\times_Z V}} X \times_Z V \times_W T \xrightarrow{\mathrm{pr}_{V\times_W T}} V \times_W T$ of a diagram $X \times_Z V \xrightarrow{\mathrm{pr}_{2V}} V \xleftarrow{\mathrm{pr}_{1V}} V \times_W T$ exists. Then, $X \xleftarrow{\mathrm{pr}_X \mathrm{pr}_{X\times_Z V}} X \times_Z V \times_W T \xrightarrow{\mathrm{pr}_{V\times_W T}} V \times_W T$ and $X \times_Z V \xleftarrow{\mathrm{pr}_{X\times_Z V}} X \times_Z V \times_W T \xrightarrow{\mathrm{pr}_{V\times_W T}\mathrm{pr}_T} T$ are limits of diagrams $X \xrightarrow{g} Z \xleftarrow{h\mathrm{pr}_{1V}} V \times_W T$ and $X \times_Z V \xrightarrow{i\mathrm{pr}_{2V}} W \xleftarrow{j} T$, respectively.

Corollary 1.3.25 *Let $f : X \to Y$, $g : X \to Z$, $h : V \to Z$, $i : V \to W$, $j : T \to W$, $k : T \to U$ be morphisms in $\mathcal{E}$ and M an object of $\mathcal{F}_Y$. The following diagram is commutative.*

$$\begin{tikzcd}
M_{[f\mathrm{pr}_X \mathrm{pr}_{X\times_Z V},\, k\mathrm{pr}_T \mathrm{pr}_{V\times_W T}]} \arrow[r, "\theta_{f,g,h\mathrm{pr}_{1V},k\mathrm{pr}_T}(M)"] \arrow[d, "\theta_{f\mathrm{pr}_X, i\mathrm{pr}_{2V},j,k}(M)"] & (M_{[f,g]})_{[h\mathrm{pr}_{1V},\, k\mathrm{pr}_T]} \arrow[d, "\theta_{h,i,j,k}(M_{[f,g]})"] \\
(M_{[f\mathrm{pr}_X,\, i\mathrm{pr}_{2V}]})_{[j,k]} \arrow[r, "\theta_{f,g,h,i}(M)_{[j,k]}"] & ((M_{[f,g]})_{[h,i]})_{[j,k]}
\end{tikzcd}$$

Proof. The assertion follows by applying the result of (1.3.24) to the following diagram.

$$\begin{tikzcd}
 & & & & X \times_Z V \times_W T \arrow[dll, "\mathrm{pr}_{X\times_Z V}"'] \arrow[drr, "\mathrm{pr}_{V\times_W T}"] & & & & \\
 & & X \times_Z V \arrow[dl, "\mathrm{pr}_X"'] \arrow[drr, "\mathrm{pr}_{2V}"] & & & & V \times_W T \arrow[dll, "\mathrm{pr}_{1V}"'] \arrow[dr, "\mathrm{pr}_T"] & & \\
 & X \arrow[dl, "f"'] \arrow[dr, "g"] & & & V \arrow[dll, "h"'] \arrow[drr, "i"] & & & T \arrow[dl, "j"'] \arrow[dr, "k"] & \\
Y & & Z & & & & W & & U
\end{tikzcd}$$

$\square$

Proposition 1.3.26 *For morphisms $f : X \to Y$, $g : X \to Z$ in $\mathcal{E}$ and an object M of $\mathcal{F}_Y$, the following morphisims of $\mathcal{F}_Z$ are identified with the identity morphism of $M_{[f,\,g]}$.*

$$\theta_{f,g,id_Z,id_Z}(M) : M_{[f\,id_X,\, id_Z g]} \to (M_{[f,\,g]})_{[id_Z,\, id_Z]}, \qquad \theta_{id_Y,id_Y,f,g}(M) : M_{[id_Y f,\, g\,id_X]} \to (M_{[id_Y,\, id_Y]})_{[f,\,g]}$$

Proof. Since $\theta_{f,g,id_Z,id_Z}(M)$ is a composition

$$M_{[f,\,g]} = M_{[f\,id_X,\,id_Z g]} \xrightarrow{\delta_{f\,id_X,\,g\,id_X,\,id_Z g,\,M}} (M_{[f\,id_X,\,g\,id_X]})_{[id_Z g,\,id_Z g]} \xrightarrow{(M_{[f,\,g]})_g} (M_{[f,\,g]})_{[id_Z,\,id_Z]} = M_{[f,\,g]}$$

and $\theta_{id_Y,id_Y,f,g}(M)$ is a composition

$$M_{[f,\,g]} = M_{[id_Y f,\,g\,id_X]} \xrightarrow{\delta_{id_Y f,\,f\,id_X,\,g\,id_X,\,M}} (M_{[id_Y f,\,id_Y f]})_{[f\,id_X,\,g\,id_X]} \xrightarrow{(M_f)_{[f,\,g]}} (M_{[id_Y,\,id_Y]})_{[f,\,g]} = M_{[f,\,g]},$$

the assertion is a direct consequence of (1.3.15). □

Lemma 1.3.27 *For a functor $D : \mathcal{P} \to \mathcal{E}$, we put $D(\tau_{01}) = j$, $D(\tau_{02}) = k$, $D(\tau_{13}) = f$, $D(\tau_{14}) = g$, $D(\tau_{24}) = h$, $D(\tau_{25}) = i$. For an object M of $\mathcal{F}_{D(3)}$, the following diagram is commutative.*

$$\begin{array}{ccc}
(fj)^*(M) & \xrightarrow{\iota_{fj,ik}(M)} & (ik)^*(M_{[fj,ik]}) \\
\downarrow{\scriptstyle j^\sharp(\iota_{f,g}(M))} & & \downarrow{\scriptstyle (ik)^*(\theta_D(M))} \\
(gj)^*(M_{[f,g]}) & \xrightarrow{k^\sharp(\iota_{h,i}(M_{[f,g]}))} & (ik)^*((M_{[f,g]})_{[h,i]})
\end{array}$$

Proof. It follows from (1.3.7) and (1) of (1.3.4) that we have

$$\begin{aligned}
k^\sharp(\iota_{h,i}(M_{[f,g]}))j^\sharp(\iota_{f,g}(M)) &= (ik)^*((M_{[f,g]})_k)\iota_{hk,ik}(M_{[f,g]})(gj)^*(M_j)\iota_{fj,gj}(M) \\
&= (ik)^*((M_{[f,g]})_k)(ik)^*((M_j)_{[hk,ik]})\iota_{hk,ik}(M_{[fj,gj]})\iota_{fj,gj}(M) \\
&= (ik)^*((M_j)_k)\iota_{hk,ik}(M_{[fj,gj]})\iota_{fj,gj}(M)
\end{aligned}$$

By the naturality of $P_{fj,ik}(M)$ and the definition of $\delta_{fj,gj,ik,M}$, the above equality implies that

$$P_{fj,ik}(M)_{(M_{[f,g]})_{[h,i]}} : \mathcal{F}_{D(0)}((fj)^*(M), (ik)^*((M_{[f,g]})_{[h,i]}) \to \mathcal{F}_{D(5)}(M_{[fj,ik]}, (M_{[f,g]})_{[h,i]})$$

maps $k^\sharp(\iota_{h,i}(M_{[f,g]}))j^\sharp(\iota_{f,g}(M))$ to $(M_j)_k\delta_{fj,gj,ik,M} = \theta_D(M)$. On the other hand, it follows from (1.3.2) that $P_{fj,ik}(M)_{(M_{[f,g]})_{[h,i]}}$ also maps $(ik)^*(\theta_D(M))\iota_{fj,ik}(M)$ to $\theta_D(M)$. □

For a morphism $g : X \to Z$, let $X \xleftarrow{\mathrm{pr}_{1X}} X \times_Z X \xrightarrow{\mathrm{pr}_{2X}} X$ be a limit of a diagram $X \xrightarrow{g} Z \xleftarrow{g} X$. We denote by $\Delta_g : X \to X \times_Z X$ the diagonal morphism, that is, the unique morphism that satisfies $\mathrm{pr}_{1X}\Delta_g = \mathrm{pr}_{2X}\Delta_g = id_X$.

Proposition 1.3.28 *For morphisms $f : X \to Y$, $g : X \to Z$, $h : X \to W$ in $\mathcal{E}$ and an object M of $\mathcal{F}_Y$, $\delta_{f,g,h,M} : M_{[f,h]} \to (M_{[f,g]})_{[g,h]}$ coincides with the following composition.*

$$M_{[f,h]} = M_{[f\mathrm{pr}_{1X}\Delta_g, h\mathrm{pr}_{2X}\Delta_g]} \xrightarrow{M_{\Delta_g}} M_{[f\mathrm{pr}_{1X}, h\mathrm{pr}_{2X}]} \xrightarrow{\theta_{f,g,g,h}(M)} (M_{[f,g]})_{[g,h]}$$

Proof. Define a functor $E : \mathcal{P} \to \mathcal{E}$ by $E(i) = X$ for $i = 0,1,2$, $E(i) = D_{f,g,g,h}(i)$ for $i = 3,4,5$ and $E(\tau_{01}) = E(\tau_{02}) = id_X$, $E(\tau_{ij}) = D_{f,g,g,h}(\tau_{ij})$ if $i \neq 0$. Then, $\theta_E(M) = \delta_{f,g,h,M} : M_{[f,h]} \to (M_{[f,g]})_{[g,h]}$ and we have a natural transformation $\lambda : E \to D$ defined by $\lambda_0 = \Delta_g$ and $\lambda_i = id_{E(i)}$ if $i \geqq 1$. It follows from (1.3.20) that $\theta_{f,g,g,h}(M)M_{\Delta_g} = \theta_E(M) = \delta_{f,g,h,M}$. □

Let $\mathcal{Q}$ be a subposet of $\mathcal{P}$ given by $\mathrm{Ob}\,\mathcal{Q} = \{0,1,2\}$. Let $D, E : \mathcal{Q} \to \mathcal{E}$ be functors and $\omega : D \to E$ a natural transformation. We put $D(\tau_{0j}) = f_j$ and $E(\tau_{0j}) = g_j$ for $j = 1,2$. For an object M of $\mathcal{F}_{E(1)}$, let $\omega_M : \omega_1^*(M)_{[f_1,f_2]} \to \omega_2^*(M_{[g_1,g_2]})$ be the image of $\iota_{g_1,g_2}(M) \in \mathcal{F}_{E(0)}(g_1^*(M), g_2^*(M_{[g_1,g_2]}))$ by the following composition of maps.

$$\begin{aligned}
\mathcal{F}_{E(0)}(g_1^*(M), g_2^*(M_{[g_1,g_2]})) &\xrightarrow{\omega_0^\sharp} \mathcal{F}_{D(0)}((g_1\omega_0)^*(M), (g_2\omega_0)^*(M_{[g_1,g_2]})) = \mathcal{F}_{D(0)}((\omega_1 f_1)^*(M), (\omega_2 f_2)^*(M_{[g_1,g_2]})) \\
&\xrightarrow{c_{\omega_1,f_1}(M)^* c_{\omega_2,f_2}(M_{[g_1,g_2]})_*^{-1}} \mathcal{F}_{D(0)}(f_1^*(\omega_1^*(M)), f_2^*(\omega_2^*(M_{[g_1,g_2]}))) \\
&\xrightarrow{P_{f_1,f_2}(\omega_1^*(M))_{\omega_2^*(M_{[g_1,g_2]})}} \mathcal{F}_{D(2)}(\omega_1^*(M)_{[f_1,f_2]}, \omega_2^*(M_{[g_1,g_2]}))
\end{aligned}$$

Remark 1.3.29 *(1) If $D(i) = E(i)$ and ω_i is the identity morphism of $D(i)$ for $i = 1,2$, then ω_M coincides with $M_{\omega_0} : M_{[f_1,f_2]} = M_{[g_1\omega_0,g_2\omega_0]} \to M_{[g_1,g_2]}$.*

(2) It follows from (1.3.2) and the definition of ω_M that the following diagram is commutative.

$$\begin{tikzcd}
f_1^*(\omega_1^*(M)) \arrow[r, "c_{\omega_1,f_1}(M)"] \arrow[d, "\iota_{f_1,f_2}(\omega_1^*(M))"] & (\omega_1 f_1)^*(M) = (g_1\omega_0)^*(M) \arrow[r, "\omega_0^\sharp(\iota_{g_1,g_2}(M))"] & (g_2\omega_0)^*(M_{[g_1,g_2]}) \arrow[d, equal] \\
f_2^*(\omega_1^*(M)_{[f_1,f_2]}) \arrow[r, "f_2^*(\omega_M)"] & f_2^*(\omega_2^*(M_{[g_1,g_2]})) \arrow[r, "c_{\omega_2,f_2}(M_{[g_1,g_2]})"] & (\omega_2 f_2)^*(M_{[g_1,g_2]})
\end{tikzcd}$$

Proposition 1.3.30 *Assume that $D(0) = E(0)$ and ω_0 is the identity morphism of $D(0)$. For an object N of $\mathcal{F}_{E(2)}$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathcal{F}_{D(0)}(g_1^*(M), g_2^*(N)) \arrow[r, "c_{\omega_2,f_2}(N)_*^{-1}"] \arrow[d, "P_{g_1,g_2}(M)_N"] & \mathcal{F}_{D(0)}(g_1^*(M), f_2^*(\omega_2^*(N))) \arrow[r, "c_{\omega_1,f_1}(M)^*"] & \mathcal{F}_{D(0)}(f_1^*(\omega_1^*(M)), f_2^*(\omega_2^*(N))) \arrow[d, "P_{f_1,f_2}(\omega_1^*(M))_{\omega_2^*(N)}"] \\
\mathcal{F}_{E(2)}(M_{[g_1,g_2]}, N) \arrow[r, "\omega_2^*"] & \mathcal{F}_{D(2)}(\omega_2^*(M_{[g_1,g_2]}), \omega_2^*(N)) \arrow[r, "\omega_M^*"] & \mathcal{F}_{D(2)}(\omega_1^*(M)_{[f_1,f_2]}, \omega_2^*(N))
\end{tikzcd}$$

Proof. First we note that $g_i = \omega_i f_i$ for $i = 1, 2$. It follows from (1.3.29) and the definition of ω_M that we have $f_2^*(\omega_M)\iota_{f_1,f_2}(\omega_1^*(M)) = c_{\omega_2,f_2}(M_{[g_1,g_2]})^{-1}\iota_{g_1,g_2}(M)c_{\omega_1,f_1}(M)$. (1.3.2) and (1.1.11) imply

$$\begin{aligned}
c_{\omega_2,f_2}(N)^{-1}P_{g_1,g_2}(M)_N^{-1}(\varphi)c_{\omega_1,f_1}(M) &= c_{\omega_2,f_2}(N)^{-1}g_2^*(\varphi)\iota_{g_1,g_2}(M)c_{\omega_1,f_1}(M) \\
&= f_2^*\omega_2^*(\varphi)c_{\omega_2,f_2}(M_{[g_1,g_2]})^{-1}\iota_{g_1,g_2}(M)c_{\omega_1,f_1}(M) \\
&= f_2^*\omega_2^*(\varphi)f_2^*(\omega_M)\iota_{f_1,f_2}(\omega_1^*(M)) = f_2^*(\omega_2^*(\varphi)\omega_M)\iota_{f_1,f_2}(\omega_1^*(M)) \\
&= P_{f_1,f_2}(\omega_1^*(M))_{\omega_2^*(N)}^{-1}(\omega_2^*(\varphi)\omega_M)
\end{aligned}$$

for $\varphi \in \mathcal{F}_{E(2)}(M_{[g_1,g_2]}, N)$, which shows that the above diagram is commutative. □

Proposition 1.3.31 *For a morphism $\varphi : M \to N$ of $\mathcal{F}_{E(1)}$, the following diagram is commutative.*

$$\begin{tikzcd}
\omega_1^*(M)_{[f_1,f_2]} \arrow[r, "\omega_M"] \arrow[d, "\omega_1^*(\varphi)_{[f_1,f_2]}"] & \omega_2^*(M_{[g_1,g_2]}) \arrow[d, "\omega_2^*(\varphi_{[g_1,g_2]})"] \\
\omega_1^*(N)_{[f_1,f_2]} \arrow[r, "\omega_N"] & \omega_2^*(N_{[g_1,g_2]})
\end{tikzcd}$$

Proof. It follows from (1.1.11), (1) of (1.3.4) and (1.1.15) that the following diagrams are commutative.

$$\begin{tikzcd}
f_1^*\omega_1^*(M) \arrow[r, "c_{\omega_1,f_1}(M)"] \arrow[d, "f_1^*\omega_1^*(\varphi)"] & (\omega_1 f_1)^*(M) = (g_1\omega_0)^*(M) \arrow[r, "\omega_0^\sharp(\iota_{g_1,g_2}(M))"] \arrow[d, "(g_1\omega_0)^*(\varphi)"] & (g_2\omega_0)^*(M_{[g_1,g_2]}) = (\omega_2 f_2)^*(M_{[g_1,g_2]}) \arrow[d, "(g_2\omega_0)^*(\varphi_{[g_1,g_2]})"] \\
f_1^*\omega_1^*(N) \arrow[r, "c_{\omega_1,f_1}(N)"] & (\omega_1 f_1)^*(N) = (g_1\omega_0)^*(N) \arrow[r, "\omega_0^\sharp(\iota_{g_1,g_2}(N))"] & (g_2\omega_0)^*(N_{[g_1,g_2]}) = (\omega_2 f_2)^*(N_{[g_1,g_2]})
\end{tikzcd}$$

$$\begin{tikzcd}
(\omega_2 f_2)^*(M_{[g_1,g_2]}) \arrow[r, "c_{\omega_2,f_2}(M_{[g_1,g_2]})^{-1}"] \arrow[d, "(\omega_2 f_2)^*(\varphi_{[g_1,g_2]})"] & f_2^*\omega_2^*(M_{[g_1,g_2]}) \arrow[d, "f_2^*\omega_2^*(\varphi_{[g_1,g_2]})"] \\
(\omega_2 f_2)^*(N_{[g_1,g_2]}) \arrow[r, "c_{\omega_2,f_2}(N_{[g_1,g_2]})^{-1}"] & f_2^*\omega_2^*(N_{[g_1,g_2]})
\end{tikzcd}$$

By applying (1.3.6) to the following commutative diagram,

$$\begin{tikzcd}
f_1^*\omega_1^*(M) \arrow[r, "c_{\omega_2,f_2}(M_{[g_1,g_2]})^{-1}\omega_0^\sharp(\iota_{g_1,g_2}(M))c_{\omega_1,f_1}(M)"] \arrow[d, "f_1^*\omega_1^*(\varphi)"] & f_2^*\omega_2^*(M_{[g_1,g_2]}) \arrow[d, "f_2^*\omega_2^*(\varphi_{[g_1,g_2]})"] \\
f_1^*\omega_1^*(N) \arrow[r, "c_{\omega_2,f_2}(N_{[g_1,g_2]})^{-1}\omega_0^\sharp(\iota_{g_1,g_2}(N))c_{\omega_1,f_1}(N)"] & f_2^*\omega_2^*(N_{[g_1,g_2]})
\end{tikzcd}$$

the assertion follows. □

Lemma 1.3.32 *Let $D, E, F : \mathcal{Q} \to \mathcal{E}$ be functors and $\omega : D \to E$, $\chi : E \to F$ natural transformations. We put $D(\tau_{0j}) = f_j$, $E(\tau_{0j}) = g_j$ and $F(\tau_{0j}) = h_j$ for $j = 1, 2$. For $M \in \operatorname{Ob}\mathcal{F}_{F(1)}$, $N \in \operatorname{Ob}\mathcal{F}_{F(2)}$ and a morphism $\varphi : h_1^*(M) \to h_2^*(N)$ of $\mathcal{F}_{F(0)}$, the following diagram is commutative.*

$$\begin{tikzcd}
\omega_0^*((\chi_1 g_1)^*(M))) \arrow[r, "c_{\chi_1 g_1,\omega_0}(M)"] \arrow[d, equal] & (\chi_1 g_1 \omega_0)^*(M) = (h_1\chi_0\omega_0)^*(M) \arrow[r, "(\chi_0\omega_0)^\sharp(\varphi)"] & (h_2\chi_0\omega_0)^*(N) \arrow[d, equal] \\
\omega_0^*((h_1\chi_0)^*(M)) \arrow[r, "\omega_0^*(\chi_0^\sharp(\varphi))"] & \omega_0^*((h_2\chi_0)^*(N)) = \omega_0^*((\chi_2 g_2)^*(N)) \arrow[r, "c_{\chi_2 g_2,\omega_0}(N)"] & (\chi_2 g_2\omega_0)^*(N)
\end{tikzcd}$$

Proof. The following diagram is commutative by (1.1.12), (1.1.16) and the definition of $\omega_0^\sharp$.

$$\begin{tikzcd}
\omega_0^*((\chi_1 g_1)^*(M))) \arrow[r, "c_{\chi_1 g_1,\omega_0}(M)"] \arrow[d, equal] & (\chi_1 g_1\omega_0)^*(M) \arrow[r, equal] \arrow[d, equal] & (h_1\chi_0\omega_0)^*(M) \arrow[d, "(\chi_0\omega_0)^\sharp(\varphi)"] \\
\omega_0^*((h_1\chi_0)^*(M)) \arrow[r, "c_{h_1\chi_0,\omega_0}(M)"] \arrow[d, "\omega_0^*(\chi_0^\sharp(\varphi))"] & (h_1\chi_0\omega_0)^*(M) \arrow[r, "\omega_0^\sharp(\chi_0^\sharp(\varphi))"] & (h_2\chi_0\omega_0)^*(N) \arrow[d, equal] \\
\omega_0^*((h_2\chi_0)^*(N)) \arrow[r, equal] \arrow[urr, "c_{h_2\chi_0,\omega_0}(N)"] & \omega_0^*((\chi_2 g_2)^*(N)) \arrow[r, "c_{\chi_2 g_2,\omega_0}(N)"] & (\chi_2 g_2\omega_0)^*(N)
\end{tikzcd}$$

□

Proposition 1.3.33 *Let $D, E, F : \mathcal{Q} \to \mathcal{E}$ be functors and M an object of $\mathcal{F}_{F(1)}$. We put $D(\tau_{0j}) = f_j$, $E(\tau_{0j}) = g_j$ and $F(\tau_{0j}) = h_j$ for $j = 1, 2$. For natural transformations $\omega : D \to E$ and $\chi : E \to F$, the following diagram is commutative.*

$$\begin{tikzcd}
\omega_1^*(\chi_1^*(M))_{[f_1,f_2]} \arrow[r, "\omega_{\chi_1^*(M)}"] \arrow[d, "c_{\chi_1,\omega_1}(M)_{[f_1,f_2]}"] & \omega_2^*(\chi_1^*(M)_{[g_1,g_2]}) \arrow[r, "\omega_2^*(\chi_M)"] & \omega_2^*(\chi_2^*(M_{[h_1,h_2]})) \arrow[d, "c_{\chi_2,\omega_2}(M_{[h_1,h_2]})"] \\
(\chi_1\omega_1)^*(M)_{[f_1,f_2]} \arrow[rr, "(\chi\omega)_M"] & & (\chi_2\omega_2)^*(M_{[h_1,h_2]})
\end{tikzcd}$$

Proof. It follows from (1.3.2) and (1.3.29) that we have

$$\begin{aligned}
P_{f_1,f_2}(\omega_1^*(\chi_1^*(M)))^{-1}_{\omega_2^*(\chi_2^*(M_{[h_1,h_2]}))}(\omega_2^*(\chi_M)\omega_{\chi_1^*(M)}) &= f_2^*(\omega_2^*(\chi_M)\omega_{\chi_1^*(M)})\iota_{f_1,f_2}(\omega_1^*(\chi_1^*(M))) \\
&= f_2^*(\omega_2^*(\chi_M))f_2^*(\omega_{\chi_1^*(M)})\iota_{f_1,f_2}(\omega_1^*(\chi_1^*(M))) \\
&= f_2^*(\omega_2^*(\chi_M))c_{\omega_2,f_2}(\chi_1^*(M)_{[g_1,g_2]})^{-1}\omega_0^\sharp(\iota_{g_1,g_2}(\chi_1^*(M)))c_{\omega_1,f_1}(\chi_1^*(M))
\end{aligned}$$

Hence it suffices to show that the following diagram is commutative by (1.3.6).

$$\begin{tikzcd}
f_1^*(\omega_1^*(\chi_1^*(M))) \arrow[r, "f_2^*(\omega_2^*(\chi_M))c_{\omega_2,f_2}(\chi_1^*(M)_{[g_1,g_2]})^{-1}\omega_0^\sharp(\iota_{g_1,g_2}(\chi_1^*(M)))c_{\omega_1,f_1}(\chi_1^*(M))"] \arrow[d, "f_1^*(c_{\chi_1,\omega_1}(M))"] & f_2^*(\omega_2^*(\chi_2^*(M_{[h_1,h_2]}))) \arrow[d, "f_2^*(c_{\chi_2,\omega_2}(M_{[h_1,h_2]}))"] \\
f_1^*(\chi_1\omega_1)^*(M) \arrow[r, "c_{\chi_2\omega_2,f_2}(M_{[h_1,h_2]})^{-1}(\chi_0\omega_0)^\sharp(\iota_{h_1,h_2}(M))c_{\chi_1\omega_1,f_1}(M)"] & f_2^*(\chi_2\omega_2)^*(M_{[h_1,h_2]})
\end{tikzcd}$$

It follows from (1.1.11) and (1.1.12) that we have

$$\begin{aligned}
&f_2^*(\omega_2^*(\chi_M))c_{\omega_2,f_2}(\chi_1^*(M)_{[g_1,g_2]})^{-1} = c_{\omega_2,f_2}(\chi_2^*(M_{[h_1,h_2]}))^{-1}(\omega_2 f_2)^*(\chi_M) = c_{\omega_2,f_2}(\chi_2^*(M_{[h_1,h_2]}))^{-1}(g_2\omega_0)^*(\chi_M) \\
&c_{\chi_1\omega_1,f_1}(M)f_1^*(c_{\chi_1,\omega_1}(M))c_{\omega_1,f_1}(\chi_1^*(M))^{-1} = c_{\chi_1,\omega_1 f_1}(M) = c_{\chi_1,g_1\omega_0}(M) \\
&c_{\chi_2\omega_2,f_2}(M_{[h_1,h_2]})f_2^*(c_{\chi_2,\omega_2}(M_{[h_1,h_2]}))c_{\omega_2,f_2}(\chi_2^*(M_{[h_1,h_2]}))^{-1} = c_{\chi_2,\omega_2 f_2}(M_{[h_1,h_2]}) = c_{\chi_2,g_2\omega_0}(M_{[h_1,h_2]}).
\end{aligned}$$

Hence the commutativity of the above diagram is equivalent to the following equality.

$$c_{\chi_2,g_2\omega_0}(M_{[h_1,h_2]})(g_2\omega_0)^*(\chi_M)\omega_0^\sharp(\iota_{g_1,g_2}(\chi_1^*(M))) = (\chi_0\omega_0)^\sharp(\iota_{h_1,h_2}(M))c_{\chi_1,g_1\omega_0}(M) \quad \cdots (*)$$

The following diagram is commutative by (1.1.11) and (1.3.29).

$$\begin{tikzcd}
\omega_0^*((h_1\chi_0)^*(M)) \arrow[rr, "\omega_0^*(\chi_0^\sharp(\iota_{h_1,h_2}(M)))"] \arrow[d, equal] & & \omega_0^*((h_2\chi_0)^*(M_{[h_1,h_2]})) \arrow[d, equal] \\
\omega_0^*((\chi_1 g_1)^*(M)) & & \omega_0^*((\chi_2 g_2)^*(M_{[h_1,h_2]})) \\
\omega_0^*(g_1^*(\chi_1^*(M))) \arrow[u, "\omega_0^*(c_{\chi_1,g_1}(M))"] \arrow[r, "\omega_0^*(\iota_{g_1,g_2}(\chi_1^*(M)))"] \arrow[d, "c_{g_1,\omega_0}(\chi_1^*(M))"] & \omega_0^*(g_2^*(\chi_1^*(M)_{[g_1,g_2]})) \arrow[r, "\omega_0^*(g_2^*(\chi_M))"] \arrow[d, "c_{g_2,\omega_0}(\chi_1^*(M)_{[g_1,g_2]})"] & \omega_0^*(g_2^*(\chi_2^*(M_{[h_1,h_2]}))) \arrow[u, "\omega_0^*(c_{\chi_2,g_2}(M_{[h_1,h_2]}))"] \arrow[d, "c_{g_2,\omega_0}(\chi_2^*(M_{[h_1,h_2]}))"] \\
(g_1\omega_0)^*(\chi_1^*(M)) \arrow[r, "\omega_0^\sharp(\iota_{g_1,g_2}(\chi_1^*(M)))"] & (g_2\omega_0)^*(\chi_1^*(M)_{[g_1,g_2]}) \arrow[r, "(g_2\omega_0)^*(\chi_M)"] & (g_2\omega_0)^*(\chi_2^*(M_{[h_1,h_2]}))
\end{tikzcd}$$

Hence the left hand side of $(*)$ equals

$$\begin{aligned}
&c_{\chi_2,g_2\omega_0}(M_{[h_1,h_2]})c_{g_2,\omega_0}(\chi_2^*(M_{[h_1,h_2]}))\omega_0^*(c_{\chi_2,g_2}(M_{[h_1,h_2]}))^{-1}\omega_0^*(\chi_0^\sharp(\iota_{h_1,h_2}(M)))\omega_0^*(c_{\chi_1,g_1}(M))c_{g_1,\omega_0}(\chi_1^*(M))^{-1}\\
&\qquad = c_{\chi_2 g_2,\omega_0}(M_{[h_1,h_2]})\omega_0^*(\chi_0^\sharp(\iota_{h_1,h_2}(M)))c_{\chi_1,g_1\omega_0}(M)^{-1}c_{\chi_1,g_1\omega_0}(M)\\
&\qquad = (\chi_0\omega_0)^\sharp(\iota_{h_1,h_2}(M))c_{\chi_1,g_1\omega_0}(M)
\end{aligned}$$

by (1.1.12) and (1.3.32) for $N = M_{[h_1,h_2]}$ and $\varphi = \iota_{h_1,h_2}(M)$. □

Proposition 1.3.34 *For functors $D, E : \mathcal{P} \to \mathcal{E}$, we put $D(\tau_{ij}) = f_{ij}$ and $E(\tau_{ij}) = g_{ij}$ and define functors $D_i, E_i : \mathcal{Q} \to \mathcal{E}$ for $i = 0, 1, 2$ as follows.*

$$\begin{array}{lllll}
D_0(0) = D(0) & D_0(1) = D(3) & D_0(2) = D(5) & D_0(\tau_{01}) = f_{13}f_{01} & D_0(\tau_{02}) = f_{25}f_{02} \\
E_0(0) = E(0) & E_0(1) = E(3) & E_0(2) = E(5) & E_0(\tau_{01}) = g_{13}g_{01} & E_0(\tau_{02}) = g_{25}g_{02} \\
D_1(0) = D(1) & D_1(1) = D(3) & D_1(2) = D(4) & D_1(\tau_{01}) = f_{13} & D_1(\tau_{02}) = f_{14} \\
E_1(0) = E(1) & E_1(1) = E(3) & E_1(2) = E(4) & E_1(\tau_{01}) = g_{13} & E_1(\tau_{02}) = g_{14} \\
D_2(0) = D(2) & D_2(1) = D(4) & D_2(2) = D(5) & D_2(\tau_{01}) = f_{24} & D_2(\tau_{02}) = f_{25} \\
E_2(0) = E(2) & E_2(1) = E(4) & E_2(2) = E(5) & E_2(\tau_{01}) = g_{24} & E_2(\tau_{02}) = g_{25}
\end{array}$$

For a natural transformation $\gamma : D \to E$, we define a natural transformations $\gamma^i : D_i \to E_i$ $(i = 0, 1, 2)$ by

$$\gamma^0_0 = \gamma_0 \quad \gamma^0_1 = \gamma_3 \quad \gamma^0_2 = \gamma_5 \quad \gamma^1_0 = \gamma_1 \quad \gamma^1_1 = \gamma_3 \quad \gamma^1_2 = \gamma_4 \quad \gamma^2_0 = \gamma_2 \quad \gamma^2_1 = \gamma_4 \quad \gamma^2_2 = \gamma_5$$

For an object M of $\mathcal{F}_{E_0(1)} = \mathcal{F}_{E(3)}$, the following diagram is commutative.

$$\begin{tikzcd}
\gamma_3^*(M)_{[f_{13}f_{01},f_{25}f_{02}]} \arrow[rr, "\gamma^0_M"] \arrow[d, "\theta_D(\gamma_3^*(M))"] & & \gamma_5^*(M_{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[d, "\gamma_5^*(\theta_E(M))"] \\
(\gamma_3^*(M)_{[f_{13},f_{14}]})_{[f_{24},f_{25}]} \arrow[r, "(\gamma^1_M)_{[f_{24},f_{25}]}"] & (\gamma_4^*(M_{[g_{13},g_{14}]}))_{[f_{24},f_{25}]} \arrow[r, "\gamma^2_{M_{[g_{13},g_{14}]}}"] & \gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})
\end{tikzcd}$$

Proof. By the naturality of $P_{f_{13}f_{01},f_{25}f_{02}}(\gamma_3^*(M))$ and the definition of γ^0_M, $\gamma_5^*(\theta_E(M))\gamma^0_M$ is the image of the following composition by $P_{f_{13}f_{01},f_{25}f_{02}}(\gamma_3^*(M))_{\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})}$.

$$\begin{aligned}
&(f_{13}f_{01})^*(\gamma_3^*(M)) \xrightarrow{c_{\gamma_3,f_{13}f_{01}}(M)} (\gamma_3 f_{13}f_{01})^*(M) = (g_{13}g_{01}\gamma_0)^*(M) \xrightarrow{\gamma_0^\sharp(\iota_{g_{13}g_{01},g_{25}g_{02}}(M))} \\
&\qquad (g_{25}g_{02}\gamma_0)^*(M_{[g_{13}g_{01},g_{25}g_{02}]}) = (\gamma_5 f_{25}f_{02})^*(M_{[g_{13}g_{01},g_{25}g_{02}]}) \xrightarrow{c_{\gamma_5,f_{25}f_{02}}(M_{[g_{13}g_{01},g_{25}g_{02}]})^{-1}} \\
&\qquad (f_{25}f_{02})^*(\gamma_5^*(M_{[g_{13}g_{01},g_{25}g_{02}]})) \xrightarrow{(f_{25}f_{02})^*(\gamma_5^*(\theta_E(M)))} (f_{25}f_{02})^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))
\end{aligned}$$

On the other hand, $\gamma^2_{M_{[g_{13},g_{14}]}}(\gamma^1_M)_{[f_{24},f_{25}]}\theta_D(\gamma_3^*(M)))$ is the image of the following composition.

$$\begin{aligned}
&(f_{13}f_{01})^*(\gamma_3^*(M)) \xrightarrow{\iota_{f_{13}f_{01},f_{25}f_{02}}(\gamma_3^*(M))} (f_{25}f_{02})^*(\gamma_3^*(M)_{[f_{13}f_{01},f_{25}f_{02}]}) \xrightarrow{(f_{25}f_{02})^*(\theta_D(\gamma_3^*(M)))} \\
&\qquad (f_{25}f_{02})^*((\gamma_3^*(M)_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}) \xrightarrow{(f_{25}f_{02})^*((\gamma^1_M)_{[f_{24},f_{25}]})} (f_{25}f_{02})^*((\gamma_4^*(M_{[g_{13},g_{14}]}))_{[f_{24},f_{25}]}) \\
&\qquad \xrightarrow{(f_{25}f_{02})^*(\gamma^2_{M_{[g_{13},g_{14}]}})} (f_{25}f_{02})^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))
\end{aligned}$$

We see that $\gamma^2_{M_{[g_{13},g_{14}]}}(\gamma^1_M)_{[f_{24},f_{25}]}\theta_D(\gamma_3^*(M)))$ is the image of the following composition by applying (1.3.18) to the first two morphisms in the above diagram.

$$\begin{aligned}(f_{13}f_{01})^*(\gamma_3^*(M)) &\xrightarrow{f_{01}^\sharp(\iota_{f_{13},f_{14}}(\gamma_3^*(M)))} (f_{14}f_{01})^*(\gamma_3^*(M)_{[f_{13},f_{14}]}) = (f_{24}f_{02})^*(\gamma_3^*(M)_{[f_{13},f_{14}]}) \\ &\xrightarrow{f_{02}^\sharp(\iota_{f_{24},f_{25}}(\gamma_3^*(M)_{[f_{13},f_{14}]}))} (f_{25}f_{02})^*((\gamma_3^*(M)_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}) \xrightarrow{(f_{25}f_{02})^*((\gamma^1_M)_{[f_{24},f_{25}]})} \\ &(f_{25}f_{02})^*((\gamma_4^*(M_{[g_{13},g_{14}]}))_{[f_{24},f_{25}]}) \xrightarrow{(f_{25}f_{02})^*(\gamma^2_{M_{[g_{13},g_{14}]}})} (f_{25}f_{02})^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))\end{aligned}$$

Hence it suffices to show that the following diagram (i) is commutative.

$$\begin{tikzcd}
(f_{13}f_{01})^*(\gamma_3^*(M)) \arrow[r, "c_{\gamma_3,f_{13}f_{01}}(M)"] \arrow[d, "f_{01}^\sharp(\iota_{f_{13},f_{14}}(\gamma_3^*(M)))"] & (\gamma_3f_{13}f_{01})^*(M) \arrow[r, equal] & (g_{13}g_{01}\gamma_0)^*(M) \arrow[d, "\gamma_0^\sharp(\iota_{g_{13}g_{01},g_{25}g_{02}}(M))"] \\
(f_{14}f_{01})^*(\gamma_3^*(M)_{[f_{13},f_{14}]}) \arrow[d, equal] & & (g_{25}g_{02}\gamma_0)^*(M_{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[d, equal] \\
(f_{24}f_{02})^*(\gamma_3^*(M)_{[f_{13},f_{14}]}) \arrow[d, "f_{02}^\sharp(\iota_{f_{24},f_{25}}(\gamma_3^*(M)_{[f_{13},f_{14}]}))"] & & (\gamma_5f_{25}f_{02})^*(M_{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[d, "c_{\gamma_5,f_{25}f_{02}}(M_{[g_{13}g_{01},g_{25}g_{02}]})^{-1}"] \\
(f_{25}f_{02})^*((\gamma_3^*(M)_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}) \arrow[d, "(f_{25}f_{02})^*((\gamma^1_M)_{[f_{24},f_{25}]})"] & & (f_{25}f_{02})^*(\gamma_5^*(M_{[g_{13}g_{01},g_{25}g_{02}]})) \arrow[d, "(f_{25}f_{02})^*(\gamma_5^*(\theta_E(M)))"] \\
(f_{25}f_{02})^*((\gamma_4^*(M_{[g_{13},g_{14}]}))_{[f_{24},f_{25}]}) \arrow[rr, "(f_{25}f_{02})^*(\gamma^2_{M_{[g_{13},g_{14}]}})"] & & (f_{25}f_{02})^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))
\end{tikzcd}$$

diagram (i)

The following diagram (ii) is commutative by (1.1.11) and the definition of $f_{02}^\sharp$.

$$\begin{tikzcd}
(f_{24}f_{02})^*(\gamma_3^*(M)_{[f_{13},f_{14}]}) \arrow[r, "c_{f_{24},f_{02}}(\gamma_3^*(M)_{[f_{13},f_{14}]})^{-1}"] \arrow[d, "f_{02}^\sharp(\iota_{f_{24},f_{25}}(\gamma_3^*(M)_{[f_{13},f_{14}]}))"] & f_{02}^*(f_{24}^*(\gamma_3^*(M)_{[f_{13},f_{14}]})) \arrow[d, "f_{02}^*(\iota_{f_{24},f_{25}}(\gamma_3^*(M)_{[f_{13},f_{14}]}))"] \\
(f_{25}f_{02})^*((\gamma_3^*(M)_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}) \arrow[d, "(f_{25}f_{02})^*((\gamma^1_M)_{[f_{24},f_{25}]})"] & f_{02}^*(f_{25}^*((\gamma_3^*(M)_{[f_{13},f_{14}]})_{[f_{24},f_{25}]})) \arrow[l, "c_{f_{25},f_{02}}((\gamma_3^*(M)_{[f_{13},f_{14}]})_{[f_{24},f_{25}]})"'] \arrow[d, "f_{02}^*(f_{25}^*((\gamma^1_M)_{[f_{24},f_{25}]}))"] \\
(f_{25}f_{02})^*((\gamma_4^*(M_{[g_{13},g_{14}]}))_{[f_{24},f_{25}]}) \arrow[d, "(f_{25}f_{02})^*(\gamma^2_{M_{[g_{13},g_{14}]}})"] & f_{02}^*(f_{25}^*(\gamma_4^*(M_{[g_{13},g_{14}]})_{[f_{24},f_{25}]}) \arrow[l, "c_{f_{25},f_{02}}(\gamma_4^*(M_{[g_{13},g_{14}]})_{[f_{24},f_{25}]})"'] \arrow[d, "f_{02}^*(f_{25}^*(\gamma^2_{M_{[g_{13},g_{14}]}}))"] \\
(f_{25}f_{02})^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})) & f_{02}^*(f_{25}^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))) \arrow[l, "c_{f_{25},f_{02}}(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))"']
\end{tikzcd}$$

diagram (ii)

It follows from (1.3.4), (1.3.2) and the definition of $\gamma^2_{M_{[g_{13},g_{14}]}}$ that the following equalities hold.

$$\begin{aligned}&f_{25}^*((\gamma^1_M)_{[f_{24},f_{25}]}))\iota_{f_{24},f_{25}}(\gamma_3^*(M)_{[f_{13},f_{14}]}) = \iota_{f_{24},f_{25}}(\gamma_4^*(M_{[g_{13},g_{14}]}))f_{24}^*(\gamma^1_M)\\ &f_{25}^*(\gamma^2_{M_{[g_{13},g_{14}]}})\iota_{f_{24},f_{25}}(\gamma_4^*(M_{[g_{13},g_{14}]})) = c_{\gamma_5,f_{25}}((M_{[g_{13},g_{14}]})_{[g_{13},g_{14}]})^{-1}\gamma_2^\sharp(\iota_{g_{24},g_{25}}(M_{[g_{13},g_{14}]}))c_{\gamma_4,f_{24}}(M_{[g_{13},g_{14}]})\end{aligned}$$

Hence the composition of the right vertical morphisms in diagram (ii) coincides with the following.

$$\begin{aligned}&f_{02}^*(f_{25}^*(\gamma^2_{M_{[g_{13},g_{14}]}}))f_{02}^*(f_{25}^*((\gamma^1_M)_{[f_{24},f_{25}]}))f_{02}^*(\iota_{f_{24},f_{25}}(\gamma_3^*(M)_{[f_{13},f_{14}]}))\\ &\quad = f_{02}^*(f_{25}^*(\gamma^2_{M_{[g_{13},g_{14}]}}))f_{02}^*(\iota_{f_{24},f_{25}}(\gamma_4^*(M_{[g_{13},g_{14}]})))f_{02}^*(f_{24}^*(\gamma^1_M))\\ &\quad = f_{02}^*(c_{\gamma_5,f_{25}}((M_{[g_{13},g_{14}]})_{[g_{13},g_{14}]})^{-1})f_{02}^*(\gamma_2^\sharp(\iota_{g_{24},g_{25}}(M_{[g_{13},g_{14}]})))f_{02}^*(c_{\gamma_4,f_{24}}(M_{[g_{13},g_{14}]}))f_{02}^*(f_{24}^*(\gamma^1_M))\end{aligned}$$

Since $f_{02}^*(f_{24}^*(\gamma^1_M))c_{f_{24},f_{02}}(\gamma_3^*(M)_{[f_{13},f_{14}]})^{-1} = c_{f_{24},f_{02}}(\gamma_4^*(M_{[g_{13},g_{14}]}))^{-1}(f_{24}f_{02})^*(\gamma^1_M)$ by (1.1.11), the commutativity of diagram (ii) implies that the composition of the right vertical morphisms and the lower horizontal morphism in diagram (i) coincides with the following composition.

$$\begin{aligned}
&(f_{13}f_{01})^*(\gamma_3^*(M)) \xrightarrow{f_{01}^\sharp(\iota_{f_{13},f_{14}}(\gamma_3^*(M)))} (f_{14}f_{01})^*(\gamma_3^*(M)_{[f_{13},f_{14}]}) \xrightarrow{(f_{14}f_{01})^*(\gamma_M^1)} (f_{14}f_{01})^*(\gamma_4^*(M_{[g_{13},g_{14}]})) = \\
&\quad (f_{24}f_{02})^*(\gamma_4^*(M_{[g_{13},g_{14}]})) \xrightarrow{c_{f_{24},f_{02}}(\gamma_4^*(M_{[g_{13},g_{14}]}))^{-1}} f_{02}^*(f_{24}^*(\gamma_4^*(M_{[g_{13},g_{14}]}))) \xrightarrow{f_{02}^*(c_{\gamma_4,f_{24}}(M_{[g_{13},g_{14}]}))} \\
&\quad f_{02}^*((\gamma_4 f_{24})^*(M_{[g_{13},g_{14}]})) = f_{02}^*((g_{24}\gamma_2)^*(M_{[g_{13},g_{14}]})) \xrightarrow{f_{02}^*(\gamma_2^\sharp(\iota_{g_{24},g_{25}}(M_{[g_{13},g_{14}]})))} \\
&\quad f_{02}^*((g_{25}\gamma_2)^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})) = f_{02}^*((\gamma_5 f_{25})^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})) \xrightarrow{f_{02}^*(c_{\gamma_5,f_{25}}((M_{[g_{13},g_{14}]})_{[g_{13},g_{14}]})^{-1})} \\
&\quad f_{02}^*(f_{25}^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))) \xrightarrow{c_{f_{25},f_{02}}(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))} (f_{25}f_{02})^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))
\end{aligned}$$

diagram (iii)

Next, we consider the composition of the upper horizontal morphism and the right vertical morphisms in diagram (i). It follows from (1.1.11) and (1.3.18) that the following diagram is commutative.

$$\begin{tikzcd}
\gamma_0^*((g_{13}g_{01})^*(M)) \arrow[r, "\gamma_0^*(g_{01}^\sharp(\iota_{g_{13},g_{14}}(M)))"] \arrow[d, "\gamma_0^*(\iota_{g_{13}g_{01},g_{25}g_{02}}(M))"] & \gamma_0^*((g_{14}g_{01})^*(M_{[g_{13},g_{14}]})) \arrow[r, equal] & \gamma_0^*((g_{24}g_{02})^*(M_{[g_{13},g_{14}]})) \arrow[d, "\gamma_0^*(g_{02}^\sharp(\iota_{g_{24},g_{25}}(M_{[g_{13},g_{14}]})))"'] \\
\gamma_0^*((g_{25}g_{02})^*(M_{[g_{13}g_{01},g_{25}g_{02}]})) \arrow[rr, "\gamma_0^*((g_{25}g_{02})^*(\theta_E(M)))"] \arrow[d, "c_{g_{25}g_{02},\gamma_0}(M_{[g_{13}g_{01},g_{25}g_{02}]})"] & & \gamma_0^*((g_{25}g_{02})^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})) \arrow[d, "c_{g_{25}g_{02},\gamma_0}((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})"'] \\
(g_{25}g_{02}\gamma_0)^*(M_{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[rr, "(g_{25}g_{02}\gamma_0)^*(\theta_E(M))"] \arrow[d, equal] & & (g_{25}g_{02}\gamma_0)^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}) \arrow[d, equal] \\
(\gamma_5 f_{25}f_{02})^*(M_{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[rr, "(\gamma_5 f_{25}f_{02})^*(\theta_E(M))"] \arrow[d, "c_{\gamma_5,f_{25}f_{02}}(M_{[g_{13}g_{01},g_{25}g_{02}]})^{-1}"] & & (\gamma_5 f_{25}f_{02})^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}) \arrow[d, "c_{\gamma_5,f_{25}f_{02}}((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})^{-1}"'] \\
(f_{25}f_{02})^*(\gamma_5^*(M_{[g_{13}g_{01},g_{25}g_{02}]})) \arrow[rr, "(f_{25}f_{02})^*(\gamma_5^*(\theta_E(M)))"] & & (f_{25}f_{02})^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))
\end{tikzcd}$$

Since $\gamma_0^\sharp(\iota_{g_{13}g_{01},g_{25}g_{02}}(M)) = c_{g_{25}g_{02},\gamma_0}(M_{[g_{13}g_{01},g_{25}g_{02}]})\gamma_0^*(\iota_{g_{13}g_{01},g_{25}g_{02}}(M))c_{g_{13}g_{01},\gamma_0}(M)^{-1}$, it follows from the above diagram that the composition of the upper horizontal morphism and the right vertical morphisms in diagram (i) coincides with the following composition.

$$\begin{aligned}
&(f_{13}f_{01})^*(\gamma_3^*(M)) \xrightarrow{c_{\gamma_3,f_{13}f_{01}}(M)} (\gamma_3 f_{13}f_{01})^*(M) = (g_{13}g_{01}\gamma_0)^*(M) \xrightarrow{c_{g_{13}g_{01},\gamma_0}(M)^{-1}} \gamma_0^*((g_{13}g_{01})^*(M)) \\
&\quad \xrightarrow{\gamma_0^*(g_{01}^\sharp(\iota_{g_{13},g_{14}}(M)))} \gamma_0^*((g_{14}g_{01})^*(M_{[g_{13},g_{14}]})) = \gamma_0^*((g_{24}g_{02})^*(M_{[g_{13},g_{14}]})) \xrightarrow{\gamma_0^*(g_{02}^\sharp(\iota_{g_{24},g_{25}}(M_{[g_{13},g_{14}]})))} \\
&\quad \gamma_0^*((g_{25}g_{02})^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})) \xrightarrow{c_{g_{25}g_{02},\gamma_0}((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})} (g_{25}g_{02}\gamma_0)^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}) = \\
&\quad (\gamma_5 f_{25}g_{02})^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}) \xrightarrow{c_{\gamma_5,f_{25}f_{02}}((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})^{-1}} (f_{25}f_{02})^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))
\end{aligned}$$

diagram (iv)

The following diagram is commutative by (1.1.11), (1.1.12) and (1.3.29).

$$\begin{tikzcd}
(f_{13}f_{01})^*(\gamma_3^*(M)) \arrow[r, "c_{\gamma_3,f_{13}f_{01}}(M)"] \arrow[d, "c_{f_{13},f_{01}}(\gamma_3^*(M))^{-1}"] & (\gamma_3 f_{13}f_{01})^*(M) \arrow[r, equal] \arrow[d, "c_{\gamma_3 f_{13},f_{01}}(M)^{-1}"'] & (g_{13}\gamma_1 f_{01})^*(M) \arrow[d, "c_{g_{13}\gamma_1,f_{01}}(M)^{-1}"'] \\
f_{01}^*(f_{13}^*(\gamma_3^*(M))) \arrow[r, "f_{01}^*(c_{\gamma_3,f_{13}}(M))"] \arrow[dd, "f_{01}^*(\iota_{f_{13},f_{14}}(\gamma_3^*(M)))"] & f_{01}^*((\gamma_3 f_{13})^*(M)) \arrow[r, equal] & f_{01}^*((g_{13}\gamma_1)^*(M)) \arrow[d, "f_{01}^*(\gamma_1^\sharp(\iota_{g_{13},g_{14}}(M)))"'] \\
& f_{01}^*((\gamma_4 f_{14})^*(M_{[g_{13},g_{14}]})) \arrow[r, equal] & f_{01}^*((g_{14}\gamma_1)^*(M_{[g_{13},g_{14}]})) \arrow[d, "c_{g_{14}\gamma_1,f_{01}}(M_{[g_{13},g_{14}]})"'] \\
f_{01}^*(f_{14}^*(\gamma_3^*(M)_{[f_{13},f_{14}]})) \arrow[r, "f_{01}^*(f_{14}^*(\gamma_M^1))"] \arrow[d, "c_{f_{14},f_{01}}(\gamma_3^*(M)_{[f_{13},f_{14}]})"] & f_{01}^*(f_{14}^*(\gamma_4^*(M_{[g_{13},g_{14}]}))) \arrow[u, "f_{01}^*(c_{\gamma_4,f_{14}}(M_{[g_{13},g_{14}]}))"] \arrow[d, "c_{f_{14},f_{01}}(\gamma_4^*(M_{[g_{13},g_{14}]}))"] & (g_{14}\gamma_1 f_{01})^*(M_{[g_{13},g_{14}]}) \arrow[d, equal] \\
(f_{14}f_{01})^*(\gamma_3^*(M)_{[f_{13},f_{14}]}) \arrow[r, "(f_{14}f_{01})^*(\gamma_M^1)"] & (f_{14}f_{01})^*(\gamma_4^*(M_{[g_{13},g_{14}]})) \arrow[r, "c_{\gamma_4,f_{14}f_{01}}(M_{[g_{13},g_{14}]})"] & (\gamma_4 f_{14}f_{01})^*(M_{[g_{13},g_{14}]})
\end{tikzcd}$$

Hence the following diagram is commutative by (1.1.12) and (1.1.16). Here we put $N = M_{[g_{13},g_{14}]}$ below.

$$\begin{tikzcd}
(f_{13}f_{01})^*(\gamma_3^*(M)) \arrow[r, "c_{\gamma_3,f_{13}f_{01}}(M)"] \arrow[d, "f_{01}^\sharp(\iota_{f_{13},f_{14}}(\gamma_3^*(M)))"] & (\gamma_3 f_{13}f_{01})^*(M) \arrow[r, equal] \arrow[d, equal] & (g_{13}g_{01}\gamma_0)^*(M) \arrow[d, "c_{g_{13}g_{01},\gamma_0}(M)^{-1}"] \arrow[ldd, "\gamma_0^\sharp(g_{01}^\sharp(\iota_{g_{13},g_{14}}(M)))"'] \\
(f_{14}f_{01})^*(\gamma_3^*(M)_{[f_{13},f_{14}]}) \arrow[d, "(f_{14}f_{01})^*(\gamma_M^1)"] & (g_{13}\gamma_1 f_{01})^*(M) \arrow[d, "f_{01}^\sharp(\gamma_1^\sharp(\iota_{g_{13},g_{14}}(M)))"'] & \gamma_0^*((g_{13}g_{01})^*(M)) \arrow[d, "\gamma_0^*(g_{01}^\sharp(\iota_{g_{13},g_{14}}(M)))"'] \\
(f_{14}f_{01})^*(\gamma_4^*(N)) \arrow[d, equal] \arrow[rdd, "c_{\gamma_4,f_{14}f_{01}}(N)"] & (g_{14}g_{01}\gamma_0)^*(N) \arrow[d, equal] & \gamma_0^*((g_{14}g_{01})^*(N)) \arrow[l, "c_{g_{14}g_{01},\gamma_0}(N)"'] \arrow[d, equal] \\
(f_{24}f_{02})^*(\gamma_4^*(N)) \arrow[d, "c_{f_{24},f_{02}}(\gamma_4^*(N))^{-1}"] & (g_{24}g_{02}\gamma_0)^*(N) \arrow[d, equal] \arrow[rdd, bend right, "\gamma_0^\sharp(g_{02}^\sharp(\iota_{g_{24},g_{25}}(N)))"] & \gamma_0^*((g_{24}g_{02})^*(N)) \arrow[l, "c_{g_{24}g_{02},\gamma_0}(N)"'] \arrow[d, "\gamma_0^*(g_{02}^\sharp(\iota_{g_{24},g_{25}}(N)))"'] \\
f_{02}^*(f_{24}^*(\gamma_4^*(N))) \arrow[d, "f_{02}^*(c_{\gamma_4,f_{24}}(N))"] & (\gamma_4 f_{14}f_{01})^*(N) \arrow[d, equal] & \gamma_0^*((g_{25}g_{02})^*(N_{[g_{24},g_{25}]})) \arrow[d, "c_{g_{25}g_{02},\gamma_0}(N_{[g_{24},g_{25}]})"'] \\
f_{02}^*((\gamma_4 f_{24})^*(N)) \arrow[d, equal] & (\gamma_4 f_{24}f_{02})^*(N) \arrow[l, "c_{\gamma_4 f_{24},f_{02}}(N)"'] \arrow[d, equal] & (g_{25}g_{02}\gamma_0)^*(N_{[g_{24},g_{25}]}) \arrow[d, equal] \\
f_{02}^*((g_{24}\gamma_2)^*(N)) \arrow[d, "f_{02}^*(\gamma_2^\sharp(\iota_{g_{24},g_{25}}(N)))"] & (g_{24}\gamma_2 f_{02})^*(N) \arrow[l, "c_{g_{24}\gamma_2,f_{02}}(N)"'] \arrow[r, "f_{02}^\sharp(\gamma_2^\sharp(\iota_{g_{24},g_{25}}(N)))"] & (\gamma_5 f_{25}f_{02})^*(N_{[g_{24},g_{25}]}) \\
f_{02}^*((g_{25}\gamma_2)^*(N_{[g_{24},g_{25}]})) \arrow[d, equal] \arrow[rru, "c_{f_{25}\gamma_2,f_{02}}(N_{[g_{24},g_{25}]})"] & & (f_{25}f_{02})^*(\gamma_5^*(N_{[g_{24},g_{25}]})) \arrow[u, "c_{\gamma_5,f_{25}f_{02}}(N_{[g_{24},g_{25}]})"'] \\
f_{02}^*((\gamma_5 f_{25})^*(N_{[g_{24},g_{25}]})) \arrow[r, "f_{02}^*(c_{\gamma_5,f_{25}}(N_{[g_{24},g_{25}]})^{-1})"] \arrow[rruu, "c_{\gamma_5 f_{25},f_{02}}(N_{[g_{24},g_{25}]})"'] & & f_{02}^*(f_{25}^*(\gamma_5^*(N_{[g_{24},g_{25}]}))) \arrow[u, "c_{f_{25},f_{02}}(\gamma_5^*(N_{[g_{24},g_{25}]}))"']
\end{tikzcd}$$

We see that the compositions of diagram (iii) and the compositions of diagram (iv) coincide, which implies the assertion. $\square$

1.4 Right fibered representable pair

Let $p : \mathcal{F} \to \mathcal{E}$ be a normalized cloven fibered category. For morphisms $f : X \to Y$, $g : X \to Z$ in $\mathcal{E}$ and an object N of $\mathcal{F}_Z$, we define a presheaf $F_N^{f,g} : \mathcal{F}_Y^{op} \to \mathcal{S}et$ on $\mathcal{F}_Y$ by $F_N^{f,g}(M) = F_{f,g}(M, N) = \mathcal{F}_X(f^*(M), g^*(N))$ for $M \in \mathrm{Ob}\,\mathcal{F}_Y$ and $F_N^{f,g}(\varphi) = F_{f,g}(\varphi, id_N) = f^*(\varphi)^*$ for $\varphi \in \mathrm{Mor}\,\mathcal{F}_Y$.

Suppose that $F_N^{f,g}$ is representable. We choose an object $N^{[f,g]}$ of $\mathcal{F}_Y$ such that there exists a natural equivalence $E_{f,g}(N) : F_N^{f,g} \to h_{N^{[f,g]}}$, where $h_{N^{[f,g]}}$ is the presheaf on $\mathcal{F}_Y$ represented by $N^{[f,g]}$. If $X = Y$ and f is the identity morphism of X, we take $g^*(N)$ as $N^{[id_X,g]}$. Hence $E_{id_X,g}(N)_M$ is the identity map of $\mathcal{F}_X(M, g^*(N))$. Let us denote by $\pi_{f,g}(N) : f^*(N^{[f,g]}) \to g^*(N)$ the morphism in $\mathcal{F}_X$ which is mapped to the identity morphism of $N^{[f,g]}$ by $E_{f,g}(N)_{N^{[f,g]}} : \mathcal{F}_X(f^*(N^{[f,g]}), g^*(N)) \to \mathcal{F}_Y(N^{[f,g]}, N^{[f,g]})$.

Definition 1.4.1 *We say that a pair (f, g) of morphisms $f : X \to Y$ and $g : X \to Z$ in $\mathcal{E}$ is a right fibered representable pair with respect to an object N of $\mathcal{F}_Z$ if the presheaf $F_N^{f,g}$ on $\mathcal{F}_Y$ is representable. If (f, g) is a right fibered representable pair with respect to all objects of $\mathcal{F}_Z$, we say that (f, g) is a right fibered representable pair.*

Remark 1.4.2 *If $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ has a right adjoint $f_! : \mathcal{F}_X \to \mathcal{F}_Y$, $F_N^{f,g} : \mathcal{F}_Y^{op} \to \mathcal{S}et$ is representable for any object N of $\mathcal{F}_Z$. In fact, $N^{[f,g]}$ is defined to be $f_!g^*(N)$ in this case and (f, g) is a right fibered representable pair for any morphism g in $\mathcal{E}$ whose domain is X. If we denote by $\mathrm{ad}^f_{M,P} : \mathcal{F}_X(f^*(M), P) \to \mathcal{F}_Y(M, f_!(P))$ the bijection which is natural in $M \in \mathrm{Ob}\,\mathcal{F}_Y$ and $P \in \mathrm{Ob}\,\mathcal{F}_X$, we have $E_{f,g}(N)_M = \mathrm{ad}^f_{M,g^*(N)} : \mathcal{F}_X(f^*(M), g^*(N)) \to \mathcal{F}_Y(M, f_!g^*(N))$. Let us denote by $\varepsilon^f : f^*f_! \to id_{\mathcal{F}_X}$ the counit of the adjunction $f^* \dashv f_!$, then we have $\pi_{f,g}(N) = \varepsilon^f_{g^*(N)} : f^*(N^{[f,g]}) = f^*f_!g^*(N) \to g^*(N)$. We note that if f^* has a right adjoint if and only if (f, id_X) is a right fibered representable pair.*

Proposition 1.4.3 *The inverse of $E_{f,g}(N)_M : \mathcal{F}_X(f^*(M), g^*(N)) \to \mathcal{F}_Y(M, N^{[f,g]})$ is given by the map defined by $\varphi \mapsto \pi_{f,g}(N)f^*(\varphi)$.*

Proof. For $\varphi \in \mathcal{F}_Y(M, N^{[f,g]})$, the following diagram commutes by naturality of $E_{f,g}(N)$.

$$\begin{tikzcd}
\mathcal{F}_X(f^*(N^{[f,g]}), g^*(N)) \arrow[r, "f^*(\varphi)^*"] \arrow[d, "E_{f,g}(N)_{N^{[f,g]}}"] & \mathcal{F}_X(f^*(M), g^*(N)) \arrow[d, "E_{f,g}(N)_M"] \\
\mathcal{F}_Y(N^{[f,g]}, N^{[f,g]}) \arrow[r, "\varphi^*"] & \mathcal{F}_Y(M, N^{[f,g]})
\end{tikzcd}$$

It follows that $E_{f,g}(N)_M$ maps $\pi_{f,g}(N)f^*(\varphi)$ to φ. □

For a morphism $\varphi : L \to N$ of $\mathcal{F}_Z$, define a natural transformation $F^{f,g}_\varphi : F^{f,g}_L \to F^{f,g}_N$ by

$$(F^{f,g}_\varphi)_M = g^*(\varphi)_* : F^{f,g}_L(M) = \mathcal{F}_X(f^*(M), g^*(L)) \to \mathcal{F}_X(f^*(M), g^*(N)) = F^{f,g}_N(M).$$

It is clear that $F^{f,g}_{\psi\varphi} = F^{f,g}_\psi F^{f,g}_\varphi$ for morphisms $\psi : N \to P$ and $\varphi : L \to N$ of $\mathcal{F}_Z$. If (f,g) is a right fibered representable pair with respect to L and M, we define $\varphi^{[f,g]} : L^{[f,g]} \to N^{[f,g]}$ by

$$\varphi^{[f,g]} = E_{f,g}(N)_{L^{[f,g]}}((F^{f,g}_\varphi)_{L^{[f,g]}}(\pi_{f,g}(L))) = E_{f,g}(N)_{L^{[f,g]}}(g^*(\varphi)\pi_{f,g}(L)) \in h_{N^{[f,g]}}(L^{[f,g]})$$

Proposition 1.4.4 *(1) The following diagrams commute for any $M \in \operatorname{Ob}\mathcal{F}_Y$.*

$$\begin{tikzcd}
f^*(L^{[f,g]}) \arrow[r, "f^*(\varphi^{[f,g]})"] \arrow[d, "\pi_{f,g}(L)"] & f^*(N^{[f,g]}) \arrow[d, "\pi_{f,g}(N)"] \\
g^*(L) \arrow[r, "g^*(\varphi)"] & g^*(N)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(L)) \arrow[r, "g^*(\varphi)_*"] \arrow[d, "E_{f,g}(L)_M"] & \mathcal{F}_X(f^*(M), g^*(N)) \arrow[d, "E_{f,g}(N)_M"] \\
\mathcal{F}_Y(M, L^{[f,g]}) \arrow[r, "\varphi^{[f,g]}_*"] & \mathcal{F}_Y(M, N^{[f,g]})
\end{tikzcd}$$

(2) For morphisms $\psi : N \to P$ and $\varphi : L \to N$ of $\mathcal{F}_Z$, we have $(\psi\varphi)^{[f,g]} = \psi^{[f,g]}\varphi^{[f,g]}$.

(3) If $g^* : \mathcal{F}_Z \to \mathcal{F}_X$ preserves monomorphisms (g^* has a left adjoint, for example) and $\varphi : L \to N$ is a monomorphism, so is $\varphi^{[f,g]} : L^{[f,g]} \to N^{[f,g]}$.

Proof. (1) We have $E_{f,g}(N)_{L^{[f,g]}}(g^*(\varphi)\pi_{f,g}(L)) = \varphi^{[f,g]}$ by the definition of $\varphi^{[f,g]}$. On the other hand, it follows from (1.4.3) that $E_{f,g}(N)_{L^{[f,g]}}(\pi_{f,g}(N)f^*(\varphi^{[f,g]})) = \varphi^{[f,g]}$. Since $E_{f,g}(N)_{L^{[f,g]}}$ is bijective, the left diagram commutes.

For $\psi \in \mathcal{F}_Y(M, L^{[f,g]})$, it follows from 1.4.3 and commutativity of the left diagram that we have

$$\begin{aligned}
g^*(\varphi)_*E_{f,g}(L)^{-1}_M(\psi) &= g^*(\varphi)\pi_{f,g}(L)f^*(\psi) = \pi_{f,g}(N)f^*(\varphi^{[f,g]})f^*(\psi) = \pi_{f,g}(N)f^*(\varphi^{[f,g]}\psi) \\
&= E_{f,g}(N)^{-1}_M(\varphi^{[f,g]}\psi) = E_{f,g}(N)^{-1}_M\varphi^{[f,g]}_*(\psi).
\end{aligned}$$

Hence the right diagram commutes.

(2) The following diagram commutes by (1).

$$\begin{tikzcd}
\mathcal{F}_X(f^*(L^{[f,g]}), g^*(L)) \arrow[r, "g^*(\varphi)_*"] \arrow[d, "E_{f,g}(L)_{L^{[f,g]}}"] & \mathcal{F}_X(f^*(L^{[f,g]}), g^*(N)) \arrow[r, "g^*(\psi)_*"] \arrow[d, "E_{f,g}(N)_{L^{[f,g]}}"] & \mathcal{F}_X(f^*(L^{[f,g]}), g^*(P)) \arrow[d, "E_{f,g}(P)_{L^{[f,g]}}"] \\
\mathcal{F}_Y(L^{[f,g]}, L^{[f,g]}) \arrow[r, "\varphi^{[f,g]}_*"] & \mathcal{F}_Y(L^{[f,g]}, N^{[f,g]}) \arrow[r, "\psi^{[f,g]}_*"] & \mathcal{F}_Y(L^{[f,g]}, P^{[f,g]})
\end{tikzcd}$$

Hence $\psi^{[f,g]}\varphi^{[f,g]} = \psi^{[f,g]}_*\varphi^{[f,g]}_*(id_{L^{[f,g]}}) = E_{f,g}(P)_{L^{[f,g]}}(g^*(\psi)g^*(\varphi)\pi_{f,g}(L)) = E_{f,g}(P)_{L^{[f,g]}}(g^*(\psi\varphi)\pi_{f,g}(L)) = (\psi\varphi)^{[f,g]}$.

(3) is a direct consequence of (1). □

Remark 1.4.5 *Suppose that $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ has a right adjoint $f_! : \mathcal{F}_X \to \mathcal{F}_Y$. For a morphism $\varphi : L \to N$ of $\mathcal{F}_Z$, we have $\varphi^{[f,g]} = f_!g^*(\varphi) : L^{[f,g]} = f_!g^*(L) \to f_!g^*(N) = N^{[f,g]}$. In fact, if we denote by $\eta^f : id_{\mathcal{F}_X} \to f_!f^*$ the unit of the adjunction $f^* \dashv f_!$, we have $\varphi^{[f,g]} = E_X(N)_{L^{[f,g]}}(g^*(\varphi)\pi_{f,g}(L)) = \mathrm{ad}^f_{L^{[f,g]},g^*(N)}\big(g^*(\varphi)\varepsilon^f_{f^*(L)}\big) = f_!g^*(\varphi)f_!\big(\varepsilon^f_{g^*(L)}\big)\eta^f_{f_!g^*(L)} = f_!g^*(\varphi)$.*

Lemma 1.4.6 *Let $\xi : f^*(L) \to g^*(M)$, $\zeta : f^*(N) \to g^*(K)$ be morphisms in $\mathcal{F}_X$ for $L, N \in \operatorname{Ob}\mathcal{F}_Y$, $M, K \in \operatorname{Ob}\mathcal{F}_Z$. Let $\varphi : L \to N$ be a morphism in $\mathcal{F}_Y$ and $\psi : M \to K$ a morphism in $\mathcal{F}_Z$. We put $\check{\xi} = E_{f,g}(L)_M(\xi)$ and $\check{\zeta} = E_{f,g}(K)_N(\zeta)$. The following left diagram commutes if and only if the right one commutes.*

$$\begin{array}{ccc} f^*(L) & \xrightarrow{\xi} & g^*(M) \\ \downarrow{\scriptstyle f^*(\varphi)} & & \downarrow{\scriptstyle g^*(\psi)} \\ f^*(N) & \xrightarrow{\zeta} & g^*(K) \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{\check{\xi}} & M^{[f,g]} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi^{[f,g]}} \\ N & \xrightarrow{\check{\zeta}} & K^{[f,g]} \end{array}$$

Proof. The following diagram is commutative by (1.4.4) and the naturality of $E_{f,g}(K)$.

$$\begin{array}{ccccc} \mathcal{F}_X(f^*(L), g^*(M)) & \xrightarrow{g^*(\psi)_*} & \mathcal{F}_X(f^*(L), g^*(K)) & \xleftarrow{f^*(\varphi)^*} & \mathcal{F}_X(f^*(N), g^*(K)) \\ \downarrow{\scriptstyle E_{f,g}(M)_L} & & \downarrow{\scriptstyle E_{f,g}(K)_L} & & \downarrow{\scriptstyle E_{f,g}(K)_N} \\ \mathcal{F}_Y(L, M^{[f,g]}) & \xrightarrow{\psi_*^{[f,g]}} & \mathcal{F}_Y(L, K^{[f,g]}) & \xleftarrow{\varphi^*} & \mathcal{F}_Y(N, K^{[f,g]}) \end{array}$$

Since $\check{\xi} = E_{f,g}(L)_M(\xi)$, $\check{\zeta} = E_{f,g}(K)_N(\zeta)$ and $E_{f,g}(K)_L$ is bijective, $g^*(\psi)\xi = g^*(\psi)_*(\xi) = f^*(\varphi)^*(\zeta) = \zeta f^*(\varphi)$ if and only if $\psi^{[f,g]}\check{\xi} = \psi_*^{[f,g]}(\check{\xi}) = \varphi^*(\check{\zeta}) = \check{\zeta}\varphi$. □

For morphisms $f : X \to Y$, $g : X \to Z$, $k : V \to X$ in $\mathcal{E}$ and $N \in \mathrm{Ob}\,\mathcal{F}_Z$, suppose that (f,g) and (fk, gk) are right fibered representable pairs with respect to N. We define a morphism $N^k : N^{[f,g]} \to N^{[fk,gk]}$ of $\mathcal{F}_Y$ by

$$N^k = E_{fk,gk}(N)_{N^{[f,g]}}(k^\sharp_{N^{[f,g]},N}(\pi_{f,g}(N))) \in \mathcal{F}_Y(N^{[f,g]}, N^{[fk,gk]}).$$

Proposition 1.4.7 *(1) The following diagram commutes for any $M \in \mathrm{Ob}\,\mathcal{F}_Y$.*

$$\begin{array}{ccc} \mathcal{F}_X(f^*(M), g^*(N)) & \xrightarrow{k^\sharp_{M,N}} & \mathcal{F}_V((fk)^*(M), (gk)^*(N)) \\ \downarrow{\scriptstyle E_{f,g}(N)_M} & & \downarrow{\scriptstyle E_{fk,gk}(N)_M} \\ \mathcal{F}_Y(M, N^{[f,g]}) & \xrightarrow{N^k_*} & \mathcal{F}_Y(M, N^{[fk,gk]}) \end{array} \qquad \begin{array}{ccc} (fk)^*(N^{[f,g]}) & \xrightarrow{k^\sharp_{N^{[f,g]},N}(\pi_{f,g}(N))} & (gk)^*(N) \\ & {\scriptstyle (fk)^*(N^k)}\searrow \quad \nearrow{\scriptstyle \pi_{fk,gk}(N)} & \\ & (fk)^*(N^{[fk,gk]}) & \end{array}$$

(2) For morphisms $f : X \to Y$, $g : X \to Z$, $k : V \to X$, $h : U \to V$ and $M \in \mathrm{Ob}\,\mathcal{F}_Y$, suppose that (f,g), (fk, gk) and (fkh, gh) are right fibered representable pairs with respect to N. Then, $k : V \to X$ and $l : U \to V$ in $\mathcal{E}$, $N^{kh} = N^h N^k$.

(3) The image of the identity morphism of $k^*(N)$ by $E_{k,k}(N)_N$ is $N^k : N = N^{[id_X, id_X]} \to N^{[k,k]}$ if $X = Z$.

(4) A composition $k^*(N) = k^*(N^{[id_X,id_X]}) \xrightarrow{k^*(N^k)} k^*(N^{[k,k]}) \xrightarrow{\pi_{k,k}(N)} k^*(N)$ is the identity morphism of $k^*(N)$ if $X = Z$.

Proof. (1) For $\varphi \in \mathcal{F}_Y(M, N^{[f,g]})$, it follows from the naturality of $k^\sharp_{M,N}$ and (1.4.3) that we have

$$\begin{aligned} k^\sharp_{M,N} E_{f,g}(N)_M^{-1}(\varphi) &= k^\sharp_{M,N}(\pi_{f,g}(N) f^*(\varphi)) = k^\sharp_{M,N} f^*(\varphi)^*(\pi_Y(N)) = f^*(\varphi)^* k^\sharp_{N^{[f,g]},N}(\pi_{f,g}(N)) \\ &= f^*(\varphi)^* E_{fk,gk}(N)^{-1}_{N^{[f,g]}}(N^k) = \pi_{fk,gk}(N) f^*(N^k) f^*(\varphi) = \pi_{fk,gk}(N) f^*(N^k \varphi) \\ &= \pi_{fk,gk}(N) f^*((N^k)_*(\varphi)) = E_{fk,gk}(N)_M^{-1}(N^k)_*(\varphi). \end{aligned}$$

The commutativity of the right diagram follows from (1.4.3) and the commutativity of the left diagram for the case $M = N^{[f,g]}$.

(2) The following diagram commutes by (1).

$$\begin{array}{ccccc} \mathcal{F}_X(f^*(N^{[f,g]}), g^*(N)) & \xrightarrow{k^\sharp_{N^{[f,g]},N}} & \mathcal{F}_V((fk)^*(N^{[f,g]}), (gk)^*(N)) & \xrightarrow{h^\sharp_{N^{[f,g]},N}} & \mathcal{F}_U((fkh)^*(N^{[f,g]}), (gkh)^*(N)) \\ \downarrow{\scriptstyle E_{f,g}(N)_{N^{[f,g]}}} & & \downarrow{\scriptstyle E_{fk,gk}(N)_{N^{[f,g]}}} & & \downarrow{\scriptstyle E_{fkh,gkh}(N)_{N^{[f,g]}}} \\ \mathcal{F}_Y(N^{[f,g]}, N^{[f,g]}) & \xrightarrow{N^k_*} & \mathcal{F}_Y(N^{[f,g]}, N^{[fk,gk]}) & \xrightarrow{N^h_*} & \mathcal{F}_Y(N^{[f,g]}, N^{[fkh,gkh]}) \end{array}$$

It follows the above diagram and (1.1.16) that

$$\begin{aligned} N^h N^k &= N^h_* N^k_*(id_{N^{[f,g]}}) = E_{fkh,gkh}(N)_{N^{[f,g]}}(h^\sharp_{N^{[f,g]},N} k^\sharp_{N^{[f,g]},N}(\pi_{f,g}(N))) \\ &= E_{fkh,gkh}(N)_{N^{[f,g]}}((kh)^\sharp_{N^{[f,g]},N}(\pi_{f,g}(N))) = N^{kh}. \end{aligned}$$

(3) Apply (1) for $M = N$, $Z = Y = X$ and $f = g = id_X$.

(4) It follows from (1.4.3) that $E_{k,k}(N)_N : \mathcal{F}_X(k^*(N), k^*(N)) \to \mathcal{F}_1(N, N^{[k,k]})$ maps $\pi_{k,k}(N) k^*(N^k)$ to $N^k : N \to N^{[k,k]}$. Thus the assertion follows from (3). □

Remark 1.4.8 *Suppose that the inverse image functors $f^*: \mathcal{F}_Y \to \mathcal{F}_X$ and $(fk)^*: \mathcal{F}_Y \to \mathcal{F}_V$ have right adjoints $f_!: \mathcal{F}_X \to \mathcal{F}_Y$ and $(fk)_!: \mathcal{F}_V \to \mathcal{F}_Y$, respectively.*

(1) Since $k^\sharp_{N^{[f,g]},N}(\pi_{f,g}(N)) = c_{g,k}(N)k^*\big(\varepsilon^f_{g^*(N)}\big)c_{f,k}(N^{[f,g]})^{-1}$ by (1.4.2) and

$$E_{fk,gk}(N)_{N^{[f,g]}} = \mathrm{ad}^{fk}_{N^{[f,g]},g^*(N)} : \mathcal{F}_V((fk)^*(N^{[f,g]}),(gk)^*(N)) \to \mathcal{F}_Y(N^{[f,g]},N^{[fk,gk]})$$

maps $\varphi \in \mathcal{F}_X((fk)^*(N^{[f,g]}),(gk)^*(N))$ to $(fk)_!(\varphi)\eta^{fk}_{N^{[f,g]}}$, $N^k: N^{[f,g]} \to N^{[fk,gk]}$ coincides with the following composition.

$$N^{[f,g]} \xrightarrow{\eta^{fk}_{N^{[f,g]}}} (fk)_!(fk)^*(N^{[f,g]}) \xrightarrow{(fk)_!(c_{f,k}(N^{[f,g]}))^{-1}} (fk)_!k^*f^*(N^{[f,g]}) = (fk)_!k^*f^*f_!g^*(N)$$
$$\xrightarrow{(fk)_!k^*\left(\varepsilon^f_{g^*(N)}\right)} (fk)_!k^*g^*(N) \xrightarrow{(fk)_!(c_{g,k}(N))} (fk)_!(gk)^*(N) = N^{[fk,gk]}$$

We remark that N^k is the adjoint of the following composition with respect to the adjunction $(fk)^* \dashv (fk)_!$.

$$(fk)^*(N^{[f,g]}) \xrightarrow{c_{f,k}(N^{[f,g]})^{-1}} k^*f^*(N^{[f,g]}) = k^*f^*f_!g^*(N) \xrightarrow{k^*(\varepsilon^f_{g^*(N)})} k^*g^*(N) \xrightarrow{c_{g,k}(N)} (gk)^*(N)$$

(2) The following diagram commutes by (1.4.7) if $X = Y = Z$ and $f = g = id_X$.

$$\begin{tikzcd}
\mathcal{F}_X(N, N^{[id_X,id_X]}) \arrow[r, "N^k_*"] \arrow[d, "(\mathrm{ad}^{id_X}_{N,id_X^*(N)})^{-1}"] & \mathcal{F}_1(N, N^{[k,k]}) \arrow[d, "(\mathrm{ad}^k_{N,k^*(N)})^{-1}"] \\
\mathcal{F}_X(id_X^*(N), id_X^*(N)) \arrow[r, "(k^\sharp)_{N,N}"] & \mathcal{F}_V(k^*(N), k^*(N))
\end{tikzcd}$$

Since id_X^* is the identity functor of $\mathcal{F}_X$, so is $id_{X!}$. Hence $N^k: N = N^{[id_X,id_X]} \to N^{[k,k]} = k_!k^*(N)$ is identified with the unit $\eta^k_N : N \to k_!k^*(N)$ of the adjunction $k^* \dashv k_!$ by the above diagram.

Proposition 1.4.9 *For morphisms $f: X \to Y$, $g: X \to Z$, $k: V \to X$ in $\mathcal{E}$ and a morphism $\varphi: L \to N$ of $\mathcal{F}_Z$, the following diagram commutes.*

$$\begin{tikzcd}
L^{[f,g]} \arrow[r, "\varphi^{[f,g]}"] \arrow[d, "L^k"] & N^{[f,g]} \arrow[d, "N^k"] \\
L^{[fk,gk]} \arrow[r, "\varphi^{[fk,gk]}"] & N^{[fk,gk]}
\end{tikzcd}$$

Proof. The following diagram commutes by the naturality of $k^\sharp$.

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(L)) \arrow[r, "k^\sharp_{M,L}"] \arrow[d, "g^*(\varphi)_*"] & \mathcal{F}_V((fk)^*(M), (gk)^*(L)) \arrow[d, "(gk)^*(\varphi)_*"] \\
\mathcal{F}_X(f^*(M), g^*(N)) \arrow[r, "k^\sharp_{M,N}"] & \mathcal{F}_V((fk)^*(M), (gk)^*(N))
\end{tikzcd}$$

Then, it follows from the commutativity of four diagrams

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(L)) \arrow[r, "E_{f,g}(L)_M"] \arrow[d, "g^*(\varphi)_*"] & \mathcal{F}_Y(M, L^{[f,g]}) \arrow[d, "\varphi^{[f,g]}_*"] \\
\mathcal{F}_X(f^*(M), g^*(N)) \arrow[r, "E_{f,g}(N)_M"] & \mathcal{F}_Y(M, N^{[f,g]})
\end{tikzcd}
\quad
\begin{tikzcd}
\mathcal{F}_Y((fk)^*(M), (gk)^*(L)) \arrow[r, "E_{fk,gk}(L)_M"] \arrow[d, "(gk)^*(\varphi)_*"] & \mathcal{F}_Y(M, L^{[fk,gk]}) \arrow[d, "\varphi^{[fk,gk]}_*"] \\
\mathcal{F}_Y((fk)^*(M), (gk)^*(N)) \arrow[r, "E_{fk,gk}(N)_M"] & \mathcal{F}_Y(M, N^{[fk,gk]})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(L)) \arrow[r, "E_{f,g}(L)_M"] \arrow[d, "k^\sharp_{M,L}"] & \mathcal{F}_Y(M, L^{[f,g]}) \arrow[d, "L^k_*"] \\
\mathcal{F}_Y((fk)^*(M), (gk)^*(L)) \arrow[r, "E_{fk,gk}(L)_M"] & \mathcal{F}_Y(M, L^{[fk,gk]})
\end{tikzcd}
\quad
\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(N)) \arrow[r, "E_{f,g}(N)_M"] \arrow[d, "k^\sharp_{M,N}"] & \mathcal{F}_Y(M, N^{[f,g]}) \arrow[d, "N^k_*"] \\
\mathcal{F}_Y((fk)^*(M), (gk)^*(N)) \arrow[r, "E_{fk,gk}(N)_M"] & \mathcal{F}_Y(M, N^{[fk,gk]})
\end{tikzcd}$$

and the fact that $E_{f,g}(L)_M : \mathcal{F}_X(f^*(M), g^*(L)) \to \mathcal{F}_Y(M, L^{[f,g]})$ is bijective that the following diagram commutes for any $M \in \mathrm{Ob}\,\mathcal{F}_Y$.

$$\begin{array}{ccc}\mathcal{F}_Y(M,L^{[f,g]}) & \xrightarrow{\varphi^{[f,g]}_*} & \mathcal{F}_Y(M,N^{[f,g]}) \\ \downarrow{\scriptstyle L^k_*} & & \downarrow{\scriptstyle N^k_*} \\ \mathcal{F}_Y(M,L^{[fk,gk]}) & \xrightarrow{\varphi^{[fk,gk]}_*} & \mathcal{F}_Y(M,N^{[fk,gk]})\end{array}$$

Thus the assertion follows. □

Remark 1.4.10 *We denote by $\varphi^k : L^{[f,g]} \to N^{[fk,gk]}$ the composition $N^k\varphi^{[f,g]} = \varphi^{[fk,gk]}L^k$. For morphisms $i : W \to T$, $j : W \to Y$, $h : U \to W$ in $\mathcal{E}$, it follows from (1.4.9) that the following diagram commutes.*

$$\begin{array}{ccc}(N^{[f,g]})^{[i,j]} & \xrightarrow{(N^k)^{[i,j]}} & (N^{[fk,gk]})^{[i,j]} \\ \downarrow{\scriptstyle (N^{[f,g]})^h} & & \downarrow{\scriptstyle (N^{[fk,gk]})^h} \\ (N^{[f,g]})^{[ih,jh]} & \xrightarrow{(N^k)^{[ih,jh]}} & (N^{[fk,gk]})^{[ih,jh]}\end{array}$$

Namely, we have $(N^{[fk,gk]})^h(N^k)^{[i,j]} = (N^k)^{[ih,jh]}(N^{[f,g]})^h$ which we denote by $(N^k)^h$ for short.

For morphisms $f : X \to Y$, $g : X \to Z$, $h : X \to W$ in $\mathcal{E}$ and $N \in \operatorname{Ob}\mathcal{F}_W$, we define a morphism $\epsilon^{f,g,h}_N : (N^{[g,h]})^{[f,g]} \to N^{[f,h]}$ of $\mathcal{F}_Y$ to be the image of $\pi_{g,h}(N)\pi_{f,g}(N^{[g,h]}) \in \mathcal{F}_X(f^*((N^{[g,h]})^{[f,g]}), h^*(N))$ by

$$E_{f,h}(N)_{(N^{[g,h]})^{[f,g]}} : \mathcal{F}_X(f^*((N^{[g,h]})^{[f,g]}), h^*(N)) \to \mathcal{F}_Y((N^{[g,h]})^{[f,g]}, N^{[f,h]}).$$

Proposition 1.4.11 *The following diagram commutes for any $M \in \operatorname{Ob}\mathcal{F}_Z$.*

$$\begin{array}{ccc}\mathcal{F}_X(f^*(M), g^*(N^{[g,h]})) & \xrightarrow{\pi_{g,h}(N)_*} & \mathcal{F}_X(f^*(M), h^*(N)) \\ \downarrow{\scriptstyle E_{f,g}(N^{[g,h]})_M} & & \downarrow{\scriptstyle E_{f,h}(N)_M} \\ \mathcal{F}_Y(M,(N^{[g,h]})^{[f,g]}) & \xrightarrow{\epsilon^{f,g,h}_{N*}} & \mathcal{F}_Y(M, N^{[f,h]})\end{array}$$

Proof. For $\varphi \in \mathcal{F}_Y(M,(N^{[g,h]})^{[f,g]})$, by the definition of $\epsilon^{f,g,h}_N$ and the naturality of $E_{f,h}(N)$, we have

$$\begin{aligned}\pi_{g,h}(N)_*E_{f,g}(N^{[g,h]})^{-1}_M(\varphi) &= \pi_{g,h}(N)\pi_{f,g}(N^{[g,h]})f^*(\varphi) = f^*(\varphi)^*E_{f,h}(N)^{-1}_{(N^{[g,h]})^{[f,g]}}(\epsilon^{f,g,h}_N) \\ &= E_{f,h}(N)^{-1}_M\varphi^*(\epsilon^{f,g,h}_N) = E_{f,h}(N)^{-1}_M\epsilon^{f,g,h}_{N*}(\varphi).\end{aligned}$$

□

We note that $\epsilon^{f,g,h}_N : (N^{[g,h]})^{[f,g]} \to N^{[f,h]}$ is the unique morphism that makes the diagram of (1.4.11) commute for any $M \in \operatorname{Ob}\mathcal{F}_W$.

Remark 1.4.12 *If $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ and $g^* : \mathcal{F}_Z \to \mathcal{F}_X$ have right adjoints $f_! : \mathcal{F}_X \to \mathcal{F}_Y$ and $g_! : \mathcal{F}_X \to \mathcal{F}_Z$, the following diagram is commutative for any $M \in \operatorname{Ob}\mathcal{F}_Y$ by the naturality of ad^f.*

$$\begin{array}{ccc}\mathcal{F}_X(f^*(M), g^*g_!h^*(N)) & \xrightarrow{\varepsilon^g_{h^*(N)*}} & \mathcal{F}_X(f^*(M), h^*(N)) \\ \downarrow{\scriptstyle \operatorname{ad}^f_{M,g^*g_!h^*(N)}} & & \downarrow{\scriptstyle \operatorname{ad}^f_{M,h^*(N)}} \\ \mathcal{F}_Y(M, f_!g^*g_!h^*(N)) & \xrightarrow{f_!(\varepsilon^g_{h^*(N)})_*} & \mathcal{F}_1(M, f_!h^*(N))\end{array}$$

It follows that $\epsilon^{f,g,h}_N = f_!\big(\varepsilon^g_{h^*(N)}\big)$.

Lemma 1.4.13 *For morphisms $f : X \to Y$, $g : X \to Z$, $h : X \to W$, $k : V \to X$ in $\mathcal{E}$ and a morphism $\varphi : M \to N$ of $\mathcal{F}_W$, the following diagrams are commutative.*

$$\begin{array}{ccc}(M^{[g,h]})^{[f,g]} & \xrightarrow{\epsilon^{f,g,h}_M} & M^{[f,h]} \\ \downarrow{\scriptstyle (\varphi^{[g,h]})^{[f,g]}} & & \downarrow{\scriptstyle \varphi^{[f,h]}} \\ (N^{[g,h]})^{[f,g]} & \xrightarrow{\epsilon^{f,g,h}_N} & N^{[f,h]}\end{array} \qquad \begin{array}{ccc}(N^{[g,h]})^{[f,g]} & \xrightarrow{\epsilon^{f,g,h}_N} & N^{[f,h]} \\ \downarrow{\scriptstyle (N^k)^k} & & \downarrow{\scriptstyle N^k} \\ (N^{[gk,hk]})^{[fk,gk]} & \xrightarrow{\epsilon^{fk,gk,hk}_N} & N^{[fk,hk]}\end{array}$$

Proof. The following diagram is commutative by (1) of (1.4.4) for any $L \in \operatorname{Ob} \mathcal{F}_Y$.

$$\begin{tikzcd}
\mathcal{F}_X(f^*(L), g^*(M^{[g,h]})) \arrow[r, "\pi_{g,h}(M)_*"] \arrow[d, "g^*(\varphi^{[g,h]})_*"] & \mathcal{F}_X(f^*(L), h^*(M)) \arrow[d, "h^*(\varphi)_*"] \\
\mathcal{F}_X(f^*(L), g^*(N^{[g,h]})) \arrow[r, "\pi_{g,h}(N)_*"] & \mathcal{F}_X(f^*(L), h^*(N))
\end{tikzcd}$$

Hence the following diagram commutes by (1.4.11) and (1) of (1.4.4).

$$\begin{tikzcd}
\mathcal{F}_Y(L, (M^{[g,h]})^{[f,g]}) \arrow[r, "\epsilon^{f,g,h}_{M*}"] \arrow[d, "(\varphi^{[g,h]})^{[f,g]}_*"] & \mathcal{F}_Y(L, M^{[f,h]}) \arrow[d, "\varphi^{[f,h]}_*"] \\
\mathcal{F}_Y(L, (N^{[g,h]})^{[f,g]}) \arrow[r, "\epsilon^{f,g,h}_{N*}"] & \mathcal{F}_Y(L, N^{[f,h]})
\end{tikzcd}$$

Thus the left diagram is commutative.

For $M \in \operatorname{Ob} \mathcal{F}_Y$ and $\xi \in \mathcal{F}_X(f^*(M), g^*(N^{[g,h]}))$, it follows from (1.4.7) and (1.1.15) that we have

$$\pi_{gk,hk}(N)(gk)^*(N^k)k^\sharp_{M,N^{[g,h]}}(\xi) = k^\sharp_{N^{[g,h]},N}(\pi_{g,h}(N))k^\sharp_{M,N^{[g,h]}}(\xi) = k^\sharp_{M,N}(\pi_{g,h}(N)\xi).$$

This shows that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(N^{[g,h]})) \arrow[r, "\pi_{g,h}(N)_*"] \arrow[d, "(gk)^*(N^k)_*k^\sharp_{M,N^{[g,h]}}"] & \mathcal{F}_X(f^*(M), g^*(N)) \arrow[d, "k^\sharp_{M,N}"] \\
\mathcal{F}_V((fk)^*(M), (gk)^*(N^{[gk,hk]})) \arrow[r, "\pi_{gk,hk}(N)_*"] & \mathcal{F}_Y((fk)^*(M), (hk)^*(N))
\end{tikzcd}$$

The following diagram commutes by (1) of (1.4.4) and (1.4.7).

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(N^{[g,h]})) \arrow[r, "k^\sharp_{M,N^{[g,h]}}"] \arrow[d, "E_{f,g}(N^{[g,h]})_M"] & \mathcal{F}_Y((fk)^*(M), (gk)^*(N^{[g,h]})) \arrow[r, "(gk)^*(N^k)_*"] \arrow[d, "E_{fk,gk}(N^{[g,h]})_M"] & \mathcal{F}_Y((fk)^*(M), (gk)^*(N^{[gk,hk]})) \arrow[d, "E_{fk,gk}(N)_M"] \\
\mathcal{F}_Y(M, (N^{[g,h]})^{[f,g]}) \arrow[r, "(N^{[g,h]})^k_*"] & \mathcal{F}_Y(M, (N^{[g,h]})^{[fk,gk]}) \arrow[r, "(N^k)^{[fk,gk]}_*"] & cf_V(M, (N^{[gk,hk]})^{[fk,gk]})
\end{tikzcd}$$

Since $(N^k)^k = (N^k)^{[fk,gk]}(N^{[g,h]})^k$, it follows from (1.4.11) and (1) of (1.4.7) that the following diagram commutes for any $M \in \operatorname{Ob} \mathcal{F}_Y$.

$$\begin{tikzcd}
\mathcal{F}_Y(M, (N^{[g,h]})^{[f,g]}) \arrow[r, "\epsilon^{f,g,h}_{N*}"] \arrow[d, "(N^k)^k_*"] & \mathcal{F}_Y(M, N^{[f,h]}) \arrow[d, "N^k_*"] \\
\mathcal{F}_Y(M, (N^{[gk,hk]})^{[fk,gk]}) \arrow[r, "\epsilon^{fk,gk,hk}_{N*}"] & \mathcal{F}_Y(M, N^{[fk,hk]})
\end{tikzcd}$$

Thus the right diagram is also commutative. □

Proposition 1.4.14 *For morphisms $f : X \to Y$, $g : X \to Z$, $h : X \to W$, $i : X \to V$ in $\mathcal{E}$ and an object N of $\mathcal{F}_V$, the following diagrams are commutative.*

$$\begin{tikzcd}
g^*((N^{[h,i]})^{[g,h]}) \arrow[r, "g^*(\epsilon^{g,h,i}_N)"] \arrow[d, "\pi_{g,h}(N^{[h,i]})"] & g^*(N^{[g,i]}) \arrow[d, "\pi_{g,i}(N)"] \\
h^*(N^{[h,i]}) \arrow[r, "\pi_{h,i}(N)"] & i^*(N)
\end{tikzcd}
\qquad
\begin{tikzcd}
((N^{[h,i]})^{[g,h]})^{[f,g]} \arrow[r, "(\epsilon^{g,h,i}_N)^{[f,g]}"] \arrow[d, "\epsilon^{f,g,h}_{N^{[h,i]}}"] & (N^{[g,i]})^{[f,g]} \arrow[d, "\epsilon^{f,g,i}_N"] \\
(N^{[h,i]})^{[f,h]} \arrow[r, "\epsilon^{f,h,i}_N"] & N^{[f,i]}
\end{tikzcd}$$

Proof. It follows from the definition of $\epsilon^{g,h,i}_N$ and (1.4.3) that

$$\pi_{h,i}(N)\pi_{g,h}(N^{[h,i]}) = E_{g,i}(N)^{-1}_{(N^{[h,i]})^{[g,h]}}(\epsilon^{g,h,i}_N) = \pi_{g,i}(N)g^*(\epsilon^{g,h,i}_N).$$

Hence the following diagram commutes for $M \in \operatorname{Ob} \mathcal{F}_Y$.

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*((N^{[h,i]})^{[g,h]})) \arrow[r, "g^*(\epsilon_N^{g,h,i})_*"] \arrow[d, "\pi_{g,h}(N^{[h,i]})_*"] & \mathcal{F}_X(f^*(M), g^*(N^{[g,i]})) \arrow[d, "\pi_{g,i}(N)_*"] \\
\mathcal{F}_X(f^*(M), h^*(N^{[h,i]})) \arrow[r, "\pi_{h,i}(N)_*"] & \mathcal{F}_X(f^*(M), i^*(N))
\end{tikzcd}$$

Therefore the following diagram commutes by (1.4.11) and (1) of (1.4.4).

$$\begin{tikzcd}
\mathcal{F}_Y(M, ((N^{[h,i]})^{[g,h]})^{[f,g]}) \arrow[r, "(\epsilon_N^{g,h,i})_*^{[f,g]}"] \arrow[d, "\epsilon^{f,g,h}_{N^{[h,i]}*}"] & \mathcal{F}_Y(M, (N^{[g,i]})^{[f,g]}) \arrow[d, "\epsilon^{f,g,i}_{N*}"] \\
\mathcal{F}_Y(M, (N^{[h,i]})^{[f,h]}) \arrow[r, "\epsilon^{f,h,i}_{N*}"] & \mathcal{F}_Y(M, N^{[f,i]})
\end{tikzcd}$$

□

Proposition 1.4.15 *For morphisms $f : X \to Y$, $g : X \to Z$ in $\mathcal{E}$ and an object N of $\mathcal{F}_Z$, the following compositions coincide with the identity morphism of $N^{[f,g]}$.*

$$N^{[f,g]} = (N^{[f,g]})^{[id_Y, id_Y]} \xrightarrow{(N^{[f,g]})^f} (N^{[f,g]})^{[f,f]} \xrightarrow{\epsilon_N^{f,f,g}} N^{[f,g]}$$

$$N^{[f,g]} = (N^{[id_Z, id_Z]})^{[f,g]} \xrightarrow{(N^g)^{[f,g]}} (N^{[g,g]})^{[f,g]} \xrightarrow{\epsilon_N^{f,g,g}} N^{[f,g]}$$

Proof. The following diagram commutes for any $M \in \mathrm{Ob}\,\mathcal{F}_Y$ by (1) of (1.4.7) and (1.4.11).

$$\begin{tikzcd}
\mathcal{F}_Y(id_Y^*(M), id_Y^*(N^{[f,g]})) \arrow[r, "f^\sharp_{M,N^{[f,g]}}"] \arrow[d, "E_{id_Y,id_Y}(N^{[f,g]})_M"] & \mathcal{F}_X(f^*(M), f^*(N^{[f,g]})) \arrow[r, "\pi_{f,g}(N)_*"] \arrow[d, "E_{f,f}(N^{[f,g]})_N"] & \mathcal{F}_X(f^*(M), g^*(N)) \arrow[d, "E_{f,g}(N)_M"] \\
\mathcal{F}_Y(M, (N^{[f,g]})^{[id_Y,id_Y]}) \arrow[r, "(N^{[f,g]})^f_*"] & \mathcal{F}_Y(M, (N^{[f,g]})^{[f,f]}) \arrow[r, "\epsilon^{f,f,g}_{N*}"] & \mathcal{F}_Y(M, N^{[f,g]})
\end{tikzcd}$$

It follows from (1.4.3) that $\epsilon^{f,f,g}_{N*}(N^{[f,g]})^f_* : \mathcal{F}_Y(M, N^{[f,g]}) = \mathcal{F}_Y(M, (N^{[f,g]})^{[id_Y,id_Y]}) \to \mathcal{F}_Y(M, N^{[f,g]})$ is the identity map of $\mathcal{F}_Y(M, N^{[f,g]})$.

The following diagram commutes for any $M \in \mathrm{Ob}\,\mathcal{F}_Y$ by (1) of (1.4.4) and (1.4.11).

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(N^{[id_Y,id_Y]})) \arrow[r, "g^*(N^g)_*"] \arrow[d, "E_{f,g}(N^{[id_Y,id_Y]})_M"] & \mathcal{F}_X(f^*(M), g^*(N^{[g,g]})) \arrow[r, "\pi_{g,g}(N)_*"] \arrow[d, "E_{f,g}(N^{[g,g]})_M"] & \mathcal{F}_X(f^*(M), g^*(N)) \arrow[d, "E_{f,g}(N)_M"] \\
\mathcal{F}_Y(M, (N^{[id_Y,id_Y]})^{[f,g]}) \arrow[r, "(N^g)^{[f,g]}_*"] & \mathcal{F}_Y(M, (N^{[g,g]})^{[f,g]}) \arrow[r, "\epsilon^{f,g,g}_{N*}"] & \mathcal{F}_Y(M, N^{[f,g]})
\end{tikzcd}$$

Since the composition of the upper horizontal maps of the above diagram coincides with the identity map of $\mathcal{F}_X(f^*(M), g^*(N))$ by (4) of (1.4.7), the composition of the lower horizontal maps of the above diagram is the identity map of $\mathcal{F}_Y(M, N^{[f,g]})$. □

Let $f : X \to Y$, $g : X \to Z$, $h : X \to W$ be morphisms in $\mathcal{E}$ and L, M, N objects of $\mathcal{F}_Y$, $\mathcal{F}_Z$, $\mathcal{F}_W$, respectively. We define a map

$$\chi^{f,g,h}_{L,M,N} : \mathcal{F}_Y(L, M^{[f,g]}) \times \mathcal{F}_Z(M, N^{[g,h]}) \to \mathcal{F}_Y(L, N^{[f,h]})$$

as follows. For $\varphi \in \mathcal{F}_Y(L, M^{[f,g]})$ and $\psi \in \mathcal{F}_Z(M, N^{[g,h]})$, let $\chi^{f,g,h}_{L,M,N}(\varphi, \psi)$ be the following composition.

$$L \xrightarrow{\varphi} M^{[f,g]} \xrightarrow{\psi^{[f,g]}} (N^{[g,h]})^{[f,g]} \xrightarrow{\epsilon_N^{f,g,h}} N^{[f,h]}$$

Proposition 1.4.16 *The following diagram is commutative.*

$$\begin{tikzcd}
\mathcal{F}_X(f^*(L), g^*(M)) \times \mathcal{F}_X(g^*(M), h^*(N)) \arrow[r, "composition"] \arrow[d, "E_{f,g}(M)_L \times E_{g,h}(N)_M"] & \mathcal{F}_X(f^*(L), h^*(N)) \arrow[d, "E_{f,h}(N)_L"] \\
\mathcal{F}_Y(L, M^{[f,g]}) \times \mathcal{F}_Z(M, N^{[g,h]}) \arrow[r, "\chi^{f,g,h}_{L,M,N}"] & \mathcal{F}_Y(L, N^{[f,h]})
\end{tikzcd}$$

Proof. For $\zeta \in \mathcal{F}_X(f^*(L), g^*(M))$ and $\xi \in \mathcal{F}_X(g^*(M), h^*(N))$, we put $\varphi = E_{f,g}(M)_L(\zeta)$ and $\psi = E_{g,h}(N)_M(\xi)$. Then, we have $\psi^{[f,g]}\varphi = E_{f,g}(N^{[g,h]})_L(g^*(\psi)\zeta)$ by (1.4.4). It follows from (1.4.11) and (1.4.3) that

$$\epsilon_N^{f,g,h}\psi^{[f,g]}\varphi = \epsilon_{N*}^{f,g,h}E_{f,g}(N^{[g,h]})_L(g^*(\psi)\zeta) = E_{f,h}(N)_L(\pi_{g,h}(N)g^*(\psi)\zeta) = E_{f,h}(N)_L(\xi\zeta).$$

Thus the result follows. □

For a functor $D : \mathcal{P} \to \mathcal{E}$ and an object N of $\mathcal{F}_{D(5)}$, we put $D(\tau_{ij}) = f_{ij}$ and define a morphism

$$\theta^D(N) : (N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} \to N^{[f_{13}f_{01},f_{25}f_{02}]}$$

of $\mathcal{F}_{D(3)}$ to be the following composition.

$$(N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} \xrightarrow{(N^{f_{02}})^{f_{01}}} (N^{[f_{24}f_{02},f_{25}f_{02}]})^{[f_{13}f_{01},f_{14}f_{01}]} \xrightarrow{\epsilon_N^{f_{13}f_{01},f_{14}f_{01},f_{25}f_{02}}} N^{[f_{13}f_{01},f_{25}f_{02}]}$$

Proposition 1.4.17 *We assume that the inverse image functors $f_{13}^* : \mathcal{F}_{D(3)} \to \mathcal{F}_{D(1)}$, $f_{24}^* : \mathcal{F}_{D(4)} \to \mathcal{F}_{D(2)}$, $(f_{13}f_{01})^* : \mathcal{F}_{D(3)} \to \mathcal{F}_{D(0)}$ and $(f_{14}f_{01})^* : \mathcal{F}_{D(4)} \to \mathcal{F}_{D(0)}$ have right adjoints $(f_{13})_! : \mathcal{F}_{D(1)} \to \mathcal{F}_{D(3)}$, $(f_{24})_! : \mathcal{F}_{D(2)} \to \mathcal{F}_{D(4)}$, $(f_{13}f_{01})_! : \mathcal{F}_{D(0)} \to \mathcal{F}_{D(3)}$ and $(f_{14}f_{01})_! : \mathcal{F}_{D(0)} \to \mathcal{F}_{D(4)}$, respectively. Let $\varepsilon^{f_{13}} : f_{13}^*(f_{13})_! \to id_{\mathcal{F}_{D(1)}}$ and $\varepsilon^{f_{24}} : f_{24}^*(f_{24})_! \to id_{\mathcal{F}_{D(2)}}$ be the counits of the adjunctions $f_{13}^* \dashv (f_{13})_!$ and $f_{24}^* \dashv (f_{24})_!$, respectively. For an object N of $\mathcal{F}_{D(5)}$,*

$$\theta^D(N) : (N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} = (f_{13})_!(f_{14}^*((f_{24})_!(f_{25}^*(N)))) \to (f_{13}f_{01})_!((f_{25}f_{02})^*(N)) = N^{[f_{13}f_{01},f_{25}f_{02}]}$$

coincides with the adjoint of the following composition with respect to the adjunction $(f_{13}f_{01})^* \dashv (f_{13}f_{01})_!$.

$$\begin{aligned}
&(f_{13}f_{01})^*((f_{13})_!(f_{14}^*((f_{24})_!(f_{25}^*(N))))) \xrightarrow{c_{f_{13},f_{01}}((f_{13})_!(f_{14}^*((f_{24})_!(f_{25}^*(N)))))^{-1}} f_{01}^*(f_{13}^*((f_{13})_!(f_{14}^*((f_{24})_!(f_{25}^*(N))))))\\
&\xrightarrow{f_{01}^*\left(\varepsilon^{f_{13}}_{f_{14}^*((f_{24})_!(f_{25}^*(N)))}\right)} f_{01}^*(f_{14}^*((f_{24})_!(f_{25}^*(N)))) \xrightarrow{c_{f_{14},f_{01}}((f_{24})_!(f_{25}^*(N)))} (f_{14}f_{01})^*((f_{24})_!(f_{25}^*(N)))\\
&= (f_{24}f_{02})^*((f_{24})_!(f_{25}^*(N))) \xrightarrow{c_{f_{24},f_{02}}((f_{24})_!(f_{25}^*(N)))^{-1}} f_{02}^*(f_{24}^*((f_{24})_!(f_{25}^*(N)))) \xrightarrow{f_{02}^*\left(\varepsilon^{f_{24}}_{f_{25}^*(N)}\right)} f_{02}^*(f_{25}^*(N))\\
&\xrightarrow{c_{f_{25},f_{02}}(N)} (f_{25}f_{02})^*(N)
\end{aligned}$$

Proof. By the definition of $\theta^D(M)$ and (1.4.12), $\theta^D(M)$ is the following composition.

$$\begin{aligned}
&(N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} \xrightarrow{(N^{[f_{24},f_{25}]})^{f_{01}}} (N^{[f_{24},f_{25}]})^{[f_{13}f_{01},f_{14}f_{01}]} = (N^{[f_{24},f_{25}]})^{[f_{13}f_{01},f_{24}f_{02}]} = (f_{13}f_{01})_!(f_{24}f_{02})^*(N^{[f_{24},f_{25}]})\\
&\xrightarrow{(f_{13}f_{01})_!(f_{24}f_{02})^*(N^{f_{02}})} (f_{13}f_{01})_!(f_{24}f_{02})^*(N^{[f_{14}f_{01},f_{25}f_{02}]}) = (f_{13}f_{01})_!(f_{24}f_{02})^*(f_{14}f_{01})_!(f_{25}f_{02})^*(N)\\
&\xrightarrow{(f_{13}f_{01})_!\left(\varepsilon^{f_{14}f_{01}}_{(f_{25}f_{02})^*(N)}\right)} (f_{13}f_{01})_!(f_{25}f_{02})^*(N) = N^{[f_{13}f_{01},f_{25}f_{02}]}
\end{aligned}$$

It follows from (1) of (1.4.8) that the adjoint of $(N^{[f_{24},f_{25}]})^{f_{01}} : (N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} \to (N^{[f_{24},f_{25}]})^{[f_{13}f_{01},f_{14}f_{01}]}$ with respect to the adjunction $(f_{13}f_{01})_* \dashv (f_{13}f_{01})_!$ is the following composition.

$$\begin{aligned}
&(f_{13}f_{01})^*((N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}) \xrightarrow{c_{f_{13},f_{01}}((N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]})^{-1}} f_{01}^*f_{13}^*((N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}) = f_{01}^*f_{13}^*(f_{13})_!f_{14}^*(N^{[f_{24},f_{25}]})\\
&\xrightarrow{f_{01}^*\left(\varepsilon^{f_{13}}_{f_{14}^*(N^{[f_{24},f_{25}]})}\right)} f_{01}^*f_{14}^*(N^{[f_{24},f_{25}]}) \xrightarrow{c_{f_{14},f_{01}}(N^{[f_{24},f_{25}]})} (f_{14}f_{01})^*(N^{[f_{24},f_{25}]})
\end{aligned}$$

It also follows from (1) of (1.4.8) that $N^{f_{02}} : N^{[f_{24},f_{25}]} \to N^{[f_{14}f_{01},f_{13}f_{01}]}$ coincides with the following composition.

$$\begin{aligned}
&N^{[f_{24},f_{25}]} \xrightarrow{\eta^{f_{24}f_{02}}_{N^{[f_{24},f_{25}]}}} (f_{24}f_{02})_!(f_{24}f_{02})^*(N^{[f_{24},f_{25}]}) \xrightarrow{(f_{24}f_{02})_!(c_{f_{24},f_{02}}(N^{[f_{24},f_{25}]}))^{-1}} (f_{24}f_{02})_!f_{02}^*f_{24}^*(N^{[f_{24},f_{25}]})\\
&= (f_{24}f_{02})_!f_{02}^*f_{24}^*(f_{24})_!f_{25}^*(N) \xrightarrow{(f_{24}f_{02})_!f_{02}^*\left(\varepsilon^{f_{24}}_{f_{25}^*(N)}\right)} (f_{24}f_{02})_!f_{02}^*f_{25}^*(N) \xrightarrow{(f_{24}f_{02})_!(c_{f_{25},f_{02}}(N))}\\
&(f_{24}f_{02})_!(f_{25}f_{02})^*(N) = N^{[f_{24}f_{02},f_{25}f_{02}]}
\end{aligned}$$

Hence if we put $\psi = c_{f_{25},f_{02}}(N)f_{02}^*(\varepsilon^{f_{24}}_{f_{25}^*(N)})c_{f_{24},f_{02}}(N^{[f_{24},f_{25}]})^{-1} : (f_{24}f_{02})^*(N^{[f_{24},f_{25}]}) \to (f_{25}f_{02})^*(N)$, the adjoint of $\theta^D(M)$ with respect to the adjunction $(f_{13}f_{01})^* \dashv (f_{13}f_{01})_!$ is the following composition.

$$
\begin{aligned}
&(f_{13}f_{01})^*((N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}) \xrightarrow{c_{f_{13},f_{01}}((N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]})^{-1}} f_{01}^*f_{13}^*((N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}) = f_{01}^*f_{13}^*(f_{13})_!f_{14}^*(N^{[f_{24},f_{25}]})\\
&\xrightarrow{f_{01}^*(\varepsilon^{f_{13}}_{f_{14}^*(N^{[f_{24},f_{25}]})})} f_{01}^*f_{14}^*(N^{[f_{24},f_{25}]}) \xrightarrow{c_{f_{14},f_{01}}(N^{[f_{24},f_{25}]})} (f_{24}f_{02})^*(N^{[f_{24},f_{25}]}) \xrightarrow{(f_{24}f_{02})^*(\eta^{f_{24}f_{02}}_{N^{[f_{24},f_{25}]}})}\\
&(f_{24}f_{02})^*(f_{24}f_{02})_!(f_{24}f_{02})^*(N^{[f_{24},f_{25}]}) \xrightarrow{(f_{24}f_{02})^*(f_{24}f_{02})_!(\psi)} (f_{24}f_{02})^*(f_{24}f_{02})_!(f_{25}f_{02})^*(N)\\
&= (f_{14}f_{01})^*(f_{14}f_{01})_!(f_{25}f_{02})^*(N) \xrightarrow{\varepsilon^{f_{14}f_{01}}_{(f_{25}f_{02})^*(N)}} (f_{25}f_{02})^*(N)
\end{aligned}
$$

By the naturality of $\varepsilon^{f_{14}f_{01}}$, the composition of the last three morphisms in the above diagram coincides with $\psi\varepsilon^{f_{14}f_{01}}_{(f_{14}f_{01})^*(N^{[f_{24},f_{25}]})}(f_{24}f_{02})^*(\eta^{f_{24}f_{02}}_{N^{[f_{24},f_{25}]}}) = \psi$, which implies the assertion. □

Proposition 1.4.18 *The following diagram is commutative.*

$$
\begin{tikzcd}
(f_{13}f_{01})^*((N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}) \arrow[rr, "(f_{13}f_{01})^*(\theta^D(N))"] \arrow[d, "f_{01}^\sharp(\pi_{f_{13},f_{14}}(N^{[f_{24},f_{25}]}))"] & & (f_{13}f_{01})^*(N^{[f_{13}f_{01},f_{25}f_{02}]}) \arrow[d, "\pi_{f_{13}f_{01},f_{25}f_{02}}(N)"] \\
(f_{14}f_{01})^*(N^{[f_{24},f_{25}]}) \arrow[r, equal] & (f_{24}f_{02})^*(N^{[f_{24},f_{25}]}) \arrow[r, "f_{02}^\sharp(\pi_{f_{24},f_{25}}(N))"] & (f_{25}f_{02})^*(N)
\end{tikzcd}
$$

Proof. By the naturality of $E_{f_{13}f_{01},f_{25}f_{02}}(N)$, $\theta^D(N)$ is the image of

$$\pi_{f_{24}f_{02},f_{25}f_{02}}(N)\pi_{f_{13}f_{01},f_{14}f_{01}}(N^{[f_{24}f_{02},f_{25}f_{02}]})(f_{13}f_{01})^*((N^{f_{02}})^{f_{01}}) : (f_{13}f_{01})^*((N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}) \to (f_{25}f_{02})^*(N)$$

by $E_{f_{13}f_{01},f_{25}f_{02}}(N)_{(N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}}$. Hence the following equality holds by (1.4.3).

$$\pi_{f_{13}f_{01},f_{25}f_{02}}(N)(f_{13}f_{01})^*(\theta^D(N)) = \pi_{f_{24}f_{02},f_{25}f_{02}}(N)\pi_{f_{13}f_{01},f_{14}f_{01}}(N^{[f_{24}f_{02},f_{25}f_{02}]})(f_{13}f_{01})^*((N^{f_{02}})^{f_{01}}) \quad \cdots (*)$$

It follows from (1.4.7), (1.1.11) and (1.4.4) that we have

$$
\begin{aligned}
&\pi_{f_{13}f_{01},f_{14}f_{01}}(N^{[f_{24}f_{02},f_{25}f_{02}]})(f_{13}f_{01})^*((N^{f_{02}})^{f_{01}})\\
&= \pi_{f_{13}f_{01},f_{14}f_{01}}(N^{[f_{24}f_{02},f_{25}f_{02}]})(f_{13}f_{01})^*((N^{[f_{24}f_{02},f_{25}f_{02}]})^{f_{01}})(f_{13}f_{01})^*((N^{f_{02}})^{[f_{13},f_{14}]})\\
&= f_{01}^\sharp(\pi_{f_{13},f_{14}}(N^{[f_{24}f_{02},f_{25}f_{02}]}))(f_{13}f_{01})^*((N^{f_{02}})^{[f_{13},f_{14}]})\\
&= c_{f_{14},f_{01}}(N^{[f_{24}f_{02},f_{25}f_{02}]})f_{01}^*(\pi_{f_{13},f_{14}}(N^{[f_{24}f_{02},f_{25}f_{02}]}))c_{f_{13},f_{01}}((N^{[f_{24}f_{02},f_{25}f_{02}]})^{[f_{13},f_{14}]})^{-1}(f_{13}f_{01})^*((N^{f_{02}})^{[f_{13},f_{14}]})\\
&= c_{f_{14},f_{01}}(N^{[f_{24}f_{02},f_{25}f_{02}]})f_{01}^*(\pi_{f_{13},f_{14}}(N^{[f_{24}f_{02},f_{25}f_{02}]}))f_{01}^*(f_{13}^*((N^{f_{02}})^{[f_{13},f_{14}]}))c_{f_{13},f_{01}}((N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]})^{-1}\\
&= c_{f_{14},f_{01}}(N^{[f_{24}f_{02},f_{25}f_{02}]})f_{01}^*(f_{14}^*(N^{f_{02}}))f_{01}^*(\pi_{f_{13},f_{14}}(N^{[f_{24},f_{25}]}))c_{f_{13},f_{01}}((N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]})^{-1}\\
&= (f_{14}f_{01})^*(N^{f_{02}})c_{f_{14},f_{01}}(N^{[f_{24},f_{25}]})f_{01}^*(\pi_{f_{13},f_{14}}(N^{[f_{24},f_{25}]}))c_{f_{13},f_{01}}((N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]})^{-1}\\
&= (f_{24}f_{02})^*(N^{f_{02}})f_{01}^\sharp(\pi_{f_{13},f_{14}}(N^{[f_{24},f_{25}]}))
\end{aligned}
$$

Therefore we have

$$(*) = \pi_{f_{24}f_{02},f_{25}f_{02}}(N)(f_{24}f_{02})^*(N^{f_{02}})f_{01}^\sharp(\pi_{f_{13},f_{14}}(N^{[f_{24},f_{25}]})) = f_{02}^\sharp(\pi_{f_{24},f_{25}}(N))f_{01}^\sharp(\pi_{f_{13},f_{14}}(N^{[f_{24},f_{25}]}))$$

which implies the assertion. □

Proposition 1.4.19 *For a morphism $\varphi : N \to N$ of $\mathcal{F}_Z$, the following diagram commutes.*

$$
\begin{tikzcd}
(M^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} \arrow[r, "\theta^D(M)"] \arrow[d, "(\varphi^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}"] & M^{[f_{13}f_{01},f_{25}f_{02}]} \arrow[d, "\varphi^{[f_{13}f_{01},f_{25}f_{02}]}"] \\
(N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} \arrow[r, "\theta^D(N)"] & N^{[f_{13}f_{01},f_{25}f_{02}]}
\end{tikzcd}
$$

Proof. The following diagram commutes by (1.4.13), (1.4.9), (1.4.4) and (1.4.7).

$$\begin{array}{ccccc}
(M^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} & \xrightarrow{(M^{f_{02}})^{f_{01}}} & (M^{[f_{13}f_{01},f_{14}f_{01}]})^{[f_{24}f_{02},f_{25}f_{02}]} & \xrightarrow{\epsilon_M^{f_{13}f_{01},f_{14}f_{01},f_{25}f_{02}}} & M^{[f_{13}f_{01},f_{25}f_{02}]} \\
\downarrow{\scriptstyle (\varphi^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}} & & \downarrow{\scriptstyle (\varphi^{[f_{13}f_{01},f_{14}f_{01}]})^{[f_{24}f_{02},f_{25}f_{02}]}} & & \downarrow{\scriptstyle \varphi^{[f_{13}f_{01},f_{25}f_{02}]}} \\
(N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} & \xrightarrow{(N^{f_{02}})^{f_{01}}} & (N^{[f_{13}f_{01},f_{14}f_{01}]})^{[f_{24}f_{02},f_{25}f_{02}]} & \xrightarrow{\epsilon_N^{f_{13}f_{01},f_{14}f_{01},f_{25}f_{02}}} & N^{[f_{13}f_{01},f_{25}f_{02}]}
\end{array}$$

Hence the assertion follows. □

Proposition 1.4.20 *Let $E:\mathcal{P}\to\mathcal{E}$ be a functor which satisfies $E(i)=D(i)$ for $i=3,4,5$ and $\lambda:D\to E$ a natural transformation which satisfies $\lambda_i=id_{D(i)}$ for $i=3,4,5$. We put $E(\tau_{ij})=g_{ij}$, then the following diagram commutes.*

$$\begin{array}{ccc}
(N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]} & \xrightarrow{\theta^E(N)} & N^{[g_{13}g_{01},g_{25}g_{02}]} \\
\downarrow{\scriptstyle (N^{\lambda_2})^{\lambda_1}} & & \downarrow{\scriptstyle N^{\lambda_0}} \\
(N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} & \xrightarrow{\theta^D(N)} & N^{[f_{13}f_{01},f_{25}f_{02}]}
\end{array}$$

Proof. Since $f_{ij}=g_{ij}\lambda_i$ for $i=1,2$, we have $f_{13}f_{01}=g_{13}\lambda_1f_{01}=g_{13}g_{01}\lambda_0$, $f_{14}f_{01}=g_{14}\lambda_1f_{01}=g_{14}g_{01}\lambda_0$ and $f_{25}f_{02}=g_{25}\lambda_2f_{02}=g_{25}g_{02}\lambda_0$. It follows from (1.4.7), (1.4.9) and (1.4.13) that

$$\begin{array}{ccccc}
(N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]} & \xrightarrow{(N^{g_{02}})^{g_{01}}} & (N^{[g_{24}g_{02},g_{25}g_{02}]})^{[g_{13}g_{01},g_{14}g_{01}]} & \xrightarrow{\epsilon_N^{g_{13}g_{01},g_{14}g_{01},g_{25}g_{02}}} & N^{[g_{13}g_{01},g_{25}g_{02}]} \\
\downarrow{\scriptstyle (N^{\lambda_2})^{\lambda_1}} & & \downarrow{\scriptstyle (N^{\lambda_0})^{\lambda_0}} & & \downarrow{\scriptstyle N^{\lambda_0}} \\
(N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} & \xrightarrow{(N^{f_{02}})^{f_{01}}} & (N^{[f_{24}f_{02},f_{25}f_{02}]})^{[f_{13}f_{01},f_{14}f_{01}]} & \xrightarrow{\epsilon_N^{f_{13}f_{01},f_{14}f_{01},f_{25}f_{02}}} & N^{[f_{13}f_{01},f_{25}f_{02}]}
\end{array}$$

is commutative. □

For morphisms $f:X\to Y$, $g:X\to Z$, $h:V\to Z$, $i:V\to W$ in $\mathcal{E}$, let $X\xleftarrow{\mathrm{pr}_X}X\times_Z V\xrightarrow{\mathrm{pr}_V}V$ be a limit of a diagram $X\xrightarrow{g}Z\xleftarrow{h}V$. We define a functor $D_{f,g,h,i}:\mathcal{P}\to\mathcal{E}$ by $D_{f,g,h,i}(0)=X\times_Z V$, $D_{f,g,h,i}(1)=X$, $D_{f,g,h,i}(2)=V$, $D_{f,g,h,i}(3)=Y$, $D_{f,g,h,i}(4)=Z$, $D_{f,g,h,i}(5)=W$ and $D_{f,g,h,i}(\tau_{01})=\mathrm{pr}_X$, $D_{f,g,h,i}(\tau_{02})=\mathrm{pr}_V$, $D_{f,g,h,i}(\tau_{13})=f$, $D_{f,g,h,i}(\tau_{14})=g$, $D_{f,g,h,i}(\tau_{24})=h$, $D_{f,g,h,i}(\tau_{25})=i$. For an object N of $\mathcal{F}_W$, we denote $\theta^{D_{f,g,h,i}}(N)$ by $\theta^{f,g,h,i}(N)$. The following facts are special cases of (1.4.19) and (1.4.20).

Proposition 1.4.21 *Let $f:X\to Y$, $g:X\to Z$, $h:V\to Z$, $i:V\to W$, $j:S\to X$, $k:T\to V$ be morphisms in $\mathcal{E}$ and $\varphi:M\to N$ a morphism in $\mathcal{F}_Z$. The following diagrams are commutative.*

$$\begin{array}{ccc}
(M^{[h,i]})^{[f,g]} & \xrightarrow{\theta^{f,g,h,i}(M)} & M^{[f\mathrm{pr}_X,i\mathrm{pr}_V]} \\
\downarrow{\scriptstyle (\varphi^{[h,i]})^{[f,g]}} & & \downarrow{\scriptstyle \varphi^{[f\mathrm{pr}_X,i\mathrm{pr}_V]}} \\
(N^{[h,i]})^{[f,g]} & \xrightarrow{\theta^{f,g,h,i}(N)} & N^{[f\mathrm{pr}_X,i\mathrm{pr}_V]}
\end{array}
\qquad
\begin{array}{ccc}
(N^{[h,i]})^{[f,g]} & \xrightarrow{\theta^{f,g,h,i}(N)} & N^{[f\mathrm{pr}_X,i\mathrm{pr}_V]} \\
\downarrow{\scriptstyle (N^k)^j} & & \downarrow{\scriptstyle N^{j\times_Z k}} \\
(N^{[hk,ik]})^{[fj,gj]} & \xrightarrow{\theta^{fj,gj,hk,ik}(N)} & N^{[fj\mathrm{pr}_S,ik\mathrm{pr}_T]}
\end{array}$$

Remark 1.4.22 *If $X\xleftarrow{\mathrm{pr}'_X}X\times'_Z V\xrightarrow{\mathrm{pr}'_V}V$ is another limit of a diagram $X\xrightarrow{g}Z\xleftarrow{h}V$, there exists unique isomorphism $l:X\times'_Z V\to X\times_Z V$ that satisfies $\mathrm{pr}'_X=\mathrm{pr}_X l$ and $\mathrm{pr}'_V=\mathrm{pr}_V l$. We denote by $\theta'^{f,g,h,i}(N):(N^{[f,g]})^{[h,i]}\to N^{[f\mathrm{pr}'_X,i\mathrm{pr}'_V]}$ the morphism in $\mathcal{F}_W$ obtained from $X\xleftarrow{\mathrm{pr}'_X}X\times'_Z V\xrightarrow{\mathrm{pr}'_V}V$. Then, $N^l:N^{[f\mathrm{pr}_X,\,i\mathrm{pr}_V]}\to N^{[f\mathrm{pr}'_X,\,i\mathrm{pr}'_V]}$ is an isomorphism and (1.4.20) implies $\theta'^{f,g,h,i}(N)=N^l\theta^{f,g,h,i}(N)$.*

Definition 1.4.23 *Let $f:X\to Y$, $g:X\to Z$, $h:V\to Z$, $i:V\to W$ be morphisms in $\mathcal{E}$ and N an object of $\mathcal{F}_Z$. We say that a quadruple (f,g,h,i) is an associative right fibered representable quadruple with respect to N if the following conditions are satisfied.*

(i) A limit $X\xleftarrow{\mathrm{pr}_X}X\times_Z V\xrightarrow{\mathrm{pr}_V}V$ of a diagram $X\xrightarrow{g}Z\xleftarrow{h}V$ exists.
(ii) (h,i) is a right fibered representable pair with respect to N.
(iii) (f,g) is a right fibered representable pair with respect to $N_{[h,i]}$.
(iv) $(f\mathrm{pr}_X,i\mathrm{pr}_V)$ is a right fibered representable pair with respect to N.
(v) $\theta^{f,g,h,i}(N):(N^{[h,i]})^{[f,g]}\to N^{[f\mathrm{pr}_X,i\mathrm{pr}_V]}$ is an isomorphism.

If (f,g,h,i) is an associative right fibered representable quadruple with respect to any object of $\mathcal{F}_Y$, we say that (f,g,h,i) is an associative right fibered representable quadruple.

Proposition 1.4.24 *Under the assumption of (1.3.24), the following diagram is commutative.*

$$\begin{tikzcd}
((N^{[j,k]})^{[h,i]})^{[f,g]} \arrow[r, "\theta^{D_1}(N)^{[f,g]}"] \arrow[d, "\theta^{D_4}(N^{[j,k]})"] & (N^{[ht,ku]})^{[f,g]} \arrow[d, "\theta^{D_3}(N)"] \\
(N^{[j,k]})^{[fr,is]} \arrow[r, "\theta^{D_2}(N)"] & N^{[frv,kuw]}
\end{tikzcd}$$

Proof. The following diagrams are commutative by (1.4.14), (1.4.13), (1.4.9), (1.4.4) and (1.4.7).

$$\begin{tikzcd}
 & ((N^{[j,k]})^{[h,i]})^{[fr,gr]} \arrow[dl, "((N^{[j,k]})^{s})^{[fr,gr]}"'] \arrow[d, "((N^{uw})^{s})^{[fr,gr]}"] \\
((N^{[j,k]})^{[hs,is]})^{[fr,gr]} \arrow[r, "((N^{uw})^{[hs,is]})^{[fr,gr]}"] \arrow[d, "\epsilon^{fr,gr,is}_{N^{[j,k]}}"] & ((N^{[juw,kuw]})^{[htw,itw]})^{[frv,grv]} \arrow[d, "\epsilon^{fr,gr,is}_{N^{[juw,kuw]}}"] \\
(N^{[j,k]})^{[fr,is]} \arrow[r, "(N^{uw})^{[fr,is]}"] & (N^{[juw,kuw]})^{[fr,is]}
\end{tikzcd}$$

$$\begin{tikzcd}
((N^{[j,k]})^{[h,i]})^{[f,g]} \arrow[r, "((N^{u})^{t})^{[f,g]}"] \arrow[d, "((N^{[j,k]})^{[h,i]})^{r}"] & ((N^{[ju,ku]})^{[ht,it]})^{[f,g]} \arrow[r, "(\epsilon^{ht,it,ku}_{N})^{[f,g]}"] \arrow[d, "((N^{[ju,ku]})^{[ht,it]})^{rv}"] & (N^{[ht,ku]})^{[f,g]} \arrow[d, "(N^{[ht,ku]})^{rv}"'] \\
((N^{[j,k]})^{[h,i]})^{[fr,gr]} \arrow[r, "((N^{u})^{t})^{v}"] \arrow[d, "((N^{uw})^{s})^{[fr,gr]}"] & ((N^{[ju,ku]})^{[ht,it]})^{[frv,grv]} \arrow[r, "(\epsilon^{ht,it,ku}_{N})^{[frv,grv]}"] \arrow[d, "((N^{w})^{w})^{[frv,grv]}"] & (N^{[ht,ku]})^{[frv,grv]} \arrow[d, "(N^{w})^{[frv,grv]}"'] \\
((N^{[juw,kuw]})^{[hs,is]})^{[fr,gr]} \arrow[r, "((N^{[juw,kuw]})^{v})^{v}"] \arrow[d, "\epsilon^{fr,gr,is}_{N^{[juw,kuw]}}"] & ((N^{[juw,kuw]})^{[htw,itw]})^{[frv,grv]} \arrow[r, "(\epsilon^{htw,itw,kuw}_{N})^{[frv,grv]}"] \arrow[d, "\epsilon^{frv,grv,itw}_{N^{[juw,kuw]}}"] & (N^{[htw,kuw]})^{[frv,grv]} \arrow[d, "\epsilon^{frv,grv,kuw}_{N}"'] \\
(N^{[juw,kuw]})^{[fr,is]} \arrow[r, "(N^{[juw,kuw]})^{v}"] & (N^{[juw,kuw]})^{[frv,itw]} \arrow[r, "\epsilon^{frv,itw,kuw}_{N}"] & N^{[frv,kuw]}
\end{tikzcd}$$

Hence the asserion follows from the definition of $\theta^{D_l}(N)$. □

For morphisms $g : X \to Z$, $h : V \to Z$, $i : V \to W$, $j : T \to W$ in $\mathcal{E}$, let $X \xleftarrow{\mathrm{pr}_X} X \times_Z V \xrightarrow{\mathrm{pr}_{2V}} V$ and $V \xleftarrow{\mathrm{pr}_{1V}} V \times_W T \xrightarrow{\mathrm{pr}_T} T$ be limits of diagrams $X \xrightarrow{g} Z \xleftarrow{h} V$ and $V \xrightarrow{i} W \xleftarrow{j} T$, respectively. We also assume that a limit $X \times_Z V \xleftarrow{\mathrm{pr}_{X\times_Z V}} X \times_Z V \times_W T \xrightarrow{\mathrm{pr}_{V\times_W T}} V \times_W T$ of a diagram $X \times_Z V \xrightarrow{\mathrm{pr}_{2V}} V \xleftarrow{\mathrm{pr}_{1V}} V \times_W T$ exists. Then, $X \xleftarrow{\mathrm{pr}_X \mathrm{pr}_{X\times_Z V}} X \times_Z V \times_W T \xrightarrow{\mathrm{pr}_{V\times_W T}} V \times_W T$ and $X \times_Z V \xleftarrow{\mathrm{pr}_{X\times_Z V}} X \times_Z V \times_W T \xrightarrow{\mathrm{pr}_{V\times_W T}\mathrm{pr}_T} T$ are limits of diagrams $X \xrightarrow{g} Z \xleftarrow{h\mathrm{pr}_{1V}} V \times_W T$ and $X \times_Z V \xrightarrow{i\mathrm{pr}_{2V}} W \xleftarrow{j} T$, respectively.

Corollary 1.4.25 *Let $f : X \to Y$, $g : X \to Z$, $h : V \to Z$, $i : V \to W$, $j : T \to W$, $k : T \to U$ be morphisms in $\mathcal{E}$ and N an object of $\mathcal{F}_U$. The following diagram is commutative.*

$$\begin{tikzcd}
((N^{[j,k]})^{[h,i]})^{[f,g]} \arrow[r, "\theta^{h,i,j,k}(N)^{[f,g]}"] \arrow[d, "\theta^{f,g,h,i}(N^{[j,k]})"] & (N^{[h\mathrm{pr}_{1V},k\mathrm{pr}_T]})^{[f,g]} \arrow[d, "\theta^{f,g,h\mathrm{pr}_{1V},k\mathrm{pr}_T}(N)"] \\
(N^{[j,k]})^{[f\mathrm{pr}_X,i\mathrm{pr}_{2V}]} \arrow[r, "\theta^{f\mathrm{pr}_X,i\mathrm{pr}_{2V},j,k}(N)"] & N^{[f\mathrm{pr}_X pr_{X\times_Z V},k\mathrm{pr}_T\mathrm{pr}_{V\times_W T}]}
\end{tikzcd}$$

Proof. The assertion follows by applying the result of (1.4.24) to the following diagram.

$$\begin{tikzcd}
 & & & X \times_Z V \times_W T \arrow[dl, "\mathrm{pr}_{X\times_Z V}"'] \arrow[dr, "\mathrm{pr}_{V\times_W T}"] & & & \\
 & & X \times_Z V \arrow[dl, "\mathrm{pr}_X"'] \arrow[dr, "\mathrm{pr}_{2V}"] & & V \times_W T \arrow[dl, "\mathrm{pr}_{1V}"'] \arrow[dr, "\mathrm{pr}_T"] & & \\
 & X \arrow[dl, "f"'] \arrow[dr, "g"] & & V \arrow[dl, "h"'] \arrow[dr, "i"] & & T \arrow[dl, "j"'] \arrow[dr, "k"] & \\
Y & & Z & & W & & U
\end{tikzcd}$$

$\square$

Proposition 1.4.26 *For morphisms $f : X \to Y$, $g : X \to Z$ in $\mathcal{E}$ and an object N of $\mathcal{F}_Z$, the following morphisims of $\mathcal{F}_Y$ are identified with the identity morphism of $N^{[f,\,g]}$.*

$$\theta^{f,g,id_Z,id_Z}(N) : (N^{[id_Z,\,id_Z]})^{[f,\,g]} \to N^{[f\,id_X,\,id_Z g]}, \qquad \theta^{id_Y,id_Y,f,g}(N) : (N^{[f,\,g]})^{[id_Y,\,id_Y]} \to N^{[id_Y f,\,g\,id_X]}$$

Proof. Since $\theta^{f,g,id_Z,id_Z}(N)$ is a composition

$$N^{[f,\,g]} = (N^{[id_Z,\,id_Z]})^{[f,\,g]} \xrightarrow{(N^g)^{[f,\,g]}} (N^{[id_Z g,\,id_Z g]})^{[f\,id_X,\,g\,id_X]} \xrightarrow{\epsilon_N^{f\,id_X,\,g\,id_X,\,id_Z g}} N^{[id_Y f,\,id_Z\,g]} = N^{[f,\,g]}$$

and $\theta^{id_Y,id_Y,f,g}(N)$ is a composition

$$N^{[f,\,g]} = (N^{[f,\,g]})^{[id_Y,\,id_Y]} \xrightarrow{(N^{[f,\,g]})^f} (N^{[f\,id_X,\,g\,id_X]})^{[id_Y f,\,id_Y f]} \xrightarrow{\epsilon^{id_Y f,\,f\,id_X,\,g\,id_X,\,N}} N^{[id_Y f,\,g\,id_X]} = N^{[f,\,g]},$$

the assertion is a direct consequence of (1.4.15). $\square$

Lemma 1.4.27 *For a functor $D : \mathcal{P} \to \mathcal{E}$, we put $D(\tau_{01}) = j$, $D(\tau_{02}) = k$, $D(\tau_{13}) = f$, $D(\tau_{14}) = g$, $D(\tau_{24}) = h$, $D(\tau_{25}) = i$. For an object N of $\mathcal{F}_{D(5)}$, the following diagram is commutative.*

$$\begin{array}{ccc}
(fj)^*((N^{[h,i]})^{[f,g]}) & \xrightarrow{j^\sharp(\pi_{f,g}(N^{[h,i]}))} & (gj)^*(N^{[h,i]}) \\
\downarrow{\scriptstyle (fj)^*(\theta^D(N))} & & \downarrow{\scriptstyle k^\sharp(\pi_{h,i}(N))} \\
(fj)^*(N^{[fj,ik]}) & \xrightarrow{\pi_{fj,ik}(N)} & (ik)^*(N)
\end{array}$$

Proof. It follows from (1.4.7) and (1) of (1.4.4) that we have

$$\begin{aligned}
k^\sharp(\pi_{h,i}(N))j^\sharp(\pi_{f,g}(N^{[h,i]})) &= \pi_{hk,ik}(N)(hk)^*(N^k)\pi_{fj,gj}(N^{[h,i]})(fj)^*((N^{[h,i]})^j) \\
&= \pi_{hk,ik}(N)\pi_{fj,gj}(N^{[hk,ik]})(fj)^*((N^k)^{[fj,gj]})(fj)^*((N^{[h,i]})^j) \\
&= \pi_{hk,ik}(N)\pi_{fj,gj}(N^{[hk,ik]})(fj)^*((N^k)^j)
\end{aligned}$$

By the naturality of $E_{fj,ik}(N)$ and the definition of $\epsilon_N^{fj,gj,ik}$,

$$E_{fj,ik}(N)_{(N^{[h,i]})^{[f,g]}} : \mathcal{F}_{D(0)}((fj)^*((N^{[h,i]})^{[f,g]}), (ik)^*(N)) \to \mathcal{F}_{D(3)}((N^{[h,i]})^{[f,g]}, N^{[fj,ik]})$$

maps $k^\sharp(\pi_{h,i}(N))j^\sharp(\pi_{f,g}(N^{[h,i]}))$ to $\epsilon_N^{fj,gj,ik}(N^k)^j = \theta^D(N)$. On the other hand, it follows from (1.4.3) that $E_{fj,ik}(N)_{(N^{[h,i]})^{[f,g]}}$ also maps $\pi_{fj,ik}(N)(fj)^*(\theta^D(N))$ to $\theta^D(N)$. $\square$

For a morphism $g : X \to Z$, let $X \xleftarrow{\mathrm{pr}_{1X}} X \times_Z X \xrightarrow{\mathrm{pr}_{2X}} X$ be a limit of a diagram $X \xrightarrow{g} Z \xleftarrow{g} X$. We denote by $\Delta_g : X \to X \times_Z X$ the diagonal morphism, that is, the unique morphism that satisfies $\mathrm{pr}_{1X}\Delta_g = \mathrm{pr}_{2X}\Delta_g = id_X$.

Proposition 1.4.28 *For morphisms $f : X \to Y$, $g : X \to Z$, $h : X \to W$ in $\mathcal{E}$ and an object N of $\mathcal{F}_W$, $\epsilon_N^{f,g,h} : (N^{[g,h]})^{[f,g]} \to N^{[f,h]}$ coincides with the following composition.*

$$(N^{[g,h]})^{[f,g]} \xrightarrow{\theta^{f,g,g,h}(N)} N^{[f\mathrm{pr}_{1X},h\mathrm{pr}_{2X}]} \xrightarrow{N^{\Delta_g}} N^{[f\mathrm{pr}_{1X}\Delta_g,h\mathrm{pr}_{2X}\Delta_g]} = N^{[f,h]}$$

Proof. Define a functor $E : \mathcal{P} \to \mathcal{E}$ by $E(i) = X$ for $i = 0, 1, 2$, $E(i) = D_{f,g,g,h}(i)$ for $i = 3, 4, 5$ and $E(\tau_{01}) = E(\tau_{02}) = id_X$, $E(\tau_{ij}) = D_{f,g,g,h}(\tau_{ij})$ if $i \neq 0$. Then, $\theta^E(N) = \epsilon_N^{f,g,h} : (N^{[g,h]})^{[f,g]} \to N^{[f,h]}$ and we have a natural transformation $\lambda : E \to D$ defined by $\lambda_0 = \Delta_g$ and $\lambda_i = id_{E(i)}$ if $i \geqq 1$. It follows from (1.4.20) that $N^{\Delta_g}\theta^{f,g,g,h}(N) = \theta^E(N) = \epsilon_N^{f,g,h}$. $\square$

Let $D, E : \mathcal{Q} \to \mathcal{E}$ be functors and N an object of $\mathcal{F}_{E(2)}$. We put $D(\tau_{0j}) = f_j$ and $E(\tau_{0j}) = g_j$ for $j = 1, 2$. For a natural transformation $\omega : D \to E$, let $\omega^N : \omega_1^*(N^{[g_1,g_2]}) \to \omega_2^*(N)^{[f_1,f_2]}$ be the image of $\pi_{g_1,g_2}(N) \in \mathcal{F}_{E(0)}(g_1^*(N^{[g_1,g_2]}), g_2^*(N))$ by the following composition of maps.

$$\begin{aligned}
\mathcal{F}_{E(0)}(g_1^*(N^{[g_1,g_2]}), g_2^*(N)) &\xrightarrow{\omega_0^\sharp} \mathcal{F}_{D(0)}((g_1\omega_0)^*(N^{[g_1,g_2]}), (g_2\omega_0)^*(N)) = \mathcal{F}_{D(0)}((\omega_1 f_1)^*(N^{[g_1,g_2]}), (\omega_2 f_2)^*(N)) \\
&\xrightarrow{c_{\omega_1,f_1}(N^{[g_1,g_2]})^* c_{\omega_2,f_2}(N)_*^{-1}} \mathcal{F}_{D(0)}(f_1^*(\omega_1^*(N^{[g_1,g_2]})), f_2^*(\omega_2^*(N))) \\
&\xrightarrow{E_{f_1,f_2}(\omega_2^*(N))_{\omega_1^*(N^{[g_1,g_2]})}} \mathcal{F}_{D(2)}(\omega_1^*(N^{[g_1,g_2]}), \omega_2^*(N)^{[f_1,f_2]})
\end{aligned}$$

Remark 1.4.29 *(1) If $D(i) = E(i)$ and ω_i is the identity morphism of $D(i)$ for $i = 1, 2$, then ω^N coincides with $N^{\omega_0} : N^{[g_1,g_2]} \to N^{[g_1\omega_0, g_2\omega_0]} = N^{[f_1,f_2]}$.*

(2) It follows from (1.4.3) and the definition of ω^N that the following diagram is commutative.

$$\begin{tikzcd}
(\omega_1 f_1)^*(N^{[g_1,g_2]}) \arrow[r, "c_{\omega_1,f_1}(N^{[g_1,g_2]})^{-1}"] \arrow[d, equal] & f_1^*(\omega_1^*(N^{[g_1,g_2]})) \arrow[r, "f_1^*(\omega^N)"] & f_1^*(\omega_2^*(N)^{[f_1,f_2]}) \arrow[d, "\pi_{f_1,f_2}(\omega_2^*(N))"] \\
(g_1\omega_0)^*(N^{[g_1,g_2]}) \arrow[r, "\omega_0^\sharp(\pi_{g_1,g_2}(N))"] & (g_2\omega_0)^*(N) = (\omega_2 f_2)^*(N) \arrow[r, "c_{\omega_2,f_2}(N)^{-1}"] & f_2^*(\omega_2^*(N))
\end{tikzcd}$$

Proposition 1.4.30 *Assume that $D(0) = E(0)$ and ω_0 is the identity morphism of $D(0)$. For an object M of $\mathcal{F}_{E(1)}$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathcal{F}_{D(0)}(g_1^*(M), g_2^*(N)) \arrow[r, "c_{\omega_2,f_2}(N)_*^{-1}"] \arrow[d, "E_{g_1,g_2}(N)_M"] & \mathcal{F}_{D(0)}(g_1^*(M), f_2^*(\omega_2^*(N))) \arrow[r, "c_{\omega_1,f_1}(M)^*"] & \mathcal{F}_{D(0)}(f_1^*(\omega_1^*(M)), f_2^*(\omega_2^*(N))) \arrow[d, "E_{f_1,f_2}(\omega_2^*(N))_{\omega_1^*(M)}"] \\
\mathcal{F}_{E(1)}(M, N^{[g_1,g_2]}) \arrow[r, "\omega_1^*"] & \mathcal{F}_{D(1)}(\omega_1^*(M), \omega_1^*(N^{[g_1,g_2]})) \arrow[r, "\omega_*^N"] & \mathcal{F}_{D(1)}(\omega_1^*(M), \omega_2^*(N)^{[f_1,f_2]})
\end{tikzcd}$$

Proof. First we note that $g_i = \omega_i f_i$ for $i = 1, 2$. It follows from (1.4.29) and the definition of ω^N that we have $\pi_{f_1,f_2}(\omega_2^*(N)) f_1^*(\omega^N) = c_{\omega_2,f_2}(N)^{-1}\pi_{g_1,g_2}(N) c_{\omega_1,f_1}(N^{[g_1,g_2]})$. (1.4.3) and (1.1.11) imply

$$\begin{aligned}
c_{\omega_2,f_2}(N)^{-1} E_{g_1,g_2}(N)_M^{-1}(\varphi) c_{\omega_1,f_1}(M) &= c_{\omega_2,f_2}(N)^{-1}\pi_{g_1,g_2}(N) g_1^*(\varphi) c_{\omega_1,f_1}(M) \\
&= c_{\omega_2,f_2}(N)^{-1}\pi_{g_1,g_2}(N) c_{\omega_2,f_2}(N^{[g_1,g_2]}) f_1^*\omega_1^*(\varphi) \\
&= \pi_{f_1,f_2}(\omega_2^*(N)) f_1^*(\omega^N) f_1^*\omega_1^*(\varphi) = \pi_{f_1,f_2}(\omega_2^*(N)) f_1^*(\omega^N\omega_1^*(\varphi)) \\
&= E_{f_1,f_2}(\omega_2^*(N))_{\omega_1^*(M)}^{-1}(\omega^N\omega_1^*(\varphi))
\end{aligned}$$

for $\varphi \in \mathcal{F}_{E(1)}(M, N^{[g_1,g_2]})$, which shows that the above diagram is commutative. □

Proposition 1.4.31 *For a morphism $\varphi : M \to N$ of $\mathcal{F}_{E(2)}$, the following diagram is commutative.*

$$\begin{tikzcd}
\omega_1^*(M^{[g_1,g_2]}) \arrow[r, "\omega^M"] \arrow[d, "\omega_1^*(\varphi^{[g_1,g_2]})"] & \omega_2^*(M)^{[f_1,f_2]} \arrow[d, "\omega_2^*(\varphi)^{[f_1,f_2]}"] \\
\omega_1^*(N^{[g_1,g_2]}) \arrow[r, "\omega^N"] & \omega_2^*(N)^{[f_1,f_2]}
\end{tikzcd}$$

Proof. It follows from (1.1.11), (1) of (1.4.4) and (1.1.15) that the following diagrams are commutative.

$$\begin{tikzcd}
f_1^*\omega_1^*(M^{[g_1,g_2]}) \arrow[r, "c_{\omega_1,f_1}(M^{[g_1,g_2]})"] \arrow[d, "f_1^*\omega_1^*(\varphi^{[g_1,g_2]})"] & (\omega_1 f_1)^*(M^{[g_1,g_2]}) = (g_1\omega_0)^*(M^{[g_1,g_2]}) \arrow[r, "\omega_0^\sharp(\pi_{g_1,g_2}(M))"] \arrow[d, "(g_1\omega_0)^*(\varphi^{[g_1,g_2]})"] & (g_2\omega_0)^*(M) \arrow[d, "(f_2\omega_0)^*(\varphi)"] \\
f_1^*\omega_1^*(N^{[g_1,g_2]}) \arrow[r, "c_{\omega_1,g_1}(N^{[g_1,g_2]})"] & (\omega_1 f_1)^*(N^{[g_1,g_2]}) = (g_1\omega_0)^*(N^{[g_1,g_2]}) \arrow[r, "\omega_0^\sharp(\pi_{g_1,g_2}(N))"] & (g_2\omega_0)^*(N)
\end{tikzcd}$$

$$\begin{tikzcd}
(g_2\omega_0)^*(M) = (\omega_2 f_2)^*(M) \arrow[r, "c_{\omega_2,f_2}(M)^{-1}"] \arrow[d, "(\omega_2 f_2)^*(\varphi)"] & f_2^*\omega_2^*(M) \arrow[d, "f_2^*\omega_2^*(\varphi)"] \\
(g_2\omega_0)^*(N) = (\omega_2 f_2)^*(N) \arrow[r, "c_{\omega_2,f_2}(N)^{-1}"] & f_2^*\omega_2^*(N)
\end{tikzcd}$$

By applying (1.4.6) to the following commutative diagram,

$$\begin{tikzcd}
f_1^*\omega_1^*(M^{[g_1,g_2]}) \arrow[r, "c_{\omega_2,f_2}(M)^{-1}\omega_0^\sharp(\pi_{g_1,g_2}(M))c_{\omega_1,f_1}(M^{[g_1,g_2]})"] \arrow[d, "f_1^*\omega_1^*(\varphi^{[g_1,g_2]})"] & f_2^*\omega_2^*(M) \arrow[d, "f_2^*\omega_2^*(\varphi)"] \\
f_1^*\omega_1^*(N^{[g_1,g_2]}) \arrow[r, "c_{\omega_2,f_2}(N)^{-1}\omega_0^\sharp(\pi_{g_1,g_2}(N))c_{\omega_1,f_1}(N^{[g_1,g_2]})"] & f_2^*\omega_2^*(N)
\end{tikzcd}$$

the assertion follows. □

Proposition 1.4.32 *Let $D, E, F : \mathcal{Q} \to \mathcal{E}$ be functors and M an object of $\mathcal{F}_{F(1)}$. We put $D(\tau_{0j}) = f_j$, $E(\tau_{0j}) = g_j$ and $F(\tau_{0j}) = h_j$ for $j = 1, 2$. For natural transformations $\omega : D \to E$ and $\chi : E \to F$, the following diagram is commutative.*

$$\begin{tikzcd}
\omega_1^*(\chi_1^*(N^{[h_1,h_2]})) \arrow[r, "\omega_1^*(\chi^N)"] \arrow[d, "c_{\chi_1,\omega_1}(N^{[h_1,h_2]})"] & \omega_1^*(\chi_2^*(N)^{[g_1,g_2]}) \arrow[r, "\omega^{\chi_2^*(N)}"] & \omega_2^*(\chi_2^*(N))^{[f_1,f_2]} \arrow[d, "c_{\chi_2,\omega_2}(N)^{[f_1,f_2]}"] \\
(\chi_1\omega_1)^*(N^{[h_1,h_2]}) \arrow[rr, "(\chi\omega)^N"] & & (\chi_2\omega_2)^*(N)^{[f_1,f_2]}
\end{tikzcd}$$

Proof. It follows from (1.4.3) and (1.4.29) that we have

$$\begin{aligned}
E_{f_1,f_2}(\omega_2^*(\chi_2^*(N)))^{-1}_{\omega_1^*(\chi_1^*(N^{[h_1,h_2]}))}(\omega^{\chi_2^*(N)}\omega_1^*(\chi^N)) &= \pi_{f_1,f_2}(\omega_2^*(\chi_2^*(N)))f_1^*(\omega^{\chi_2^*(N)}\omega_1^*(\chi^N)) \\
&= \pi_{f_1,f_2}(\omega_2^*(\chi_2^*(N)))f_1^*(\omega^{\chi_2^*(N)})f_1^*(\omega_1^*(\chi^N)) \\
&= c_{\omega_2,f_2}(\chi_2^*(N))^{-1}\omega_0^\sharp(\pi_{g_1,g_2}(\chi_2^*(N)))c_{\omega_1,f_1}(\chi_2^*(N)^{[g_1,g_2]})f_1^*(\omega_1^*(\chi^N))
\end{aligned}$$

Hence it suffices to show that the following diagram is commutative by (1.4.6).

$$\begin{tikzcd}
f_1^*(\omega_1^*(\chi_1^*(N^{[h_1,h_2]}))) \arrow[r, "c_{\omega_2,f_2}(\chi_2^*(N))^{-1}\omega_0^\sharp(\pi_{g_1,g_2}(\chi_2^*(N)))c_{\omega_1,f_1}(\chi_2^*(N)^{[g_1,g_2]})f_1^*(\omega_1^*(\chi^N))"] \arrow[d, "f_1^*(c_{\chi_1,\omega_1}(N^{[h_1,h_2]}))"] & f_2^*(\omega_2^*(\chi_2^*(N))) \arrow[d, "f_2^*(c_{\chi_2,\omega_2}(N))"] \\
f_1^*(\chi_1\omega_1)^*(N^{[h_1,h_2]}) \arrow[r, "c_{\chi_2\omega_2,f_2}(N)^{-1}(\chi_0\omega_0)^\sharp(\pi_{h_1,h_2}(N))c_{\chi_1\omega_1,f_1}(N^{[h_1,h_2]})"] & f_2^*(\chi_2\omega_2)^*(N)
\end{tikzcd}$$

It follows from (1.1.11) and (1.1.12) that we have

$$\begin{gathered}
c_{\omega_1,f_1}(\chi_2^*(N^{[g_1,g_2]}))f_1^*(\omega_1^*(\chi^N)) = (\omega_1 f_1)^*(\chi^N)c_{\omega_1,f_1}(\chi_1^*(N^{[h_1,h_2]})) = (g_1\omega_0)^*(\chi^N)c_{\omega_1,f_1}(\chi_1^*(N^{[h_1,h_2]})) \\
c_{\chi_2\omega_2,f_2}(N)f_2^*(c_{\chi_2,\omega_2}(N))c_{\omega_2,f_2}(\chi_2^*(N))^{-1} = c_{\chi_2,\omega_2 f_2}(N) = c_{\chi_2,g_2\omega_0}(N) \\
c_{\chi_1\omega_1,f_1}(N^{[h_1,h_2]})f_1^*(c_{\chi_1,\omega_1}(N^{[h_1,h_2]}))c_{\omega_1,f_1}(\chi_1^*(N^{[h_1,h_2]}))^{-1} = c_{\chi_1,\omega_1 f_1}(N^{[h_1,h_2]}) = c_{\chi_1,g_1\omega_0}(N^{[h_1,h_2]}).
\end{gathered}$$

Hence the commutativity of the above diagram is equivalent to the following equality.

$$c_{\chi_2,g_2\omega_0}(N)\omega_0^\sharp(\pi_{g_1,g_2}(\chi_2^*(N)))(g_1\omega_0)^*(\chi^N) = (\chi_0\omega_0)^\sharp(\pi_{h_1,h_2}(N))c_{\chi_1,g_1\omega_0}(N^{[h_1,h_2]}) \quad \cdots (*)$$

The following diagram is commutative by (1.1.11) and (1.3.29).

$$\begin{tikzcd}
\omega_0^*((h_1\chi_0)^*(N^{[h_1,h_2]})) \arrow[rr, "\omega_0^*(\chi_0^\sharp(\pi_{h_1,h_2}(N)))"] \arrow[d, equal] & & \omega_0^*((h_2\chi_0)^*(N)) \arrow[d, equal] \\
\omega_0^*((\chi_1 g_1)^*(N^{[h_1,h_2]})) & & \omega_0^*((\chi_2 g_2)^*(N)) \\
\omega_0^*(g_1^*(\chi_1^*(N^{[h_1,h_2]}))) \arrow[u, "\omega_0^*(c_{\chi_1,g_1}(N^{[h_1,h_2]}))"] \arrow[r, "\omega_0^*(g_1^*(\chi_N))"] \arrow[d, "c_{g_1,\omega_0}(\chi_1^*(N^{[h_1,h_2]}))"] & \omega_0^*(g_1^*(\chi_2^*(N)^{[g_1,g_2]})) \arrow[r, "\omega_0^*(\pi_{g_1,g_2}(\chi_2^*(N)))"] \arrow[d, "c_{g_1,\omega_0}(\chi_2^*(N)^{[g_1,g_2]})"] & \omega_0^*(g_2^*(\chi_2^*(N))) \arrow[u, "\omega_0^*(c_{\chi_2,g_2}(N))"] \arrow[d, "c_{g_2,\omega_0}(\chi_2^*(N))"] \\
(g_1\omega_0)^*(\chi_1^*(N^{[h_1,h_2]})) \arrow[r, "(g_1\omega_0)^*(\chi^N)"] & (g_1\omega_0)^*(\chi_2^*(N)^{[g_1,g_2]}) \arrow[r, "\omega_0^\sharp(\pi_{g_1,g_2}(\chi_2^*(N)))"] & (g_2\omega_0)^*(\chi_2^*(N))
\end{tikzcd}$$

Hence the left hand side of $(*)$ equals

$$\begin{aligned}
&c_{\chi_2,g_2\omega_0}(N)c_{g_2,\omega_0}(\chi_2^*(N))\omega_0^*(c_{\chi_2,g_2}(N))^{-1}\omega_0^*(\chi_0^\sharp(\pi_{h_1,h_2}(N)))\omega_0^*(c_{\chi_1,g_1}(N^{[h_1,h_2]}))c_{g_1,\omega_0}(\chi_1^*(N^{[h_1,h_2]}))^{-1} \\
&\quad = c_{\chi_2 g_2,\omega_0}(N)\omega_0^*(\chi_0^\sharp(\pi_{h_1,h_2}(N)))c_{\chi_1 g_1,\omega_0}(N^{[h_1,h_2]})^{-1}c_{\chi_1,g_1\omega_0}(N^{[h_1,h_2]}) \\
&\quad = (\chi_0\omega_0)^\sharp(\pi_{h_1,h_2}(N))c_{\chi_1,g_1\omega_0}(N^{[h_1,h_2]})
\end{aligned}$$

by (1.1.12) and (1.3.32) for $M = N^{[h_1,h_2]}$ and $\varphi = \pi_{h_1,h_2}(N)$. □

Proposition 1.4.33 *For functors $D, E : \mathcal{P} \to \mathcal{E}$, we put $D(\tau_{ij}) = f_{ij}$ and $E(\tau_{ij}) = g_{ij}$ and define functors $D_i, E_i : \mathcal{Q} \to \mathcal{E}$ for $i = 0, 1, 2$ as follows.*

$$\begin{array}{lllll}
D_0(0) = D(0) & D_0(1) = D(3) & D_0(2) = D(5) & D_0(\tau_{01}) = f_{13}f_{01} & D_0(\tau_{02}) = f_{25}f_{02} \\
E_0(0) = E(0) & E_0(1) = E(3) & E_0(2) = E(5) & E_0(\tau_{01}) = g_{13}g_{01} & E_0(\tau_{02}) = g_{25}g_{02} \\
D_1(0) = D(1) & D_1(1) = D(3) & D_1(2) = D(4) & D_1(\tau_{01}) = f_{13} & D_1(\tau_{02}) = f_{14} \\
E_1(0) = E(1) & E_1(1) = E(3) & E_1(2) = E(4) & E_1(\tau_{01}) = g_{13} & E_1(\tau_{02}) = g_{14} \\
D_2(0) = D(2) & D_2(1) = D(4) & D_2(2) = D(5) & D_2(\tau_{01}) = f_{24} & D_2(\tau_{02}) = f_{25} \\
E_2(0) = E(2) & E_2(1) = E(4) & E_2(2) = E(5) & E_2(\tau_{01}) = g_{24} & E_2(\tau_{02}) = g_{25}
\end{array}$$

For a natural transformation $\gamma : D \to E$, we define a natural transformations $\gamma^i : D_i \to E_i$ $(i = 0, 1, 2)$ by

$$\gamma^0_0 = \gamma_0 \quad \gamma^0_1 = \gamma_3 \quad \gamma^0_2 = \gamma_5 \quad \gamma^1_0 = \gamma_1 \quad \gamma^1_1 = \gamma_3 \quad \gamma^1_2 = \gamma_4 \quad \gamma^2_0 = \gamma_2 \quad \gamma^2_1 = \gamma_4 \quad \gamma^2_2 = \gamma_5$$

For an object N of $\mathcal{F}_{E_0(2)} = \mathcal{F}_{E(5)}$, the following diagram is commutative.

$$\begin{tikzcd}
\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}) \arrow[r, "\gamma^{1N^{[g_{24},g_{25}]}}"] \arrow[d, "\gamma_3^*(\theta^D(N))"] & (\gamma_4^*(N^{[g_{24},g_{25}]}))^{[f_{13},f_{14}]} \arrow[r, "(\gamma^{2N})^{[f_{13},f_{14}]}"] & (\gamma_5^*(N)^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} \arrow[d, "\theta^E(\gamma_5^*(N))"] \\
\gamma_3^*(N^{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[rr, "\gamma^{0N}"] & & \gamma_5^*(N)^{[f_{13}f_{01},f_{25}f_{02}]}
\end{tikzcd}$$

Proof. By the naturality of $E_{f_{13}f_{01},f_{25}f_{02}}(\gamma_5^*(N))$ and the definition of γ^{0N}, $\gamma^{0N}\gamma_3^*(\theta^D(N))$ is the image of the following composition by $E_{f_{13}f_{01},f_{25}f_{02}}(\gamma_5^*(N))_{\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})}$.

$$(f_{13}f_{01})^*(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \xrightarrow{(f_{13}f_{01})^*(\gamma_3^*(\theta^D(N)))} (f_{13}f_{01})^*(\gamma_3^*(N^{[g_{13}g_{01},g_{25}g_{02}]})) \xrightarrow{c_{\gamma_3,f_{13}f_{01}}(N^{[g_{13}g_{01},g_{25}g_{02}]})}$$
$$(\gamma_3 f_{13}f_{01})^*(N^{[g_{13}g_{01},g_{25}g_{02}]}) = (g_{13}g_{01}\gamma_0)^*(N^{[g_{13}g_{01},g_{25}g_{02}]}) \xrightarrow{\gamma_0^\sharp(\pi_{g_{13}g_{01},g_{25}g_{02}}(N))} (g_{25}g_{02}\gamma_0)^*(N)$$
$$= (\gamma_5 f_{25}f_{02})^*(N) \xrightarrow{c_{\gamma_5,f_{25}f_{02}}(N)^{-1}} (f_{25}f_{02})^*(\gamma_5^*(N))$$

On the other hand, $\theta^E(\gamma_5^*(N))(\gamma^{2N})^{[f_{13},f_{14}]}\gamma^{1N^{[g_{24},g_{25}]}}$ is the image of the following composition.

$$(f_{13}f_{01})^*(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \xrightarrow{(f_{13}f_{01})^*(\gamma^{1N^{[g_{24},g_{25}]}})} (f_{13}f_{01})^*((\gamma_4^*(N^{[g_{24},g_{25}]}))^{[f_{13},f_{14}]}) \xrightarrow{(f_{13}f_{01})^*((\gamma^{2N})^{[f_{13},f_{14}]})}$$
$$(f_{13}f_{01})^*((\gamma_5^*(N)^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}) \xrightarrow{(f_{13}f_{01})^*(\theta^E(\gamma_5^*(N)))} (f_{13}f_{01})^*(\gamma_5^*(N)^{[f_{13}f_{01},f_{25}f_{02}]})$$
$$\xrightarrow{\pi_{f_{13}f_{01},f_{25}f_{02}}(\gamma_5^*(N))} (f_{25}f_{02})^*(\gamma_5^*(N))$$

We see that $\theta^E(\gamma_5^*(N))(\gamma^{2N})^{[f_{13},f_{14}]}\gamma^{1N^{[g_{24},g_{25}]}}$ is the image of the following composition by applying (1.4.18) to the last two morphisms in the above diagram.

$$(f_{13}f_{01})^*(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \xrightarrow{(f_{13}f_{01})^*(\gamma^{1N^{[g_{24},g_{25}]}})} (f_{13}f_{01})^*((\gamma_4^*(N^{[g_{24},g_{25}]}))^{[f_{13},f_{14}]}) \xrightarrow{(f_{13}f_{01})^*((\gamma^{2N})^{[f_{13},f_{14}]})}$$
$$(f_{13}f_{01})^*((\gamma_5^*(N)^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}) \xrightarrow{f_{01}^\sharp(\pi_{f_{13},f_{14}}((\gamma_5^*(N)^{[f_{24},f_{25}]})))} (f_{14}f_{01})^*(\gamma_5^*(N)^{[f_{24},f_{25}]})$$
$$= (f_{24}f_{02})^*(\gamma_5^*(N)^{[f_{24},f_{25}]}) \xrightarrow{f_{02}^\sharp(\pi_{f_{24},f_{25}}(\gamma_5^*(N)))} (f_{25}f_{02})^*(\gamma_5^*(N))$$

Hence it suffices to show that the following diagram (i) is commutative.

$$\begin{tikzcd}
(f_{13}f_{01})^*(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \arrow[rr, "(f_{13}f_{01})^*(\gamma^{1N^{[g_{24},g_{25}]}})"] \arrow[d, "(f_{13}f_{01})^*(\gamma_3^*(\theta^D(N)))"] & & (f_{13}f_{01})^*((\gamma_4^*(N^{[g_{24},g_{25}]}))^{[f_{13},f_{14}]}) \arrow[d, "(f_{13}f_{01})^*((\gamma^{2N})^{[f_{13},f_{14}]})"] \\
(f_{13}f_{01})^*(\gamma_3^*(N^{[g_{13}g_{01},g_{25}g_{02}]})) \arrow[d, "c_{\gamma_3,f_{13}f_{01}}(N^{[g_{13}g_{01},g_{25}g_{02}]})"] & & (f_{13}f_{01})^*((\gamma_5^*(N)^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}) \arrow[d, "f_{01}^\sharp(\pi_{f_{13},f_{14}}((\gamma_5^*(N)^{[f_{24},f_{25}]})))"] \\
(\gamma_3 f_{13}f_{01})^*(N^{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[d, equal] & \text{diagram } (i) & (f_{14}f_{01})^*(\gamma_5^*(N)^{[f_{24},f_{25}]}) \arrow[d, equal] \\
(g_{13}g_{01}\gamma_0)^*(N^{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[d, "\gamma_0^\sharp(\pi_{g_{13}g_{01},g_{25}g_{02}}(N))"] & & (f_{24}f_{02})^*(\gamma_5^*(N)^{[f_{24},f_{25}]}) \arrow[d, "f_{02}^\sharp(\pi_{f_{24},f_{25}}(\gamma_5^*(N)))"] \\
(g_{25}g_{02}\gamma_0)^*(N) \arrow[r, equal] & (\gamma_5 f_{25}f_{02})^*(N) \arrow[r, "c_{\gamma_5,f_{25}f_{02}}(N)^{-1}"] & (f_{25}f_{02})^*(\gamma_5^*(N))
\end{tikzcd}$$

The following diagram (ii) is commutative by (1.1.11) and the definition of $f_{01}^\sharp$.

$$
\begin{tikzcd}
f_{01}^*(f_{13}^*(\gamma_3^*(N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \arrow[r, "c_{f_{13},f_{01}}(\gamma_3^*(N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})"] \arrow[d, "f_{01}^*(f_{13}^*(\gamma^{1N^{[g_{24},g_{25}]}}))"] & (f_{13}f_{01})^*(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \arrow[d, "(f_{13}f_{01})^*(\gamma^{1N^{[g_{24},g_{25}]}})"] \\
f_{01}^*(f_{13}^*(\gamma_4^*(N^{[g_{24},g_{25}]})^{[f_{13},f_{14}]})) \arrow[r, "c_{f_{13},f_{01}}(\gamma_4^*(N^{[g_{24},g_{25}]})^{[f_{13},f_{14}]})"] \arrow[d, "f_{01}^*(f_{13}^*((\gamma^{2N})^{[f_{13},f_{14}]}))"] & (f_{13}f_{01})^*(\gamma_4^*(N^{[g_{24},g_{25}]})^{[f_{13},f_{14}]}) \arrow[d, "(f_{13}f_{01})^*((\gamma^{2N})^{[f_{13},f_{14}]})"] \\
f_{01}^*(f_{13}^*((\gamma_5^*(N)^{[f_{24},f_{25}]})^{[f_{13},f_{14}]})) \arrow[r, "c_{f_{13},f_{01}}((\gamma_5^*(N)^{[f_{24},f_{25}]})^{[f_{13},f_{14}]})"] \arrow[d, "f_{01}^*(\pi_{f_{13},f_{14}}(\gamma_5^*(N)^{[f_{24},f_{25}]}))"] & (f_{13}f_{01})^*((\gamma_5^*(N)^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}) \arrow[d, "f_{01}^\sharp(\pi_{f_{13},f_{14}}(\gamma_5^*(N)^{[f_{24},f_{25}]}))"] \\
f_{01}^*(f_{14}^*(\gamma_5^*(N)^{[f_{24},f_{25}]})) \arrow[r, "c_{f_{14},f_{01}}(\gamma_5^*(N)^{[f_{24},f_{25}]})"] & (f_{14}f_{01})^*(\gamma_5^*(N)^{[f_{24},f_{25}]})
\end{tikzcd}
$$

diagram (ii)

It follows from (1.4.4), (1.4.3) and the definition of $\gamma^{1N^{[g_{24},g_{25}]}}$ that the following equalities hold.

$$\pi_{f_{13},f_{14}}(\gamma_5^*(N)^{[f_{24},f_{25}]})f_{13}^*((\gamma^{2N})^{[f_{13},f_{14}]}) = f_{14}^*(\gamma^{2N})\pi_{f_{13},f_{14}}(\gamma_4^*(N^{[g_{24},g_{25}]}))$$
$$\pi_{f_{13},f_{14}}(\gamma_4^*(N^{[g_{24},g_{25}]}))f_{13}^*(\gamma^{1N^{[g_{24},g_{25}]}}) = c_{\gamma_4,f_{14}}(N^{[g_{24},g_{25}]})^{-1}\gamma_1^\sharp(\pi_{g_{13},g_{14}}(N^{[g_{24},g_{25}]}))c_{\gamma_3,f_{13}}((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})$$

Hence the composition of the left vertical morphisms in diagram (ii) coincides with the following.

$$
\begin{aligned}
&f_{01}^*(\pi_{f_{13},f_{14}}(\gamma_5^*(N)^{[f_{24},f_{25}]}))f_{01}^*(f_{13}^*((\gamma^{2N})^{[f_{13},f_{14}]}))f_{01}^*(f_{13}^*(\gamma^{1N^{[g_{24},g_{25}]}}))\\
&= f_{01}^*(f_{14}^*(\gamma^{2N}))f_{01}^*(\pi_{f_{13},f_{14}}(\gamma_4^*(N^{[g_{24},g_{25}]})))f_{01}^*(f_{13}^*(\gamma^{1N^{[g_{24},g_{25}]}}))\\
&= f_{01}^*(f_{14}^*(\gamma^{2N}))f_{01}^*(c_{\gamma_4,f_{14}}(N^{[g_{24},g_{25}]})^{-1})f_{01}^*(\gamma_1^\sharp(\pi_{g_{13},g_{14}}(N^{[g_{24},g_{25}]})))f_{01}^*(c_{\gamma_3,f_{13}}((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}))
\end{aligned}
$$

Since $c_{f_{14},f_{01}}(\gamma_5^*(N)^{[f_{24},f_{25}]})f_{01}^*(f_{14}^*(\gamma^{2N})) = (f_{14}f_{01})^*(\gamma^{2N})c_{f_{14},f_{01}}(\gamma_4^*(N^{[g_{24},g_{25}]}))$ by (1.1.11), the commutativity of diagram (ii) implies that the composition of the upper horizontal morphism and the right vertical morphisms in diagram (i) coincides with the following composition.

$$
\begin{aligned}
&(f_{13}f_{01})^*(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \xrightarrow{c_{f_{13},f_{01}}(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}))^{-1}} f_{01}^*(f_{13}^*(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})))\\
&\xrightarrow{f_{01}^*(c_{\gamma_3,f_{13}}((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}))} f_{01}^*((\gamma_3f_{13})^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) = f_{01}^*((g_{13}\gamma_1)^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}))\\
&\xrightarrow{f_{01}^*(\gamma_1^\sharp(\pi_{g_{13},g_{14}}(N^{[g_{24},g_{25}]})))} f_{01}^*((g_{14}\gamma_1)^*(N^{[g_{24},g_{25}]})) = f_{01}^*((\gamma_4f_{14})^*(N^{[g_{24},g_{25}]})) \xrightarrow{f_{01}^*(c_{\gamma_4,f_{14}}(N^{[g_{24},g_{25}]})^{-1})}\\
&f_{01}^*(f_{14}^*(\gamma_4^*(N^{[g_{24},g_{25}]}))) \xrightarrow{c_{f_{14},f_{01}}(\gamma_4^*(N^{[g_{24},g_{25}]}))} (f_{14}f_{01})^*(\gamma_4^*(N^{[g_{24},g_{25}]})) \xrightarrow{(f_{14}f_{01})^*(\gamma^{2N})}\\
&(f_{14}f_{01})^*(\gamma_5^*(N)^{[f_{24},f_{25}]}) = (f_{24}f_{02})^*(\gamma_5^*(N)^{[f_{24},f_{25}]}) \xrightarrow{f_{02}^\sharp(\pi_{f_{24},f_{25}}(\gamma_5^*(N)))} (f_{25}f_{02})^*(\gamma_5^*(N))
\end{aligned}
$$

diagram (iii)

Next, we consider the composition of the left vertical morphisms and the lower horizontal morphism in diagram (i). It follows from (1.1.11) and (1.4.18) that the following diagram is commutative.

$$
\begin{tikzcd}
(f_{13}f_{01})^*(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \arrow[rr, "(f_{13}f_{01})^*(\gamma_3^*(\theta^D(N)))"] \arrow[d, "c_{\gamma_3,f_{13}f_{01}}((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})"] & & (f_{13}f_{01})^*(\gamma_3^*(N^{[g_{13}g_{01},g_{25}g_{02}]})) \arrow[d, "c_{\gamma_3,f_{13}f_{01}}(N^{[g_{13}g_{01},g_{25}g_{02}]})"'] \\
(\gamma_3f_{13}f_{01})^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}) \arrow[rr, "(\gamma_3f_{13}f_{01})^*(\theta^D(N))"] \arrow[d, equal] & & (\gamma_3f_{13}f_{01})^*(N^{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[d, equal] \\
(g_{13}g_{01}\gamma_0)^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}) \arrow[rr, "(g_{13}g_{01}\gamma_0)^*(\theta^D(N))"] \arrow[d, "c_{g_{13}g_{01},\gamma_0}((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})"] & & (g_{13}g_{01}\gamma_0)^*(N^{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[d, "c_{g_{13}g_{01},\gamma_0}(N^{[g_{13}g_{01},g_{25}g_{02}]})^{-1}"'] \\
\gamma_0^*((g_{13}g_{01})^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \arrow[rr, "\gamma_0^*((g_{13}g_{01})^*(\theta^D(N)))"] \arrow[d, "\gamma_0^*(g_{01}^\sharp(\pi_{g_{13},g_{14}}(N^{[g_{24},g_{25}]})))"] & & \gamma_0^*((g_{13}g_{01})^*(N^{[g_{13}g_{01},g_{25}g_{02}]})) \arrow[d, "\gamma_0^*(\pi_{g_{13}g_{01},g_{25}g_{02}}(N))"'] \\
\gamma_0^*((g_{14}g_{01})^*(N^{[g_{24},g_{25}]})) \arrow[r, equal] & \gamma_0^*((g_{24}g_{02})^*(N^{[g_{24},g_{25}]})) \arrow[r, "\gamma_0^*(g_{02}^\sharp(\pi_{g_{24},g_{25}}(N)))"] & \gamma_0^*((g_{25}g_{02})^*(N))
\end{tikzcd}
$$

Since $\gamma_0^\sharp(\pi_{g_{13}g_{01},g_{25}g_{02}}(N)) = c_{g_{25}g_{02},\gamma_0}(N)\gamma_0^*(\pi_{g_{13}g_{01},g_{25}g_{02}}(N))c_{g_{13}g_{01},\gamma_0}(N^{[g_{13}g_{01},g_{25}g_{02}]})^{-1}$, it follows from the above diagram that the composition of the left vertical morphisms and the lower horizontal morphism of diagram (i) coincides with the following composition.

$$
\begin{aligned}
&(f_{13}f_{01})^*(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \xrightarrow{c_{\gamma_3,f_{13}f_{01}}((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})} (\gamma_3 f_{13}f_{01})^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})\\
&= (g_{13}g_{01}\gamma_0)^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}) \xrightarrow{c_{g_{13}g_{01},\gamma_0}((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})} \gamma_0^*((g_{13}g_{01})^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}))\\
&\xrightarrow{\gamma_0^*(g_{01}^\sharp(\pi_{g_{13},g_{14}}(N^{[g_{24},g_{25}]})))} \gamma_0^*((g_{14}g_{01})^*(N^{[g_{24},g_{25}]})) = \gamma_0^*((g_{24}g_{02})^*(N^{[g_{24},g_{25}]})) \xrightarrow{\gamma_0^*(g_{02}^\sharp(\pi_{g_{24},g_{25}}(N)))}\\
&\gamma_0^*((g_{25}g_{02})^*(N)) \xrightarrow{c_{g_{25}g_{02},\gamma_0}(N)} (g_{25}g_{02}\gamma_0)^*(N) = (\gamma_5 f_{25}f_{02})^*(N) \xrightarrow{c_{\gamma_5,f_{25}f_{02}}(N)^{-1}} (f_{25}f_{02})^*(\gamma_5^*(N))
\end{aligned}
$$

diagram (iv)

The following diagram is commutative by (1.1.11), (1.1.12) and (1.4.29).

$$
\begin{tikzcd}
f_{02}^*((g_{24}\gamma_2)^*(N^{[g_{24},g_{25}]})) \arrow[r, "c_{g_{24}\gamma_2,f_{02}}(N^{[g_{24},g_{25}]})"] \arrow[d, equal] \arrow[dr, "f_{02}^*(\gamma_2^\sharp(\pi_{g_{24},g_{25}}(N)))"'] & (g_{24}\gamma_2 f_{02})^*(N^{[g_{24},g_{25}]}) \arrow[r, "f_{02}^\sharp(\gamma_2^\sharp(\pi_{g_{24},g_{25}}(N)))"] & (\gamma_5 f_{25}f_{02})^*(N) \\
f_{02}^*((\gamma_4 f_{24})^*(N^{[g_{24},g_{25}]})) & f_{02}^*((g_{25}\gamma_2)^*(N)) \arrow[r, equal] & f_{02}^*((\gamma_5 f_{25})^*(N)) \arrow[u, "c_{\gamma_5 f_{25},f_{02}}(N)"'] \\
f_{02}^*(f_{24}^*(\gamma_4^*(N^{[g_{24},g_{25}]}))) \arrow[u, "f_{02}^*(c_{\gamma_4,f_{24}}(N^{[g_{24},g_{25}]}))"'] \arrow[r, "f_{02}^*(f_{24}^*(\gamma^{2N}))"] & f_{02}^*(f_{25}^*(\gamma_5^*(N)^{[f_{24},f_{25}]})) \arrow[r, "f_{02}^*(\pi_{f_{24},f_{25}}(\gamma_5^*(N)))"] & f_{02}^*(f_{25}^*(\gamma_5^*(N))) \arrow[u, "f_{02}^*(c_{\gamma_5,f_{25}}(N))"] \\
(f_{24}f_{02})^*(\gamma_4^*(N^{[g_{24},g_{25}]})) \arrow[u, "c_{f_{24},f_{02}}(\gamma_4^*(N^{[g_{24},g_{25}]}))^{-1}"'] \arrow[r, "(f_{24}f_{02})^*(\gamma^{2N})"] & (f_{25}f_{02})^*(\gamma_5^*(N)^{[f_{24},f_{25}]}) \arrow[u, "c_{f_{25},f_{02}}(\gamma_5^*(N)^{[f_{24},f_{25}]})^{-1}"'] \arrow[r, "f_{02}^\sharp(\pi_{f_{24},f_{25}}(\gamma_5^*(N)))"] & (f_{25}f_{02})^*(\gamma_5^*(N)) \arrow[u, "c_{f_{25},f_{02}}(\gamma_5^*(N))^{-1}"] \arrow[uuu, bend right=60, "c_{\gamma_5,f_{25}f_{02}}(N)"']
\end{tikzcd}
$$

We note that, by (1.1.12), $c_{\gamma_4,f_{24}f_{02}}(M) : (f_{24}f_{02})^*(\gamma_4^*(M)) \to (\gamma_4 f_{24}f_{02})^*(M)$ coincides with a commposition $c_{g_{24}\gamma_2,f_{02}}(N^{[g_{24},g_{25}]})c_{f_{25},f_{02}}(\gamma_5^*(N)^{[f_{24},f_{25}]})c_{f_{25},f_{02}}(\gamma_5^*(N))^{-1}$. Hence the following diagram is commutative by (1.1.12) and (1.1.16). Here we put $M = N^{[g_{24},g_{25}]}$ and $L = (N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}$ below.

$$
\begin{tikzcd}
(f_{13}f_{01})^*(\gamma_3^*(L)) \arrow[r, "c_{\gamma_3,f_{13}f_{01}}(L)"] \arrow[d, "c_{f_{13},f_{01}}(\gamma_3^*(L))^{-1}"] & (\gamma_3 f_{13}f_{01})^*(L) \arrow[r, equal] \arrow[d, equal] & (g_{13}g_{01}\gamma_0)^*(L) \arrow[d, "c_{g_{13}g_{01},\gamma_0}(L)"] \arrow[ddl, "\gamma_0^\sharp(g_{01}^\sharp(\pi_{g_{13},g_{14}}(M)))"'] \\
f_{01}^*(f_{13}^*(\gamma_3^*(L))) \arrow[d, "f_{01}^*(c_{\gamma_3,f_{13}}(L))"] & (g_{13}\gamma_1 f_{01})^*(L) \arrow[d, "f_{01}^\sharp(\gamma_1^\sharp(\pi_{g_{13},g_{14}}(M)))"'] & \gamma_0^*((g_{13}g_{01})^*(L)) \arrow[d, "\gamma_0^*(g_{01}^\sharp(\pi_{g_{13},g_{14}}(M)))"'] \\
f_{01}^*((\gamma_3 f_{13})^*(L)) \arrow[d, equal] & (g_{14}g_{01}\gamma_0)^*(M) \arrow[d, equal] & \gamma_0^*((g_{14}g_{01})^*(M)) \arrow[l, "c_{g_{14}g_{01},\gamma_0}(M)"'] \arrow[d, equal] \\
f_{01}^*((g_{13}\gamma_1)^*(L)) \arrow[d, "f_{01}^*(\gamma_1^\sharp(\pi_{g_{13},g_{14}}(M)))"] \arrow[uur, bend left, "c_{g_{13}\gamma_1,f_{01}}(L)"'] & (g_{24}g_{02}\gamma_0)^*(M) \arrow[d, equal] \arrow[ddr, "\gamma_0^\sharp(g_{02}^\sharp(\pi_{g_{24},g_{25}}(N)))"'] & \gamma_0^*((g_{24}g_{02})^*(M)) \arrow[l, "c_{g_{24}g_{02},\gamma_0}(M)"'] \arrow[d, "\gamma_0^*(g_{02}^\sharp(\pi_{g_{24},g_{25}}(N)))"'] \\
f_{01}^*((g_{14}\gamma_1)^*(M)) \arrow[d, equal] \arrow[uur, bend right, "c_{g_{14}\gamma_1,f_{01}}(M)"] & (\gamma_4 f_{14}f_{01})^*(M) \arrow[d, equal] & \gamma_0^*((g_{25}g_{02})^*(N)) \arrow[d, "c_{g_{25}g_{02},\gamma_0}(N)"] \\
f_{01}^*((\gamma_4 f_{14})^*(M)) \arrow[d, "f_{01}^*(c_{\gamma_4,f_{14}}(M)^{-1})"] \arrow[uur, "c_{\gamma_4 f_{14},f_{01}}(M)"'] & (\gamma_4 f_{24}f_{02})^*(M) & (g_{25}g_{02}\gamma_0)^*(N) \arrow[d, equal] \\
f_{01}^*(f_{14}^*(\gamma_4^*(M))) \arrow[d, "c_{f_{14},f_{01}}(\gamma_4^*(M))"'] & & (\gamma_5 f_{25}f_{02})^*(N) \\
(f_{14}f_{01})^*(\gamma_4^*(M)) \arrow[r, equal] \arrow[d, "(f_{14}f_{01})^*(\gamma^{2N})"] \arrow[uuur, "c_{\gamma_4,f_{14}f_{01}}(M)"'] & (f_{24}f_{02})^*(\gamma_4^*(M)) \arrow[uu, "c_{\gamma_4,f_{24}f_{02}}(M)"] \arrow[d, "(f_{24}f_{02})^*(\gamma^{2N})"] & \\
(f_{14}f_{01})^*(\gamma_5^*(N)^{[f_{24},f_{25}]}) \arrow[r, equal] & (f_{24}f_{02})^*(\gamma_5^*(N)^{[f_{24},f_{25}]}) \arrow[r, "f_{02}^\sharp(\pi_{f_{24},f_{25}}(\gamma_5^*(N)))"] & (f_{25}f_{02})^*(\gamma_5^*(N)) \arrow[uu, "c_{\gamma_5,f_{25}f_{02}}(N)"']
\end{tikzcd}
$$

We see that the compositions of diagram (iii) and the compositions of diagram (iv) coincide, which implies the assertion. $\square$

1.5 Two-sided fibered representable pair

Proposition 1.5.1 *Let $p : \mathcal{F} \to \mathcal{E}$ be a normalized cloven fibered category and $f : X \to Y$, $g : X \to Z$ morphisms in $\mathcal{E}$.*

(1) Suppose that (f, g) is a right fibered representable pair. If a morphism $\varphi : M \to N$ of $\mathcal{F}_Y$ is an epimorphism and (f, g) is a left fibered representable pair with respect to M and N, then $\varphi_{[f,g]} : M_{[f,g]} \to N_{[f,g]}$ is an epimorphism in $\mathcal{F}_Z$.

(2) Suppose that (f, g) is a left fibered representable pair. If a morphism $\varphi : M \to N$ of $\mathcal{F}_Z$ is a monomorphism and (f, g) is a right fibered representable pair with respect to M and N, then $\varphi^{[f,g]} : M^{[f,g]} \to N^{[f,g]}$ is a monomorphism in $\mathcal{F}_Y$.

Proof. (1) The following diagram commutes by (1.3.4) and the naturality of $E_{f,g}(K)$.

$$\begin{tikzcd}
\mathcal{F}_Z(N_{[f,g]}, K) \arrow[d, "\varphi^{[f,g]*}"] & \mathcal{F}_X(f^*(N), g^*(K)) \arrow[l, "P_{f,g}(N)_K"'] \arrow[r, "E_{f,g}(K)_N"] \arrow[d, "f^*(\varphi)^*"] & \mathcal{F}_Y(N, K^{[f,g]}) \arrow[d, "\varphi^*"] \\
\mathcal{F}_Z(M_{[f,g]}, K) & \mathcal{F}_X(f^*(M), f^*(K)) \arrow[l, "P_{f,g}(M)_K"'] \arrow[r, "E_{f,g}(K)_M"] & \mathcal{F}_Y(M, K^{[f,g]})
\end{tikzcd}$$

Since $\varphi^* : \mathcal{F}_Y(N, K^{[f,g]}) \to \mathcal{F}_Y(M, K^{[f,g]})$ is injective by the assumption, it follows from the above diagram that $\varphi^{[f,g]*} : \mathcal{F}_Z(N_{[f,g]}, K) \to \mathcal{F}_Z(M_{[f,g]}, K)$ is also injective.

(2) The following diagrams commute by (1.4.4) and the naturality of $P_{f,g}(K)$.

$$\begin{tikzcd}
\mathcal{F}_Y(K, M^{[f,g]}) \arrow[d, "\varphi_*^{[f,g]}"] & \mathcal{F}_X(f^*(K), g^*(M)) \arrow[l, "E_{f,g}(M)_K"'] \arrow[r, "P_{f,g}(K)_M"] \arrow[d, "g^*(\varphi)_*"] & \mathcal{F}_Z(K_{[f,g]}, M) \arrow[d, "\varphi_*"] \\
\mathcal{F}_Y(K, N^{[f,g]}) & \mathcal{F}_{f,g}(f^*(K), g^*(N)) \arrow[l, "E_{f,g}(N)_K"'] \arrow[r, "P_{f,g}(K)_N"] & \mathcal{F}_Z(K_{[f,g]}, N)
\end{tikzcd}$$

Since $\varphi_* : \mathcal{F}_1(K_{[f,g]}, M) \to \mathcal{F}_1(K_{[f,g]}, N)$ is injective by the assumption, it follows from the above diagram that $\varphi^{[f,g]} : \mathcal{F}_1(K, M^{[f,g]}) \to \mathcal{F}_1(K, N^{[f,g]})$ is also injective. □

Proposition 1.5.2 *Let $p : \mathcal{F} \to \mathcal{T}$ be a normalized cloven fibered category and $f : X \to Y$, $g : X \to Z$ morphisms in $\mathcal{E}$.*

(1) Suppose that (f, g) is a right fibered representable pair and that (f, g) is a left fibered representable pair with respect to objects L, M, N of $\mathcal{F}_Y$. If $\lambda : N \to L$ is a coequalizer of morphisms $\varphi, \psi : M \to N$ of $\mathcal{F}_Y$, then $\lambda_{[f,g]} : N_{[f,g]} \to L_{[f,g]}$ is a coequalizer of morphisms $\varphi_{[f,g]}, \psi_{[f,g]} : M_{[f,g]} \to N_{[f,g]}$.

(2) Suppose that (f, g) is a left fibered representable pair and that (f, g) is a right fibered representable pair with respect to objects L, M, N of $\mathcal{F}_Z$. If $\lambda : L \to M$ is an equalizer of morphisms $\varphi, \psi : M \to N$ of $\mathcal{F}_Z$, then $\lambda^{[f,g]} : L^{[f,g]} \to M^{[f,g]}$ is an equalizer of morphisms $\varphi^{[f,g]}, \psi^{[f,g]} : M^{[f,g]} \to N^{[f,g]}$.

Proof. (1) The following diagrams commute for $\xi = \varphi, \psi$ by (1.3.4) and the naturality of $E_{f,g}(K)$.

$$\begin{tikzcd}
\mathcal{F}_Z(L_{[f,g]}, K) \arrow[d, "(\lambda_{[f,g]})^*"] & \mathcal{F}_X(f^*(L), g^*(K)) \arrow[l, "P_{f,g}(L)_K"'] \arrow[r, "E_{f,g}(K)_L"] \arrow[d, "f^*(\lambda)^*"] & \mathcal{F}_Y(L, K^{[f,g]}) \arrow[d, "\lambda^*"] \\
\mathcal{F}_Z(N_{[f,g]}, K) \arrow[d, "(\xi_{[f,g]})^*"] & \mathcal{F}_X(f^*(N), g^*(K)) \arrow[l, "P_{f,g}(N)_K"'] \arrow[r, "E_{f,g}(K)_N"] \arrow[d, "f^*(\xi)^*"] & \mathcal{F}_Y(N, K^{[f,g]}) \arrow[d, "\xi^*"] \\
\mathcal{F}_Z(M_{[f,g]}, K) & \mathcal{F}_X(f^*(M), g^*(K)) \arrow[l, "P_{f,g}(M)_K"'] \arrow[r, "E_{f,g}(K)_M"] & \mathcal{F}_Y(M, K^{[f,g]})
\end{tikzcd}$$

Since $\lambda^* : \mathcal{F}_Y(L, K^{[f,g]}) \to \mathcal{F}_Y(N, K^{[f,g]})$ is an equalizer of maps $\varphi^*, \psi^* : \mathcal{F}_Y(N, K^{[f,g]}) \to \mathcal{F}_Y(M, K^{[f,g]})$, it follows from the above diagrams that $(\lambda_{[f,g]})^* : \mathcal{F}_Z(L_{[f,g]}, K) \to \mathcal{F}_Z(N_{[f,g]}, K)$ is an equalizer of maps $(\varphi_{[f,g]})^*, (\psi_{[f,g]})^* : \mathcal{F}_Z(N_{[f,g]}, K) \to \mathcal{F}_Z(M_{[f,g]}, K)$.

(2) The following diagrams commute for $\xi = \varphi, \psi$ by (1.4.4) and the naturality of $P_{f,g}(K)$.

$$\begin{array}{ccccc}
\mathcal{F}_Y(K, L^{[f,g]}) & \xleftarrow{E_{f,g}(L)_K} & \mathcal{F}_X(f^*(K), g^*(L)) & \xrightarrow{P_{f,g}(K)_L} & \mathcal{F}_Z(K_{[f,g]}, L) \\
\downarrow{\scriptstyle \lambda_*^{[f,g]}} & & \downarrow{\scriptstyle g^*(\lambda)_*} & & \downarrow{\scriptstyle \lambda_*} \\
\mathcal{F}_Y(K, M^{[f,g]}) & \xleftarrow{E_{f,g}(M)_K} & \mathcal{F}_X(f^*(K), g^*(M)) & \xrightarrow{P_{f,g}(K)_M} & \mathcal{F}_Z(K_{[f,g]}, M) \\
\downarrow{\scriptstyle \xi_*^{[f,g]}} & & \downarrow{\scriptstyle g^*(\xi)_*} & & \downarrow{\scriptstyle \xi_*} \\
\mathcal{F}_Y(K, N^{[f,g]}) & \xleftarrow{E_{f,g}(N)_K} & \mathcal{F}_X(f^*(K), g^*(N)) & \xrightarrow{P_{f,g}(K)_N} & \mathcal{F}_Z(K_{[f,g]}, N)
\end{array}$$

Since $\lambda_* : \mathcal{F}_Z(K_{[f,g]}, L) \to \mathcal{F}_Z(K_{[f,g]}, M)$ is an equalizer of maps $\varphi_*, \psi_* : \mathcal{F}_Z(K_{[f,g]}, M) \to \mathcal{F}_Z(K_{[f,g]}, N)$, it follows from the above diagrams that $\lambda_* : \mathcal{F}_Y(K, L^{[f,g]}) \to \mathcal{F}_Y(K, M^{[f,g]})$ is an equalizer of maps $\varphi_*^{[f,g]}, \psi_*^{[f,g]} : \mathcal{F}_Y(K, M^{[f,g]}) \to \mathcal{F}_Y(K, N^{[f,g]})$. □

Proposition 1.5.3 *For a functor $D : \mathcal{P} \to \mathcal{E}$, we put $D(\tau_{01}) = j$, $D(\tau_{02}) = k$, $D(\tau_{13}) = f$, $D(\tau_{14}) = g$, $D(\tau_{24}) = h$, $D(\tau_{25}) = i$. For objects M of $\mathcal{F}_{D(3)}$ and N of $\mathcal{F}_{D(5)}$, we assume the following.*
(i) (f, g) and (fj, ik) are left fibered representable pairs with respect to M.
(ii) (h, i) and (fj, ik) are right fibered representable pairs with respect to N.
(iii) (f, g) is a right fibered representable pair with respect to $N^{[h,i]}$.
(iv) (h, i) is a left fibered representable pair with respect to $M_{[f,g]}$.
Then, the following diagram is commutative.

$$\begin{array}{ccccc}
\mathcal{F}_{D(5)}((M_{[f,g]})_{[h,i]}, N) & \xrightarrow{\theta_D(M)^*} & \mathcal{F}_{D(5)}(M_{[fj,ik]}, N) & \xrightarrow{P_{fj,ik}(M)_N^{-1}} & \mathcal{F}_{D(0)}((fj)^*(M), (ik)^*(N)) \\
\downarrow{\scriptstyle P_{h,i}(M_{[f,g]})_N^{-1}} & & & & \downarrow{\scriptstyle E_{fj,ik}(N)_M} \\
\mathcal{F}_{D(2)}(h^*(M_{[f,g]}), i^*(N)) & & & & \mathcal{F}_{D(3)}(M, N^{[fj,ik]}) \\
\downarrow{\scriptstyle E_{h,i}(N)_{M_{[f,g]}}} & & & & \uparrow{\scriptstyle \theta^D(N)_*} \\
\mathcal{F}_{D(4)}(M_{[f,g]}, N^{[h,i]}) & \xrightarrow{P_{f,g}(M)_{N^{[h,i]}}^{-1}} & \mathcal{F}_{D(1)}(f^*(M), g^*(N^{[h,i]})) & \xrightarrow{E_{f,g}(N^{[h,i]})_M} & \mathcal{F}_{D(3)}(M, (N^{[h,i]})^{[f,g]})
\end{array}$$

Proof. For $\varphi \in \mathcal{F}_{D(5)}((M_{[f,g]})_{[h,i]}, N)$, we put $\psi = E_{h,i}(N)_{M_{[f,g]}} P_{h,i}(M_{[f,g]})_N^{-1}(\varphi) : M_{[f,g]} \to N^{[h,i]}$ and $\xi = E_{f,g}(N^{[h,i]})_M P_{f,g}(M)_{N^{[h,i]}}^{-1}(\psi) : M \to (N^{[h,i]})^{[f,g]}$. It follows from (1.3.2) and (1.4.3) that the following diagrams commute.

$$\begin{array}{ccc}
f^*(M) & \xrightarrow{\iota_{f,g}(M)} & g^*(M_{[f,g]}) \\
\downarrow{\scriptstyle f^*(\xi)} & & \downarrow{\scriptstyle g^*(\psi)} \\
f^*((N^{[h,i]})^{[f,g]}) & \xrightarrow{\pi_{f,g}(N^{[h,i]})} & g^*(N^{[h,i]})
\end{array}
\qquad
\begin{array}{ccc}
h^*(M_{[f,g]}) & \xrightarrow{\iota_{h,i}(M_{[f,g]})} & i^*((M_{[f,g]})_{[h,i]}) \\
\downarrow{\scriptstyle h^*(\psi)} & & \downarrow{\scriptstyle i^*(\varphi)} \\
h^*(N^{[h,i]}) & \xrightarrow{\pi_{h,i}(N)} & i^*(N)
\end{array}$$

By applying $j^\sharp$ to the above left diagram and $k^\sharp$ to the right one, we have the following commutative diagram by (1.1.15).

$$\begin{array}{ccccccc}
(fj)^*(M) & \xrightarrow{j^\sharp(\iota_{f,g}(M))} & (gj)^*(M_{[f,g]}) & = & (hk)^*(M_{[f,g]}) & \xrightarrow{k^\sharp(\iota_{h,i}(M_{[f,g]}))} & (ik)^*((M_{[f,g]})_{[h,i]}) \\
\downarrow{\scriptstyle (fj)^*(\xi)} & & \downarrow{\scriptstyle (gj)^*(\psi)} & & \downarrow{\scriptstyle (hk)^*(\psi)} & & \downarrow{\scriptstyle (ik)^*(\varphi)} \\
(fj)^*((N^{[h,i]})^{[f,g]}) & \xrightarrow{j^\sharp(\pi_{f,g}(N^{[h,i]}))} & (gj)^*(N^{[h,i]}) & = & (hk)^*(N^{[h,i]}) & \xrightarrow{k^\sharp(\pi_{h,i}(N))} & (ik)^*(N)
\end{array}$$

Hence, by (1.3.27) and (1.4.27), the following diagram commutes.

$$\begin{array}{ccc}
(fj)^*(M) & \xrightarrow{(ik)^*(\theta_D(M))\iota_{fj,ik}(M)} & (ik)^*((M_{[f,g]})_{[h,i]}) \\
\downarrow{\scriptstyle (fj)^*(\xi)} & & \downarrow{\scriptstyle (ik)^*(\varphi)} \\
(fj)^*((N^{[h,i]})^{[f,g]}) & \xrightarrow{\pi_{fj,ik}(N)(fj)^*(\theta^D(N))} & (ik)^*(N)
\end{array}$$

By (1.3.2) and (1.4.3), we have

$$P_{fj,ik}(M)_N((ik)^*(\varphi)(ik)^*(\theta_D(M))\iota_{fj,ik}(M)) = P_{fj,ik}(M)_N((ik)^*(\varphi\,\theta_D(M))\iota_{fj,ik}(M)) = \varphi\,\theta_D(N)$$
$$E_{fj,ik}(N)_M(\pi_{fj,ik}(N)(fj)^*(\theta^D(N))(fj)^*(\xi)) = E_{fj,ik}(N)_M(\pi_{fj,ik}(N)(fj)^*(\theta^D(N)\,\xi)) = \theta^D(N)\,\xi.$$

This shows that $P_{fj,ik}(M)_N^{-1}(\varphi\,\theta_D(N)) = E_{fj,ik}(N)_M^{-1}(\theta^D(N)\,\xi)$, which implies the result. $\square$

Definition 1.5.4 *We say that (f,g) is a two-sided fibered representable pair if (f,g) is a left and right fibered representable pair.*

Remark 1.5.5 *If (f,g), (h,i) and (fj,ik) are two-sided fibered representable pairs, (1.5.3) implies that $\theta_D(M) : M_{[fj,ik]} \to (M_{[f,g]})_{[h,i]}$ is an isomorphism for all object M of $\mathcal{F}_{D(3)}$ if and only if $\theta^D(N) : (N^{[h,i]})^{[f,g]} \to N^{[fj,ik]}$ is an isomorphism for all object N of $\mathcal{F}_{D(5)}$.*

2 Examples of fibered categories

2.1 Fibered category of affine modules

Let K_* be a graded commutative algebra. We denote by $\mathcal{A}lg_{K_*}$ the category of graded K_*-algebras and homomorphisms between them. We also denote by $\mathcal{M}od_{K_*}$ the category of graded left K_*-modules and homomorphisms which preserve degrees. For an object R_* of $\mathcal{A}lg_{K_*}$, we denote by $\eta_{R_*} : K_* \to R_*$ the unit of R_* and by $\mu_{R_*} : R_* \otimes_{K_*} R_* \to R_*$ is the map induced by the product of R_*.

Let $\mathcal{C}$ be a subcategory of $\mathcal{A}lg_{K_*}$ and $\mathcal{M}$ a subcategory of $\mathcal{M}od_{K_*}$.

Condition 2.1.1 *We assume $\mathcal{M}$ satisfies the following conditions.*

(∗) *If a morphism $S_* \to R_*$ of $\mathcal{C}$ and a right S_*-module structure on $M_* \in \mathrm{Ob}\,\mathcal{M}$ are given, then $M_* \otimes_{S_*} R_*$ is an object of $\mathcal{M}$.*

Definition 2.1.2 *We define a category $\mathcal{M}od(\mathcal{C},\mathcal{M})$ as follows.* $\mathrm{Ob}\,\mathcal{M}od(\mathcal{C},\mathcal{M})$ *consists of triples (R_*, M_*, α) where $R_* \in \mathrm{Ob}\,\mathcal{C}$, $M_* \in \mathrm{Ob}\,\mathcal{M}$ and $\alpha : M_* \otimes_{K_*} R_* \to M_*$ is a right R_*-module structure of M_*. A morphism from (R_*, M_*, α) to (S_*, N_*, β) is a pair (λ, φ) of morphisms $\lambda \in \mathcal{C}(R_*, S_*)$ and $\varphi \in \mathcal{M}(M_*, N_*)$ such that the following diagram commutes.*

$$\begin{tikzcd} M_* \otimes_{K_*} R_* \arrow[r, "\alpha"] \arrow[d, "\varphi \otimes_{K_*} \lambda"] & M_* \arrow[d, "\varphi"] \\ N_* \otimes_{K_*} R_* \arrow[r, "\beta"] & N_* \end{tikzcd}$$

Composition of $(\lambda, \varphi) : (R_*, M_*, \alpha) \to (S_*, N_*, \beta)$ and $(\nu, \psi) : (S_*, N_*, \beta) \to (T_*, L_*, \gamma)$ is defined to be $(\nu\lambda, \psi\varphi)$.

Define functors $p_{\mathcal{C}} : \mathcal{M}od(\mathcal{C},\mathcal{M}) \to \mathcal{C}$ and $p_{\mathcal{M}} : \mathcal{M}od(\mathcal{C},\mathcal{M}) \to \mathcal{M}$ by $p_{\mathcal{C}}(R_*, M_*, \alpha) = R_*$, $p_{\mathcal{C}}(\lambda, \varphi) = \lambda$ and $p_{\mathcal{M}}(R_*, M_*, \alpha) = M_*$, $p_{\mathcal{M}}(\lambda, \varphi) = \varphi$.

For $R_* \in \mathrm{Ob}\,\mathcal{C}$, we denote by $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$ a subcategory of $\mathcal{M}od(\mathcal{C},\mathcal{M})$ consisting of objects which map to R_* by $p_{\mathcal{C}}$ and morphisms which map the identity morphism of R_* by $p_{\mathcal{C}}$. Hence $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$ is a subcategory of the category of right R_*-modules.

Proposition 2.1.3 *If $\mathcal{C}$ and $\mathcal{M}$ are complete, so is $\mathcal{M}od(\mathcal{C},\mathcal{M})$.*

Proof. For a functor $D : \mathcal{I} \to \mathcal{M}od(\mathcal{C},\mathcal{M})$, we assume that limits of $p_{\mathcal{C}}D : \mathcal{I} \to \mathcal{C}$ and $p_{\mathcal{M}}D : \mathcal{I} \to \mathcal{M}$ exist. Let $\left(A_* \xrightarrow{\rho_i} p_{\mathcal{C}}D(i)\right)_{i \in \mathrm{Ob}\,\mathcal{I}}$ be a limiting cone of $p_{\mathcal{C}}D : \mathcal{I} \to \mathcal{C}$ and $\left(L_* \xrightarrow{\pi_i} p_{\mathcal{M}}D(i)\right)_{i \in \mathrm{Ob}\,\mathcal{I}}$ a limiting cone of $p_{\mathcal{M}}D : \mathcal{I} \to \mathcal{M}$. For $i \in \mathrm{Ob}\,\mathcal{I}$ and $(\tau : i \to j) \in \mathrm{Mor}\,\mathcal{I}$, we put $D(i) = (R_{i*}, M_{i*}, \alpha_i)$ and $D(\tau) = (\lambda_\tau, \varphi_\tau)$. Since the following diagram commutes for any $(\tau : i \to j) \in \mathrm{Mor}\,\mathcal{I}$, there exists unique morphism $\lambda : L_* \otimes_{K_*} A_* \to L_*$ satisfying $\pi_i\lambda = \alpha_i(\pi_i \otimes_{K_*} \rho_i)$ for any $i \in \mathrm{Ob}\,\mathcal{I}$.

$$\begin{tikzcd} L_* \otimes_{K_*} A_* \arrow[r, "\pi_i \otimes_{K_*} \rho_i"] \arrow[dr, "\pi_j \otimes_{K_*} \rho_j"] & M_{i*} \otimes_{K_*} R_{i*} \arrow[r, "\alpha_i"] \arrow[d, "\varphi_\tau \otimes_{K_*} \lambda_\tau"] & M_{i*} \arrow[d, "\varphi_\tau"] \\ & M_{j*} \otimes_{K_*} R_{j*} \arrow[r, "\alpha_j"] & M_j \end{tikzcd}$$

It can be verified that (A_*, L_*, λ) is an object of $\mathcal{M}od(\mathcal{C},\mathcal{M})$ and that $\left((A_*, L_*, \lambda) \xrightarrow{(\rho_i, \pi_i)} D(i)\right)_{i \in \mathrm{Ob}\,\mathcal{I}}$ is a limiting cone of D. □

Proposition 2.1.4 *$p_{\mathcal{C}}^{op} : \mathcal{M}od(\mathcal{C},\mathcal{M})^{op} \to \mathcal{C}^{op}$ is a fibered category.*

Proof. For a morphism $\lambda : S_* \to R_*$ of $\mathcal{C}$ and $\boldsymbol{N} = (S_*, N_*, \beta) \in \mathrm{Ob}\,\mathcal{M}od(\mathcal{C},\mathcal{M})$, let $i_\lambda(\boldsymbol{N}) : N_* \to N_* \otimes_{S_*} R_*$ be a map defined by $i_\lambda(\boldsymbol{N})(x) = x \otimes 1$ and $\beta_\lambda : (N_* \otimes_{S_*} R_*) \otimes_{K_*} R_* \to R_* \otimes_{S_*} N_*$ the following composition.

$$(N_* \otimes_{S_*} R_*) \otimes_{K_*} R_* \xrightarrow{\cong} N_* \otimes_{S_*} (R_* \otimes_{K_*} R_*) \xrightarrow{id_{N_*} \otimes_{S_*} \mu_{R_*}} N_* \otimes_{S_*} R_*$$

Since the following diagram commutes, $(\lambda, i_\lambda(\boldsymbol{N})) : (S_*, N_*, \beta) \to (R_*, N_* \otimes_{S_*} R_*, \beta_\lambda)$ is a morphism in $\mathcal{M}od(\mathcal{C},\mathcal{M})$.

$$\begin{tikzcd}
N_*\otimes_{K_*} S_* \arrow[r, "\beta"] \arrow[d, "i_\lambda(\boldsymbol{N})\otimes_{K_*}\lambda"] & N_* \arrow[d, "i_\lambda(\boldsymbol{N})"] \\
(N_*\otimes_{S_*} R_*)\otimes_{K_*} R_* \arrow[r, "\beta_\lambda"] & N_*\otimes_{S_*} R_*
\end{tikzcd}$$

A map $(\lambda, i_\lambda(\boldsymbol{N}))_* : \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{R_*}((R_*, M_*, \alpha), (R_*, N_*\otimes_{S_*} R_*, \beta_\lambda)) \to \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{\lambda}((R_*, M_*, \alpha), (S_*, N_*, \beta))$ given by $(\lambda, i_\lambda(\boldsymbol{N}))_*((id_{R_*}, \varphi)) = (\lambda, \varphi i_\lambda(\boldsymbol{N}))$ is bijective. In fact, if $(\lambda, \psi) : (S_*, N_*, \beta) \to (R_*, M_*, \alpha)$ is an element of $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{\lambda}((R_*, M_*, \alpha), (S_*, N_*, \beta))$, since $\psi\beta = \alpha(\psi\otimes_{K_*}\lambda) : N_*\otimes_{K_*} S_* \to M_*$, we have

$$\begin{aligned}\alpha(\psi\otimes_{K_*} id_{R_*})(z\otimes\lambda(y)x) &= \alpha(\psi(z)\otimes\lambda(y)x) = \alpha(\alpha(\psi(z)\otimes\lambda(y))\otimes x)\\ &= \alpha(\psi\beta(y\otimes z)\otimes x) = \alpha(\psi\otimes_{K_*} id_{R_*})(\beta(z\otimes y)\otimes x)\end{aligned}$$

for $x\in R_*$, $y\in S_*$ and $z\in N_*$. Hence there exists unique morphism $\tilde{\psi} : N_*\otimes_{S_*} R_* \to M_*$ that makes the following diagram commute. Here, $\otimes_\lambda : N_*\otimes_{K_*} R_* \to N_*\otimes_{S_*} R_*$ denotes the quotient map.

$$\begin{tikzcd}
N_*\otimes_{K_*} R_* \arrow[r, "\psi\otimes_{K_*} id_{R_*}"] \arrow[d, "\otimes_\lambda"] & M_*\otimes_{K_*} R_* \arrow[d, "\alpha"] \\
N_*\otimes_{S_*} R_* \arrow[r, "\tilde{\psi}"] & M_*
\end{tikzcd}$$

Then, a correspondence $(\lambda, \psi) \mapsto (id_{R_*}, \tilde{\psi})$ gives the inverse of $(\lambda, i_\lambda(\boldsymbol{N}))_*$. In fact, since

$$\begin{tikzcd}
N_*\otimes_{K_*} R_* \arrow[r, "i_\lambda(\boldsymbol{N})\otimes_{K_*} id_{R_*}"] \arrow[dr, "\otimes_\lambda"'] & N_*\otimes_{K_*} R_*\otimes_{S_*} R_* \arrow[r, "\varphi\otimes_{K_*} id_{R_*}"] \arrow[d, "\beta_\lambda"] & M_*\otimes_{K_*} R_* \arrow[d, "\alpha"] \\
 & N_*\otimes_{S_*} R_* \arrow[r, "\varphi"] & M_*
\end{tikzcd}$$

commutes for $(id_{R_*}, \varphi) \in \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{R_*}((R_*, M_*, \alpha), (R_*, N_*\otimes_{S_*} R_*, \beta_\lambda))$, the correspondence $(\lambda, \psi) \mapsto (id_{R_*}, \tilde{\psi})$ is a left inverse of $(\lambda, i_\lambda(\boldsymbol{N}))_*$. For $(\lambda, \psi) \in \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{\lambda}((R_*, M_*, \alpha), (S_*, N_*, \beta))$ and $x\in N_*$, since

$$\tilde{\psi} i_\lambda(\boldsymbol{N})(x) = \tilde{\psi}(x\otimes_{S_*} 1) = \tilde{\psi}\otimes_\lambda(x\otimes_{K_*} 1) = \alpha(\psi\otimes_{K_*} id_{R_*})(x\otimes_{K_*} 1) = \psi(x),$$

it follows that the correspondence $(\lambda, \psi) \mapsto (id_{R_*}, \tilde{\psi})$ is a right inverse of $(\lambda, i_\lambda(\boldsymbol{N}))_*$. Thus $(\lambda, i_\lambda(\boldsymbol{N}))$ is a cartesian morphism and $p^{op}_{\mathcal{C}} : \mathcal{M}od(\mathcal{C},\mathcal{M})^{op} \to \mathcal{C}^{op}$ is a prefibered category. We set $\lambda^*(\boldsymbol{N}) = (R_*, N_*\otimes_{S_*} R_*, \beta_\lambda)$ and $\boldsymbol{\alpha}_\lambda(\boldsymbol{N}) = (\lambda, i_\lambda(\boldsymbol{N})) : \lambda^*(\boldsymbol{N}) \to \boldsymbol{N}$ in $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}$.

For morphisms $\lambda : S_* \to R_*$, $\nu : T_* \to S_*$ of $\mathcal{C}$ and $\boldsymbol{L} = (T_*, L_*, \gamma) \in \operatorname{Ob}\mathcal{M}od(\mathcal{C},\mathcal{M})$, there is an isomorphism $c_{\nu,\lambda}(\boldsymbol{N}) : L_*\otimes_{T_*} R_* \to (L_*\otimes_{T_*} S_*)\otimes_{S_*} R_*$ given by $c_{\nu,\lambda}(\boldsymbol{N})(w\otimes x) = w\otimes 1\otimes x$. We put $\boldsymbol{c}_{\nu,\lambda}(\boldsymbol{N}) = (id_{R_*}, c_{\nu,\lambda}(\boldsymbol{N}))$. Then, $\boldsymbol{c}_{\nu,\lambda}(\boldsymbol{N}) : \lambda^*\nu^*(\boldsymbol{N}) \to (\lambda\nu)^*(\boldsymbol{N})$ is an isomorphism in $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{R_*}$ and the following diagram commutes.

$$\begin{tikzcd}
\lambda^*\nu^*(\boldsymbol{N}) \arrow[r, "\boldsymbol{\alpha}_\lambda(\nu^*(\boldsymbol{N}))"] \arrow[d, "\boldsymbol{c}_{\nu,\lambda}(\boldsymbol{N})"] & \nu^*(\boldsymbol{N}) \arrow[d, "\boldsymbol{\alpha}_\nu(\boldsymbol{N})"] \\
(\lambda\nu)^*(\boldsymbol{N}) \arrow[r, "\boldsymbol{\alpha}_{\lambda\nu}(\boldsymbol{N})"] & \boldsymbol{N}
\end{tikzcd}$$

Therefore $p^{op}_{\mathcal{C}} : \mathcal{M}od(\mathcal{C},\mathcal{M})^{op} \to \mathcal{C}^{op}$ is a fibered category. □

Proposition 2.1.5 *For a morphism $\lambda : S_* \to R_*$ of $\mathcal{C}$, $\lambda^* : \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{S_*} \to \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{R_*}$ has a left adjoint.*

Proof. Define a functor $\lambda_* : \mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*} \to \mathcal{M}od(\mathcal{C},\mathcal{M})_{S_*}$ as follows. For $(R_*, M_*, \alpha) \in \operatorname{Ob}\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$, set $\lambda_*(R_*, M_*, \alpha) = (S_*, M_*, \alpha(id_{M_*}\otimes_{K_*}\lambda))$. For $(id_{R_*}, \psi) \in \mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}((R_*, L_*, \gamma), (R_*, M_*, \alpha))$, we set $\lambda_*(id_{R_*}, \psi) = (id_{S_*}, \psi)$. It is clear that $(id_{S_*}, \varphi) \in \mathcal{M}od(\mathcal{C},\mathcal{M})_{S_*}((S_*, N_*, \beta), \lambda_*(R_*, M_*, \alpha))$ if and only if $(\lambda, \varphi) \in \mathcal{M}od(\mathcal{C},\mathcal{M})_\lambda((S_*, N_*, \beta), (R_*, M_*, \alpha))$. It follows from the proof of (2.1.4) that we have a natural bijection $(\lambda, i_\lambda(\boldsymbol{N}))^* : \mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}(\lambda^*(S_*, N_*, \beta), (R_*, M_*, \alpha)) \to \mathcal{M}od(\mathcal{C},\mathcal{M})_\lambda((S_*, N_*, \beta), (R_*, M_*, \alpha))$. Thus a correspondence $(id_{R_*}, \varphi) \mapsto (id_{S_*}, \varphi\, i_\lambda(\boldsymbol{N}))$ gives a bijection

$$\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}(\lambda^*(S_*, N_*, \beta), (R_*, M_*, \alpha)) \to \mathcal{M}od(\mathcal{C},\mathcal{M})_{S_*}((S_*, N_*, \beta), \lambda_*(R_*, M_*, \alpha))$$

which is natural. Hence λ_* is a right adjoint of $\lambda^* : \mathcal{M}od(\mathcal{C},\mathcal{M})_{S_*} \to \mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$. □

Remark 2.1.6 *Let $\lambda : S_* \to R_*$ be a morphism in $\mathcal{C}$.*

(1) *The unit $\varepsilon(\lambda) : id_{\mathcal{M}od(\mathcal{C},\mathcal{M})_{S_*}} \to \lambda_*\lambda^*$ is given as follows. For an object $\boldsymbol{N} = (S_*, N_*, \beta)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{S_*}$, $\varepsilon(\lambda)_{\boldsymbol{N}} : \boldsymbol{N} \to \lambda_*\lambda^*(\boldsymbol{N})$ is defined to be*

$$(id_{S_*}, i_\lambda(\boldsymbol{N})) : (S_*, N_*, \beta) \to (S_*, N_* \otimes_{S_*} R_*, \beta_\lambda(id_{N_*\otimes_{S_*}R_*} \otimes_{K_*} \lambda))$$

(2) *The counit $\eta(\lambda) : \lambda^*\lambda_* \to id_{\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}}$ is given as follows. For an object $\boldsymbol{M} = (R_*, M_*, \alpha)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$, we put $\alpha' = \alpha(id_{M_*} \otimes_{K_*} \lambda)$. Then, we have $\lambda^*(\lambda_*(\boldsymbol{M})) = (R_*, M_* \otimes_{S_*} R_*, \alpha'_\lambda)$. Let us denote by $\bar{\alpha} : M_* \otimes_{R_*} R_* \to M_*$ the isomorphism induced by α. $\eta(\lambda)_{\boldsymbol{M}} : \lambda^*(\lambda_*(\boldsymbol{M})) \to \boldsymbol{M}$ is defined to be*

$$(id_{R_*}, \bar{\alpha}\otimes_\lambda) : (R_*, M_* \otimes_{S_*} R_*, \alpha'_\lambda) \to (R_*, M_*, \alpha).$$

We assume that K_* is an object of $\mathcal{C}$ in the following proposition. Then, K_* is an initial object of $\mathcal{C}$.

Proposition 2.1.7 *Let $\boldsymbol{M} = (K_*, M_*, \alpha)$ be an object of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{K_*}$.*

(1) The cartesian section $s_{\boldsymbol{M}} : \mathcal{C}^{op} \to \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}$ of $p_{\mathcal{C}}^{op} : \mathcal{M}od(\mathcal{C},\mathcal{M})^{op} \to \mathcal{C}^{op}$ associated with $\boldsymbol{M}$ is given as follows. Put $s_{\boldsymbol{M}}(R_*) = \eta_{R_*}^*(\boldsymbol{M}) = (R_*, M_* \otimes_{K_*} R_*, \alpha_{\eta_{R_*}})$ for $R_* \in \mathrm{Ob}\,\mathcal{C}$. For a morphism $\lambda : S_* \to R_*$ of $\mathcal{C}^{op}$, $s_{\boldsymbol{M}}(\lambda) \in \mathcal{M}od(\mathcal{C},\mathcal{M})_\lambda^{op}(s_{\boldsymbol{M}}(S_*), s_{\boldsymbol{M}}(R_*))$ is defined by

$$s_{\boldsymbol{M}}(\lambda) = (\lambda, id_{M_*} \otimes_{K_*} \lambda) : (S_*, M_* \otimes_{K_*} S_*,\ \alpha_{\eta_{S_*}}) \to (R_*, M_* \otimes_{K_*} R_*,\ \alpha_{\eta_{R_*}}).$$

(2) For a morphism $\lambda : S_* \to R_*$ of $\mathcal{C}^{op}$, Then, the morphism

$$(s_{\boldsymbol{M}})_\lambda : s_{\boldsymbol{M}}(S_*) = (S_*, M_* \otimes_{K_*} S_*,\ \alpha_{\eta_{S_*}}) \to (S_*, (M_* \otimes_{K_*} R_*) \otimes_{R_*} S_*,\ (\alpha_{\eta_{R_*}})_\lambda) = \lambda^*(s_{\boldsymbol{M}}(R_*))$$

of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{S_*}^{op}$ coincides with $(id_{S_*}, c_{\eta_{R_*},\lambda}(\boldsymbol{M})^{-1})$. Here, $c_{\eta_{R_*},\lambda}(\boldsymbol{M})^{-1} : (M_* \otimes_{K_*} R_*) \otimes_{R_*} S_* \to M_* \otimes_{K_*} S_*$ is given by $c_{\eta_{R_*},\lambda}(\boldsymbol{M})^{-1}(x \otimes r \otimes s) = x \otimes \lambda(r)s$.

(3) For morphisms $\lambda : S_* \to R_*$ and $\nu : S_* \to T_*$ of $\mathcal{C}^{op}$, the morphism $(s_{\boldsymbol{M}})_{\lambda,\nu} : \lambda^*(s_{\boldsymbol{M}}(R_*)) \to \nu^*(s_{\boldsymbol{M}}(T_*))$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{S_*}^{op}$ is given by $(id_{S_*}, c_{\eta_{T_*},\nu}(\boldsymbol{M})^{-1} c_{\eta_{R_*},\lambda}(\boldsymbol{M}))$.

Proof. The assertions follow from (1.1.22), (1.1.23) and the definition of $p_{\mathcal{C}}^{op} : \mathcal{M}od(\mathcal{C},\mathcal{M})^{op} \to \mathcal{C}^{op}$. □

Proposition 2.1.8 *Let $\lambda : R_* \to S_*$ and $\nu : T_* \to S_*$ be morphisms in $\mathcal{C}$.*

(1) *For an object $\boldsymbol{M} = (R_*, M_*, \alpha)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$, $\boldsymbol{M}_{[\lambda,\nu]}$ is given by*

$$\boldsymbol{M}_{[\lambda,\nu]} = \nu_*(\lambda^*(\boldsymbol{M})) = (T_*, M_* \otimes_{R_*} S_*, \alpha_\lambda(id_{M_*\otimes_{R_*}S_*} \otimes_{K_*} \nu)).$$

(2) *For an object $\boldsymbol{M} = (R_*, M_*, \alpha)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$, we define $i_{\lambda,\nu}(\boldsymbol{M}) : (M_* \otimes_{R_*} S_*) \otimes_{T_*} S_* \to M_* \otimes_{R_*} S_*$ by $i_{\lambda,\nu}(\boldsymbol{M})(x \otimes s \otimes t) = x \otimes st$. Then,*

$$\iota_{\lambda,\nu}(\boldsymbol{M}) : \nu^*(\boldsymbol{M}_{[\lambda,\nu]}) = (S_*, (M_* \otimes_{R_*} S_*) \otimes_{T_*} S_*, \beta_\nu) \to (S_*, M_* \otimes_{R_*} S_*, \alpha_\lambda) = \lambda^*(\boldsymbol{M})$$

is given by $\iota_{\lambda,\nu}(\boldsymbol{M}) = (id_{S_*}, i_{\lambda,\nu}(\boldsymbol{M}))$. Here we put $\beta = \alpha_\lambda(id_{M_*\otimes_{R_*}S_*} \otimes_{K_*} \nu) : (M_* \otimes_{R_*} S_*) \otimes_{K_*} T_* \to M_* \otimes_{R_*} S_*$.

(3) *For an object $\boldsymbol{M}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$ and an object $\boldsymbol{N}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{T_*}$,*

$$P_{\lambda,\nu}(\boldsymbol{M})_{\boldsymbol{N}} : \mathcal{M}od(\mathcal{C},\mathcal{M})_{S_*}(\nu^*(\boldsymbol{N}), \lambda^*(\boldsymbol{M})) \to \mathcal{M}od(\mathcal{C},\mathcal{M})_{T_*}(\boldsymbol{N}, \boldsymbol{M}_{[\lambda,\nu]})$$

maps (id_{S_*}, φ) to $(id_{T_*}, \varphi i_\nu(\boldsymbol{N}))$.

(4) *For a morphism $\boldsymbol{\varphi} = (id_{R_*}, \varphi) : \boldsymbol{M} \to \boldsymbol{N}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$, $\boldsymbol{\varphi}_{[\lambda,\nu]} : \boldsymbol{M}_{[\lambda,\nu]} \to \boldsymbol{N}_{[\lambda,\nu]}$ is given by $\nu_*(\lambda^*(\boldsymbol{\varphi})) = (id_{T_*}, \varphi \otimes_{R_*} id_{S_*})$.*

(5) *For a morphisms $\gamma : S_* \to A_*$ of $\mathcal{C}$,*

$$\boldsymbol{M}_\gamma : \boldsymbol{M}_{[\lambda,\nu]} = (T_*, M_* \otimes_{R_*} S_*, \alpha_\lambda(id_{M_*\otimes_{R_*}S_*} \otimes_{K_*} \nu)) \to (T_*, M_* \otimes_{R_*} A_*, \alpha_{\gamma\lambda}(id_{M_*\otimes_{R_*}A_*} \otimes_{K_*} \gamma\nu)) = \boldsymbol{M}_{[\gamma\lambda,\gamma\nu]}$$

is given by $\boldsymbol{M}_\gamma = (id_{T_*}, id_{M_*} \otimes_{R_*} \gamma)$.

Proof. (1) The assertion follows from (2.1.4), (2.1.5) and (1.3.3).

(2) Since $\iota_{\lambda,\nu}(\boldsymbol{M}) = (\eta_\nu)_{\lambda^*(\boldsymbol{M})}$ by (1.3.3), the assertion follows from and (2.1.6).

(3) The assertion follows from (1.3.3) and (2.1.5).

(4) This is a direct consequence of (1.3.5).

(5) The assertion can be verified from (1.3.8) and (2.1.6). □

Proposition 2.1.9 *For morphisms $\lambda : R_* \to S_*$, $\nu : T_* \to S_*$, $\gamma : A_* \to S_*$ of $\mathcal{C}$ and an object $\boldsymbol{M} = (R_*, M_*, \alpha)$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{R_*}$, define a map $\tilde{\delta}_{\lambda,\nu,\gamma,\boldsymbol{M}} : (M_* \otimes_{R_*} S_*) \otimes_{T_*} S_* \to M_* \otimes_{R_*} S_*$ by $\tilde{\delta}_{\lambda,\nu,\gamma,\boldsymbol{M}}(x \otimes s \otimes t) = x \otimes st$. Then, $\delta_{\lambda,\nu,\gamma,\boldsymbol{M}} : (\boldsymbol{M}_{[\lambda,\nu]})_{[\nu,\gamma]} \to \boldsymbol{M}_{[\lambda,\gamma]}$ is given by $\delta_{\lambda,\nu,\gamma,\boldsymbol{M}} = (id_{A_*}, \tilde{\delta}_{\lambda,\nu,\gamma,\boldsymbol{M}})$.*

Proof. First we note that it follows from (1) of (2.1.8) that $(\boldsymbol{M}_{[\lambda,\nu]})_{[\nu,\gamma]}$ is given as follows.

$$(\boldsymbol{M}_{[\lambda,\nu]})_{[\nu,\gamma]} = (T_*, M_* \otimes_{R_*} S_*, \tilde{\alpha})_{[\nu,\gamma]} = (A_*, (M_* \otimes_{R_*} S_*) \otimes_{T_*} S_*, \tilde{\alpha}_\nu(id_{(M_* \otimes_{R_*} S_*) \otimes_{T_*} S_*} \otimes_{K_*} \gamma))$$

Here we put $\tilde{\alpha} = \alpha_\lambda(id_{M_* \otimes_{R_*} S_*} \otimes_{K_*} \nu)$. Since $\delta_{\lambda,\nu,\gamma,\boldsymbol{M}} = \gamma_*\big(\eta(\nu)_{\lambda^*(\boldsymbol{M})}\big)$ by (1.3.12), the assertion follows from (2) of (2.1.6). □

Proposition 2.1.10 *For a functor $D : \mathcal{P} \to \mathcal{C}^{op}$, we put $D(i) = R_{i*}$ $(i = 0, 1, 2, 3, 4, 5)$, $D(\tau_{ij}) = \lambda_{ij}$ $((i, j) = (0, 1), (0, 2), (1, 3), (1, 4), (2, 4), (2, 5))$. For an object $\boldsymbol{M} = (R_{3*}, M_*, \alpha)$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{R_{3*}}$, we define*

$$\tilde{\theta}_D(\boldsymbol{M}) : (M_* \otimes_{R_{3*}} R_{1*}) \otimes_{R_{4*}} R_{2*} \to M_* \otimes_{R_{3*}} R_{0*}$$

by $\tilde{\theta}_D(\boldsymbol{M})(x \otimes s \otimes t) = x \otimes \lambda_{01}(s)\lambda_{02}(t)$. Then, $\theta_D(\boldsymbol{M}) : (\boldsymbol{M}_{[\lambda_{13},\lambda_{14}]})_{[\lambda_{24},\lambda_{25}]} \to \boldsymbol{M}_{[\lambda_{01}\lambda_{13},\lambda_{02}\lambda_{25}]}$ is given by $\theta_D(\boldsymbol{M}) = (id_{R_{5*}}, \tilde{\theta}_D(\boldsymbol{M}))$. Hence if $R_{0*} = R_{1*} \otimes_{R_{4*}} R_{2*}$ and $\lambda_{01} : R_{1*} \to R_{0*}$, $\lambda_{02} : R_{2*} \to R_{0*}$ are given by $\lambda_{01}(s) = s \otimes 1$, $\lambda_{02}(t) = 1 \otimes t$, then $\theta_D(\boldsymbol{M})$ is an isomorphism in $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{R_{5*}}$.

Proof. Put $\tilde{\alpha} = \alpha_{\lambda_{13}}(id_{M_* \otimes_{R_{3*}} R_{1*}} \otimes_{K_*} \lambda_{14})$ and $\hat{\alpha} = \alpha_{\lambda_{01}\lambda_{13}}(id_{M_* \otimes_{R_{3*}} R_{0*}} \otimes_{K_*} \lambda_{01}\lambda_{14})$. Then, we have the following equalities by (1) of (2.1.8).

$$\begin{aligned}
(\boldsymbol{M}_{[\lambda_{13},\lambda_{14}]})_{[\lambda_{24},\lambda_{25}]} &= (R_{5*}, (M_* \otimes_{R_{3*}} R_{1*}) \otimes_{R_{4*}} R_{2*}, \tilde{\alpha}_{\lambda_{24}}(id_{(M_* \otimes_{R_{3*}} R_{1*}) \otimes_{R_{4*}} R_{2*}} \otimes_{K_*} \lambda_{25})) \\
(\boldsymbol{M}_{[\lambda_{01}\lambda_{13},\lambda_{01}\lambda_{14}]})_{[\lambda_{02}\lambda_{24},\lambda_{02}\lambda_{25}]} &= (R_{5*}, (M_* \otimes_{R_{3*}} R_{0*}) \otimes_{R_{4*}} R_{0*}, \hat{\alpha}_{\lambda_{02}\lambda_{24}}(id_{(M_* \otimes_{R_{3*}} R_{1*}) \otimes_{R_{4*}} R_{2*}} \otimes_{K_*} \lambda_{25})) \\
\boldsymbol{M}_{[\lambda_{01}\lambda_{13},\lambda_{02}\lambda_{25}]} &= (R_{5*}, M_* \otimes_{R_{3*}} R_{0*}, \alpha_{\lambda_{01}\lambda_{13}}(id_{M_* \otimes_{R_{3*}} R_{0*}} \otimes_{K_*} \lambda_{02}\lambda_{25}))
\end{aligned}$$

Since $\theta_D(\boldsymbol{M})$ is defined to be a composition

$$(\boldsymbol{M}_{[\lambda_{13},\lambda_{14}]})_{[\lambda_{24},\lambda_{25}]} \xrightarrow{(\boldsymbol{M}_{\lambda_{01}})_{\lambda_{02}}} (\boldsymbol{M}_{[\lambda_{01}\lambda_{13},\lambda_{01}\lambda_{14}]})_{[\lambda_{02}\lambda_{24},\lambda_{02}\lambda_{25}]} \xrightarrow{\delta_{\lambda_{01}\lambda_{13},\lambda_{01}\lambda_{14},\lambda_{02}\lambda_{25},\boldsymbol{M}}} \boldsymbol{M}_{[\lambda_{01}\lambda_{13},\lambda_{02}\lambda_{25}]},$$

the assertion follows from (3) of (2.1.5) and (2.1.9). □

Remark 2.1.11 *For morphisms $\lambda : R_* \to S_*$, $\nu : T_* \to S_*$, $\kappa : T_* \to A_*$, $\rho : B_* \to A_*$ of $\mathcal{C}$, assume that maps $\iota_1 : S_* \to S_* \otimes_{T_*} A_*$ and $\iota_2 : A_* \to S_* \otimes_{T_*} A_*$ defined by $\iota_1(s) = s \otimes 1$, $\iota_2(a) = 1 \otimes a$ are morphisms in $\mathcal{C}$. Then, if we define $\tilde{\theta}_{\lambda,\nu,\kappa,\rho}(\boldsymbol{M}) : (M_* \otimes_{R_*} S_*) \otimes_{T_*} A_* \to M_* \otimes_{R_*} (S_* \otimes_{T_*} A_*)$ by $\tilde{\theta}_{\lambda,\nu,\kappa,\rho}(\boldsymbol{M}) = (x \otimes s) \otimes t = x \otimes (s \otimes t)$, $\theta_{\lambda,\nu,\kappa,\rho}(\boldsymbol{M}) = (id_{B_*}, \tilde{\theta}_{\lambda,\nu,\kappa,\rho}(\boldsymbol{M}))$ is an isomorphism in $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{A_*}$, namely $(\lambda, \nu, \kappa, \rho)$ is an associative left fibered representable quadruple.*

Proposition 2.1.12 *For functor $D, E : \mathcal{Q} \to \mathcal{C}^{op}$ and a natural transformation $\omega : D \to E$, we put $D(i) = R_{i*}$, $E(i) = S_{i*}$ $(i = 0, 1, 2)$, $D(\tau_{0i}) = \lambda_i$, $E(\tau_{0i}) = \nu_i$ $(i = 1, 2)$. For an object $\boldsymbol{M} = (S_{1*}, M_*, \alpha)$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{S_{1*}}$, define a map $\tilde{\omega}_{\boldsymbol{M}} : (M_* \otimes_{S_{1*}} S_{0*}) \otimes_{S_{2*}} R_{2*} \to (M_* \otimes_{S_{1*}} R_{1*}) \otimes_{R_{1*}} R_{0*}$ by $\tilde{\omega}_{\boldsymbol{M}}(x \otimes s \otimes r) = x \otimes 1 \otimes \omega_0(s)\lambda_2(r)$. Then, $\omega_{\boldsymbol{M}} : \omega_2^*(\boldsymbol{M}_{[\nu_1,\nu_2]}) \to \omega_1^*(\boldsymbol{M})_{[\lambda_1,\lambda_2]}$ is given by $\omega_{\boldsymbol{M}} = (id_{R_{2*}}, \tilde{\omega}_{\boldsymbol{M}})$.*

Proof. Put $\tilde{\alpha} = \alpha_{\nu_1}(id_{S_{0*} \otimes_{S_{1*}} M_*} \otimes_{K_*} \nu_2)$. It follows from (1) of (2.1.8) that we have

$$\begin{aligned}
\omega_2^*(\boldsymbol{M}_{[\nu_1,\nu_2]}) &= \omega_2^*(S_{2*}, M_* \otimes_{S_{1*}} S_{0*}, \tilde{\alpha}) = (R_{2*}, (M_* \otimes_{S_{1*}} S_{0*}) \otimes_{S_{1*}} R_{2*}, \tilde{\alpha}_{\omega_2}) \\
\omega_1^*(\boldsymbol{M})_{[\lambda_1,\lambda_2]} &= (R_{1*}, M_* \otimes_{S_{1*}} R_{1*}, \alpha_{\omega_1})_{[\lambda_1,\lambda_2]} \\
&= (R_{2*}, (M_* \otimes_{S_{1*}} R_{1*}) \otimes_{R_{1*}} R_{0*}, (\alpha_{\omega_1})_{\lambda_1}(id_{M_* \otimes_{S_{1*}} R_{1*}) \otimes_{R_{1*}} R_{0*}} \otimes_{K_*} \lambda_2).
\end{aligned}$$

Define $i_{\nu_1,\nu_2,\omega_0}(\boldsymbol{M}) : (M_* \otimes_{S_{1*}} S_{0*}) \otimes_{S_{2*}} R_{0*} \to M_* \otimes_{S_{1*}} R_{0*}$ by $i_{\nu_1,\nu_2,\omega_0}(\boldsymbol{M})(x \otimes s \otimes r) = x \otimes \omega_0(s)r$. It follows from (2) of (2.1.8) that $\omega_0^\sharp(\iota_{\nu_1,\nu_2}(\boldsymbol{M})) : (\lambda_2\omega_2)^*(\boldsymbol{M}_{[\nu_1,\nu_2]}) = (\omega_0\nu_2)^*(\boldsymbol{M}_{[\nu_1,\nu_2]}) \to (\omega_0\nu_1)^*(\boldsymbol{M}) = (\lambda_1\omega_1)^*(\boldsymbol{M})$ is given by $\omega_0^\sharp(\iota_{\nu_1,\nu_2}(\boldsymbol{M})) = (id_{R_{0*}}, i_{\nu_1,\nu_2,\omega_0}(\boldsymbol{M}))$. Hence

$$\boldsymbol{c}_{\omega_1,\lambda_1}(\boldsymbol{M})\omega_0^\sharp(\iota_{\nu_1,\nu_2}(\boldsymbol{M}))\boldsymbol{c}_{\omega_2,\lambda_2}(\boldsymbol{M}_{[\nu_1,\nu_2]})^{-1} : \lambda_2^*(\omega_2^*(\boldsymbol{M}_{[\nu_1,\nu_2]})) \to \lambda_1^*(\omega_1^*(\boldsymbol{M}))$$

is equal to $(id_{R_{0*}}, c_{\omega_1,\lambda_1}(\boldsymbol{M})i_{\nu_1,\nu_2,\omega_0}(\boldsymbol{M})c_{\omega_2,\lambda_2}(\boldsymbol{M}_{[\nu_1,\nu_2]})^{-1})$. Thus, by the definition of $\omega_{\boldsymbol{M}}$, we have

$$\omega_{\boldsymbol{M}} = (id_{R_{2*}}, c_{\omega_1,\lambda_1}(\boldsymbol{M})i_{\nu_1,\nu_2,\omega_0}(\boldsymbol{M})c_{\omega_2,\lambda_2}(\boldsymbol{M}_{[\nu_1,\nu_2]})^{-1}i_{\lambda_2}(\omega_2^*(\boldsymbol{M}_{[\nu_1,\nu_2]})))$$

and it can be verified that

$$c_{\omega_1,\lambda_1}(\boldsymbol{M})i_{\nu_1,\nu_2,\omega_0}(\boldsymbol{M})c_{\omega_2,\lambda_2}(\boldsymbol{M}_{[\nu_1,\nu_2]})^{-1}i_{\lambda_2}(\omega_2^*(\boldsymbol{M}_{[\nu_1,\nu_2]})) : (M_*\otimes_{S_{1*}} S_{0*})\otimes_{S_{1*}} R_{2*} \to (M_*\otimes_{S_{1*}} R_{1*})\otimes_{R_{1*}} R_{0*}$$

maps $x\otimes s\otimes r$ to $x\otimes 1\otimes \omega_0(s)\lambda_2(r)$. □

The following assertion is a direct consequence of (2.1.8).

Proposition 2.1.13 *For morphisms $\lambda : R_* \to S_*$ and $\nu : T_* \to S_*$ of $\mathcal{A}lg_{K_*}$, $[\lambda,\nu]_* : \mathcal{M}od(\mathcal{A}lg_{K_*}, \mathcal{M}od_{K_*})_{R_*} \to \mathcal{M}od(\mathcal{A}lg_{K_*}, \mathcal{M}od_{K_*})_{T_*}$ preserves coequalizers. It preserves equalizers if λ is flat.*

2.2 Fibered category of functorial modules

Definition 2.2.1 *For a functor $F : \mathcal{C} \to \mathcal{S}et$, we define a functor $U_F : \mathcal{C}_F \to \mathcal{C}$ by $U_F(R_*,\rho) = R_*$ and $U_F(\lambda : (R_*,\rho) \to (S_*,\sigma)) = (\lambda : S_* \to R_*)$. A functor $M : \mathcal{C}_F \to \mathcal{M}od(\mathcal{C},\mathcal{M})$ is called an F-module if M satisfies $p_{\mathcal{C}}M = U_F$. A natural transformation $\varphi : M \to N$ of F-modules is called a morphism in F-modules if φ satisfies $p_{\mathcal{C}}(\varphi_{(R_*,\rho)}) = id_{R_*}$ for $(R_*,\rho) \in \mathrm{Ob}\,\mathcal{C}_F$. We denote by $\mathcal{M}od(F)$ the category of F-modules and morphisms in F-modules.*

We put $\mathcal{E} = \mathrm{Funct}(\mathcal{C}, \mathcal{S}et)$. For a morphism $f : G \to F$ of $\mathcal{E}$, define a functor $\tilde{f} : \mathcal{C}_G \to \mathcal{C}_F$ by $\tilde{f}(R_*,\rho) = (R_*, f_{R_*}(\rho))$ for $(R_*,\rho) \in \mathrm{Ob}\,\mathcal{C}_G$ and $\tilde{f}(\lambda : (R_*,\rho) \to (S_*,\sigma)) = (\lambda : (R_*, f_{R_*}(\rho)) \to (S_*, f_{S_*}(\sigma)))$. Define a functor $f^* : \mathcal{M}od(F) \to \mathcal{M}od(G)$ by $f^*(M) = M\tilde{f}$ and $f^*(\varphi)_{(R_*,\rho)} = \varphi_{\tilde{f}(R_*,\rho)} = \varphi_{(R_*,f_{R_*}(\rho))}$ for $(R_*,\rho) \in \mathrm{Ob}\,\mathcal{C}_G$. Note that $(gf)^* = f^*g^* : \mathcal{M}od(H) \to \mathcal{M}od(G)$ holds for morphisms $f : G \to F$ and $g : F \to H$ of $\mathrm{Funct}(\mathcal{C}, \mathcal{S}et)$ and that id_F^* is the identity functor of $\mathcal{M}od(F)$.

We define a category $\mathcal{MOD}$ as follows. Objects of $\mathcal{MOD}$ are pairs (F,M) of $F \in \mathrm{Ob}\,\mathcal{E}$ and an F-module M. A morphism $(G,N) \to (F,M)$ is a pair (f,φ) of a morphism $f : G \to F$ of $\mathcal{E}$ and a morphism in G-modules $\varphi : f^*(M) \to N$. Composition of morphisms $(f,\varphi) : (G,N) \to (F,M)$ and $(g,\psi) : (F,M) \to (H,L)$ is defined to be $(gf, \varphi f^*(\psi))$.

Define a functor $p_{\mathcal{E}} : \mathcal{MOD} \to \mathcal{E}$ by $p_{\mathcal{E}}(F,M) = F$ and $p_{\mathcal{E}}(f,\varphi) = f$. Then, for each $F \in \mathrm{Ob}\,\mathcal{E}$, the subcategory $\mathcal{MOD}_F$ of $\mathcal{MOD}$ consisting of objects of the form (F,M) and morphisms in the form (id_F, φ) is identified with the opposite category $\mathcal{M}od(F)^{op}$ of F-modules.

Proposition 2.2.2 *$p_{\mathcal{E}} : \mathcal{MOD} \to \mathcal{E}$ is a fibered category.*

Proof. For a morphism $f : G \to F$ of $\mathcal{E}$ and $(F,M) \in \mathrm{Ob}\,\mathcal{MOD}_F$, it is clear that a map

$$(f, id_{f^*(M)})_* : \mathcal{MOD}_G((G,N),(G,f^*(M))) \to \mathcal{MOD}_f((G,N),(F,M))$$

which maps (id_G,φ) to (f,φ) is bijective. Thus $(f, id_{f^*(M)}) : (G, f^*(M)) \to (F,M)$ is a cartesian morphism and $p_{\mathcal{E}} : \mathcal{MOD} \to \mathcal{E}$ is a prefibered category. We set $f^*(F,M) = (G, f^*(M))$ and $\alpha_f(F,M) = (f, id_{f^*(M)})$.

For morphisms $f : G \to F$, $g : F \to H$ of $\mathcal{E}$ and $(H,L) \in \mathrm{Ob}\,\mathcal{MOD}_H$, we note that $f^*g^*(H,L) = f^*(F, g^*(L)) = (G, f^*(g^*(L))) = (G, (gf)^*(L)) = (gf)^*(H,L)$. Define $c_{g,f}(H,L)$ to be the identity morphism of $f^*g^*(H,L) = (gf)^*(H,L)$. Then, the following diagram commutes.

$$\begin{tikzcd} f^*g^*(H,L) \arrow[r, "\alpha_f(g^*(H,L))"] \arrow[d, "c_{g,f}(H,L)"] & g^*(H,L) \arrow[d, "\alpha_g(H,L)"] \\ (fg)^*(H,L) \arrow[r, "\alpha_{fg}(H,L)"] & (H,L) \end{tikzcd}$$

Therefore $p_{\mathcal{E}} : \mathcal{MOD} \to \mathcal{E}$ is a fibered category. □

Remark 2.2.3 *(1) For a morphism $f : G \to F$ of $\mathcal{E}$, the functor $f^* : \mathcal{MOD}_F \to \mathcal{MOD}_G$ is given by $f^*(F,M) = (G, f^*(M))$ and $f^*(id_F,\varphi) = (id_G, f^*(\varphi))$ for $M \in \mathcal{M}od(F)$ and $\varphi \in \mathcal{M}od(F)(M,N)$.*

(2) A morphism $(f,\varphi) : (G,N) \to (F,M)$ of $\mathcal{MOD}$ is cartesian if and only if $\varphi : f^*(M) \to N$ is an isomorphism in F-modules.

Proposition 2.2.4 *$\mathcal{MOD}$ has coproducts.*

Proof. Let $((F_i, M_i))_{i\in I}$ be a family of objects of $\mathcal{MOD}$. Put $F = \coprod_{i\in I} F_i$ and we denote by $\iota_i : F_i \to F$ be the canonical morphism. Define an F-module $M : \mathcal{C}_F \to \mathcal{M}od(\mathcal{C}, \mathcal{M})$ as follows. For $(R_*, \rho) \in \mathrm{Ob}\,\mathcal{C}_F$, we set $M(R_*, \rho) = M_i(R_*, \rho)$ if $\rho \in F_i(R_*)$. If $\lambda : (R_*, \rho) \to (S_*, \sigma)$ is a morphism in $\mathcal{C}_F$ such that $\rho \in F_i(R_*)$, then $\sigma = F(\lambda)(\rho) = F_i(\lambda)(\rho) \in F_i(S_*)$. We define $M(\lambda) : M(R_*, \rho) \to M(S_*, \sigma)$ by $M(\lambda) = M_i(\lambda)$ if $\rho \in F_i(R_*)$. We note that, if (R_*, ρ) is an F_i-model, then $\iota_i^*(M)(R_*, \rho) = M(R_*, (\iota_i)_{R_*}(\rho)) = M_i(R_*, \rho)$. Define a morphism $\boldsymbol{\iota}_i : \iota_i^*(M) \to M_i$ of F_i-modules by $(\boldsymbol{\iota}_i)_{(R_*,\rho)} = id_{M_i(R_*,\rho)} : \iota_i^*(M)(R_*, \rho) \to M_i(R_*, \rho)$.

Let $((g_i, \boldsymbol{\gamma}_i) : (F_i, M_i) \to (G, N))_{i\in I}$ be a family of morphism in $\mathcal{MOD}$. There exists unique morphism $g : F \to G$ satisfying $g\iota_i = g_i$ for any $i \in I$. Since $g^*(N)(R_*, \rho) = N(R_*, g_{R_*}(\iota_i)_{R_*}(\rho)) = N(R_*, (g_i)_{R_*}(\rho)) = g_i^*(N)(R_*, \rho)$ for $(R_*, \rho) \in \mathrm{Ob}\,\mathcal{C}_F$ if $\rho \in F_i(R_*)$, we define a morphism $\boldsymbol{\gamma} : g^*(N) \to M$ of F-modules by $\boldsymbol{\gamma}_{(R_*,\rho)} = (\boldsymbol{\gamma}_i)_{(R_*,\rho)}$. Since $\iota_i^* g^*(N)(R_*, \rho) = N(R_*, g_{R_*}(\iota_i)_{R_*}(\rho)) = N(R_*, (g_i)_{R_*}(\rho))$ if $\rho \in F_i(R_*)$, it follows $(\boldsymbol{\iota}_i\iota_i^*(\boldsymbol{\gamma}))_{(R_*,\rho)} = (\boldsymbol{\iota}_i)_{(R_*,\rho)}\iota_i^*(\boldsymbol{\gamma})_{(R_*,\rho)} = \boldsymbol{\gamma}_{(R_*,(\iota_i)_{R_*}(\rho))} = (\boldsymbol{\gamma}_i)_{(R_*,\rho)}$, thai is, $\boldsymbol{\iota}_i\iota_i^*(\boldsymbol{\gamma}) = \boldsymbol{\gamma}_i$. Hence we have $(g, \boldsymbol{\gamma})(\iota_i, \boldsymbol{\iota}_i) = (g_i, \boldsymbol{\gamma}_i)$. Suppose that a morphism $(g', \boldsymbol{\gamma}') : (F, M) \to (G, N)$ also satisfies $(g', \boldsymbol{\gamma}')(\iota_i, \boldsymbol{\iota}_i) = (g_i, \boldsymbol{\gamma}_i)$ for any $i \in I$. Since $g'\iota_i = g\iota_i$ for all $i \in I$, it follows $g' = g$. Then, we have

$$\boldsymbol{\gamma}'_{(R_*,(\iota_i)_{R_*}(\rho))} = \iota_i^*(\boldsymbol{\gamma}')_{(R_*,\rho)} = (\boldsymbol{\iota}_i)_{(R_*,\rho)}\iota_i^*(\boldsymbol{\gamma}')_{(R_*,\rho)} = (\boldsymbol{\gamma}_i)_{(R_*,\rho)} = (\boldsymbol{\iota}_i)_{(R_*,\rho)}\iota_i^*(\boldsymbol{\gamma})_{(R_*,\rho)} = \boldsymbol{\gamma}_{(R_*,(\iota_i)_{R_*}(\rho))}$$

for any $i \in I$ and $(R_*, \rho) \in \mathcal{C}_{F_i}$. Therefore $\boldsymbol{\gamma}' = \boldsymbol{\gamma}$. □

The following assertion is straightforward.

Lemma 2.2.5 *For $R_* \in \mathrm{Ob}\,\mathcal{C}$, let $(\boldsymbol{M}_i)_{i\in I}$ be a family of objects of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{R_*}$ and put $\boldsymbol{M}_i = (R_*, M_{i*}, \alpha_i)$. Assume that a coproduct $\coprod_{i\in I} M_{i*}$ in $\mathcal{M}$ exists and we denote by $\iota_j : M_{j*} \to \coprod_{i\in I} M_{i*}$ the inclusion map to j-summand for $j \in I$. Let $\alpha : \left(\coprod_{i\in I} M_{i*}\right) \otimes_{K_*} R_* \to \coprod_{i\in I} M_{i*}$ be the unique map that makes the following diagram commute for any $j \in I$.*

$$\begin{array}{ccc}
M_{j*} \otimes_{K_*} R_* & \xrightarrow{\alpha_j} & M_{j*} \\
\downarrow{\scriptstyle \iota_j \otimes_{K_*} id_{R_*}} & & \downarrow{\scriptstyle \iota_j} \\
\left(\coprod_{i\in I} M_{i*}\right) \otimes_{K_*} R_* & \xrightarrow{\alpha} & \coprod_{i\in I} M_{i*}
\end{array}$$

Then $\left(R_*, \coprod_{i\in I} M_{i*}, \alpha\right)$ is a coproduct of $(\boldsymbol{M}_i)_{i\in I}$ in $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{R_*}$. Hence if $\mathcal{M}$ has coproducts, $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{R_*}$ has coproducts for any $R_* \in \mathrm{Ob}\,\mathcal{C}$.

Proposition 2.2.6 *If $\mathcal{M}$ has coproducts, $f^* : \mathcal{M}od(F) \to \mathcal{M}od(G)$ has a left adjoint for any morphism $f : G \to F$ of $\mathcal{E}$.*

Proof. Let $N : \mathcal{C}_G \to \mathcal{M}od(\mathcal{C}, \mathcal{M})$ be a G-module. For $(R_*, \rho) \in \mathrm{Ob}\,\mathcal{C}_F$, we put

$$f_!(N)(R_*, \rho) = \coprod_{\kappa \in f_{R_*}^{-1}(\rho)} N(R_*, \kappa).$$

Here, $\coprod_{\kappa \in f_{R_*}^{-1}(\rho)} N(R_*, \kappa)$ denotes a coproduct in $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{R_*}$. We also denote by

$$\iota_f(N)_\nu : N(R_*, \nu) \longrightarrow \coprod_{\kappa \in f_{R_*}^{-1}(\rho)} N(R_*, \kappa)$$

the inclusion morphism into ν-summand. If $\lambda \in \mathcal{C}_F((R_*, \rho), (S_*, \sigma))$, then $F(U_F(\lambda))(\rho) = \sigma$ and it follows that $\kappa \in f_{R_*}^{-1}(\rho)$ implies $G(U_F(\lambda))(\kappa) \in f_{S_*}^{-1}(\sigma)$. For $\kappa \in G(R_*)$, let $\lambda_\kappa \in \mathcal{C}_G((R_*, \kappa), (S_*, G(U_F(\lambda))(\kappa)))$ be the morphism satisfying $U_G(\lambda_\kappa) = U_F(\lambda)$. Let

$$f_!(N)(\lambda) : f_!(N)(R_*, \rho) = \coprod_{\kappa \in f_{R_*}^{-1}(\rho)} N(R_*, \kappa) \longrightarrow \coprod_{\nu \in f_{S_*}^{-1}(\sigma)} N(S_*, \nu) = f_!(N)(S_*, \sigma)$$

be the unique morphism that make the following diagram commute for any $\kappa \in f_{R_*}^{-1}(\rho)$.

$$\begin{CD}
N(R_*,\kappa) @>{N(\lambda_\kappa)}>> N(S_*,G(\lambda)(\kappa))\\
@V{\iota_f(N)_\kappa}VV @VV{\iota_f(N)_{G(\lambda)(\kappa)}}V\\
\coprod_{\kappa\in f_{R_*}^{-1}(\rho)} N(R_*,\kappa) @>{f_!(N)(\lambda)}>> \coprod_{\nu\in f_{S_*}^{-1}(\sigma)} N(S_*,\nu)
\end{CD}$$

For a morphism $\varphi : M \to N$ of G-modules, we define a morphism $f_!(\varphi) : f_!(M) \to f_!(N)$ of F-modules as follows. For $(R_*,\rho) \in \mathrm{Ob}\,\mathcal{C}_F$, let

$$f_!(\varphi)_{(R_*,\rho)} : f_!(M)(R_*,\rho) = \coprod_{\kappa\in f_{R_*}^{-1}(\rho)} M(R_*,\kappa) \longrightarrow \coprod_{\kappa\in f_{R_*}^{-1}(\rho)} N(R_*,\kappa) = f_!(N)(R_*,\rho)$$

be the unique morphism that makes the following diagram commute.

$$\begin{CD}
M(R_*,\kappa) @>{\varphi_{(R_*,\kappa)}}>> N(R_*,\kappa)\\
@V{\iota_f(M)_\kappa}VV @VV{\iota_f(N)_\kappa}V\\
\coprod_{\kappa\in f_{R_*}^{-1}(\rho)} M(R_*,\kappa) @>{f_!(\varphi)_{(R_*,\rho)}}>> \coprod_{\kappa\in f_{R_*}^{-1}(\rho)} N(R_*,\kappa)
\end{CD}$$

We define a map $\mathrm{Ad} : \mathcal{M}od(G)(N, f^*(M)) \to \mathcal{M}od(F)(f_!(N), M)$ as follows. For $\varphi \in \mathcal{M}od(G)(N, f^*(M))$ and $(R_*,\rho) \in \mathrm{Ob}\,\mathcal{C}_F$, let

$${}^t\varphi_{(R_*,\rho)} : f_!(N)(R_*,\rho) = \coprod_{\kappa\in f_{R_*}^{-1}(\rho)} N(R_*,\kappa) \to M(R_*,\rho)$$

be the unique morphism that makes the following diagram commute for every $\kappa \in f_{R_*}^{-1}(\rho)$.

$$\begin{CD}
N(R_*,\kappa) @>{\varphi_{(R_*,\kappa)}}>> M(R_*,f_{R_*}(\kappa))\\
@V{\iota_f(N)_\kappa}VV @|\\
\coprod_{\kappa\in f_{R_*}^{-1}(\rho)} N(R_*,\kappa) @>{{}^t\varphi_{(R_*,\rho)}}>> M(R_*,\rho)
\end{CD}$$

Then, the naturality of φ implies the naturality of ${}^t\varphi$. Put $\mathrm{Ad}(\varphi) = {}^t\varphi$. The inverse of Ad is given as follows. For $\psi \in \mathcal{M}od(F)(f_!(N), M)$ and $(T_*,\tau) \in \mathrm{Ob}\,\mathcal{C}_G$, let $\tilde{\psi}_{(T_*,\tau)} : N(T_*,\tau) \to M(T_*, f_{T_*}(\tau)) = f^*(M)(T_*,\tau)$ be the following composition.

$$N(T_*,\tau) \xrightarrow{\iota_f(N)_\tau} \coprod_{\kappa\in f_{T_*}^{-1}(f_{T_*}(\tau))} N(T_*,\kappa) = f_!(N)(T_*, f_{T_*}(\tau)) \xrightarrow{\psi_{(T_*,f_{T_*}(\tau))}} M(T_*, f_{T_*}(\tau))$$

Then, the naturality of ψ implies the naturality of $\tilde{\psi}$. Put $\mathrm{Ad}^{-1}(\psi) = \tilde{\psi}$. $\square$

Remark 2.2.7 *The unit $\bar{\eta}^f : id_{\mathcal{M}od(G)} \to f^*f_!$ and the counit $\bar{\varepsilon}^f : f_!f^* \to id_{\mathcal{M}od(F)}$ are given as follows. For $N \in \mathrm{Ob}\,\mathcal{M}od(G)$ and $(T_*,\tau) \in \mathrm{Ob}\,\mathcal{C}_G$,*

$$(\bar{\eta}^f_N)_{(T_*,\tau)} : N(T_*,\tau) \longrightarrow \coprod_{\kappa\in f_{T_*}^{-1}(f_{T_*}(\tau))} N(T_*,\kappa) = f_!(N)(T_*, f_{T_*}(\tau)) = f^*f_!(N)(T_*,\tau)$$

is the inclusion morphism $\iota_f(N)_\tau$ into τ-summand. For $M \in \mathrm{Ob}\,\mathcal{M}od(F)$ and $(R_*,\rho) \in \mathrm{Ob}\,\mathcal{C}_F$,

$$(\bar{\varepsilon}^f_M)_{(R_*,\rho)} : f_!f^*(M)(R_*,\rho) = \coprod_{\kappa\in f_{R_*}^{-1}(\rho)} M(R_*, f_{R_*}(\kappa)) \longrightarrow M(R_*,\rho)$$

is the morphism induced by the identity morphism of $M(R_*,\rho)$.

Since $\mathcal{MOD}_F$ is identified with $\mathcal{M}od(F)^{op}$ and the inverse image functor $f^* : \mathcal{MOD}_F \to \mathcal{MOD}_G$ is identified with the functor $(f^*)^{op} : \mathcal{M}od(F)^{op} \to \mathcal{M}od(G)^{op}$, (2.2.6) implies the following result.

Corollary 2.2.8 *If $\mathcal{M}$ has coproducts, the inverse image functor $f^* : \mathcal{M}OD_F \to \mathcal{M}OD_G$ has a right adjoint for any morphism $f : G \to F$ of $\mathcal{E}$.*

Remark 2.2.9 *The unit $\eta^f : id_{\mathcal{M}od_F} \to f_! f^*$ and the counit $\varepsilon^f : f^* f_! \to id_{\mathcal{M}od_G}$ of the adjunction $f^* \dashv f_!$ are given as follows. For $M \in \mathrm{Ob}\,\mathcal{M}od(F)$, $\eta^f_{(F,M)} = (id_F, \bar{\varepsilon}^f_M) : (F, M) \to (F, f^* f_!(M)) = f^* f_!(F, M)$. For $N \in \mathrm{Ob}\,\mathcal{M}od(G)$, $\varepsilon^f_{(G,N)} : f_! f^*(G, N) = (id_G, \bar{\eta}^f_N) : (G, f_! f^*(N)) \to (G, N)$.*

Proposition 2.2.10 *Suppose that $\mathcal{M}$ is complete. For any morphism $f : G \to F$ of $\mathcal{E}$, $f^* : \mathcal{M}od(F) \to \mathcal{M}od(G)$ has a right adjoint.*

Proof. Let N be a G-module. For $(T_*, t) \in \mathrm{Ob}\,\mathcal{C}_G$, we put $N(T_*, t) = (T_*, N_{(T_*,t)*}, \mu_{(T_*,t)})$. Then, we have $p_{\mathcal{M}} NQ\langle\alpha, (T_*, t)\rangle = p_{\mathcal{M}} N(T_*, t) = N_{(T_*,t)*}$ for $(R_*, x) \in \mathrm{Ob}\,\mathcal{C}_F$ and $\langle\alpha, (T_*, t)\rangle \in \mathrm{Ob}\,((R_*, x){\downarrow}\tilde{f})$. Let

$$\left(N_f(R_*, x)_* \xrightarrow{\pi_{\langle\alpha,(T_*,t)\rangle}} p_{\mathcal{M}} NQ\langle\alpha, (T_*, t)\rangle\right)_{\langle\alpha,(T_*,t)\rangle\in\mathrm{Ob}\,((R_*,x)\downarrow\tilde{f})}$$

be a limiting cone of composition $((R_*, x){\downarrow}\tilde{f}) \xrightarrow{Q} \mathcal{C}_G \xrightarrow{N} \mathcal{M}od(\mathcal{C}, \mathcal{M}) \xrightarrow{p_{\mathcal{M}}} \mathcal{M}$. Let $\tau : \langle\alpha, (T_*, t)\rangle \to \langle\beta, (S_*, s)\rangle$ be a morphism in $((R_*, x){\downarrow}\tilde{f})$ and put $NQ(\tau) = (\tau, \tilde{\tau})$. Then, we have $p_{\mathcal{M}} NQ(\tau)\pi_{\langle\alpha,(T_*,t)\rangle} = \pi_{\langle\beta,(S_*,s)\rangle}$, $\tau U_F(\alpha) = U_F(\beta)$ and the following diagram commutes.

$$\begin{array}{ccc}
N_{(T_*,t)*} \otimes_{K_*} T_* & \xrightarrow{\mu_{(T_*,t)}} & N_{(T_*,t)*} \\
\Big\downarrow{\scriptstyle \tilde{\tau}\otimes_{K_*}\tau} & & \Big\downarrow{\scriptstyle \tilde{\tau}} \\
N_{(S_*,s)*} \otimes_{K_*} S_* & \xrightarrow{\mu_{(S_*,s)}} & N_{(S_*,s)*}
\end{array}$$

Thus we have

$$\begin{aligned}
p_{\mathcal{M}} NQ(\tau)\mu_{(T_*,t)}(\pi_{\langle\alpha,(T_*,t)\rangle} \otimes_{K_*} U_F(\alpha)) &= \tilde{\tau}\mu_{(T_*,t)}(\pi_{\langle\alpha,(T_*,t)\rangle} \otimes_{K_*} U_F(\alpha)) \\
&= \mu_{(S_*,s)}(\tilde{\tau} \otimes_{K_*} \tau)(\pi_{\langle\alpha,(T_*,t)\rangle} \otimes_{K_*} U_F(\alpha)) \\
&= \mu_{(S_*,s)}(p_{\mathcal{M}} NQ(\tau)\pi_{\langle\alpha,(T_*,t)\rangle} \otimes_{K_*} \tau U_F(\alpha)) \\
&= \mu_{(S_*,s)}(\pi_{\langle\beta,(S_*,s)\rangle} \otimes_{K_*} U_F(\beta)).
\end{aligned}$$

Hence $\left(N_f(R_*, x)_* \otimes_{K_*} R_* \xrightarrow{\mu_{(T_*,t)}(\pi_{\langle\alpha,(T_*,t)\rangle}\otimes_{K_*} U_F(\alpha))} N_{(T_*,t)*}\right)_{\langle\alpha,(T_*,t)\rangle\in\mathrm{Ob}\,((R_*,x)\downarrow\tilde{f})}$ is a cone of $p_{\mathcal{M}} NQ$ and there exists unique map $\rho_{(R_*,x)} : N_f(R_*, x)_* \otimes_{K_*} R_* \to N_f(R_*, x)_*$ satisfying

$$\pi_{\langle\alpha,(T_*,t)\rangle}\rho_{(R_*,x)} = \mu_{(T_*,t)}(\pi_{\langle\alpha,(T_*,t)\rangle} \otimes_{K_*} U_F(\alpha))$$

for any object $\langle\alpha, (T_*, t)\rangle$ of $((R_*, x){\downarrow}\tilde{f})$. Let $\nu_{T_*} : T_* \otimes_{K_*} T_* \to T_*$ be the multiplication of T_*. Then

$$\begin{aligned}
\pi_{\langle\alpha,(T_*,t)\rangle}\rho_{(R_*,x)}(\rho_{(R_*,x)} \otimes_{K_*} id_{R_*}) &= \mu_{(T_*,t)}(\pi_{\langle\alpha,(T_*,t)\rangle} \otimes_{K_*} U_F(\alpha))(\rho_{(R_*,x)} \otimes_{K_*} id_{R_*}) \\
&= \mu_{(T_*,t)}(\pi_{\langle\alpha,(T_*,t)\rangle}\rho_{(R_*,x)} \otimes_{K_*} U_F(\alpha)) \\
&= \mu_{(T_*,t)}(\mu_{(T_*,t)}(\pi_{\langle\alpha,(T_*,t)\rangle} \otimes_{K_*} U_F(\alpha)) \otimes_{K_*} U_F(\alpha)) \\
&= \mu_{(T_*,t)}(\mu_{(T_*,t)} \otimes_{K_*} id_{T_*})(\pi_{\langle\alpha,(T_*,t)\rangle} \otimes_{K_*} U_F(\alpha) \otimes_{K_*} U_F(\alpha)) \\
&= \mu_{(T_*,t)}(id_{N_{(T_*,t)*}} \otimes_{K_*} \nu_{T_*})(\pi_{\langle\alpha,(T_*,t)\rangle} \otimes_{K_*} U_F(\alpha) \otimes_{K_*} U_F(\alpha)) \\
&= \mu_{(T_*,t)}(\pi_{\langle\alpha,(T_*,t)\rangle} \otimes_{K_*} U_F(\alpha))(id_{N_f(R_*,x)_*} \otimes_{K_*} \nu_{R_*}) \\
&= \pi_{\langle\alpha,(T_*,t)\rangle}\rho_{(R_*,x)}(id_{N_f(R_*,x)_*} \otimes_{K_*} \nu_{R_*})
\end{aligned}$$

for any $\langle\alpha, (T_*, t)\rangle \in \mathrm{Ob}\,((R_*, x){\downarrow}\tilde{f})$. Therefore $\rho_{(R_*,x)}(\rho_{(R_*,x)} \otimes_{K_*} id_{R_*}) = \rho_{(R_*,x)}(id_{N_f(R_*,x)_*} \otimes_{K_*} \nu_{R_*})$. For a K_*-module N_* and a K_*-algebra R_*, let $i_{N_*,R_*} : N_* \to N_* \otimes_{K_*} R_*$ be a map defined by $i_{N_*,R_*}(x) = x \otimes_{K_*} 1$. Then, for any $\langle\alpha, (T_*, t)\rangle \in \mathrm{Ob}\,((R_*, x){\downarrow}\tilde{f})$, we have

$$\begin{aligned}
\pi_{\langle\alpha,(T_*,t)\rangle}\rho_{(R_*,x)} i_{N_f(R_*,x)_*,R_*} &= \mu_{(T_*,t)}(\pi_{\langle\alpha,(T_*,t)\rangle} \otimes_{K_*} U_F(\alpha)) i_{N_f(R_*,x)_*,R_*} \\
&= \mu_{(T_*,t)} i_{N_{(T_*,t)*},T_*}\pi_{\langle\alpha,(T_*,t)\rangle} = \pi_{\langle\alpha,(T_*,t)\rangle}.
\end{aligned}$$

Thus $\rho_{(R_*,x)} i_{N_f(R_*,x)_*,R_*} = id_{N_f(R_*,x)_*}$ and $\rho_{(R_*,x)} : N_f(R_*,x)_* \otimes_{K_*} R_* \to N_f(R_*,x)_*$ is a right R_*-module structure of $N_f(R_*,x)_*$. We note that $(U_F(\alpha), \pi_{\langle\alpha,(T_*,t)\rangle}) : (R_*, N_f(R_*,x)_*, \rho_{(R_*,x)}) \to (T_*, N_{(T_*,t)*}, \mu_{(T_*,t)})$ is a morphism in $\mathcal{M}od(\mathcal{C},\mathcal{M})$.

Recall that a morphism $\gamma : (S_*,y) \to (R_*,x)$ of $\mathcal{C}_F$ defines a functor $(\gamma{\downarrow}id_{\tilde f}) : ((R_*,x){\downarrow}\tilde f) \to ((S_*,y){\downarrow}\tilde f)$ by $(\gamma{\downarrow}id_{\tilde f})\langle\alpha,(T_*,t)\rangle = \langle\alpha\gamma,(T_*,t)\rangle$. Hence we have a cone

$$\left(N_f(R_*,x)_* \xrightarrow{\pi_{(\gamma\downarrow id_{\tilde f})\langle\alpha,(T_*,t)\rangle}} p_{\mathcal{M}}NQ(\gamma{\downarrow}id_{\tilde f})\langle\alpha,(T_*,t)\rangle\right)_{\langle\alpha,(T_*,t)\rangle\in\mathrm{Ob}\,((R_*,x)\downarrow\tilde f)}.$$

Since $p_{\mathcal{M}}NQ(\gamma{\downarrow}id_{\tilde f})\langle\alpha,(T_*,t)\rangle = p_{\mathcal{M}}N(T_*,t)$ for any $\langle\alpha,(T_*,t)\rangle \in ((R_*,x){\downarrow}\tilde f)$, there exists unique morphism $N_f(\gamma) : N_f(S_*,y)_* \to N_f(R_*,x)_*$ such that $\pi_{\langle\alpha,(T_*,t)\rangle}N_f(\gamma) = \pi_{(\gamma\downarrow id_{\tilde f})\langle\alpha,(T_*,t)\rangle}$ for any $\langle\alpha,(T_*,t)\rangle \in \mathrm{Ob}\,((R_*,x){\downarrow}\tilde f)$. It is easy to verify that this choice of $N_f(\gamma)$ makes N_f a functor. Since

$$\begin{aligned}
\pi_{\langle\alpha,(T_*,t)\rangle}\rho_{(R_*,x)}(N_f(\gamma)\otimes_{K_*}U_F(\gamma)) &= \mu_{(T_*,t)}(\pi_{\langle\alpha,(T_*,t)\rangle}\otimes_{K_*}U_F(\alpha))(N_f(\gamma)\otimes_{K_*}U_F(\gamma))\\
&= \mu_{(T_*,t)}(\pi_{\langle\alpha,(T_*,t)\rangle}N_f(\gamma)\otimes_{K_*}U_F(\alpha\gamma))\\
&= \mu_{(T_*,t)}(\pi_{(\gamma\downarrow id_{\tilde f})\langle\alpha,(T_*,t)\rangle}\otimes_{K_*}U_F(\alpha\gamma))\\
&= \pi_{(\gamma\downarrow id_{\tilde f})\langle\alpha,(T_*,t)\rangle}\rho_{(S_*,y)} = \pi_{\langle\alpha,(T_*,t)\rangle}N_f(\gamma)\rho_{(S_*,y)}
\end{aligned}$$

for any $\langle\alpha,(T_*,t)\rangle \in \mathrm{Ob}\,((R_*,x){\downarrow}\tilde f)$, we have $\rho_{(R_*,x)}(N_f(\gamma)\otimes_{K_*}U_F(\gamma)) = N_f(\gamma)\rho_{(S_*,y)}$, in other words, $(U_F(\gamma), N_f(\gamma)) : (S_*, N_f(S_*,y)_*, \rho_{(S_*,y)}) \to (R_*, N_f(R_*,x)_*, \rho_{(R_*,x)})$ is a morphism in $\mathcal{M}od(\mathcal{C},\mathcal{M})$. We define an F-module $f_*(N)$ by $f_*(N)(R_*,x) = (R_*, N_f(R_*,x)_*, \rho_{(R_*,x)})$ and $f_*(N)(\gamma) = (U_F(\gamma), N_f(\gamma))$.

For each $(T_*,t) \in \mathrm{Ob}\,\mathcal{C}_G$, we define a morphism $\tilde\varepsilon_{(T_*,t)} : f_*(N)\tilde f(T_*,t) \to N(T_*,t)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})$ by $\tilde\varepsilon_{(T_*,t)} = (id_{T_*}, \pi_{\langle id_{\tilde f(T_*,t)},(T_*,t)\rangle})$. We note that a morphism $\lambda : (T_*,t) \to (S_*,s)$ of $\mathcal{C}_G$ defines a morphism $\lambda : \langle id_{\tilde f(T_*,t)},(T_*,t)\rangle \to \langle\tilde f(\lambda),(S_*,s)\rangle$ of $(\tilde f(T_*,t){\downarrow}\tilde f)$. It follows from the definition of $f_*(N)\tilde f(\lambda) : f_*(N)\tilde f(T_*,t) \to f_*(N)\tilde f(S_*,s)$ that

$$\begin{aligned}
\tilde\varepsilon_{(S_*,s)}f_*(N)\tilde f(\lambda) &= (id_{S_*}, \pi_{\langle id_{\tilde f(S_*,s)},(S_*,s)\rangle})(U_F(\tilde f(\lambda)), N_f(\tilde f(\lambda))) = (U_G(\lambda), \pi_{\langle id_{\tilde f(S_*,s)},(S_*,s)\rangle}N_f(\tilde f(\lambda)))\\
&= (U_G(\lambda), \pi_{\langle id_{\tilde f(S_*,s)},(S_*,s)\rangle}N_f(\tilde f(\lambda))) = (U_G(\lambda), \pi_{(\tilde f(\lambda)\downarrow id_{\tilde f})\langle id_{\tilde f(S_*,s)},(S_*,s)\rangle})\\
&= (U_G(\lambda), \pi_{\langle\tilde f(\lambda),(S_*,s)\rangle}) = (U_G(\lambda), p_{\mathcal{M}}NQ(\lambda)\pi_{\langle id_{\tilde f(T_*,t)},(T_*,t)\rangle}) = N(\lambda)\tilde\varepsilon_{(T_*,t)}.
\end{aligned}$$

Therefore we have a morphism $\tilde\varepsilon : f_*(N)\tilde f \to N$ of F-modules.

Let $M : \mathcal{C}_F \to \mathcal{M}od(\mathcal{C},\mathcal{M})$ be an F-module and $\zeta : M\tilde f \to N$ a morphism in G-modules. For $(R_*,x) \in \mathrm{Ob}\,\mathcal{C}_F$, we put $M(R_*,x) = (R_*, M_{(R_*,x)*}, \chi_{(R_*,x)})$. If $\varphi : \langle\alpha,(T_*,t)\rangle \to \langle\beta,(S_*,s)\rangle$ is a morphism in $((R_*,x){\downarrow}\tilde f)$, since

$$NQ(\varphi)\zeta_{(T_*,t)}M(\alpha) = \zeta_{(S_*,s)}M\tilde fQ(\varphi)M(\alpha) = \zeta_{(S_*,s)}M(\tilde f(Q(\varphi))\alpha) = \zeta_{(S_*,s)}M(\beta),$$

$\left(M(R_*,x) \xrightarrow{\zeta_{(T_*,t)}M(\alpha)} NQ\langle\alpha,(T_*,t)\rangle\right)_{\langle\alpha,(T_*,t)\rangle\in\mathrm{Ob}\,((R_*,x)\downarrow\tilde f)}$ is a cone of $NQ : ((R_*,x){\downarrow}\tilde f) \to \mathcal{M}od(\mathcal{C},\mathcal{M})$. We have unique morphism $\bar\zeta_{(R_*,x)} : M_{(R_*,x)*} \to N_f(R_*,x)_*$ such that $\pi_{\langle\alpha,(T_*,t)\rangle}\bar\zeta_{(R_*,x)} = p_{\mathcal{M}}(\zeta_{(T_*,t)}M(\alpha))$ for any $\langle\alpha,(T_*,t)\rangle \in \mathrm{Ob}\,((R_*,x){\downarrow}\tilde f)$. Define $\check\zeta_{(R_*,x)} : M(R_*,x) \to f_*(N)(R_*,x)$ by $\check\zeta_{(R_*,x)} = (id_{R_*}, \bar\zeta_{(R_*,x)})$. Let $\gamma : (L_*,y) \to (R_*,x)$ be a morphism in $\mathcal{C}_F$. For each $\langle\alpha,(T_*,t)\rangle \in \mathrm{Ob}\,((R_*,x){\downarrow}\tilde f)$, since

$$\begin{aligned}
\pi_{\langle\alpha,(T_*,t)\rangle}\bar\zeta_{(R_*,x)}p_{\mathcal{M}}(M(\gamma)) &= p_{\mathcal{M}}(\zeta_{(T_*,t)}M(\alpha))p_{\mathcal{M}}(M(\gamma)) = p_{\mathcal{M}}(\zeta_{(T_*,t)}M(\alpha\gamma))\\
&= \pi_{(\gamma\downarrow id_{\tilde f})\langle\alpha,(T_*,t)\rangle}\bar\zeta_{(L_*,y)} = \pi_{\langle\alpha,(T_*,t)\rangle}N_f(\gamma)\bar\zeta_{(L_*,y)},
\end{aligned}$$

we have $\bar\zeta_{(R_*,x)}p_{\mathcal{M}}(M(\gamma)) = N_f(\gamma)\bar\zeta_{(L_*,y)}$, which implies the naturality of $\check\zeta$. Since diagrams

$$\begin{tikzcd}
N_f(R_*,x)_*\otimes_{K_*}R_* \arrow[r, "\pi_{\langle\alpha,(T_*,t)\rangle}\otimes_{K_*}U_F(\alpha)"] \arrow[d, "\rho_{(R_*,x)}"] & N_{(T_*,t)*}\otimes_{K_*}T_* \arrow[d, "\mu_{(T_*,t)}"] \\
N_f(R_*,x)_* \arrow[r, "\pi_{\langle\alpha,(T_*,t)\rangle}"] & N_{(T_*,t)*}
\end{tikzcd}$$

$$\begin{tikzcd}
M_{(R_*,x)*}\otimes_{K_*}R_* \arrow[r, "p_{\mathcal{M}}(M(\alpha))\otimes_{K_*}U_F(\alpha)"] \arrow[d, "\chi_{(R_*,x)}"] & M_{\tilde f(T_*,t)*}\otimes_{K_*}T_* \arrow[r, "p_{\mathcal{M}}(\zeta_{(T_*,t)})\otimes_{K_*}id_{T_*}"] \arrow[d, "\chi_{\tilde f(T_*,t)}"] & N_{(T_*,t)*}\otimes_{K_*}T_* \arrow[d, "\mu_{(T_*,t)}"] \\
M_{(R_*,x)*} \arrow[r, "p_{\mathcal{M}}(M(\alpha))"] & M_{\tilde f(T_*,t)*} \arrow[r, "p_{\mathcal{M}}(\zeta_{(T_*,t)})"] & N_{(T_*,t)*}
\end{tikzcd}$$

commute for any $(R_*, x) \in \mathrm{Ob}\,\mathcal{C}_F$ and $\langle \alpha, (T_*, t)\rangle \in \mathrm{Ob}\,((R_*, x)\downarrow\tilde{f})$, we have

$$\begin{aligned}
\pi_{\langle\alpha,(T_*,t)\rangle}\rho_{(R_*,x)}(\bar{\zeta}_{(R_*,x)} \otimes_{K_*} id_{R_*}) &= \mu_{(T_*,t)}(\pi_{\langle\alpha,(T_*,t)\rangle} \otimes_{K_*} U_F(\alpha))(\bar{\zeta}_{(R_*,x)} \otimes_{K_*} id_{R_*}) \\
&= \mu_{(T_*,t)}(\pi_{\langle\alpha,(T_*,t)\rangle}\bar{\zeta}_{(R_*,x)} \otimes_{K_*} U_F(\alpha)) \\
&= \mu_{(T_*,t)}(p_{\mathcal{M}}(\zeta_{(T_*,t)})p_{\mathcal{M}}(M(\alpha)) \otimes_{K_*} U_F(\alpha)) \\
&= \mu_{(T_*,t)}(p_{\mathcal{M}}(\zeta_{(T_*,t)}) \otimes_{K_*} id_{T_*})(p_{\mathcal{M}}(M(\alpha)) \otimes_{K_*} U_F(\alpha)) \\
&= p_{\mathcal{M}}(\zeta_{(T_*,t)}M(\alpha))\chi_{(R_*,x)} = \pi_{\langle\alpha,(T_*,t)\rangle}\bar{\zeta}_{(R_*,x)}\chi_{(R_*,x)}.
\end{aligned}$$

It follows $\rho_{(R_*,x)}(\bar{\zeta}_{(R_*,x)} \otimes_{K_*} id_{R_*}) = \bar{\zeta}_{(R_*,x)}\chi_{(R_*,x)}$, that is, $\check{\zeta} : M \to f_*(N)$ is a morphism in F-modules. Thus we have a map

$$ad_{M,N} : \mathcal{M}od(G)(M\tilde{f}, N) \to \mathcal{M}od(F)(M, f_*(N))$$

which maps ζ to $\check{\zeta}$.

Finally, we show that $ad_{M,N}$ is the inverse of the map $\mathcal{M}od(F)(M, f_*(N)) \to \mathcal{M}od(G)(M\tilde{f}, N)$ given by $\xi \mapsto \tilde{\varepsilon}\xi_{\tilde{f}}$. For $\zeta \in \mathcal{M}od(G)(M\tilde{f}, N)$ and $(T_*, t) \in \mathrm{Ob}\,\mathcal{C}_G$, we have

$$\tilde{\varepsilon}_{(T_*,t)}ad_{N,M}(\zeta)_{\tilde{f}(T_*,t)} = (id_{T_*}, \pi_{\langle id_{\tilde{f}(T_*,t)},(T_*,t)\rangle}\bar{\zeta}_{\tilde{f}(T_*,t)}) = (id_{T_*}, p_{\mathcal{M}}(\zeta_{(T_*,t)}M(id_{\tilde{f}(T_*,t)}))) = \zeta_{(T_*,t)}.$$

For $\xi \in \mathcal{M}od(F)(M, f_*(N))$ and $(R_*, x) \in \mathrm{Ob}\,\mathcal{C}_F$, we put $\bar{\xi}_{(R_*,x)} = p_{\mathcal{M}}(\xi_{(R_*,x)}) : p_{\mathcal{M}}(M(R_*, x)) \to N_f(R_*, x)_*$ and $\bar{\zeta}_{(R_*,x)} = p_{\mathcal{M}}(ad_{M,N}(\tilde{\varepsilon}\xi_{\tilde{f}})_{(R_*,x)}) : p_{\mathcal{M}}(M(R_*, x)) \to N_f(R_*, x)_*$. For each $\langle \alpha, (T_*, t)\rangle \in \mathrm{Ob}\,((R_*, x)\downarrow\tilde{f})$, by the naturality of ξ, it follows that

$$\begin{aligned}
\pi_{\langle\alpha,(T_*,t)\rangle}\bar{\zeta}_{(R_*,x)} &= p_{\mathcal{M}}(\tilde{\varepsilon}_{(T_*,t)}\xi_{\tilde{f}(T_*,t)}M(\alpha)) = p_{\mathcal{M}}(\tilde{\varepsilon}_{(T_*,t)}f_*(N)(\alpha)\xi_{(R_*,x)}) = \pi_{\langle id_{\tilde{f}(T_*,t)},(T_*,t)\rangle}N_f(\alpha)\bar{\xi}_{(R_*,x)} \\
&= \pi_{(\alpha\downarrow id_{\tilde{f}})\langle id_{\tilde{f}(T_*,t)},(T_*,t)\rangle}\bar{\xi}_{(R_*,x)} = \pi_{\langle\alpha,(T_*,t)\rangle}p_{\mathcal{M}}(\xi_{(R_*,x)})
\end{aligned}$$

and this implies $p_{\mathcal{M}}(ad_{M,N}(\tilde{\varepsilon}\xi_{\tilde{f}})_{(R_*,x)}) = p_{\mathcal{M}}(\xi_{(R_*,x)})$ for any $(R_*, x) \in \mathrm{Ob}\,\mathcal{C}_F$. Therefore $ad_{M,N}(\tilde{\varepsilon}\xi_{\tilde{f}}) = \xi$. □

Corollary 2.2.11 *$p_{\mathcal{E}} : \mathcal{MOD} \to \mathcal{E}$ is a bifibered category if $\mathcal{M}$ is complete.*

Remark 2.2.12 *The unit $\hat{\eta}(f) : id_{\mathcal{M}od(F)} \to f_*f^*$ of the adjunction $f^* \dashv f_*$ is given as follows. Let M be an F-module. For an object (R_*, x) of $\mathcal{C}_F$ and a morphism $\lambda : (R_*, x) \to (S_*, y)$, we set $M(R_*, x) = (R_*, M_{(R_*,x)*}, \alpha_{(R_*,x)})$ and $M(\lambda) = (\lambda, M_\lambda) : (R_*, M_{(R_*,x)*}, \alpha_{(R_*,x)}) \to (S_*, M_{(S_*,y)*}, \alpha_{(S_*,y)})$. Suppose that $\tau : \langle \alpha, (S_*, s)\rangle \to \langle \beta, (T_*, t)\rangle$ is a morphism in $((R_*, x)\downarrow\tilde{f})$, then the following diagram is commutative.*

$$\begin{tikzcd}
 & (R_*, x) \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
(S_*, f_{S_*}(s)) \arrow[rr, "\tilde{f}(Q(\tau))"] & & (T_*, f_{T_*}(t))
\end{tikzcd}$$

It follows that

$$\left(M_{(R_*,x)*} \xrightarrow{M_{P\langle\alpha,(T_*,t)\rangle} = M_\alpha} M_{(T_*,f_{T_*}(t))*} = p_{\mathcal{M}}M\tilde{f}Q\langle\alpha,(T_*,t)\rangle\right)_{\langle\alpha,(T_*,t)\rangle \in \mathrm{Ob}\,((R_*,x)\downarrow\tilde{f})}$$

is a cone of composition $((R_*, x)\downarrow\tilde{f}) \xrightarrow{Q} \mathcal{C}_G \xrightarrow{\tilde{f}} \mathcal{C}_F \xrightarrow{M} \mathcal{M}od(\mathcal{C}, \mathcal{M}) \xrightarrow{p_{\mathcal{M}}} \mathcal{M}$. Since

$$\left(f^*(M)_f(R_*, x)_* \xrightarrow{\pi_{\langle\alpha,(T_*,t)\rangle}} p_{\mathcal{M}}M\tilde{f}Q\langle\alpha,(T_*,t)\rangle\right)_{\langle\alpha,(T_*,t)\rangle \in \mathrm{Ob}\,((R_*,x)\downarrow\tilde{f})}$$

is a limiting cone of $p_{\mathcal{M}}M\tilde{f}Q$, there exist unique morphism $(\hat{\eta}^f_M)_{(R_*,x)} : M_{(R_*,x)*} \to f^*(M)_f(R_*, x)_*$ that makes the following diagram commutative for every $\langle\alpha,(T_*,t)\rangle \in \mathrm{Ob}\,((R_*, x)\downarrow\tilde{f})$.

$$\begin{tikzcd}
M_{(R_*,x)*} \arrow[rr, "(\hat{\eta}^f_M)_{(R_*,x)}"] \arrow[dr, "M_{P\langle\alpha,(T_*,t)\rangle}"'] & & f^*(M)_f(R_*, x)_* \arrow[dl, "\pi_{\langle\alpha,(T_*,t)\rangle}"] \\
 & p_{\mathcal{M}}M\tilde{f}Q\langle\alpha,(T_*,t)\rangle &
\end{tikzcd}$$

We define $\hat{\eta}(f)_M : M \to f_*f^*(M)$ by $(\hat{\eta}(f)_M)_{(R_*,x)} = (id_{R_*}, (\hat{\eta}^f_M)_{(R_*,x)}) : M(R_*, x) \to f_*f^*(M)(R_*, x)$.

2.3 Associativity of the fibered category of functorial modules

Suppose that $\mathcal{M}$ has coproducts. Let $f : F \to G$ and $g : F \to H$ be morphisms in $\mathcal{E}$ and (H, N) an object of $\mathcal{MOD}_H$. It follows from (2.2.8) and (1.4.2) that the presheaf $F^{f,g}_{(H,N)} : \mathcal{MOD}_G^{op} \to \mathcal{S}et$ on $\mathcal{MOD}_G$ is representable. For an object (H, N) of $\mathcal{MOD}_H$, it follows from (1.4.2) that we have

$$(H, N)^{[f,g]} = f_!g^*(H, N) = (G, f_!(N\tilde{g})).$$

Since $\tilde{g}(R_*, \kappa) = (R_*, g_{R_*}(\kappa))$ for $(R_*, \kappa) \in \mathrm{Ob}\,\mathcal{C}_H$, $f_!g^*(N) = f_!(N\tilde{g}) \in \mathrm{Ob}\,\mathcal{M}od(G)$ is given by

$$f_!(N\tilde{g})(R_*, \rho) = \coprod_{\kappa \in f_{R_*}^{-1}(\rho)} N\tilde{g}(R_*, \kappa) = \coprod_{\kappa \in f_{R_*}^{-1}(\rho)} N(R_*, g_{R_*}(\kappa))$$

for $(R_*, \rho) \in \mathrm{Ob}\,\mathcal{C}_G$.

Let $\varphi : N \to M$ be a morphism in H-modules. It follows from (1.4.5) that

$$(id_H, \varphi)^{[f,g]} : (H, M)^{[f,g]} = (H, f_!(M\tilde{g})) \to (H, f_!(N\tilde{g})) = (H, N)^{[f,g]}$$

is given by $(id_H, \varphi)^{[f,g]} = (id_H, f_!g^*(\varphi))$. For $(R_*, \rho) \in \mathrm{Ob}\,\mathcal{C}_F$ and $\nu \in f_{R_*}^{-1}(\rho)$, if we denote by

$$\iota_f(g^*(N))_\nu : N(R_*, g_{R_*}(\nu)) = N\tilde{g}(R_*, \nu) \longrightarrow \coprod_{\kappa \in f_{R_*}^{-1}(\rho)} N\tilde{g}(R_*, \kappa) = f_!(N\tilde{g})(R_*, \rho)$$

the inclusion morphism to ν-summand, the following diagram commutes

$$\begin{tikzcd}
N(R_*, g_{R_*}(\nu)) \arrow[r, "\varphi_{(R_*, g_{R_*}(\nu))}"] \arrow[d, "\iota_f(g^*(N))_\nu"] & M(R_*, g_{R_*}(\nu)) \arrow[d, "\iota_f(g^*(M))_\nu"] \\
f_!(N\tilde{g})(R_*, \rho) \arrow[r, "f_!g^*(\varphi)_{(R_*, \rho)}"] & f_!(M\tilde{g})(R_*, \rho)
\end{tikzcd}$$

Let $h : L \to F$ be a morphism in $\mathcal{E}$ and N an H-module. For $(R_*, \rho) \in \mathrm{Ob}\,\mathcal{C}_G$, we define a morphism

$$N^h_{(R_*,\rho)} : (fh)_!(N\tilde{h}\tilde{g})(R_*, \rho) = \coprod_{\kappa \in (fh)_{R_*}^{-1}(\rho)} N(R_*, g_{R_*}h_{R_*}(\kappa)) \longrightarrow \coprod_{\kappa \in f_{R_*}^{-1}(\rho)} N(R_*, g_{R_*}(\kappa)) = f_!(N\tilde{g})(R_*, \rho)$$

of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{R_*}$ to be unique homomorphism that makes the following diagram commute for any $\nu \in (fh)_{R_*}^{-1}(\rho)$.

$$\begin{tikzcd}
N(R_*, (gh)_{R_*}(\nu)) \arrow[r, "id_{N(R_*, (g_{R_*}h_{R_*})(\nu))}"] \arrow[d, "\iota_{fh}((gh)^*(N))_\nu"] & N(R_*, g_{R_*}(h_{R_*}(\nu))) \arrow[d, "\iota_f(g^*(N))_{h_{R_*}(\nu)}"] \\
(fh)_!(N\widetilde{gh})(R_*, \rho) \arrow[r, "N^h_{(R_*,\rho)}"] & f_!(N\tilde{g})(R_*, \rho)
\end{tikzcd}$$

Let $\lambda : (R_*, \rho) \to (S_*, \gamma)$ be a morphism in $\mathcal{C}_G$ and ν an element of $(fh)_{R_*}^{-1}(\rho)$. Since $H(\lambda)g_{R_*} = g_{S_*}F(\lambda)$ and $F(\lambda)h_{R_*} = h_{S_*}L(\lambda)$, $U_G(\lambda) : R_* \to S_*$ defines a morphism

$$\lambda_\nu : (R_*, g_{R*}h_{R_*}(\nu)) \to (S_*, H(\lambda)(g_{R_*}h_{R_*}(\nu))) = (S_*, g_{S_*}h_{S_*}(L(\lambda)(\nu)))$$

of $\mathcal{C}_H$. We also note that

$$\begin{aligned}(fh)_{S_*}(L(\lambda)(\nu)) &= f_{S_*}(h_{S_*}(L(\lambda)(\nu))) = f_{S_*}(F(\lambda)(h_{R_*}(\nu))) = G(\lambda)(f_{R_*}(h_{R_*}(\nu))) = G(\lambda)((fh)_{R_*}(\nu)) \\ &= G(\lambda)(\rho) = \gamma \in G(S_*).\end{aligned}$$

By the definition of $N^h_{(R_*,\rho)}$ and $N^h_{(S_*,\gamma)}$, we have the following equalities.

$$N^h_{(R_*,\rho)}\iota_{fh}((gh)^*(N))_\nu = \iota_f(g^*(N))_{h_{R_*}(\nu)},$$
$$N^h_{(S_*,\gamma)}\iota_{fh}((gh)^*(N))_{L(\lambda)(\nu)} = \iota_f(g^*(N))_{h_{S_*}(L(\lambda)(\nu))}$$

Hence the left rectangle of the following diagram commutes by the definition of $(fh)_!(N\widetilde{gh})(\lambda)$ and the outer rectangle of the following diagram commutes by the definition of $f_!(N\tilde{g})(\lambda)$. Thus the right rectangle of the following diagram is commutative.

$$\begin{array}{ccccc}
N(R_*, g_{R*}h_{R_*}(\nu)) & \xrightarrow{\iota_{fh}((gh)^*(N))_\nu} & (fh)_!(N\widetilde{gh})(R_*,\rho) & \xrightarrow{N^h_{(R_*,\rho)}} & f_!(N\tilde{g})(R_*,\rho) \\
\downarrow N(\lambda_\nu) & & \downarrow (fh)_!(N\widetilde{gh})(\lambda) & & \downarrow f_!(N\tilde{g})(\lambda) \\
N(S_*, g_{S_*}h_{S_*}(L(\lambda)(\nu))) & \xrightarrow{\iota_{fh}((gh)^*(N))_{h_{S_*}(L(\lambda)(\nu))}} & (fh)_!(N\widetilde{gh})(S_*,\gamma) & \xrightarrow{N^h_{(S_*,\gamma)}} & f_!(N\tilde{g})(S_*,\gamma)
\end{array}$$

Hence we have a morphism $N^h : (fh)_!(N\tilde{g}\tilde{h}) \to f_!(N\tilde{g})$ of G-modules.

Proposition 2.3.1 *Let $f : F \to G$, $g : F \to H$ and $h : L \to F$ be morphisms in $\mathcal{E}$ and (H, N) an object of $\mathcal{MOD}_H$. The morphism*

$$(H,N)^h : (H,N)^{[f,g]} = (G, f_!(N\tilde{g})) \to (G, f_!(N\tilde{g})) = (H,N)^{[fh,gh]}$$

of $\mathcal{MOD}_G$ is given by $(H,N)^h = (id_G, N^h)$.

Proof. It follows from (1) of (1.4.8) that $(H,N)^h$ is the following composition.

$$(H,N)^{[f,g]} = f_!g^*(H,N) \xrightarrow{\eta^{fh}_{f_!g^*(H,N)}} (fh)_!(fh)^*((H,N)^{[f,g]}) = (fh)_!h^*f^*f_!g^*(H,N)$$
$$\xrightarrow{(fh)_!h^*\left(\varepsilon^f_{g^*(H,N)}\right)} (fh)_!h^*g^*(H,N) = (fh)_!(gh)^*(H,N) = (H,N)^{[fh,gh]}$$

We recall from (2.2.9) that

$$\eta^{fh}_{f_!g^*(H,N)} = (id_G, \bar{\varepsilon}^{fh}_{f_!(N\tilde{g})}) : (G, f_!(N\tilde{g})) \longrightarrow (G, (fh)_!(f_!(N\tilde{g})\widetilde{fh}))$$
$$(fh)_!h^*\left(\varepsilon^f_{g^*(H,N)}\right) = (id_G, (fh)_!h^*(\bar{\eta}^f_{N\tilde{g}})) : (G, (fh)_!h^*(f_!(N\tilde{g})\tilde{f})) \longrightarrow (G, (fh)_!(N\widetilde{gh})).$$

It follows from (2.2.7) that

$$\left(\bar{\varepsilon}^{fh}_{f_!(N\tilde{g})}\right)_{(R_*,\rho)} : (fh)_!(f_!(N\tilde{g})\widetilde{fh})(R_*,\rho) = \coprod_{\kappa\in(fh)^{-1}_{R_*}(\rho)} f_!(N\tilde{g})(R_*, (fh)_{R_*}(\kappa)) \longrightarrow f_!(N\tilde{g})(R_*,\rho)$$

is the morphism induced by the identity morphism of $f_!(N\tilde{g})(R_*,\rho)$ for $(R_*,\rho) \in \operatorname{Ob}\mathcal{C}_G$ and that

$$h^*(\bar{\eta}^f_{N\tilde{g}})_{(R_*,\nu)} : (N\widetilde{gh})(R_*,\nu) = N(R_*, g_{R_*}(h_{R_*}(\nu))) \longrightarrow \coprod_{\kappa\in f^{-1}_{R_*}(f_{R_*}(h_{R_*}(\rho)))} N(R_*, g_{R_*}(\kappa)) = f_!(N\tilde{g})\widetilde{fh}(R_*,\rho)$$

coincides with the inclusion morphism $\iota_f(N\tilde{g})_{h_{R_*}(\nu)}$ to $h_{R_*}(\nu)$-summand for $\nu \in (fh)^{-1}_{R_*}(\rho)$. For $(R_*,\rho) \in \operatorname{Ob}\mathcal{C}_G$, we have

$$(fh)_!(N\widetilde{gh})(R_*,\rho) = \coprod_{\nu\in(fh)^{-1}_{R_*}(\rho)} N(R_*, (gh)_{R_*}(\nu))$$
$$(fh)_!(f_!(N\tilde{g})\widetilde{fh})(R_*,\rho) = \coprod_{\nu\in(fh)^{-1}_{R_*}(\rho)} f_!(N\tilde{g})(R_*, (fh)_{R_*}(\nu))$$

and the following diagram is commutative for $\nu \in (fh)^{-1}_{R_*}(\rho)$.

$$\begin{tikzcd}
N(R_*, (gh)_{R_*}(\nu)) \arrow[r, "h^*(\bar{\eta}^f_{N\tilde{g}})_{(R_*,\nu)}"] \arrow[d, "\iota_{fh}(N\widetilde{gh})_\nu"] & f_!(N\tilde{g})(R_*, (fh)_{R_*}(\nu)) \arrow[d, "\iota_{fh}(f_!(N\tilde{g})\widetilde{fh})_\nu"] \arrow[dr, bend left, "id_{f_!(N\tilde{g})(R_*,\rho)}"] & \\
(fh)_!(N\widetilde{gh})(R_*,\rho) \arrow[r, "\left((fh)_!h^*(\bar{\eta}^f_{N\tilde{g}})\right)_{(R_*,\rho)}"] & (fh)_!(f_!(N\tilde{g})\widetilde{fh})(R_*,\rho) \arrow[r, "\left(\bar{\varepsilon}^{fh}_{f_!(N\tilde{g})}\right)_{(R_*,\rho)}"] & f_!(N\tilde{g})(R_*,\rho)
\end{tikzcd}$$

Thus a composition

$$(fh)_!(N\widetilde{gh})(R_*,\rho) \xrightarrow{\left((fh)_!h^*\left(\bar{\eta}^f_{N\tilde{g}}\right)\right)_{(R_*,\rho)}} (fh)_!(f_!(N\tilde{g})\widetilde{fh})(R_*,\rho) \xrightarrow{\left(\bar{\varepsilon}^{fh}_{f_!(N\tilde{g})}\right)_{(R_*,\rho)}} f_!(N\tilde{g})(R_*,\rho)$$

maps ν-summand of $(fh)_!(N\widetilde{gh})(R_*,\rho)$ to $h_{R_*}(\nu)$-summand of $f_!(N\tilde{g})(R_*,\rho)$ and this implies the assertion. □

Lemma 2.3.2 *Let $f : F_1 \to F_3$, $g : F_1 \to F_4$, $h : F_2 \to F_4$, $i : F_2 \to F_5$, $j : G_1 \to F_1$ and $k : G_2 \to F_2$ be morphisms in $\mathcal{E}$. For an F_5-module N, a morphism*

$$((F_5, N)^k)^j : ((F_5, N)^{[h,i]})^{[f,g]} = (F_3, f_!(h_!(N\tilde{i})\tilde{g})) \to (F_3, (fj)_!((hk)_!(N\widetilde{ik})\widetilde{gj})) = ((F_5, N)^{[hk,ik]})^{[fj,gj]}$$

is given by $((F_5, N)^k)^j = (id_{F_3}, f_!g^*(N^k)((hk)_!(N\widetilde{ik}))^j)$.

Proof. Since $(F_5, N)^{[hk,ik]} = (F_4, (hk)_!(N\widetilde{ik}))$, we have $((F_5, N)^{[hk,ik]})^j = (id_{F_3}, ((hk)_!(N\widetilde{ik}))^j)$ by (2.3.1). We also have $((F_5, N)^k)^{[f,g]} = (id_{F_4}, N^k)^{[f,g]} = (id_{F_3}, f_!g^*(N^k))$. Hence $((F_5, N)^k)^j = ((F_5, N)^{[hk,ik]})^j((F_5, N)^k)^{[f,g]}$ implies the assertion. $\square$

We investigate the morphism $f_!g^*(N^k)((hk)_!(N\widetilde{ik}))^j : (fj)_!((hk)_!(N\widetilde{ik})\widetilde{gj}) \to f_!(h_!(N\tilde{i})\tilde{g})$ below. If we put $M = (hk)_!(N\widetilde{ik})$, the following diagram is commutative for $(R_*, \rho) \in \mathrm{Ob}\,\mathcal{C}_{F_3}$, $\kappa \in (fh)^{-1}_{R_*}(\rho)$ and $\nu \in (hk)^{-1}_{R_*}((gj)_{R_*}(\kappa))$.

$$\begin{tikzcd}
N(R_*, (ik)_{R_*}(\nu)) \arrow[r, "id_{N(R_*,(ik)_{R_*}(\nu))}"] \arrow[d, "\iota_{hk}(N\widetilde{ik})_\nu"] & N(R_*, (ik)_{R_*}(\nu)) \arrow[r, "id_{N(R_*,(ik)_{R_*}(\nu))}"] \arrow[d, "\iota_{hk}(N\widetilde{ik})_\nu"] & N(R_*, (ik)_{R_*}(\nu)) \arrow[d, "\iota_h(N)_{k_{R_*}(\nu)}"] \\
M(R_*, (gj)_{R_*}(\kappa)) \arrow[r, "id_{M(R_*,(gj)_{R_*}(\kappa))}"] \arrow[d, "\iota_{fj}(M\widetilde{gj})_\kappa"] & M(R_*, g_{R_*}j_{R_*}(\kappa)) \arrow[r, "g^*(N^k)_{(R_*,j_{R_*}(\kappa))}"] \arrow[d, "\iota_f(M\tilde{g})_{j_{R_*}(\kappa)}"] & h_!(N\tilde{i})(R_*, g_{R_*}j_{R_*}(\kappa)) \arrow[d, "\iota_f((h_!(N\tilde{i}))\tilde{g})_{j_{R_*}(\kappa)}"] \\
(fj)_!(M\widetilde{gj})(R_*, \rho) \arrow[r, "M^j_{(R_*,\rho)}"] & f_!(M\tilde{g})(R_*, \rho) \arrow[r, "f_!g^*(N^k)_{(R_*,\rho)}"] & f_!(h_!(N\tilde{i})\tilde{g})(R_*, \rho)
\end{tikzcd}$$

We note that the following equalities hold for $(R_*, \rho) \in \mathrm{Ob}\,\mathcal{C}_{F_3}$.

$$\begin{aligned}
(fj)_!((hk)_!(N\widetilde{ik})\widetilde{gj})(R_*, \rho) &= \coprod_{\kappa \in (fh)^{-1}_{R_*}(\rho)} (hk)_!(N\widetilde{ik})(R_*, (gj)_{R_*}(\kappa)) \\
&= \coprod_{\kappa \in (fh)^{-1}_{R_*}(\rho)} \coprod_{\nu \in (hk)^{-1}_{R_*}((gj)_{R_*}(\kappa))} N(R_*, (ik)_{R_*}(\nu)) \\
f_!(h_!(N\tilde{i})\tilde{g})(R_*, \rho) &= \coprod_{\kappa \in f^{-1}_{R_*}(\rho)} h_!(N\tilde{i})(R_*, g_{R_*}(\kappa)) \\
&= \coprod_{\kappa \in f^{-1}_{R_*}(\rho)} \coprod_{\nu \in h^{-1}_{R_*}(g_{R_*}(\kappa))} N(R_*, i_{R_*}(\nu))
\end{aligned}$$

For $\kappa \in (fh)^{-1}_{R_*}(\rho)$ and $\nu \in (hk)^{-1}_{R_*}((gj)_{R_*}(\kappa))$, $\iota_{fj}((hk)_!(N\widetilde{ik})\widetilde{gj})_\kappa \iota_{hk}(N\widetilde{ik})_\nu$ is the inclusion morphism to "κ-ν-summand" of $(fj)_!((hk)_!(N\widetilde{ik})\widetilde{gj})(R_*, \rho)$ and $\iota_f((h_!(N\tilde{i}))\tilde{g})_{j_{R_*}(\kappa)}\iota_h(N)_{k_{R_*}(\nu)}$ is the inclusion morphism to "$j_{R_*}(\kappa)$-$k_{R_*}(\nu)$-summand" of $f_!(h_!(N\tilde{i})\tilde{g})(R_*, \rho)$. Hence it follows from the above diagram that

$$f_!g^*(N^k)_{(R_*,\rho)}(hk)_!(N\widetilde{ik})^j_{(R_*,\rho)} : (fj)_!((hk)_!(N\widetilde{ik})\widetilde{gj})(R_*, \rho) \longrightarrow f_!(h_!(N\tilde{i})\tilde{g})(R_*, \rho)$$

maps "κ-ν-summand" of $(fj)_!((hk)_!(N\widetilde{ik})\widetilde{gj})(R_*, \rho)$ to "$j_{R_*}(\kappa)$-$k_{R_*}(\nu)$-summand" of $f_!(h_!(N\tilde{i})\tilde{g})(R_*, \rho)$.

For morphisms $f : F \to G$, $g : F \to H$, $h : F \to L$ of $\mathcal{E}$ and an L-module N,

$$\epsilon^{f,g,h}_{(L,N)} : ((L, N)^{[g,h]})^{[f,g]} \longrightarrow (L, N)^{[f,h]}$$

is described as follows. First of all, recall that

$$\begin{aligned}
((L, N)^{[g,h]})^{[f,g]} &= f_!g^*g_!h^*(L, N) = (G, f_!(g_!(N\tilde{h})\tilde{g})) \\
(L, N)^{[f,h]} &= f_!h^*(L, N) = (G, f_!(N\tilde{h})).
\end{aligned}$$

It follows from (1.4.12) and (2.2.9) that

$$\epsilon^{f,g,h}_{(L,N)} = f_!\big(\varepsilon^g_{h^*(L,N)}\big) = \big(id_G, f_!\big(\bar{\eta}^g_{h^*(N)}\big)\big) : (G, f_!(g_!(N\tilde{h})\tilde{g})) \longrightarrow (G, f_!(N\tilde{h})).$$

Since $\big(\bar{\eta}^g_{h^*(N)}\big)_{(R_*,\nu)} : (N\tilde{h})(R_*, \nu) \longrightarrow \coprod_{\kappa \in g^{-1}_{R_*}(g_{R_*}(\nu))} N(R_*, h_{R_*}(\kappa)) = g^*g_!h^*(N)(R_*, \nu)$ is the inclusion morphism $\iota_g(N\tilde{h})_\nu$ to ν-component of $g^*g_!h^*(N)(R_*, \nu)$ for $(R_*, \nu) \in \mathrm{Ob}\,\mathcal{C}_F$ and the following diagram commutes.

$$
\begin{tikzcd}
(N\tilde{h})(R_*,\nu) \arrow[r, "\left(\bar{\eta}^{g}_{h^*(N)}\right)_{(R_*,\nu)}"] \arrow[d, "\iota_f(N\tilde{h})_\nu"] & g^*g_!(N\tilde{h})(R_*,\nu) \arrow[d, "\iota_f(g^*g_!(N\tilde{h}))_\nu"] \\
f_!(N\tilde{h})(R_*,\rho) \arrow[r, "f_!\left(\bar{\eta}^{g}_{h^*(N)}\right)_{(R_*,\rho)}"] & f_!(g^*g_!(N\tilde{h}))(R_*,\rho)
\end{tikzcd}
$$

Since $f_!(g_!(N\tilde{h})\tilde{g})(R_*,\rho) = \coprod_{\kappa\in f_{R_*}^{-1}(\rho)} \coprod_{\nu\in g_{R_*}^{-1}(g_{R_*}(\kappa))} N(R_*,h_{R_*}(\nu))$ and $\iota_f(g^*g_!(N\tilde{h}))_{(R_*,\nu)}\big(\bar{\eta}^{g}_{h^*(N)}\big)_{(R_*,\nu)}$ is the inclusion morphism to "ν-ν-summand", $f_!\big(\bar{\eta}^{g}_{h^*(N)}\big)_{(R_*,\rho)}$ maps ν-summand of $f_!(N\tilde{h})(R_*,\rho)$ to "ν-ν-summand" of $f_!(g_!(N\tilde{g})\tilde{g})(R_*,\rho)$.

Proposition 2.3.3 *Suppose that $\mathcal{M}$ has coproducts. Then, for any morphisms $f: F_1\to F_3$, $g: F_1\to F_4$, $h: F_2\to F_4$, $i: F_2\to F_5$ of $\mathcal{E}$, (f_1,f_2,f_3,f_4) is an associative left fibered representable quadruple. Namely,*

$$\theta^{f,g,h,i}(F_5,N) : ((F_5,N)^{[h,i]})^{[f,g]} = f_!g^*h_!i^*(F_5,N) \longrightarrow (f\mathrm{pr}_{F_1})_!(i\mathrm{pr}_{F_2})^*(F_5,N) = (F_5,N)^{[f\mathrm{pr}_{F_1},i\mathrm{pr}_{F_2}]}$$

is an isomorphism for any F_5-module N.

Proof. We recall that $\theta^{f,g,h,i}(F_5,N)$ is defined to be the following composition.

$$((F_5,N)^{[h,i]})^{[f,g]} \xrightarrow{((F_5,N)^{\mathrm{pr}_{F_2}})^{\mathrm{pr}_{F_1}}} ((F_5,N)^{[h\mathrm{pr}_{F_2},i\mathrm{pr}_{F_2}]})^{[f\mathrm{pr}_{F_1},g\mathrm{pr}_{F_1}]} \xrightarrow{\epsilon^{f\mathrm{pr}_{F_1},g\mathrm{pr}_{F_1},i\mathrm{pr}_{F_2}}_{(F_5,N)}} (F_5,N)^{[f\mathrm{pr}_{F_1},i\mathrm{pr}_{F_2}]}$$

Note that we have the following equalities.

$$
\begin{aligned}
((F_5,N)^{[h,i]})^{[f,g]} &= f_!g^*h_!i^*(F_5,N) = (F_3, f_!(h_!(N\tilde{i})\tilde{g})) \\
((F_5,N)^{[h\mathrm{pr}_{F_2},i\mathrm{pr}_{F_2}]})^{[f\mathrm{pr}_{F_1},g\mathrm{pr}_{F_1}]} &= (f\mathrm{pr}_{F_1})_!(g\mathrm{pr}_{F_1})^*(h\mathrm{pr}_{F_2})_!(i\mathrm{pr}_{F_2})^*(F_5,N) \\
&= (F_3, (f\mathrm{pr}_{F_1})_!((g\mathrm{pr}_{F_1})_!(N\widetilde{g\mathrm{pr}}_{F_1})\widetilde{i\mathrm{pr}}_{F_2})) \\
(F_5,N)^{[f\mathrm{pr}_{F_1},i\mathrm{pr}_{F_2}]} &= (f\mathrm{pr}_{F_1})_!(i\mathrm{pr}_{F_2})^*(F_5,N) = (F_3,(f\mathrm{pr}_{F_1})_!(N\widetilde{i\mathrm{pr}}_{F_2})) \\
\epsilon^{f\mathrm{pr}_{F_1},g\mathrm{pr}_{F_1},i\mathrm{pr}_{F_2}}_{(F_5,N)} &= (f\mathrm{pr}_{F_1})_!\big(\varepsilon^{g\mathrm{pr}_{F_1}}_{(i\mathrm{pr}_{F_2})^*(F_5,N)}\big) = \big(id_{F_3}, (f\mathrm{pr}_{F_1})_!\big(\bar{\eta}^{g\mathrm{pr}_{F_1}}_{(i\mathrm{pr}_{F_2})^*(N)}\big)\big) \\
((F_5,N)^{\mathrm{pr}_{F_2}})^{\mathrm{pr}_{F_1}} &= (id_{F_3}, f_!g^*(N^{\mathrm{pr}_{F_2}})((h\mathrm{pr}_{F_2})_!(N\widetilde{i\mathrm{pr}}_{F_2}))^{\mathrm{pr}_{F_1}})
\end{aligned}
$$

The following diagram (i) is commutative for any $(R_*,\rho)\in \mathrm{Ob}\,\mathcal{C}_{F_3}$ and $(\kappa,\nu)\in (f\mathrm{pr}_{F_1})^{-1}_{R_*}(\rho)$.

$$
\begin{tikzcd}
(N\widetilde{i\mathrm{pr}}_{F_2})(R_*,(\kappa,\nu)) \arrow[r, "\iota_{f\mathrm{pr}_{F_1}}(N\widetilde{i\mathrm{pr}}_{F_2})_{(\kappa,\nu)}"] \arrow[d, "\left(\bar{\eta}^{g\mathrm{pr}_{F_1}}_{(i\mathrm{pr}_{F_2})^*(N)}\right)_{(R_*,(\kappa,\nu))}"] & (f\mathrm{pr}_{F_1})_!(N\widetilde{i\mathrm{pr}}_{F_2})(R_*,\rho) \arrow[d, "(f\mathrm{pr}_{F_1})_!\left(\bar{\eta}^{g\mathrm{pr}_{F_1}}_{(i\mathrm{pr}_{F_2})^*(N)}\right)_{(R_*,\rho)}"] \\
(g\mathrm{pr}_{F_1})_!(N\widetilde{i\mathrm{pr}}_{F_2})\widetilde{g\mathrm{pr}}_{F_1}(R_*,(\kappa,\nu)) \arrow[r, "\iota_{f\mathrm{pr}_{F_1}}((g\mathrm{pr}_{F_1})_!(N\widetilde{i\mathrm{pr}}_{F_2})\widetilde{g\mathrm{pr}}_{F_1})_{(\kappa,\nu)}"] \arrow[d, equal] & (f\mathrm{pr}_{F_1})_!((g\mathrm{pr}_{F_1})_!(N\widetilde{i\mathrm{pr}}_{F_2})\widetilde{g\mathrm{pr}}_{F_1})(R_*,\rho) \arrow[d, equal] \\
(h\mathrm{pr}_{F_2})_!(N\widetilde{i\mathrm{pr}}_{F_2})\widetilde{g\mathrm{pr}}_{F_1}(R_*,(\kappa,\nu)) \arrow[r, "\iota_{f\mathrm{pr}_{F_1}}((h\mathrm{pr}_{F_2})_!(N\widetilde{i\mathrm{pr}}_{F_2})\widetilde{g\mathrm{pr}}_{F_1})_{(\kappa,\nu)}"] \arrow[d, "id_{(h\mathrm{pr}_{F_2})_!(N\widetilde{i\mathrm{pr}}_{F_2})\widetilde{g\mathrm{pr}}_{F_1}(R_*,(\kappa,\nu))}"] & (f\mathrm{pr}_{F_1})_!((h\mathrm{pr}_{F_2})_!(N\widetilde{i\mathrm{pr}}_{F_2})\widetilde{g\mathrm{pr}}_{F_1})(R_*,\rho) \arrow[d, "((h\mathrm{pr}_{F_2})_!(N\widetilde{i\mathrm{pr}}_{F_2}))^{\mathrm{pr}_{F_1}}_{(R_*,\rho)}"] \\
(h\mathrm{pr}_{F_2})_!(N\widetilde{i\mathrm{pr}}_{F_2})\widetilde{g\mathrm{pr}}_{F_1}(R_*,(\kappa,\nu)) \arrow[r, "\iota_f((h\mathrm{pr}_{F_2})_!(N\widetilde{i\mathrm{pr}}_{F_2})\tilde{g})_{\mathrm{pr}_{F_1 R_*}(\kappa,\nu)}"] \arrow[d, "N^{\mathrm{pr}_{F_2}}_{\widetilde{g\mathrm{pr}}_{F_1}(R_*,(\kappa,\nu))}"] & f_!((h\mathrm{pr}_{F_2})_!(N\widetilde{i\mathrm{pr}}_{F_2})\tilde{g})(R_*,\rho) \arrow[d, "f_!g^*(N^{\mathrm{pr}_{F_2}})_{(R_*,\rho)}"] \\
h_!(N\tilde{i})\widetilde{g\mathrm{pr}}_{F_1}(R_*,(\kappa,\nu)) \arrow[r, "\iota_f(h_!(N\tilde{i})\tilde{g})_{\mathrm{pr}_{F_1 R_*}(\kappa,\nu)}"] & f_!(h_!(N\tilde{i})\tilde{g})(R_*,\rho)
\end{tikzcd}
$$

diagram (i)

$\big(\bar{\eta}^{g\mathrm{pr}_{F_1}}_{(i\mathrm{pr}_{F_2})^*(N)}\big)_{(R_*,(\kappa,\nu))}$ is the inclusion morphism to (κ,ν)-summand of

$$(g\mathrm{pr}_{F_1})_!(N\widetilde{i\mathrm{pr}}_{F_2})\widetilde{g\mathrm{pr}}_{F_1}(R_*,(\kappa,\nu)) = \coprod_{(\gamma,\chi)\in(g\mathrm{pr}_{F_1})^{-1}_{R_*}((g\mathrm{pr}_{F_1})(\kappa,\nu))} (N\widetilde{i\mathrm{pr}}_{F_2})(R_*,(\gamma,\chi))$$

and $N^{\mathrm{pr}_{F_2}}_{\widetilde{g\mathrm{pr}}_{F_1}(R_*,(\kappa,\nu))} : (h\mathrm{pr}_{F_2})_!(N\widetilde{i\mathrm{pr}}_{F_2})\widetilde{g\mathrm{pr}}_{F_1}(R_*,(\kappa,\nu)) \to h_!(N\tilde{i})\widetilde{g\mathrm{pr}}_{F_1}(R_*,(\kappa,\nu))$ maps (κ,ν)-summand of $(h\mathrm{pr}_{F_2})_!(N\widetilde{i\mathrm{pr}}_{F_2})\widetilde{g\mathrm{pr}}_{F_1}(R_*,(\kappa,\nu))$ to ν-summand of

$$h_!(N\tilde{i})\widetilde{g\mathrm{pr}}_{F_1}(R_*,(\kappa,\nu)) = \coprod_{\chi\in h^{-1}_{R_*}(g_{R_*}(\kappa))} N(R_*, i_{R_*}(\chi))$$

which is mapped by $\iota_f(h_!(N\tilde{i})\tilde{g})_{\mathrm{pr}_{F_1 R_*}(\kappa,\nu)} : h_!(N\tilde{i})\widetilde{g\mathrm{pr}}_{F_1}(R_*,(\kappa,\nu)) \to f_!(h_!(N\tilde{i})\tilde{g})(R_*,\rho)$ to "κ-ν-summand" of $f_!(h_!(N\tilde{i})\tilde{g})(R_*,\rho) = \coprod_{\gamma\in f^{-1}_{R_*}(\rho)} \coprod_{\chi\in h^{-1}_{R_*}(g_{R_*}(\gamma))} N(R_*, i_{R_*}(\chi))$. Moreover, we note that $(\gamma,\chi) \in (f\mathrm{pr}_{F_1})^{-1}_{R_*}(\rho)$ if and only if "$(\gamma,\chi) \in (F_1 \times_{F_4} F_2)(R_*)$ and $\gamma \in f^{-1}_{R_*}(\rho)$" which is equivalent to "$\chi \in h^{-1}_{R_*}(g_{R_*}(\gamma))$ and $\gamma \in f^{-1}_{R_*}(\rho)$". Hence the following diagram is commutative and the composition of the right vertical morphisms in diagram (i) is an isomorphism.

$$\begin{tikzcd}
(N\widetilde{i\mathrm{pr}}_{F_2})(R_*,(\kappa,\nu)) \arrow[r, "\iota_{f\mathrm{pr}_{F_1}}(N\widetilde{i\mathrm{pr}}_{F_2})_{(\kappa,\nu)}"] \arrow[d, "id_{N(R_*,i_{R_*}(\nu))}"] & \coprod_{(\gamma,\chi)\in(f\mathrm{pr}_{F_1})^{-1}_{R_*}(\rho)} (N\widetilde{i\mathrm{pr}}_{F_2})(R_*,(\gamma,\chi)) \arrow[d, "\text{the composition of the right vertical morphisms in diagram } (i)"'] \\
N(R_*, i_{R_*}(\nu)) \arrow[r, "\text{inclusion to } \kappa\text{-}\nu\text{-summand}"'] & \coprod_{\gamma\in f^{-1}_{R_*}(\rho)} \coprod_{\chi\in h^{-1}_{R_*}(g_{R_*}(\gamma))} N(R_*, i_{R_*}(\chi))
\end{tikzcd}$$

Thus $\theta^{f,g,h,i}(F_5,N) = \big(id_{F_3}, f_!g^*(N^{\mathrm{pr}_{F_2}})((h\mathrm{pr}_{F_2})_!(N\widetilde{i\mathrm{pr}}_{F_2}))^{\mathrm{pr}_{F_1}}(f\mathrm{pr}_{F_1})_!\big(\bar{\eta}^{g\mathrm{pr}_{F_1}}_{(i\mathrm{pr}_{F_2})^*(N)}\big)\big)$ is an isomorphism. □

Proposition 2.3.4 *Suppose that $\mathcal{M}$ has coproducts and is complete. For morphisms $f : F_1 \to F_3$, $g : F_1 \to F_4$, $h : F_2 \to F_4$ and $i : F_2 \to F_5$ of $\mathcal{E}$, (f,g,h,i) is an associative left and right fibered representable quadruple.*

Proof. Clearly, $\mathcal{E}$ has finite limis with terminal object $1 = h_{K_*}$. It follows from (2.2.8) and (1.4.2) that the presheaf $F^{f,g}_N$ on $\mathcal{F}_G$ is representable for any morphisms $f : F \to G$, $g : F \to H$ of $\mathcal{E}$ and $N \in \mathrm{Ob}\,\mathcal{F}_H$. It follows from (2.2.11) and (1.3.3) that the presheaf $F_{f,g,M}$ on $\mathcal{F}^{op}_H$ is representable for any morphisms $f : F \to G$, $g : F \to H$ of $\mathcal{E}$ and $M \in \mathrm{Ob}\,\mathcal{F}_G$. Then, assertion follows from (1.5.5) and (2.3.3). □

2.4 Fibered category of morphisms

Let $\mathcal{E}$ be a category with finite limits. Suppose that $X \xleftarrow{\pi_f} E \times_Y X \xrightarrow{f_\pi} E$ is a limit of a diagram $X \xrightarrow{f} Y \xleftarrow{\pi} E$ in $\mathcal{E}$. For morphisms $\varphi : V \to E$ and $\psi : V \to X$ of $\mathcal{E}$ which satisfy $\pi\varphi = f\psi$, we denote by $(\varphi,\psi) : V \to E \times_Y X$ the unique morphism that satisfy $f_\pi(\varphi,\psi) = \varphi$ and $\pi_f(\varphi,\psi) = \psi$. Suppose moreover that $Z \xleftarrow{\rho_g} F \times_W X \xrightarrow{g_\rho} E$ is a limit of a diagram $Z \xrightarrow{g} W \xleftarrow{\rho} F$. If morphisms $\kappa : E \to F$, $h : X \to Z$ and $i : Y \to W$ in $\mathcal{E}$ satisfy $\rho\kappa = i\pi$ and $gh = if$, we denote $(\kappa f_\pi, h\pi_f)$ by $\kappa \times_i h$. If $Y = W$ and i is the identity morphism id_Y of Y, $\kappa \times_i h$ is denoted by $\kappa \times_Y h$.

$$\begin{tikzcd}
V \arrow[dr, dashed, "(\varphi,\psi)"] \arrow[drr, bend left, "\varphi"] \arrow[ddr, bend right, "\psi"'] & & \\
& E \times_Y X \arrow[r, "f_\pi"] \arrow[d, "\pi_f"] & E \arrow[d, "\pi"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
E \times_Y X \arrow[r, "f_\pi"] \arrow[d, "\pi_f"] \arrow[drr, dashed, "\kappa\times_i h"] & E \arrow[d, "\pi"] \arrow[drr, "\kappa"] & & \\
X \arrow[r, "f"] \arrow[drr, "h"'] & Y \arrow[drr, "i"] & F \times_W Z \arrow[r, "g_\rho"] \arrow[d, "\rho_g"] & F \arrow[d, "\rho"] \\
& & Z \arrow[r, "g"] & W
\end{tikzcd}$$

Lemma 2.4.1 *Under the above setting, $(\kappa \times_i h)(\varphi,\psi) = (\kappa\varphi, h\psi)$ holds.*

Proof. The equality follows from the commutativity of the following diagram.

$$\begin{tikzcd}
V \arrow[drr, bend left, "\varphi"] \arrow[dr, "(\varphi,\psi)"] \arrow[ddr, bend right, "\psi"'] & & & & \\
& E \times_Y X \arrow[r, "f_\pi"] \arrow[d, "\pi_f"] \arrow[drr, "\kappa\times_i h"] & E \arrow[d, "\pi"] \arrow[drr, "\kappa"] & & \\
& X \arrow[r, "f"] \arrow[drr, "h"'] & Y \arrow[drr, "i"] & F \times_W Z \arrow[r, "g_\rho"'] \arrow[d, "\rho_g"] & F \arrow[d, "\rho"] \\
& & & Z \arrow[r, "g"] & W
\end{tikzcd}$$

□

Proposition 2.4.2 *For morphisms $f : X \to Y$, $g : Z \to X$ of $\mathcal{E}$ and an object $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ of $\mathcal{E}^{(2)}$, consider the following cartesian squares.*

$$\begin{tikzcd}
F \times_Y X \arrow[r, "f_\rho"] \arrow[d, "\rho_f"] & F \arrow[d, "\rho"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
(F \times_Y X) \times_X Z \arrow[r, "g_{\rho_f}"] \arrow[d, "(\rho_f)_g"] & F \times_Y X \arrow[d, "\rho_f"] \\
Z \arrow[r, "g"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
F \times_Y Z \arrow[r, "(fg)_\rho"] \arrow[d, "\rho_{fg}"] & F \arrow[d, "\rho"] \\
Z \arrow[r, "fg"] & Y
\end{tikzcd}$$

The unique morphism $(id_F \times_Y g, \rho_{fg}) : F \times_Y Z \to (F \times_Y X) \times_X Z$ that makes the following left diagram commute is an isomorphism whose inverse is the unique morphism $(f_\rho g_{\rho_f}, (\rho_f)_g) : (F \times_Y X) \times_X Z \to F \times_Y Z$ that makes the following right diagram commute.

$$\begin{tikzcd}
F \times_Y Z \arrow[dr, "(id_F\times_Y g, \rho_{fg})"] \arrow[drr, bend left, "id_F\times_Y g"] \arrow[drrr, bend left, "(fg)_\rho"] \arrow[ddr, bend right, "\rho_{fg}"'] & & & \\
& (F \times_Y X) \times_X Z \arrow[r, "g_{\rho_f}"] \arrow[d, "(\rho_f)_g"] & F \times_Y X \arrow[r, "f_\rho"] \arrow[d, "\rho_f"] & F \arrow[d, "\rho"] \\
& Z \arrow[r, "g"] & X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
(F \times_Y X) \times_X Z \arrow[r, "g_{\rho_f}"] \arrow[dr, "(f_\rho g_{\rho_f}, (\rho_f)_g)"] \arrow[ddr, bend right, "(\rho_f)_g"'] & F \times_Y X \arrow[dr, bend left, "f_\rho"] & \\
& F \times_Y Z \arrow[r, "(fg)_\rho"] \arrow[d, "\rho_{fg}"] & F \arrow[d, "\rho"] \\
& Z \arrow[r, "fg"] & Y
\end{tikzcd}$$

Proof. Since the outer rectangle of the following diagram is cartesian, the assertion follows.

$$\begin{tikzcd}
(F \times_Y X) \times_X Z \arrow[r, "g_{\rho_f}"] \arrow[d, "(\rho_f)_g"] & F \times_Y X \arrow[r, "f_\rho"] \arrow[d, "\rho_f"] & F \arrow[d, "\rho"] \\
Z \arrow[r, "g"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

□

Let Δ^1 be a category given by $\mathrm{Ob}\,\Delta^1 = \{0, 1\}$ and $\mathrm{Mor}\,\Delta^1 = \{id_0, id_1, 0 \to 1\}$. For a category $\mathcal{E}$, we set $\mathcal{E}^{(2)} = \mathrm{Funct}(\Delta^1, \mathcal{E})$. Then, an object of $\mathcal{E}^{(2)}$ is identified with a morphism $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ of $\mathcal{E}$ and a morphism from $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ to $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ of $\mathcal{E}^{(2)}$ is identified with a pair $\langle \varphi, f \rangle$ of morphisms $\varphi : E \to F$ and $f : X \to Y$ of $\mathcal{E}$ satisfying $\rho\varphi = f\pi$.

Proposition 2.4.3 *([6], p.182, a)) Suppose that $\mathcal{E}$ is a category with finite limits. Let $p : \mathcal{E}^{(2)} \to \mathcal{E}$ be the evaluation functor ev_1 at 1. Then, $p : \mathcal{E}^{(2)} \to \mathcal{E}$ is a fibered category.*

Proof. For a morphism $f : X \to Y$ of $\mathcal{E}$ and an object $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ of $\mathcal{E}^{(2)}_Y$, consider the following cartesian square.

$$\begin{tikzcd}
F \times_Y X \arrow[r, "f_\rho"] \arrow[d, "\rho_f"] & F \arrow[d, "\rho"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

For an object $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ of $\mathcal{E}^{(2)}_X$, a morphism $\langle f, f_\rho \rangle : (F \times_Y X \xrightarrow{\rho_f} X) \to (F \xrightarrow{\rho} Y)$ of $\mathcal{E}^{(2)}$ induces a bijection

$$\mathcal{E}^{(2)}_X((E \xrightarrow{\pi} X), (F \times_Y X \xrightarrow{\rho_f} X)) \to \mathcal{E}^{(2)}_f((E \xrightarrow{\pi} X), (F \xrightarrow{\rho} Y)).$$

In fact, the inverse of the above map is given by $\langle \varphi, f\rangle \mapsto ((\varphi,\pi), id_X)$. Hence $\langle f_\rho, f\rangle$ is a cartesian morphism and we have a functor $f^* : \mathcal{E}^{(2)}_Y \to \mathcal{E}^{(2)}_X$ which is given as follows. $f^*(\boldsymbol{F}) = (F \times_Y X \xrightarrow{\rho_f} X)$ for an object $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ of $\mathcal{E}^{(2)}_Y$. For a morphism $\langle \varphi, id_Y\rangle : \boldsymbol{F} \to \boldsymbol{G}$ of $\mathcal{E}^{(2)}_Y$, $f^*(\langle \varphi, id_Y\rangle) : f^*(\boldsymbol{F}) \to f^*(\boldsymbol{G})$ is defined to be $\langle \varphi \times_Y id_X, id_X\rangle$, where $\boldsymbol{G} = (G \xrightarrow{\lambda} Y)$.

Under the settings of (2.4.2), we define $c_{f,g}(\boldsymbol{F}) : g^*f^*(\boldsymbol{F}) \to (fg)^*(\boldsymbol{F})$ by $c_{f,g}(\boldsymbol{F}) = \langle (f_\rho g_{\rho_f}, (\rho_f)_g), id_Z\rangle$ which is an isomorphism in $\mathcal{E}^{(2)}_Z$ by (2.4.2). Hence $p : \mathcal{E}^{(2)} \to \mathcal{E}$ is a fibered category. □

Since $\mathcal{E}^{(2)}_{1_\mathcal{E}}$ is identified with $\mathcal{E}$ by a correspondence $(X \xrightarrow{o_X} 1_\mathcal{E}) \leftrightarrow X$, it follows from (1.1.22) that cartesian sections of $p : \mathcal{E}^{(2)} \to \mathcal{E}$ are given as follows.

Proposition 2.4.4 *For an object X of $\mathcal{E}$, define a functor $s_X : \mathcal{E} \to \mathcal{E}^{(2)}$ by $s_X(Y) = (X \times Y \xrightarrow{\mathrm{pr}_Y} Y)$ and $s_X(f : Y \to Z) = \langle id_X \times f, f\rangle$. Then, s_X is a cartesian section of $p : \mathcal{E}^{(2)} \to \mathcal{E}$. If s is a cartesian section of $p : \mathcal{E}^{(2)} \to \mathcal{E}$, put $X = s(1_\mathcal{E})$, then s is naturally equivalent to s_X.*

Remark 2.4.5 *(1) We define a functor $s_\mathcal{E} : \mathcal{E} \to \mathcal{E}^{(2)}$ by $s_\mathcal{E}(X) = (X \xrightarrow{id_X} X)$ and $s_\mathcal{E}(f : X \to Y) = \langle f, f\rangle$. Then, $s_\mathcal{E}$ is a cartesian section, in fact, $s_\mathcal{E}$ is naturally equivalent to $s_{1_\mathcal{E}}$. We call $s_\mathcal{E}$ the canonical cartesian section of $p : \mathcal{E}^{(2)} \to \mathcal{E}$.*

(2) For an object X of $\mathcal{E}$, we consider the cartesian section $s_X : \mathcal{E} \to \mathcal{E}^{(2)}$ of $p : \mathcal{E}^{(2)} \to \mathcal{E}$ defined in (2.4.4). For a morphism $f : Y \to Z$ in $\mathcal{E}$, the lower rectangles of the following diagram are cartesian.

$$\begin{tikzcd}
X\times Y \arrow[dr, "{(id_X\times f,\mathrm{pr}_Y)}"] \arrow[drr, bend left, "id_X\times f"] \arrow[drrr, bend left, "\mathrm{pr}_X"] \arrow[ddr, bend right, "\mathrm{pr}_Y"'] & & & \\
& (X\times Z)\times_Z Y \arrow[r, "f_{\mathrm{pr}_Z}"] \arrow[d, "(\mathrm{pr}_Z)_f"] & X\times Z \arrow[r, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_Z"] & X \arrow[d, "o_X"] \\
& Y \arrow[r, "f"] & Z \arrow[r, "o_Z"] & 1_\mathcal{E}
\end{tikzcd}
\qquad
\begin{tikzcd}
(X\times Z)\times_Z Y \arrow[rr, "f_{\mathrm{pr}_Z}"] \arrow[dr, "{(\mathrm{pr}_X f_{\mathrm{pr}_Z},(\mathrm{pr}_Z)_f)}"] \arrow[ddr, bend right, "(\mathrm{pr}_Z)_f"'] & & X\times Z \arrow[d, "\mathrm{pr}_X"] \\
& X\times Y \arrow[r, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_Y"] & X \arrow[d, "o_X"] \\
& Y \arrow[r, "o_Y"] & 1_\mathcal{E}
\end{tikzcd}$$

Since the morphism $(s_X)_f : s_X(Y) = (X\times Y \xrightarrow{\mathrm{pr}_Y} Y) \to ((X\times Z)\times_Z Y \xrightarrow{(\mathrm{pr}_Z)_f} Y) = f^*(s_X(Z))$ in $\mathcal{E}^{(2)}_Y$ defined in (1.1.23) coincides with $c_{o_z,f}(X)^{-1}$, it is given by $\langle (id_X \times f, \mathrm{pr}_Y), id_Y\rangle$. Hence the inverse $(s_X)_f^{-1}$ of $(s_X)_f$ is given by $\langle (\mathrm{pr}_X f_{\mathrm{pr}_Z}, (\mathrm{pr}_Z)_f), id_Y\rangle$. For a morphism $g : Y \to W$ in $\mathcal{E}$, since a composition

$$(X\times Z)\times_Z Y \xrightarrow{(\mathrm{pr}_X f_{\mathrm{pr}_Z},(\mathrm{pr}_Z)_f)} X\times Y \xrightarrow{(id_X\times g,\,\mathrm{pr}_Y)} (X\times W)\times_W Y$$

coincides with $((\mathrm{pr}_X f_{\mathrm{pr}_Z}, g(\mathrm{pr}_Z)_f), (\mathrm{pr}_Z)_f)$, a morphism $(s_X)_{f,g} = (s_X)_g (s_X)_f^{-1} : f^*(s_X(Z)) \to g^*(s_X(Z))$ in $\mathcal{E}^{(2)}_Y$ is given by $\langle ((\mathrm{pr}_X f_{\mathrm{pr}_Z}, g(\mathrm{pr}_Z)_f), (\mathrm{pr}_Z)_f), id_Y\rangle$.

Lemma 2.4.6 *Let $f : X \to Y$, $g : Z \to X$, $h : W \to Y$, $i : Z \to W$ be morphisms in $\mathcal{E}$ which satisfy $fg = hi$. For an object $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ of $\mathcal{E}^{(2)}_Y$, suppose that each rectangle of the following diagrams is cartesian.*

$$\begin{tikzcd}
(F\times_Y X)\times_X Z \arrow[r, "g_{\rho_f}"] \arrow[d, "(\rho_f)_g"] & F\times_Y X \arrow[r, "f_\rho"] \arrow[d, "\rho_f"] & F \arrow[d, "\rho"] \\
Z \arrow[r, "g"] & X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
(F\times_Y W)\times_W Z \arrow[r, "i_{\rho_h}"] \arrow[d, "(\rho_h)_i"] & F\times_Y W \arrow[r, "h_\rho"] \arrow[d, "\rho_h"] & F \arrow[d, "\rho"] \\
Z \arrow[r, "i"] & W \arrow[r, "h"] & Y
\end{tikzcd}$$

Then, a composition $g^*(f^*(\boldsymbol{F})) \xrightarrow{c_{f,g}(\boldsymbol{F})} (fg)^*(\boldsymbol{F}) = (hi)^*(\boldsymbol{F}) \xrightarrow{c_{h,i}(\boldsymbol{F})^{-1}} i^*(h^*(\boldsymbol{F}))$ is given as follows.

$$\langle ((f_\rho g_{\rho_f}, i(\rho_f)_g), (\rho_f)_g), id_Z\rangle : g^*(f^*(\boldsymbol{F})) = ((F\times_Y X)\times_X Z \xrightarrow{(\rho_f)_g} Z) \to ((F\times_Y W)\times_W Z \xrightarrow{(\rho_h)_i} Z) = i^*(h^*(\boldsymbol{F}))$$

Proof. Consider the following cartesian squares.

$$\begin{tikzcd}
F\times_Y Z \arrow[r, "(fg)_\rho"] \arrow[d, "\rho_{fg}"] & F \arrow[d, "\rho"] \\
Z \arrow[r, "fg"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
F\times_Y Z \arrow[r, "(hi)_\rho"] \arrow[d, "\rho_{hi}"] & F \arrow[d, "\rho"] \\
Z \arrow[r, "hi"] & Y
\end{tikzcd}$$

Since $c_{f,g}(\boldsymbol{F}) = \langle (f_\rho g_{\rho_f}, (\rho_f)_g), id_Z\rangle$ and $c_{h,i}(\boldsymbol{F})^{-1} = \langle (id_F \times_Y i, \rho_{hi}), id_Z\rangle$, we have the following equality

$$\begin{aligned} c_{h,i}(\boldsymbol{F})^{-1}c_{f,g}(\boldsymbol{F}) &= \langle (id_F \times_Y i, \rho_{hi})(f_\rho g_{\rho_f}, (\rho_f)_g), id_Z\rangle = \langle ((id_F \times_Y i)(f_\rho g_{\rho_f}, (\rho_f)_g), \rho_{hi}(f_\rho g_{\rho_f}, (\rho_f)_g), id_Z\rangle \\ &= \langle ((f_\rho g_{\rho_f}, i(\rho_f)_g), (\rho_f)_g), id_Z\rangle \end{aligned}$$

by (2.4.1). $\square$

Let $f : X \to Y$, $g : X \to Z$, $k : V \to X$ be morphisms in $\mathcal{E}$ and $\boldsymbol{E} = (E \xrightarrow{\pi} Y)$, $\boldsymbol{F} = (F \xrightarrow{\rho} Z)$ objects of $\mathcal{E}^{(2)}_Y$, $\mathcal{E}^{(2)}_Z$, respectively. Consider the following cartesian squares.

$$\begin{tikzcd} E \times_Y X \arrow[r, "f_\pi"] \arrow[d, "\pi_f"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} E \times_Y V \arrow[r, "(fk)_\pi"] \arrow[d, "\pi_{fk}"'] & E \arrow[d, "\pi"] \\ V \arrow[r, "fk"] & Y \end{tikzcd} \qquad \begin{tikzcd} F \times_Z X \arrow[r, "g_\rho"] \arrow[d, "\rho_g"'] & F \arrow[d, "\rho"] \\ X \arrow[r, "g"] & Z \end{tikzcd} \qquad \begin{tikzcd} (F \times_Z X) \times_X V \arrow[r, "k_{\rho_g}"] \arrow[d, "(\rho_g)_k"'] & F \times_Z X \arrow[d, "\rho_g"] \\ V \arrow[r, "k"] & X \end{tikzcd}$$

Then, there exists unique morphism $id_E \times_Y k : E \times_Y V \to E \times_Y X$ that satisfies $f_\pi(id_E \times_Y k) = (fk)_\pi$ and $\pi_f(id_E \times_Y k) = k\pi_{fk}$. The natural transformation $k^\sharp : F_{f,g} \to F_{fk,gk}$ defined in the paragraph before (1.1.15) is described as follows.

Proposition 2.4.7 *$k^\sharp_{\boldsymbol{E},\boldsymbol{F}} : \mathcal{E}^{(2)}_X(f^*(\boldsymbol{E}), g^*(\boldsymbol{F})) \to \mathcal{E}^{(2)}_V((fk)^*(\boldsymbol{E}), (gk)^*(\boldsymbol{F}))$ maps $\langle \varphi, id_X\rangle \in \mathcal{E}^{(2)}_X(f^*(\boldsymbol{E}), g^*(\boldsymbol{F}))$ to $\langle (g_\rho\varphi(id_E \times_Y k), \pi_{fk}), id_V\rangle$.*

Proof. By the proof of (2.4.3), $c_{f,k}(\boldsymbol{E}) = \langle (id_F \times_Y k, \pi_{fk}), id_V\rangle$ and $c_{g,k}(\boldsymbol{F})^{-1} = \langle (g_\rho k_{\rho_g}, (\rho_g)_k), id_V\rangle$ hold. Hence we have $k^\sharp_{\boldsymbol{E},\boldsymbol{F}}(\langle \varphi, id_X\rangle) = \langle (g_\rho k_{\rho_g}, (\rho_g)_k)(\varphi \times_X id_V)(id_E \times_Y k, \pi_{fk}), id_V\rangle$. It follows from (2.4.1) that the following equalities hold.

$$\begin{aligned} (g_\rho k_{\rho_g}, (\rho_g)_k)(\varphi \times_X id_V)(id_E \times_Y k, \pi_{fk}) &= (g_\rho k_{\rho_g}, (\rho_g)_k)(\varphi(id_E \times_Y k), \pi_{fk}) \\ &= (g_\rho k_{\rho_g}(\varphi(id_E \times_Y k), \pi_{fk}), (\rho_g)_k(\varphi(id_E \times_Y k), \pi_{fk})) \\ &= (g_\rho\varphi(id_E \times_Y k), \pi_{fk}) \end{aligned}$$

$$\begin{tikzcd} E \times_Y V \arrow[rrrd, bend left, "(fk)_\pi"] \arrow[rrd, "id_E \times_Y k"] \arrow[rd, "(id_E \times_Y k, \pi_{fk})"'] \arrow[rdd, bend right, "\pi_{fk}"'] & & & & (F \times_Z X) \times_X V \arrow[r, "k_{\rho_g}"] \arrow[rd, "(g_\rho k_{\rho_g}, (\rho_g)_k)"'] \arrow[rdd, bend right, "(\rho_g)_k"'] & F \times_Z X \arrow[rd, "g_\rho"] & \\ & (E \times_Y X) \times_X V \arrow[r, "k_{\pi_f}"'] \arrow[d, "(\pi_f)_k"] \arrow[rrru, dotted, "\varphi \times_X id_V"] & E \times_Y X \arrow[r, "f_\pi"'] \arrow[d, "\pi_f"] \arrow[rrru, dotted, "\varphi"] & E \arrow[d, "\pi"] & & F \times_Z V \arrow[r, "(gk)_\rho"] \arrow[d, "\rho_{gk}"] & F \arrow[d, "\rho"] \\ & V \arrow[r, "k"] & X \arrow[r, "f"] & Y & & V \arrow[r, "gk"] & Z \end{tikzcd}$$

$\square$

Proposition 2.4.8 *The fibered category $p : \mathcal{E}^{(2)} \to \mathcal{E}$ given in (2.4.3) is a bifibered category.*

Proof. For a morphism $f : X \to Y$ of $\mathcal{E}$, define a functor $f_* : \mathcal{E}^{(2)}_X \to \mathcal{E}^{(2)}_Y$ by $f_*(\boldsymbol{E}) = (E \xrightarrow{f\pi} Y)$ for $\boldsymbol{E} = (E \xrightarrow{\pi} X) \in \operatorname{Ob}\mathcal{E}^{(2)}_X$ and $f_*(\langle \varphi, id_X\rangle) = \langle \varphi, id_Y\rangle$ for a morphism $\langle \varphi, id_X\rangle : \boldsymbol{E} \to \boldsymbol{F}$ of $\mathcal{E}^{(2)}_X$.

For $\boldsymbol{F} = (F \xrightarrow{\rho} Y) \in \operatorname{Ob}\mathcal{E}^{(2)}_Y$, let $F \xleftarrow{f_\rho} F \times_Y X \xrightarrow{\rho_f} X$ be a limit of a diagram $F \xrightarrow{\rho} Y \xleftarrow{f} X$. Then, for an object $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ of $\mathcal{E}^{(2)}$, we have

$$\mathcal{E}^{(2)}_Y(f_*(\boldsymbol{E}), \boldsymbol{F}) = \{\langle \varphi, id_Y\rangle \,|\, \varphi \in \mathcal{E}(E, F),\ \rho\varphi = f\pi\}, \quad \mathcal{E}^{(2)}_X(\boldsymbol{E}, f^*(\boldsymbol{F})) = \{\langle \psi, id_X\rangle \,|\, \psi \in \mathcal{E}(E, F \times_Y X),\ \rho_f\psi = \pi\}$$

and define a map $\Psi : \mathcal{E}^{(2)}_X(\boldsymbol{E}, f^*(\boldsymbol{F})) \to \mathcal{E}^{(2)}_Y(f_*(\boldsymbol{E}), \boldsymbol{F})$ by $\Psi(\langle \psi, id_X\rangle) = \langle f_\rho\psi, id_Y\rangle$. Since the inverse of Ψ is given by $\Psi^{-1}(\langle \varphi, id_Y\rangle) = \langle (\varphi, \pi), id_X\rangle$, Ψ is bijective and f_* is a left adjoint of f^*. $\square$

Remark 2.4.9 *The counit $\varepsilon_f : f_*f^* \to id_{\mathcal{E}^{(2)}_Y}$ of the above adjunction is given by $(\varepsilon_f)_{\boldsymbol{F}} = \langle f_\rho, id_Y\rangle : f_*f^*(\boldsymbol{F}) = (F \times_Y X \xrightarrow{f\rho_f} Y) \to \boldsymbol{F}$ for an object $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ of $\mathcal{E}^{(2)}_Y$. The unit $\eta_f : id_{\mathcal{E}^{(2)}_X} \to f^*f_*$ is given as follows. For an object $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ of $\mathcal{E}^{(2)}_X$, let $E \xleftarrow{f_{f\pi}} E \times_Y X \xrightarrow{(f\pi)_f} X$ be a limit of $E \xrightarrow{f\pi} Y \xleftarrow{f} X$. Then, $(\eta_f)_{\boldsymbol{E}} = \langle (id_E, \pi), id_X\rangle : \boldsymbol{E} \to (E \times_Y X \xrightarrow{\pi_f} X) = f^*f_*(\boldsymbol{E})$.*

$$\begin{tikzcd}
E \arrow[dr, "(id_E,\pi)"] \arrow[drr, bend left, "id_E"] \arrow[ddr, bend right, "\pi"'] & & \\
& E\times_Y X \arrow[r, "f_{f\pi}"] \arrow[d, "(f\pi)_f"] & E \arrow[d, "f\pi"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

We consider the bifibered category $p : \mathcal{E}^{(2)} \to \mathcal{E}$ for the rest of this subsection. The following fact is a direct consequence of (1.3.3) and (2.4.9).

Proposition 2.4.10 *Let $f : X \to Y$, $g : X \to Z$ be morphisms in $\mathcal{E}$ and $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$, $\boldsymbol{G} = (G \xrightarrow{\rho} Z)$ objects of $\mathcal{E}^{(2)}_Y$, $\mathcal{E}^{(2)}_Z$, respectively. Suppose that the following diagrams are cartesian.*

$$\begin{tikzcd}
F\times_Y X \arrow[r, "f_\rho"] \arrow[d, "\rho_f"] & F \arrow[d, "\rho"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
G\times_Z X \arrow[r, "g_\pi"] \arrow[d, "\pi_g"] & G \arrow[d, "\pi"] \\
X \arrow[r, "g"] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
(F\times_Y X)\times_Z X \arrow[r, "g_{g\rho_f}"] \arrow[d, "(g\rho_f)_g"] & F\times_Y X \arrow[d, "g\rho_f"] \\
X \arrow[r, "g"] & Z
\end{tikzcd}$$

(1) (f, g) is a left fibered representable pair, namely, $\boldsymbol{F}_{[f,g]} = g_* f^*(\boldsymbol{F})$.

(2) $P_{f,g}(\boldsymbol{F})_{\boldsymbol{G}} : \mathcal{E}^{(2)}_X(f^*(\boldsymbol{F}), g^*(\boldsymbol{G})) \to \mathcal{E}^{(2)}_Z(\boldsymbol{F}_{[f,g]}, \boldsymbol{G})$ maps $\langle \varphi, id_X\rangle : (F\times_Y X \xrightarrow{\rho_f} X) \to (G\times_Z X \xrightarrow{\pi_g} X)$ to $\langle g_\pi \varphi, id_Z\rangle : (F\times_Y X \xrightarrow{g\rho_f} Z) \to (G \xrightarrow{\pi} Z)$.

(3) $\iota_{f,g}(\boldsymbol{F}) = (\eta_g)_{f^*(\boldsymbol{F})} : f^*(\boldsymbol{F}) \to g^* g_* f^*(\boldsymbol{F}) = g^*(\boldsymbol{F}_{[f,g]})$ is given by

$$\langle (id_{F\times_Y X}, \rho_f), id_X\rangle : (F\times_Y X \xrightarrow{\rho_f} X) \to ((F\times_Y X)\times_Y X \xrightarrow{(g\rho_f)_g} X).$$

$$\begin{tikzcd}
F\times_Y X \arrow[dr, "{(id_{F\times_Y X},\rho_f)}"] \arrow[drr, bend left, "id_{F\times_Y X}"] \arrow[ddd, bend right, "\rho_f"'] & & & \\
& (F\times_Y X)\times_Z X \arrow[r, "g_{g\rho_f}"] \arrow[dd, "(g\rho_f)_g"] & F\times_Y X \arrow[r, "f_\rho"] \arrow[d, "\rho_f"] & F \arrow[d, "\rho"] \\
& & X \arrow[r, "f"] \arrow[d, "g"] & Y \\
& X \arrow[r, "g"] & Z &
\end{tikzcd}$$

(4) $P_{f,g}(\boldsymbol{F})^{-1}_{\boldsymbol{G}} : \mathcal{E}^{(2)}_Z(\boldsymbol{F}_{[f,g]}, \boldsymbol{G}) \to \mathcal{E}^{(2)}_X(f^*(\boldsymbol{F}), g^*(\boldsymbol{G}))$ maps $\langle \psi, id_Z\rangle$ to $\langle (\psi, \rho_f), id_X\rangle$, where $\psi : F\times_Y X \to G$.

We have the following result from (1.3.5) and (1.3.8).

Proposition 2.4.11 *Let $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ be an object of $\mathcal{E}^{(2)}_Y$ and $f : X \to Y$, $g : X \to Z$ morphisms in $\mathcal{E}$. Let $X \xleftarrow{\rho_f} F\times_Y X \xrightarrow{f_\rho} F$ be a limit of $X \xrightarrow{f} Y \xleftarrow{\rho} F$.*

(1) For an object $\boldsymbol{E} = (E \xrightarrow{\pi} Y)$ of $\mathcal{E}^{(2)}_Y$, let $X \xleftarrow{\pi_f} E\times_Y X \xrightarrow{f_\pi} E$ be a limit of $X \xrightarrow{f} Y \xleftarrow{\pi} E$. For a morphism $\boldsymbol{\varphi} = \langle \varphi, id_X\rangle : \boldsymbol{E} \to \boldsymbol{F}$ of $\mathcal{E}^{(2)}_Y$, $\boldsymbol{\varphi}_{[f,g]} : \boldsymbol{E}_{[f,g]} \to \boldsymbol{F}_{[f,g]}$ is given by $\boldsymbol{\varphi}_{[f,g]} = \langle \varphi\times_Y id_X, id_Z\rangle$.

(2) For a morphism $k : V \to X$ of $\mathcal{E}$, let $V \xleftarrow{\rho_{fk}} F\times_Y V \xrightarrow{(fk)_\rho} F$ be a limit of $V \xrightarrow{fk} Y \xleftarrow{\rho} F$. Then, $\boldsymbol{F}_k : \boldsymbol{F}_{[fk,gk]} \to \boldsymbol{F}_{[f,g]}$ is given by $\boldsymbol{F}_k = \langle id_F \times_Y k, id_Z\rangle$.

$$\begin{tikzcd}
& E\times_Y X \arrow[rr, "f_\pi"] \arrow[dr, dashed, "\varphi\times_Y id_X"] \arrow[ddr, "\pi_f"'] & & E \arrow[d, "\varphi"'] \arrow[dd, bend left, "\pi"] \\
& & F\times_Y X \arrow[r, "f_\rho"] \arrow[d, "\rho_f"] & F \arrow[d, "\rho"'] \\
Z & & X \arrow[ll, "g"'] \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
F\times_Y V \arrow[d, "\rho_{fk}"'] \arrow[dr, dashed, "id_F\times_Y k"] \arrow[drr, bend left, "(fk)_\rho"] & & \\
V \arrow[dr, "k"] & F\times_Y X \arrow[r, "f_\rho"] \arrow[d, "\rho_f"] & F \arrow[d, "\rho"] \\
Z & X \arrow[l, "g"'] \arrow[r, "f"] & Y
\end{tikzcd}$$

It follows from (1.3.12) and (2.4.9) that we have the following fact.

Proposition 2.4.12 *Let $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ be an object of $\mathcal{E}^{(2)}_Y$. For morphisms $f : X \to Y$, $g : X \to Z$, $h : X \to W$ of $\mathcal{E}$, let $X \xleftarrow{\rho_f} F \times_Y X \xrightarrow{f_\rho} F$ be a limit of $X \xrightarrow{f} Z \xleftarrow{\rho} F$, $X \xleftarrow{\mathrm{pr}_2} X \times_Y X \xrightarrow{\mathrm{pr}_1} X$ a limit of $X \xrightarrow{g} Z \xleftarrow{g} X$ and $X \xleftarrow{(g\rho_f)_g} (F \times_Y X) \times_Z X \xrightarrow{g_{g\rho_f}} F \times_Y X$ a limit of $X \xrightarrow{g} Z \xleftarrow{g\rho_f} F \times_Y X$. Then, $\delta_{f,g,h,\boldsymbol{F}} = h_*\big((\eta_g)_{f^*(\boldsymbol{F})}\big) : \boldsymbol{F}_{[f,h]} = h_* f^*(\boldsymbol{F}) \to h_* g^* g_* f^*(\boldsymbol{F}) = (\boldsymbol{F}_{[f,g]})_{[g,h]}$ is given by*

$$\langle (id_{F\times_Y X}, \rho_f), id_W\rangle : (F \times_Y X \xrightarrow{h\rho_f} W) \to ((F \times_Y X) \times_Z X \xrightarrow{h\mathrm{pr}_2(\rho_f \times_Y id_X)} W).$$

$$\begin{tikzcd}
F \times_Y X \arrow[drrr, bend left, "id_{F\times_Y X}"] \arrow[dr, "(id_{F\times_Y X}, \rho_f)"] \arrow[dddr, bend right, "\rho_f"'] & & & & \\
 & (F \times_Y X) \times_Z X \arrow[rr, "g_{g\rho_f}"] \arrow[dd, "(g\rho_f)_g"] \arrow[dr, "\rho_f \times_Z id_X"] & & F \times_Y X \arrow[r, "f_\rho"] \arrow[d, "\rho_f"] & F \arrow[d, "\rho"] \\
 & & X \times_Z X \arrow[r, "\mathrm{pr}_1"] \arrow[dl, "\mathrm{pr}_2"'] & X \arrow[r, "f"] \arrow[d, "g"] & Y \\
W & X \arrow[l, "h"'] \arrow[rr, "g"] & & Z &
\end{tikzcd}$$

For a functor $D : \mathcal{P} \to \mathcal{E}$ and an object $\boldsymbol{F} = (F \xrightarrow{\rho} D(3))$ of $\mathcal{E}^{(2)}_{D(3)}$, we put $D(\tau_{ij}) = f_{ij}$ and consider the following cartesian squares.

$$\begin{tikzcd}
F \times_{D(3)} D(1) \arrow[r, "(f_{13})_\rho"] \arrow[d, "\rho_{f_{13}}"] & F \arrow[d, "\rho"] \\
D(1) \arrow[r, "f_{13}"] & D(3)
\end{tikzcd}
\qquad
\begin{tikzcd}
F \times_{D(3)} D(0) \arrow[r, "(f_{13}f_{01})_\rho"] \arrow[d, "\rho_{f_{13}f_{01}}"] & F \arrow[d, "\rho"] \\
D(0) \arrow[r, "f_{13}f_{01}"] & D(3)
\end{tikzcd}$$

$$\begin{tikzcd}
(F \times_{D(3)} D(0)) \times_{D(4)} D(0) \arrow[r, "(f_{14}f_{01})_{f_{14}f_{01}\rho_{f_{13}f_{01}}}"] \arrow[d, "(f_{14}f_{01}\rho_{f_{13}f_{01}})_{f_{14}f_{01}}"] & F \times_{D(3)} D(0) \arrow[d, "f_{14}f_{01}\rho_{f_{13}f_{01}}"] \\
D(0) \arrow[r, "f_{14}f_{01}"] & D(4)
\end{tikzcd}$$

Then, we have uniuqe morphisms $id_F \times_{D(3)} f_{01} : F \times_{D(3)} D(0) \to F \times_{D(3)} D(1)$ and $(id_{F\times_{D(3)}D(0)}, \rho_{f_{13}f_{01}}) : F \times_{D(3)} D(0) \to (F \times_{D(3)} D(0)) \times_{D(4)} D(0)$ that make the following diagrams commute.

$$\begin{tikzcd}
F \times_{D(3)} D(0) \arrow[dr, dashed, "id_F \times_{D(3)} f_{01}"] \arrow[drr, bend left, "(f_{13}f_{01})_\rho"] \arrow[dd, "\rho_{f_{13}f_{01}}"'] & & \\
 & F \times_{D(3)} D(1) \arrow[r, "(f_{13})_\rho"'] \arrow[d, "\rho_{f_{13}}"] & F \arrow[d, "\rho"] \\
D(0) \arrow[r, "f_{01}"] & D(1) \arrow[r, "f_{13}"] & D(3)
\end{tikzcd}$$

$$\begin{tikzcd}
F \times_{D(3)} D(0) \arrow[dr, dashed, "(id_{F\times_{D(3)}D(0)}, \rho_{f_{13}f_{01}})"] \arrow[drr, bend left, "id_{F\times_{D(3)}D(0)}"] \arrow[ddr, bend right, "\rho_{f_{13}f_{01}}"'] & & \\
 & (F \times_{D(3)} D(0)) \times_{D(4)} D(0) \arrow[r, "(f_{24}f_{02})_{f_{14}f_{01}\rho_{f_{13}f_{01}}}"] \arrow[d, "(f_{14}f_{01}\rho_{f_{13}f_{01}})_{f_{24}f_{02}}"] & F \times_{D(3)} D(0) \arrow[d, "f_{14}f_{01}\rho_{f_{13}f_{01}}"] \\
 & D(0) \arrow[r, "f_{24}f_{02} = f_{14}f_{01}"] & D(4)
\end{tikzcd}$$

Then, it follows from (2.4.12) that $\delta_{f_{13}f_{01},f_{14}f_{01},f_{25}f_{02},\boldsymbol{F}} : \boldsymbol{F}_{[f_{13}f_{01},f_{25}f_{02}]} \to (\boldsymbol{F}_{[f_{13}f_{01},f_{14}f_{01}]})_{[f_{24}f_{02},f_{25}f_{02}]}$ is given by $\delta_{f_{13}f_{01},f_{14}f_{01},f_{25}f_{02},\boldsymbol{F}} = \langle (id_{F\times_{D(3)}D(0)}, \rho_{f_{13}f_{01}}), id_{D(5)}\rangle$, where

$$\boldsymbol{F}_{[f_{13}f_{01},f_{25}f_{02}]} = (F \times_{D(3)} D(0) \xrightarrow{f_{25}f_{02}\rho_{f_{13}f_{01}}} D(5))$$
$$(\boldsymbol{F}_{[f_{13}f_{01},f_{14}f_{01}]})_{[f_{24}f_{02},f_{25}f_{02}]} = ((F \times_{D(3)} D(0)) \times_{D(4)} D(0) \xrightarrow{f_{25}f_{02}(f_{14}f_{01}\rho_{f_{13}f_{01}})_{f_{24}f_{02}}} D(5)).$$

Consider the following cartesian squares.

$$\begin{tikzcd}
(F \times_{D(3)} D(0)) \times_{D(4)} D(2) \arrow[r, "(f_{24})_{f_{14}f_{01}\rho_{f_{13}f_{01}}}"] \arrow[d, "(f_{14}f_{01}\rho_{f_{13}f_{01}})_{f_{24}}"] & F \times_{D(3)} D(0) \arrow[d, "f_{14}f_{01}\rho_{f_{13}f_{01}}"] \\
D(2) \arrow[r, "f_{24}"] & D(4)
\end{tikzcd}$$

$$\begin{tikzcd}
(F \times_{D(3)} D(0)) \times_{D(4)} D(0) \arrow[r, "(f_{24}f_{02})_{f_{14}f_{01}\rho_{f_{13}f_{01}}}"] \arrow[d, "(f_{14}f_{01}\rho_{f_{13}f_{01}})_{f_{24}f_{02}}"] & F \times_{D(3)} D(0) \arrow[d, "f_{14}f_{01}\rho_{f_{13}f_{01}}"] \\
D(0) \arrow[r, "f_{24}f_{02}"] & D(4)
\end{tikzcd}$$

Then, we have uniuqe morphism $id_{F\times_{D(3)}D(0)} \times_{D(4)} f_{02} : (F \times_{D(3)} D(0)) \times_{D(4)} D(0) \to (F \times_{D(3)} D(0)) \times_{D(4)} D(2)$ that makes the following diagram commute.

$$\begin{tikzcd}
(F \times_{D(3)} D(0)) \times_{D(4)} D(0) \arrow[rrd, bend left, "(f_{24}f_{02})_{f_{14}f_{01}\rho_{f_{13}f_{01}}}"] \arrow[rd, dashed, "id_{F\times_{D(3)}D(0)} \times_{D(4)} f_{02}"] \arrow[dd, "(f_{14}f_{01}\rho_{f_{13}f_{01}})_{f_{24}f_{02}}"] & & \\
& (F \times_{D(3)} D(0)) \times_{D(4)} D(2) \arrow[r, "(f_{24})_{f_{14}f_{01}\rho_{f_{13}f_{01}}}"] \arrow[d, "(f_{14}f_{01}\rho_{f_{13}f_{01}})_{f_{24}}"] & F \times_{D(3)} D(0) \arrow[d, "f_{14}f_{01}\rho_{f_{13}f_{01}}"] \\
D(0) \arrow[r, "f_{02}"] & D(2) \arrow[r, "f_{24}"] & D(4)
\end{tikzcd}$$

It follows from (2) of (2.4.11) that $(\boldsymbol{F}_{[f_{13}f_{01},f_{14}f_{01}]})_{f_{02}} : (\boldsymbol{F}_{[f_{13}f_{01},f_{14}f_{01}]})_{[f_{24}f_{02},f_{25}f_{02}]} \to (\boldsymbol{F}_{[f_{13}f_{01},f_{14}f_{01}]})_{[f_{24},f_{25}]}$ is given by $(\boldsymbol{F}_{[f_{13}f_{01},f_{14}f_{01}]})_{f_{02}} = \langle id_{F\times_{D(3)}D(0)} \times_{D(4)} f_{02}, id_{D(5)}\rangle$, where

$$(\boldsymbol{F}_{[f_{13}f_{01},f_{14}f_{01}]})_{[f_{24},f_{25}]} = ((F \times_{D(3)} D(0)) \times_{D(4)} D(2) \xrightarrow{f_{25}(f_{14}f_{01}\rho_{f_{13}f_{01}})_{f_{24}}} D(5)).$$

We also consider the following cartesian squares.

$$\begin{tikzcd}
(F \times_{D(3)} D(1)) \times_{D(4)} D(2) \arrow[r, "(f_{24})_{f_{14}\rho_{f_{13}}}"] \arrow[d, "(f_{14}\rho_{f_{13}})_{f_{24}}"] & F \times_{D(3)} D(1) \arrow[d, "f_{14}\rho_{f_{13}}"] \\
D(2) \arrow[r, "f_{24}"] & D(4)
\end{tikzcd}$$

$$\begin{tikzcd}
(F \times_{D(3)} D(0)) \times_{D(4)} D(2) \arrow[r, "(f_{24})_{f_{14}\rho_{f_{13}}(id_F \times_{D(3)} f_{01})}"] \arrow[d, "(f_{14}\rho_{f_{13}}(id_F \times_{D(3)} f_{01}))_{f_{24}}"] & F \times_{D(3)} D(0) \arrow[d, "f_{14}\rho_{f_{13}}(id_F \times_{D(3)} f_{01})"] \\
D(2) \arrow[r, "f_{24}"] & D(4)
\end{tikzcd}$$

Then, we have uniuqe morphism $(id_F \times_{D(3)} f_{01}) \times_{D(4)} id_{D(2)} : (F \times_{D(3)} D(0)) \times_{D(4)} D(2) \to (F \times_{D(3)} D(1)) \times_{D(4)} D(2)$ that makes the following diagram commute.

$$\begin{tikzcd}
(F \times_{D(3)} D(0)) \times_{D(4)} D(2) \arrow[rr, "(f_{24})_{f_{14}\rho_{f_{13}}(id_F \times_{D(3)} f_{01})}"] \arrow[rd, dashed, "(id_F \times_{D(3)} f_{01}) \times_{D(4)} id_{D(2)}"] \arrow[rdd, bend right, "(f_{14}\rho_{f_{13}}(id_F \times_{D(3)} f_{01}))_{f_{24}}"'] & & F \times_{D(3)} D(0) \arrow[d, "id_F \times_{D(3)} f_{01}"] \\
& (F \times_{D(3)} D(1)) \times_{D(4)} D(2) \arrow[r, "(f_{24})_{f_{14}\rho_{f_{13}}}"] \arrow[d, "(f_{14}\rho_{f_{13}})_{f_{24}}"] & F \times_{D(3)} D(1) \arrow[d, "f_{14}\rho_{f_{13}}"] \\
& D(2) \arrow[r, "f_{24}"] & D(4)
\end{tikzcd}$$

It follows from (1) of (2.4.11) that $(\boldsymbol{F}_{f_{01}})_{[f_{24},f_{25}]} : (\boldsymbol{F}_{[f_{13}f_{01},f_{14}f_{01}]})_{[f_{24},f_{25}]} \to (\boldsymbol{F}_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}$ is given by $(\boldsymbol{F}_{f_{01}})_{[f_{24},f_{25}]} = \langle (id_F \times_{D(3)} f_{01}) \times_{D(4)} id_{D(2)}, id_{D(5)}\rangle$, where

$$(\boldsymbol{F}_{[f_{13},f_{14}]})_{[f_{24},f_{25}]} = ((F \times_{D(3)} D(1)) \times_{D(4)} D(2) \xrightarrow{f_{25}(f_{14}\rho_{f_{13}})_{f_{24}}} D(5)).$$

Proposition 2.4.13 *The morphism $\theta_D(\boldsymbol{F}) : \boldsymbol{F}_{[f_{13}f_{01},f_{25}f_{02}]} \to (\boldsymbol{F}_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}$ is given by*

$$\theta_D(\boldsymbol{F}) = \langle (id_F \times_{D(3)} f_{01}, f_{02}\rho_{f_{13}f_{01}}), id_{D(5)}\rangle.$$

Proof. The following diagram is commutative by (2.4.1).

$$
\begin{tikzcd}
 & (F \times_{D(3)} D(0)) \times_{D(4)} D(0) \arrow[d, "id_{F\times_{D(3)}D(0)} \times_{D(4)} f_{02}"] \\
F \times_{D(3)} D(0) \arrow[ur, "(id_{F\times_{D(3)}D(0)},\, \rho_{f_{13}f_{01}})"] \arrow[r, "(id_{F\times_{D(3)}D(0)},\, f_{02}\rho_{f_{13}f_{01}})"] \arrow[dr, "(id_F \times_{D(3)} f_{01},\, f_{02}\rho_{f_{13}f_{01}})"'] & (F \times_{D(3)} D(0)) \times_{D(4)} D(2) \arrow[d, "(id_F \times_{D(3)} f_{01}) \times_{D(4)} id_{D(2)}"] \\
 & (F \times_{D(3)} D(1)) \times_{D(4)} D(2)
\end{tikzcd}
$$

Since $(\boldsymbol{F}_{f_{01}})_{f_{02}} = (\boldsymbol{F}_{f_{01}})_{[f_{24},f_{25}]}(\boldsymbol{F}_{[f_{13}f_{01},f_{14}f_{01}]})_{f_{02}}$ and $\theta_D(\boldsymbol{F})$ is a composition

$$
\boldsymbol{F}_{[f_{13}f_{01},f_{25}f_{02}]} \xrightarrow{\delta_{f_{13}f_{01},f_{14}f_{01},f_{25}f_{02},\boldsymbol{F}}} (\boldsymbol{F}_{[f_{13}f_{01},f_{14}f_{01}]})_{[f_{24}f_{02},f_{25}f_{02}]} \xrightarrow{(\boldsymbol{F}_{f_{01}})_{f_{02}}} (\boldsymbol{F}_{[f_{13},f_{14}]})_{[f_{24},f_{25}]},
$$

the assertion follows from the argument above. $\square$

Remark 2.4.14 *Suppose that the outer trapezoid and the lower rectangle of the following diagram are cartesian. There is unique morphism* $\rho_{f_{13}} \times_{D(4)} id_{D(2)} : (F \times_{D(3)} D(1)) \times_{D(4)} D(2) \to D(1) \times_{D(4)} D(2)$ *that makes the following diagram commute and the upper trapezoid is cartesian.*

$$
\begin{tikzcd}
(F \times_{D(3)} D(1)) \times_{D(4)} D(2) \arrow[rr, "(f_{24})_{f_{14}\rho_{f_{13}}}"] \arrow[dr, dashed, "\rho_{f_{13}} \times_{D(4)} id_{D(2)}"] \arrow[ddr, "(f_{14}\rho_{f_{13}})_{f_{24}}"'] & & F \times_{D(3)} D(1) \arrow[d, "\rho_{f_{13}}"] \\
 & D(1) \times_{D(4)} D(2) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & D(1) \arrow[d, "f_{14}"] \\
 & D(2) \arrow[r, "f_{24}"] & D(4)
\end{tikzcd}
$$

Thus the following diagram is commutative. Since the upper trapezoid of the above diagram is cartesian and $(f_{14}\rho_{f_{13}})_{f_{24}} = \mathrm{pr}_2(\rho_{f_{13}} \times_{D(4)} id_{D(2)})$*,* $(id_F \times_{D(3)} f_{01}, f_{02}\rho_{f_{13}f_{01}}) : F \times_{D(3)} D(0) \to (F \times_{D(3)} D(1)) \times_{D(4)} D(2)$ *coincides with* $(id_F \times_{D(3)} f_{01}, (f_{01}, f_{02})\rho_{f_{13}f_{01}})$*. We also note that since* $\mathrm{pr}_1(f_{01}, f_{02}) = f_{01}$*, the left parallelogram of the following diagram is cartesian . Hence if* $(f_{01}, f_{02}) : D(0) \to D(1) \times_{D(4)} D(2)$ *is an isomorphism, so is* $(id_F \times_{D(3)} f_{01}, f_{02}\rho_{f_{13}f_{01}})$*.*

$$
\begin{tikzcd}
F \times_{D(3)} D(0) \arrow[d, "\rho_{f_{13}f_{01}}"] \arrow[dr, "(id_F \times_{D(3)} f_{01},f_{02}\rho_{f_{13}f_{01}})"] \arrow[drr, bend left, "id_F \times_{D(3)} f_{01}"] \arrow[drrr, bend left, "(f_{13}f_{01})_\rho"] & & & \\
D(0) \arrow[dr, "(f_{01}, f_{02})"] \arrow[ddr, bend right, "f_{02}"'] & (F \times_{D(3)} D(1)) \times_{D(4)} D(2) \arrow[r, "(f_{24})_{f_{14}\rho_{f_{13}}}"] \arrow[d, "\rho_{f_{13}} \times_{D(4)} id_{D(2)}"] & F \times_{D(3)} D(1) \arrow[r, "(f_{13})_\rho"'] \arrow[d, "\rho_{f_{13}}"] & F \arrow[d, "\rho"] \\
 & D(1) \times_{D(4)} D(2) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & D(1) \arrow[r, "f_{13}"] \arrow[d, "f_{14}"] & D(3) \\
 & D(2) \arrow[r, "f_{24}"] & D(4) &
\end{tikzcd}
$$

Proposition 2.4.15 *For any morphisms* $f : X \to Y$*,* $g : X \to Z$*,* $h : V \to Z$*,* $i : V \to W$ *in* $\mathcal{E}$*,* (f, g, h, i) *is an associative left fibered representable quadruple.*

Proof. Let $V \xleftarrow{\mathrm{pr}_V} X \times_Z V \xrightarrow{\mathrm{pr}_X} X$ be a limit of $V \xrightarrow{h} Z \xleftarrow{g} X$ and define a functor $D : \mathcal{P} \to \mathcal{E}$ by $D(0) = X \times_Z V$, $D(1) = X$, $D(2) = V$, $D(3) = Y$, $D(4) = Z$, $D(5) = W$ and $D(\tau_{01}) = \mathrm{pr}_X$, $D(\tau_{02}) = \mathrm{pr}_V$, $D(\tau_{13}) = f$, $D(\tau_{14}) = g$, $D(\tau_{24}) = h$, $D(\tau_{25}) = i$. In other words, $f_{01} = \mathrm{pr}_X$, $f_{02} = \mathrm{pr}_V$, $f_{13} = f$, $f_{14} = g$, $f_{24} = h$, $f_{25} = i$. Then, $(f_{01}, f_{02}) = (\mathrm{pr}_X, \mathrm{pr}_V) : D(0) = X \times_Z V \to X \times_Z V = D(1) \times_{D(3)} D(2)$ is the identity morphism, hence an isomorphism. For an object $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ of $\mathcal{E}^{(2)}$, it follows from (2.4.13) and (2.4.14) that $\theta_{f,g,h,i}(\boldsymbol{F}) = \langle(id_F \times_Y \mathrm{pr}_X, \mathrm{pr}_V \rho_{f\mathrm{pr}_X}), id_W\rangle : (F \times_Y (X \times_Z V) \xrightarrow{i\mathrm{pr}_V \rho_{f\mathrm{pr}_X}} W) \to ((F \times_Y X) \times_Z V \xrightarrow{i(g\rho_f)_h} W)$ is an isomorphism. $\square$

Remark 2.4.16 *For an object* $\boldsymbol{F}=(F\xrightarrow{\rho}Y)$ *of* $\mathcal{E}^{(2)}$*, consider the following cartesian squares.*

$$\begin{tikzcd} F\times_Y X \arrow[r, "f_\rho"] \arrow[d, "\rho_f"] & F \arrow[d, "\rho"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \quad \begin{tikzcd} X\times_Z V \arrow[r, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_V"] & X \arrow[d, "g"] \\ V \arrow[r, "h"] & Z \end{tikzcd} \quad \begin{tikzcd} F\times_Y(X\times_Z V) \arrow[r, "(f\mathrm{pr}_X)_\rho"] \arrow[d, "\rho_{f\mathrm{pr}_X}"] & F \arrow[d, "\rho"] \\ X\times_Z V \arrow[r, "f\mathrm{pr}_X"] & Y \end{tikzcd} \quad \begin{tikzcd} (F\times_Y X)\times_Z V \arrow[r, "h_{g\rho_f}"] \arrow[d, "(g\rho_f)_h"] & F\times_Y X \arrow[d, "g\rho_f"] \\ V \arrow[r, "h"] & Z \end{tikzcd}$$

Then, we have the following commutative diagrams.

$$\begin{tikzcd} (F\times_Y X)\times_Z V \arrow[r, "h_{g\rho_f}"] \arrow[d, "\rho_f\times_Z id_V"] \arrow[dd, bend right, "(g\rho_f)_h"'] & F\times_Y X \arrow[d, "\rho_f"] \\ X\times_Z V \arrow[r, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_V"] & X \arrow[d, "g"] \\ V \arrow[r, "h"] & Z \end{tikzcd} \qquad \begin{tikzcd} (F\times_Y X)\times_Z V \arrow[rrr, "h_{g\rho_f}"] \arrow[dr, dashed, "(f_\rho h_{g\rho_f},\rho_f\times_Z id_V)"] \arrow[ddr, "\rho_f\times_Z id_V"'] & & & F\times_Y X \arrow[dr, "f_\rho"] \arrow[dd, "\rho_f"] & \\ & F\times_Y(X\times_Z V) \arrow[urr, "id_F\times_Y\mathrm{pr}_X"] \arrow[rrrr, "(f\mathrm{pr}_X)_\rho"] \arrow[d, "\rho_{f\mathrm{pr}_X}"] & & & & F \arrow[d, "\rho"] \\ & X\times_Z V \arrow[rr, "\mathrm{pr}_X"] & & X \arrow[r, "f"] & Y \end{tikzcd}$$

Hence the inverse of $\theta_{f,g,h,i}(\boldsymbol{F})$ *is given by* $\langle(f_\rho h_{g\rho_f},\rho_f\times_Z id_V),id_W\rangle$.

Let $D,E:\mathcal{Q}\to\mathcal{E}$ be functors and $\omega:D\to E$ a natural transformation. We put $D(\tau_{0j})=f_j$ and $E(\tau_{0j})=g_j$ for $j=1,2$. For an object $\boldsymbol{F}=(F\xrightarrow{\rho}E(1))$ of $\mathcal{E}^{(2)}_{E(1)}$, we consider the following cartesian squares.

$$\begin{tikzcd} F\times_{E(1)}E(0) \arrow[r, "(g_1)_\rho"] \arrow[d, "\rho_{g_1}"] & F \arrow[d, "\rho"] \\ E(0) \arrow[r, "g_1"] & E(1) \end{tikzcd} \quad \begin{tikzcd} F\times_{E(1)}D(0) \arrow[r, "(g_1\omega_0)_\rho"] \arrow[d, "\rho_{g_1\omega_0}=\rho_{\omega_1 f_1}"] & F \arrow[d, "\rho"] \\ D(0) \arrow[r, "g_1\omega_0"] & E(1) \end{tikzcd} \quad \begin{tikzcd} (F\times_{E(1)}E(0))\times_{E(2)}E(0) \arrow[r, "(g_2)_{g_2\rho_{g_1}}"] \arrow[d, "(g_2\rho_{g_1})_{g_2}"] & F\times_{E(1)}E(0) \arrow[d, "g_2\rho_{g_1}"] \\ E(0) \arrow[r, "g_2"] & E(2) \end{tikzcd}$$

Lemma 2.4.17 *The image of* $\iota_{g_1,g_2}(\boldsymbol{F})$ *by the map*

$$\omega_0^\sharp:\mathcal{E}^{(2)}_{E(0)}(g_1^*(\boldsymbol{F}),g_2^*(\boldsymbol{F}_{[g_1,g_2]}))\to\mathcal{E}^{(2)}_{D(0)}((g_1\omega_0)^*(\boldsymbol{F}),(g_2\omega_0)^*(\boldsymbol{F}_{[g_1,g_2]}))=\mathcal{E}^{(2)}_{D(0)}((\omega_1 f_1)^*(\boldsymbol{F}),(\omega_2 f_2)^*(\boldsymbol{F}_{[g_1,g_2]}))$$

is $\langle(id_F\times_{E(1)}\omega_0,\rho_{g_1\omega_0}),id_{D(0)}\rangle:(F\times_{E(1)}D(0)\xrightarrow{\rho_{\omega_1 f_1}}D(0))\to((F\times_{E(1)}E(0))\times_{E(2)}D(0)\xrightarrow{(g_2\rho_{g_1})_{\omega_2 f_2}}D(0))$.

Proof. We recall from (2.4.10) that $\iota_{g_1,g_2}(\boldsymbol{F}):g_1^*(\boldsymbol{F})\to g_2^*(\boldsymbol{F}_{[g_1,g_2]})$ is given by

$$\langle(id_{F\times_{E(1)}E(0)},\rho_{g_1}),id_{E(0)}\rangle:(F\times_{E(1)}E(0)\xrightarrow{\rho_{g_1}}E(0))\to((F\times_{E(1)}E(0))\times_{E(1)}E(0)\xrightarrow{(g_2\rho_{g_1})_{g_2}}E(0)).$$

Hence (2.4.7) implies the following.

$$\begin{aligned}\omega_0^\sharp(\iota_{g_1,g_2}(\boldsymbol{F}))&=\langle((g_2)_{g_2\rho_{g_1}}(id_{F\times_{E(1)}E(0)},\rho_{g_1})(id_F\times_{E(1)}\omega_0),\rho_{g_1\omega_0}),id_{D(0)}\rangle\\&=\langle((g_2)_{g_2\rho_{g_1}}(id_F\times_{E(1)}\omega_0,\rho_{g_1}(id_F\times_{E(1)}\omega_0)),\rho_{g_1\omega_0}),id_{D(0)}\rangle\\&=\langle((g_2)_{g_2\rho_{g_1}}(id_F\times_{E(1)}\omega_0,\omega_0\rho_{g_1\omega}),\rho_{g_1\omega_0}),id_{D(0)}\rangle=\langle(id_F\times_{E(1)}\omega_0,\rho_{g_1\omega_0}),id_{D(0)}\rangle\end{aligned}$$ □

We consider the following diagrams, where the left one is cartesian and the both rectangles of the left one are cartesian.

$$\begin{tikzcd} F\times_{E(1)}D(0) \arrow[r, "(\omega_1 f_1)_\rho"] \arrow[d, "\rho_{\omega_1 f_1}"] & F \arrow[d, "\rho"] \\ D(0) \arrow[r, "\omega_1 f_1"] & E(1) \end{tikzcd} \qquad \begin{tikzcd} (F\times_{E(1)}D(1))\times_{D(1)}D(0) \arrow[r, "(f_1)_{\rho_{\omega_1}}"] \arrow[d, "(\rho_{\omega_1})_{f_1}"] & F\times_{E(1)}D(1) \arrow[r, "(\omega_1)_\rho"] \arrow[d, "\rho_{\omega_1}"] & F \arrow[d, "\rho"] \\ D(0) \arrow[r, "f_1"] & D(1) \arrow[r, "\omega_1"] & E(1) \end{tikzcd}$$

Thus we have an isomorphism $((\omega_1)_\rho(f_1)_{\rho_{\omega_1}},(\rho_{\omega_1})_{f_1}):(F\times_{E(1)}D(1))\times_{D(1)}D(0)\to F\times_{E(1)}D(0)$ which make the following diagram commute.

$$\begin{tikzcd}
(F\times_{E(1)}D(1))\times_{D(1)}D(0) \arrow[dr, "{((\omega_1)_\rho(f_1)_{\rho_{\omega_1}},(\rho_{\omega_1})_{f_1})}"] \arrow[drr, bend left, "(\omega_1)_\rho(f_1)_{\rho_{\omega_1}}"] \arrow[ddr, bend right, "(\rho_{\omega_1})_{f_1}"'] & & \\
& F\times_{E(1)}D(0) \arrow[r, "(\omega_1 f_1)_\rho"] \arrow[d, "\rho_{\omega_1 f_1}"] & F \arrow[d, "\rho"] \\
& D(0) \arrow[r, "\omega_1 f_1"] & E(1)
\end{tikzcd}$$

Suppose that each rectangles of the following diagram is cartesian.

$$\begin{tikzcd}
((F\times_{E(1)}E(0))\times_{E(2)}D(2))\times_{D(2)}D(0) \arrow[r, "(f_2)_{(g_2\rho_{g_1})_{\omega_2}}"] \arrow[dd, "((g_2\rho_{g_1})_{\omega_2})_{f_2}"] & (F\times_{E(1)}E(0))\times_{E(2)}D(2) \arrow[r, "(\omega_2)_{g_2\rho_{g_1}}"] \arrow[dd, "(g_2\rho_{g_1})_{\omega_2}"] & F\times_{E(1)}E(0) \arrow[r, "(g_1)_\rho"] \arrow[d, "\rho_{g_1}"] & F \arrow[d, "\rho"] \\
& & E(0) \arrow[r, "g_1"] \arrow[d, "g_2"] & E(1) \\
D(0) \arrow[r, "f_2"] & D(2) \arrow[r, "\omega_2"] & E(2) &
\end{tikzcd}$$

We also consider the following cartesian square.

$$\begin{tikzcd}
(F\times_{E(1)}E(0))\times_{E(2)}D(0) \arrow[r, "(\omega_2 f_2)_{g_2\rho_{g_1}}"] \arrow[d, "(g_2\rho_{g_1})_{\omega_2 f_2}"] & F\times_{E(1)}E(0) \arrow[d, "g_2\rho_{g_1}"] \\
D(0) \arrow[r, "\omega_2 f_2"] & E(2)
\end{tikzcd}$$

Then we have an isomorphism

$$(id_{F\times_{E(1)}E(0)}\times_{E(2)}f_2,(g_2\rho_{g_1})_{\omega_2 f_2}):(F\times_{E(1)}E(0))\times_{E(2)}D(0)\to((F\times_{E(1)}E(0))\times_{E(2)}D(2))\times_{D(2)}D(0).$$

Hence $c_{\omega_1,f_1}(\boldsymbol{F})^*c_{\omega_2,f_2}(\boldsymbol{F}_{[g_1,g_2]})_*^{-1}:\mathcal{E}^{(2)}_{D(0)}((\omega_1f_1)^*(\boldsymbol{F}),(\omega_2f_2)^*(\boldsymbol{F}_{[g_1,g_2]}))\to\mathcal{E}^{(2)}_{D(0)}(f_1^*(\omega_1^*(\boldsymbol{F})),f_2^*(\omega_2^*(\boldsymbol{F}_{[g_1,g_2]})))$ maps $\omega_0^\sharp(\iota_{g_1,g_2}(\boldsymbol{F}))$ to $\langle(id_{F\times_{E(1)}E(0)}\times_{E(2)}f_2,(g_2\rho_{g_1})_{\omega_2f_2})(id_F\times_{E(1)}\omega_0,\rho_{g_1\omega_0})((\omega_1)_\rho(f_1)_{\rho_{\omega_1}},(\rho_{\omega_1})_{f_1}),id_{D(0)}\rangle$. On the other hand, since

$$\begin{aligned}
&(f_2)_{(g_2\rho_{g_1})_{\omega_2}}(id_{F\times_{E(1)}E(0)}\times_{E(2)}f_2,(g_2\rho_{g_1})_{\omega_2f_2})(id_F\times_{E(1)}\omega_0,\rho_{g_1\omega_0})((\omega_1)_\rho(f_1)_{\rho_{\omega_1}},(\rho_{\omega_1})_{f_1})\\
&\qquad=(id_{F\times_{E(1)}E(0)}\times_{E(2)}f_2)(id_F\times_{E(1)}\omega_0,\rho_{\omega_1f_1})((\omega_1)_\rho(f_1)_{\rho_{\omega_1}},(\rho_{\omega_1})_{f_1})\\
&\qquad=(id_F\times_{E(1)}\omega_0,f_2\rho_{\omega_1f_1})((\omega_1)_\rho(f_1)_{\rho_{\omega_1}},(\rho_{\omega_1})_{f_1})\\
&\qquad=((id_F\times_{E(1)}\omega_0)((\omega_1)_\rho(f_1)_{\rho_{\omega_1}},(\rho_{\omega_1})_{f_1}),f_2\rho_{\omega_1f_1}((\omega_1)_\rho(f_1)_{\rho_{\omega_1}},(\rho_{\omega_1})_{f_1}))\\
&\qquad=(((\omega_1)_\rho(f_1)_{\rho_{\omega_1}},\omega_0(\rho_{\omega_1})_{f_1}),f_2(\rho_{\omega_1})_{f_1})\\
&((g_2\rho_{g_1})_{\omega_2})_{f_2}(id_{F\times_{E(1)}E(0)}\times_{E(2)}f_2,(g_2\rho_{g_1})_{\omega_2f_2})(id_F\times_{E(1)}\omega_0,\rho_{g_1\omega_0})((\omega_1)_\rho(f_1)_{\rho_{\omega_1}},(\rho_{\omega_1})_{f_1})\\
&\qquad=(g_2\rho_{g_1})_{\omega_2f_2}(id_F\times_{E(1)}\omega_0,\rho_{\omega_1f_1})((\omega_1)_\rho(f_1)_{\rho_{\omega_1}},(\rho_{\omega_1})_{f_1})\\
&\qquad=\rho_{\omega_1f_1}((\omega_1)_\rho(f_1)_{\rho_{\omega_1}},(\rho_{\omega_1})_{f_1})=(\rho_{\omega_1})_{f_1},
\end{aligned}$$

we have $c_{\omega_1,f_1}(\boldsymbol{F})^*c_{\omega_2,f_2}(\boldsymbol{F}_{[g_1,g_2]})_*^{-1}(\omega_0^\sharp(\iota_{g_1,g_2}(\boldsymbol{F})))=\langle((((\omega_1)_\rho(f_1)_{\rho_{\omega_1}},\omega_0(\rho_{\omega_1})_{f_1}),f_2(\rho_{\omega_1})_{f_1}),(\rho_{\omega_1})_{f_1}),id_{D(0)}\rangle$. Since the following diagram is commutative, it follows from the proof of (2.4.8) that the the image of the above element by a map $P_{f_1,f_2}(\omega_1^*(\boldsymbol{F}))_{\omega_2^*(\boldsymbol{F}_{[g_1,g_2]})}:\mathcal{E}^{(2)}_{D(0)}(f_1^*(\omega_1^*(\boldsymbol{F})),f_2^*(\omega_2^*(\boldsymbol{F}_{[g_1,g_2]})))\to\mathcal{E}^{(2)}_{D(2)}(\omega_1^*(\boldsymbol{F})_{[f_1,f_2]},\omega_2^*(\boldsymbol{F}_{[g_1,g_2]}))$ is given by $\langle(((\omega_1)_\rho(f_1)_{\rho_{\omega_1}},\omega_0(\rho_{\omega_1})_{f_1}),f_2(\rho_{\omega_1})_{f_1}),id_{D(2)}\rangle$.

$$\begin{tikzcd}
(F\times_{E(1)}D(1))\times_{D(1)}D(0) \arrow[r, "{((((\omega_1)_\rho(f_1)_{\rho_{\omega_1}},\omega_0(\rho_{\omega_1})_{f_1}),f_2(\rho_{\omega_1})_{f_1}),(\rho_{\omega_1})_{f_1})}"] \arrow[d, "(\rho_{\omega_1})_{f_1}"] \arrow[dr, "{(((\omega_1)_\rho(f_1)_{\rho_{\omega_1}},\omega_0(\rho_{\omega_1})_{f_1}),f_2(\rho_{\omega_1})_{f_1})}"] & ((F\times_{E(1)}E(0))\times_{E(2)}D(2))\times_{D(2)}D(0) \arrow[d, "(f_2)_{(g_2\rho_{g_1})_{\omega_2}}"] \\
D(0) \arrow[d, "f_2"] & (F\times_{E(1)}E(0))\times_{E(2)}D(2) \arrow[dl, "(g_2\rho_{g_1})_{\omega_2}"] \\
D(2) &
\end{tikzcd}$$

Recall that $\omega_{\boldsymbol F} : \omega_1^*(\boldsymbol F)_{[f_1,f_2]} \to \omega_2^*(\boldsymbol F_{[g_1,g_2]})$ is the image of $\iota_{g_1,g_2}(\boldsymbol F) \in \mathcal{E}^{(2)}_{E(0)}(g_1^*(\boldsymbol F), g_2^*(\boldsymbol F_{[g_1,g_2]}))$ by the following composition of maps.

$$\begin{aligned}
\mathcal{E}^{(2)}_{E(0)}(g_1^*(\boldsymbol F), g_2^*(\boldsymbol F_{[g_1,g_2]})) &\xrightarrow{\omega_0^\sharp} \mathcal{E}^{(2)}_{D(0)}((g_1\omega_0)^*(\boldsymbol F),(g_2\omega_0)^*(\boldsymbol F_{[g_1,g_2]})) = \mathcal{E}^{(2)}_{D(0)}((\omega_1 f_1)^*(\boldsymbol F),(\omega_2 f_2)^*(\boldsymbol F_{[g_1,g_2]}))\\
&\xrightarrow{c_{\omega_1,f_1}(\boldsymbol F)^* c_{\omega_2,f_2}(\boldsymbol F_{[g_1,g_2]})_*^{-1}} \mathcal{E}^{(2)}_{D(0)}(f_1^*(\omega_1^*(\boldsymbol F)), f_2^*(\omega_2^*(\boldsymbol F_{[g_1,g_2]})))\\
&\xrightarrow{P_{f_1,f_2}(\omega_1^*(\boldsymbol F))_{\omega_2^*(\boldsymbol F_{[g_1,g_2]})}} \mathcal{E}^{(2)}_{D(2)}(\omega_1^*(\boldsymbol F)_{[f_1,f_2]}, \omega_2^*(\boldsymbol F_{[g_1,g_2]}))
\end{aligned}$$

Hence the above arguments imply the following result.

Proposition 2.4.18 *Let $D, E : \mathcal{Q} \to \mathcal{E}$ be functors, $\omega : D \to E$ a natural transformation and $\boldsymbol F = (F \xrightarrow{\rho} E(1))$ an object of $\mathcal{E}^{(2)}_{E(1)}$. Put $D(\tau_{0j}) = f_j$ and $E(\tau_{0j}) = g_j$ for $j = 1,2$ and suppose that each rectangle of the following diagrams is cartesian.*

$$\begin{tikzcd}
(F\times_{E(1)} D(1))\times_{D(1)} D(0) \arrow[r, "(f_1)_{\rho_{\omega_1}}"] \arrow[d, "(\rho_{\omega_1})_{f_1}"] & F\times_{E(1)} D(1) \arrow[r, "(\omega_1)_\rho"] \arrow[d, "\rho_{\omega_1}"] & F \arrow[d, "\rho"] \\
D(0) \arrow[r, "f_1"] & D(1) \arrow[r, "\omega_1"] & E(1)
\end{tikzcd}$$

$$\begin{tikzcd}
(F\times_{E(1)} E(0))\times_{E(2)} D(2) \arrow[r, "(\omega_2)_{g_2\rho_{g_1}}"] \arrow[dd, "(g_2\rho_{g_1})_{\omega_2}"] & F\times_{E(1)} E(0) \arrow[r, "(g_1)_\rho"] \arrow[d, "\rho_{g_1}"] & F \arrow[d, "\rho"] \\
 & E(0) \arrow[r, "g_1"] \arrow[d, "g_2"] & E(1) \\
D(2) \arrow[r, "\omega_2"] & E(2) &
\end{tikzcd}$$

Then, $\omega_1^*(\boldsymbol F)_{[f_1,f_2]}$, $\omega_2^*(\boldsymbol F_{[g_1,g_2]})$ and $\omega_{\boldsymbol F} : \omega_1^*(\boldsymbol F)_{[f_1,f_2]} \to \omega_2^*(\boldsymbol F_{[g_1,g_2]})$ are given by

$$\begin{aligned}
\omega_1^*(\boldsymbol F)_{[f_1,f_2]} &= ((F\times_{E(1)} D(1))\times_{D(1)} D(0) \xrightarrow{f_2(\rho_{\omega_1})_{f_1}} D(2))\\
\omega_2^*(\boldsymbol F_{[g_1,g_2]}) &= ((F\times_{E(1)} E(0))\times_{E(2)} D(2) \xrightarrow{(g_2\rho_{g_1})_{\omega_2}} D(2))
\end{aligned}$$

and $\omega_{\boldsymbol F} = \langle (((\omega_1)_\rho (f_1)_{\rho_{\omega_1}}, \omega_0(\rho_{\omega_1})_{f_1}), f_2(\rho_{\omega_1})_{f_1}) : (F\times_{E(1)} D(1))\times_{D(1)} D(0) \to (F\times_{E(1)} E(0))\times_{E(2)} D(2), id_{D(2)}\rangle$, respectively.

$$\begin{tikzcd}
 & (F\times_{E(1)} D(1))\times_{D(1)} D(0) \arrow[rr, "(f_1)_{\rho_{\omega_1}}"] \arrow[dl, dashed, "(((\omega_1)_\rho (f_1)_{\rho_{\omega_1}}, \omega_0(\rho_{\omega_1})_{f_1}), f_2(\rho_{\omega_1})_{f_1})"'] \arrow[dr, "((\omega_1)_\rho (f_1)_{\rho_{\omega_1}}, \omega_0(\rho_{\omega_1})_{f_1})"] \arrow[dd, "(\rho_{\omega_1})_{f_1}"] & & F\times_{E(1)} D(1) \arrow[d, "(\omega_1)_\rho"] \\
(F\times_{E(1)} E(0))\times_{E(2)} D(2) \arrow[rr, "(\omega_2)_{g_2\rho_{g_1}}"] \arrow[ddd, "(g_2\rho_{g_1})_{\omega_2}"] & & F\times_{E(1)} E(0) \arrow[r, "(g_1)_\rho"] \arrow[d, "\rho_{g_1}"] & F \arrow[d, "\rho"] \\
 & & E(0) \arrow[r, "g_1"] \arrow[dd, "g_2"] & E(1) \\
 & D(0) \arrow[ur, "\omega_0"] \arrow[dl, "f_2"'] & & \\
D(2) \arrow[rr, "\omega_2"] & & E(2) &
\end{tikzcd}$$

2.5 Locally cartesian closed category

In this subsection, we assume that $\mathcal{E}$ is a locally cartesian closed category. For a morphism $f : X \to Y$ in $\mathcal{E}$, we denote by $f_! : \mathcal{E}^{(2)}_X \to \mathcal{E}^{(2)}_Y$ a right adjoint of the inverse image functor $f^* : \mathcal{E}^{(2)}_Y \to \mathcal{E}^{(2)}_X$.

For objects $\boldsymbol E = (E \xrightarrow{\pi} X)$, $\boldsymbol E' = (E' \xrightarrow{\pi'} X)$ of $\mathcal{E}^{(2)}_X$ and a morphism $\boldsymbol\varphi = \langle \varphi, id_X\rangle : \boldsymbol E \to \boldsymbol E'$, we put $f_!(\boldsymbol E) = (E^f \xrightarrow{\pi^f} Y)$ and $f_!(\boldsymbol\varphi) = \langle \varphi^f, id_Y\rangle$. Let us denote by $\boldsymbol\eta^f : id_{\mathcal{E}^{(2)}_Y} \to f_! f^*$ and $\boldsymbol\varepsilon^f : f^* f_! \to id_{\mathcal{E}^{(2)}_X}$ the unit and the counit of the adjunction $f^* \dashv f_!$, respectively. For an object $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ of $\mathcal{E}^{(2)}_Y$ and an object $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ of $\mathcal{E}^{(2)}_X$, we put

$$\boldsymbol{\eta}^f_{\boldsymbol{F}} = \langle \eta^f_{\boldsymbol{F}}, id_Y\rangle : \boldsymbol{F} = (F \xrightarrow{\rho} Y) \to ((F\times_Y X)^f \xrightarrow{(\rho_f)^f} Y) = f_! f^*(\boldsymbol{F})$$
$$\boldsymbol{\varepsilon}^f_{\boldsymbol{E}} = \langle \varepsilon^f_{\boldsymbol{E}}, id_X\rangle : f^*(f_!(\boldsymbol{E})) = (E^f \times_Y X \xrightarrow{(\pi^f)_f} X) \to (E \xrightarrow{\pi} X).$$

Here $F \xleftarrow{f_\rho} F\times_Y X \xrightarrow{\rho_f} X$ is a limit of $F \xrightarrow{\rho} Y \xleftarrow{f} X$ and $E^f \xleftarrow{f_{\pi^f}} E^f\times_Y X \xrightarrow{(\pi^f)_f} X$ is a limit of $E^f \xrightarrow{\pi^f} Y \xleftarrow{f} X$. The following fact is a direct consequence of (1.4.2).

Proposition 2.5.1 *Let $f : X \to Y$, $g : X \to Z$ be morphisms in $\mathcal{E}$ and $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$, $\boldsymbol{G} = (G \xrightarrow{\pi} Z)$ objects of $\operatorname{Ob}\mathcal{E}^{(2)}_Y$, $\operatorname{Ob}\mathcal{E}^{(2)}_Z$, respectively. Suppose that $F \xleftarrow{f_\rho} F\times_Y X \xrightarrow{\rho_f} X$ is a limit of $F \xrightarrow{\rho} Y \xleftarrow{f} X$ and that $G \xleftarrow{g_\pi} G\times_Z X \xrightarrow{\pi_g} X$ is a limit of $G \xrightarrow{\pi} Z \xleftarrow{g} X$.*

(1) (f,g) is a right fibered representable pair, namely, $\boldsymbol{G}^{[f,g]} = f_!(g^*(\boldsymbol{G})) = ((G\times_Z X)^f \xrightarrow{\pi^f_g} Y)$.

(2) $E_{f,g}(\boldsymbol{G})_{\boldsymbol{F}} : \mathcal{E}^{(2)}_X(f^*(\boldsymbol{F}), g^*(\boldsymbol{G})) \to \mathcal{E}^{(2)}_Y(\boldsymbol{F}, \boldsymbol{G}^{[f,g]})$ maps $\boldsymbol{\varphi} = \langle \varphi, id_X\rangle$ to $f_!(\boldsymbol{\varphi})\boldsymbol{\eta}^f_{\boldsymbol{F}} = \langle \varphi^f \eta^f_{\boldsymbol{F}}, id_Y\rangle$.

(3) Let $(G\times_Z X)^f \xleftarrow{f_{\pi^f_g}} (G\times_Z X)^f \times_Y X \xrightarrow{(\pi^f_g)_f} X$ be a limit of $(G\times_Z X)^f \xrightarrow{\pi^f_g} Y \xleftarrow{f} X$. Then, $\pi_{f,g}(\boldsymbol{G}) : f^*(\boldsymbol{G}^{[f,g]}) \to g^*(\boldsymbol{G})$ is given by $\boldsymbol{\varepsilon}^f_{g^*(\boldsymbol{G})} = \langle \varepsilon^f_{g^*(\boldsymbol{G})}, id_X\rangle : \big((G\times_Z X)^f \times_Y X \xrightarrow{(\pi^f_g)_f} X\big) \to (G\times_Z X \xrightarrow{\pi_g} X)$.

We have the following result from (1.4.5).

Proposition 2.5.2 *Let $\boldsymbol{G} = (G \xrightarrow{\pi} Z)$ and $\boldsymbol{H} = (H \xrightarrow{\rho} Z)$ be an object of $\mathcal{E}^{(2)}_Z$ and $g : X \to Z$ a morphism in $\mathcal{E}$. Let $X \xleftarrow{\pi_g} G\times_Z X \xrightarrow{g_\pi} G$ be a limit of $X \xrightarrow{g} Z \xleftarrow{\pi} G$ and $X \xleftarrow{\rho_h} H\times_Z X \xrightarrow{h_\rho} G$ a limit of $X \xrightarrow{h} Z \xleftarrow{\rho} H$. For a morphism $\boldsymbol{\varphi} = \langle \varphi, id_X\rangle : \boldsymbol{G} \to \boldsymbol{G}$ of $\mathcal{E}^{(2)}_Z$, $\boldsymbol{\varphi}^{[f,g]} : \boldsymbol{G}^{[f,g]} \to \boldsymbol{H}^{[f,g]}$ is given by $\boldsymbol{\varphi}^{[f,g]} = \langle (\varphi\times_Y id_X)^f, id_Y\rangle$.*

$$\begin{tikzcd}
G\times_Z X \arrow[rr, "g_\pi"] \arrow[dr, dashed, "\varphi\times_Z id_X"] \arrow[ddr, "\pi_g"'] & & G \arrow[d, "\varphi"'] \arrow[dd, bend left, "\pi"] \\
& H\times_Z X \arrow[r, "h_\rho"] \arrow[d, "\rho_h"] & H \arrow[d, "\rho"'] \\
& X \arrow[r, "g"] & Z
\end{tikzcd}$$

Let $\boldsymbol{G} = (G \xrightarrow{\pi} Z)$ be an object of $\mathcal{E}^{(2)}_Z$ and $g : X \to Z$, $k : V \to X$ morphisms in $\mathcal{E}$. Consider the following cartesian squares.

$$\begin{tikzcd}
G\times_Z X \arrow[r, "g_\pi"] \arrow[d, "\pi_g"] & G \arrow[d, "\pi"] \\
X \arrow[r, "g"] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
(G\times_Z X)\times_X V \arrow[r, "k_{\pi_g}"] \arrow[d, "(\pi_g)_k"] & G\times_Z X \arrow[d, "\pi_g"] \\
V \arrow[r, "k"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
G\times_Z V \arrow[r, "(gk)_\pi"] \arrow[d, "\pi_{gk}"] & G \arrow[d, "\pi"] \\
V \arrow[r, "gk"] & Z
\end{tikzcd}$$

There exists unique morphism $(g_\pi k_{\pi_g}, \pi_{gk}) : (G\times_Z X)\times_X V \to G\times_Z V$ that makes the following diagram commute and $(g_\pi k_{\pi_g}, \pi_{gk})$ is an isomorphism.

$$\begin{tikzcd}
G\times_Z V \arrow[rrr, bend left, "(gk)_\pi"] \arrow[rr, bend left=15, "id_G\times_Z k"] \arrow[ddr, bend right, "\pi_{gk}"'] & & & \\
& (G\times_Z X)\times_X V \arrow[ul, "(g_\pi k_{\pi_g}, \pi_{gk})"', "\cong"] \arrow[r, "k_{\pi_g}"] \arrow[d, "(\pi_g)_k"] & G\times_Z X \arrow[r, "g_\pi"] \arrow[d, "\pi_g"] & G \arrow[d, "\pi"] \\
& V \arrow[r, "k"] & X \arrow[r, "g"] & Z
\end{tikzcd}$$

Consider the following cartesian squares.

$$
\begin{array}{ccc}
\begin{tikzcd}
(G \times_Z X)^f \times_Y X \arrow[r, "f_{\pi_g^f}"] \arrow[d, "(\pi_g^f)_f"] & (G \times_Z X)^f \arrow[d, "\pi_g^f"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
&
\begin{tikzcd}
((G \times_Z X)^f \times_Y X) \times_X V \arrow[r, "k_{(\pi_g^f)_f}"] \arrow[d, "((\pi_g^f)_f)_k"] & (G \times_Z X)^f \times_Y X \arrow[d, "(\pi_g^f)_f"] \\
V \arrow[r, "k"] & X
\end{tikzcd}
\end{array}
$$

$$
\begin{tikzcd}
(G \times_Z X)^f \times_Y X \arrow[r, "(fk)_{\pi_g^f}"] \arrow[d, "(\pi_g^f)_{fk}"] & (G \times_Z X)^f \arrow[d, "\pi_g^f"] \\
V \arrow[r, "fk"] & Y
\end{tikzcd}
$$

There exists unique morphism $(id_{(G\times_Z X)^f} \times_Y k, (\pi_g^f)_{fk}) : ((G \times_Z X)^f \times_Y V \to ((G \times_Z X)^f \times_Y X) \times_X V$ that makes the following diagram commute and $(id_{(G\times_Z X)^f} \times_Y k, (\pi_g^f)_{fk})$ is an isomorphism.

$$
\begin{tikzcd}
(G \times_Z X)^f \times_Y V \arrow[dr, "(id_{(G\times_Z X)^f \times_Y k, (\pi_g^f)_{fk})}", "\cong"'] \arrow[ddr, bend right, "(\pi_g^f)_{fk}"'] \arrow[drr, bend left, "id_{(G\times_Z X)^f} \times_Y k"] \arrow[drrr, bend left=20, "(fk)_{\pi_g^f}"] & & & \\
& ((G \times_Z X)^f \times_Y X) \times_X V \arrow[r, "k_{(\pi_g^f)_f}"] \arrow[d, "((\pi_g^f)_f)_k"] & (G \times_Z X)^f \times_Y X \arrow[r, "f_{\pi_g^f}"] \arrow[d, "(\pi_g^f)_f"] & (G \times_Z X)^f \arrow[d, "\pi_g^f"] \\
& V \arrow[r, "k"] & X \arrow[r, "f"] & Y
\end{tikzcd}
$$

We also have the following result from (1.4.8).

Proposition 2.5.3 *Let $G^k : (G \times_Z X)^f \to (G \times_Z V)^{fk}$ be the following composition.*

$$
(G \times_Z X)^f \xrightarrow{\eta^{fk}_{\boldsymbol{G}[f,g]}} ((G \times_Z X)^f \times_Y V)^{fk} \xrightarrow{\left(id_{(G\times_Z X)^f \times_Y k, (\pi_g^f)_{fk}}\right)^{fk}} (((G \times_Z X)^f \times_Y X) \times_X V)^{fk}
$$
$$
\xrightarrow{\left(\varepsilon^f_{g^*(\boldsymbol{G})} \times_X id_V\right)^{fk}} ((G \times_Z X) \times_X V)^{fk} \xrightarrow{\left(g_\pi k_{\pi_g}, \pi_{gk}\right)^{fk}} (G \times_Z V)^{fk}
$$

Then $\boldsymbol{G}^k : \boldsymbol{G}^{[f,g]} \to \boldsymbol{G}^{[fk,gk]}$ is given by $\boldsymbol{G}^k = \langle G^k, id_Y\rangle$.

It follows from (1.4.12) that we have the following fact.

Proposition 2.5.4 *For morphisms $f : X \to Y$, $g : X \to Z$, $h : X \to W$ of $\mathcal{E}$ and an object $\boldsymbol{G} = (G \xrightarrow{\pi} W)$ of $\mathcal{E}^{(2)}_W$, let $X \xleftarrow{\pi_h} G \times_W X \xrightarrow{h_\pi} G$ be a limit of $X \xrightarrow{h} W \xleftarrow{\pi} G$ and $X \xleftarrow{(\pi_h^g)_g} (G \times_W X)^g \times_Z X \xrightarrow{g_{\pi_h^g}} (G \times_W X)^g$ a limit of $X \xrightarrow{g} Z \xleftarrow{\pi_h^g} (G \times_W X)^g$. Then, $\epsilon^{f,g,h}_{\boldsymbol{G}} = f_!\big(\varepsilon^g_{h^*(\boldsymbol{G})}\big) : (\boldsymbol{G}_{[g,h]})_{[f,g]} = f_!g^*g_!h^*(\boldsymbol{G}) \to f_!h^*(\boldsymbol{G}) = \boldsymbol{G}_{[f,h]}$ is given by*

$$
\langle(\varepsilon^g_{h^*(\boldsymbol{G})})^f, id_Y\rangle : (((G \times_W X)^g \times_Z X)^f \xrightarrow{(\pi_h^g)_g^f} Y) \to ((G \times_W X)^f \xrightarrow{\pi_h^f} Y).
$$

For a functor $D : \mathcal{P} \to \mathcal{E}$ and an object $\boldsymbol{G} = (F \xrightarrow{\pi} D(5))$ of $\mathcal{E}^{(2)}_{D(5)}$, we put $D(\tau_{ij}) = f_{ij}$. We have the following result from (1.5.5) and (2.4.15).

Proposition 2.5.5 *Suppose that the following diagram is cartesian.*

$$
\begin{tikzcd}
D(0) \arrow[r, "f_{02}"] \arrow[d, "f_{01}"] & D(2) \arrow[d, "f_{24}"] \\
D(1) \arrow[r, "f_{14}"] & D(4)
\end{tikzcd}
$$

Then, $\theta^D(\boldsymbol{G}) : (\boldsymbol{G}^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} \to \boldsymbol{G}^{[f_{13}f_{01},f_{25}f_{02}]}$ is an isomorphism.

It follows from (1.4.17) that

$$
\theta^D(\boldsymbol{G}) : (\boldsymbol{G}^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} = (f_{13})_!(f_{14}^*((f_{24})_!(f_{25}^*(\boldsymbol{G})))) \to (f_{13}f_{01})_!((f_{25}f_{02})^*(\boldsymbol{G})) = \boldsymbol{G}^{[f_{13}f_{01},f_{25}f_{02}]}
$$

is the following composition.

$$
\begin{aligned}
&(f_{13})_!(f_{14}^*((f_{24})_!(f_{25}^*(\boldsymbol{G})))) \xrightarrow{\boldsymbol{\eta}^{f_{13}f_{01}}_{(f_{13})_!(f_{14}^*((f_{24})_!(f_{25}^*(\boldsymbol{G}))))}} (f_{13}f_{01})_!((f_{13}f_{01})^*((f_{13})_!(f_{14}^*((f_{24})_!(f_{25}^*(\boldsymbol{G}))))))\\
&\xrightarrow{(f_{13}f_{01})_!(c_{f_{13},f_{01}}((f_{13})_!(f_{14}^*((f_{24})_!(f_{25}^*(\boldsymbol{G})))))^{-1})} (f_{13}f_{01})_!(f_{01}^*(f_{13}^*((f_{13})_!(f_{14}^*((f_{24})_!(f_{25}^*(\boldsymbol{G})))))))\\
&\xrightarrow{(f_{13}f_{01})_!\left(f_{01}^*\left(\boldsymbol{\varepsilon}^{f_{13}}_{f_{14}^*((f_{24})_!(f_{25}^*(\boldsymbol{G})))}\right)\right)} (f_{13}f_{01})_!(f_{01}^*(f_{14}^*((f_{24})_!(f_{25}^*(\boldsymbol{G}))))) \xrightarrow{(f_{13}f_{01})_!(c_{f_{14},f_{01}}((f_{24})_!(f_{25}^*(\boldsymbol{G}))))}\\
&(f_{13}f_{01})_!((f_{14}f_{01})^*((f_{24})_!(f_{25}^*(\boldsymbol{G})))) = (f_{13}f_{01})_!((f_{24}f_{02})^*((f_{24})_!(f_{25}^*(\boldsymbol{G})))) \xrightarrow{(f_{13}f_{01})_!(c_{f_{24},f_{02}}((f_{24})_!(f_{25}^*(\boldsymbol{G})))^{-1})}\\
&(f_{13}f_{01})_!(f_{02}^*(f_{24}^*((f_{24})_!(f_{25}^*(\boldsymbol{G}))))) \xrightarrow{(f_{13}f_{01})_!\left(f_{02}^*\left(\boldsymbol{\varepsilon}^{f_{24}}_{f_{25}^*(\boldsymbol{G})}\right)\right)} (f_{13}f_{01})_!(f_{02}^*(f_{25}^*(\boldsymbol{G}))) \xrightarrow{(f_{13}f_{01})_!(c_{f_{25},f_{02}}(\boldsymbol{G}))}\\
&(f_{13}f_{01})_!((f_{25}f_{02})^*(\boldsymbol{G}))
\end{aligned}
$$

We describe each morphism which appears in the above composition below. First, consider the following cartesian squares.

$$
\begin{tikzcd}
G \times_{D(5)} D(2) \arrow[r, "(f_{25})_\pi"] \arrow[d, "\pi_{f_{25}}"] & G \arrow[d, "\pi"] \\
D(2) \arrow[r, "f_{25}"] & D(5)
\end{tikzcd}
\qquad
\begin{tikzcd}
(G \times_{D(5)} D(2))^{f_{24}} \times_{D(4)} D(1) \arrow[r, "(f_{14})_{\pi^{f_{24}}_{f_{25}}}"] \arrow[d, "\left(\pi^{f_{24}}_{f_{25}}\right)_{f_{14}}"] & (G \times_{D(5)} D(2))^{f_{24}} \arrow[d, "\pi^{f_{24}}_{f_{25}}"] \\
D(1) \arrow[r, "f_{14}"] & D(4)
\end{tikzcd}
$$

Then, we have $\boldsymbol{G}^{[f_{24},f_{25}]} = (f_{24})_!(f_{25}^*(\boldsymbol{G})) = \left((G \times_{D(5)} D(2))^{f_{24}} \xrightarrow{\pi^{f_{24}}_{f_{25}}} D(4)\right)$ and

$$
(\boldsymbol{G}^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} = (f_{13})_!(f_{14}^*((f_{24})_!(f_{25}^*(\boldsymbol{G})))) = \left(((G \times_{D(5)} D(2))^{f_{24}} \times_{D(4)} D(1))^{f_{13}} \xrightarrow{\left(\pi^{f_{24}}_{f_{25}}\right)_{f_{14}}^{f_{13}}} D(3)\right).
$$

Put $H = (G \times_{D(5)} D(2))^{f_{24}} \times_{D(4)} D(1)$ and $\rho = \left(\pi^{f_{24}}_{f_{25}}\right)_{f_{14}}$. Suppose that the following diagram is cartesian.

$$
\begin{tikzcd}
H^{f_{13}} \times_{D(3)} D(0) \arrow[r, "(f_{13}f_{01})_{\rho^{f_{13}}}"] \arrow[d, "\rho^{f_{13}}_{f_{13}f_{01}}"] & H^{f_{13}} \arrow[d, "\rho^{f_{13}}"] \\
D(0) \arrow[r, "f_{13}f_{01}"] & D(3)
\end{tikzcd}
$$

Hence $((\boldsymbol{G}^{[f_{24},f_{25}]})^{[f_{13},f_{14}]})^{[f_{13}f_{01},f_{13}f_{01}]} = (f_{13}f_{01})_!((f_{13}f_{01})^*((f_{13})_!(f_{14}^*((f_{24})_!(f_{25}^*(\boldsymbol{G}))))))$ is

$$
\left((H^{f_{13}} \times_{D(3)} D(0))^{f_{13}f_{01}} \xrightarrow{\left(\rho^{f_{13}}_{f_{13}f_{01}}\right)^{f_{13}f_{01}}} D(3)\right)
$$

and $\boldsymbol{\eta}^{f_{13}f_{01}}_{(\boldsymbol{G}^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}} : (\boldsymbol{G}^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} \to ((\boldsymbol{G}^{[f_{24},f_{25}]})^{[f_{13},f_{14}]})^{[f_{13}f_{01},f_{13}f_{01}]}$ is given as follows.

$$
\left\langle \eta^{f_{13}f_{01}}_{(\boldsymbol{G}^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}} : H^{f_{13}} \to (H^{f_{13}} \times_{D(3)} D(0))^{f_{13}f_{01}}, id_{D(3)} \right\rangle \tag{2.5.1}
$$

Consider the following diagram whose rectangles are cartesian squares.

$$
\begin{tikzcd}
(H^{f_{13}} \times_{D(3)} D(1)) \times_{D(1)} D(0) \arrow[r, "(f_{01})_{\rho^{f_{13}}_{f_{13}}}"] \arrow[d, "\left(\rho^{f_{13}}_{f_{13}}\right)_{f_{01}}"] & H^{f_{13}} \times_{D(3)} D(1) \arrow[r, "(f_{13})_{\rho^{f_{13}}}"] \arrow[d, "\rho^{f_{13}}_{f_{13}}"] & H^{f_{13}} \arrow[d, "\rho^{f_{13}}"] \\
D(0) \arrow[r, "f_{01}"] & D(1) \arrow[r, "f_{13}"] & D(3)
\end{tikzcd}
$$

We have $(f_{13}f_{01})_!(f_{01}^*(f_{13}^*((\boldsymbol{G}^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}))) = \left(((H^{f_{13}} \times_{D(3)} D(1)) \times_{D(1)} D(0))^{f_{13}f_{01}} \xrightarrow{\left(\rho^{f_{13}}_{f_{13}}\right)^{f_{13}f_{01}}_{f_{01}}} D(3)\right)$ and an isomorphism $(f_{13}f_{01})_!(c_{f_{13},f_{01}}((\boldsymbol{G}^{[f_{24},f_{25}]})^{[f_{13},f_{14}]})^{-1})$ of $\mathcal{E}^{(2)}_{D(3)}$ from $(f_{13}f_{01})_!((f_{13}f_{01})^*((\boldsymbol{G}^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}))$ to $(f_{13}f_{01})_!(f_{01}^*(f_{13}^*((\boldsymbol{G}^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}))) = (f_{13}f_{01})_!(f_{01}^*(f_{13}^*((f_{13})_!(f_{14}^*(\boldsymbol{G}^{[f_{24},f_{25}]})))))$ is given below.

$$
\langle (id_{H^{f_{13}}} \times_{D(3)} f_{01}, \rho_{f_{13}f_{01}})^{f_{13}f_{01}} : (H^{f_{13}} \times_{D(3)} D(0))^{f_{13}f_{01}} \to ((H^{f_{13}} \times_{D(3)} D(1)) \times_{D(1)} D(0))^{f_{13}f_{01}}, id_{D(3)} \rangle \tag{2.5.2}
$$

Consider the following cartesian square.

$$\begin{tikzcd}
H\times_{D(1)}D(0) \arrow[r, "(f_{01})_\rho"] \arrow[d, "\rho_{f_{01}}"] & H \arrow[d, "\rho"] \\
D(0) \arrow[r, "f_{01}"] & D(1)
\end{tikzcd}$$

Then $(f_{13}f_{01})_!(f_{01}^*(f_{14}^*(\boldsymbol{G}^{[f_{24},f_{25}]}))) = \Big((H\times_{D(1)}D(0))^{f_{13}f_{01}} \xrightarrow{\rho_{f_{01}}^{f_{13}f_{01}}} D(3)\Big)$ and

$$(f_{13}f_{01})_!\big(f_{01}^*\big(\boldsymbol{\varepsilon}^{f_{13}}_{f_{14}^*(\boldsymbol{G}^{[f_{24},f_{25}]})}\big)\big) : (f_{13}f_{01})_!(f_{01}^*(f_{13}^*((f_{13})_!(f_{14}^*(\boldsymbol{G}^{[f_{24},f_{25}]}))))) \to (f_{13}f_{01})_!(f_{01}^*(f_{14}^*(\boldsymbol{G}^{[f_{24},f_{25}]})))$$

is given as follows.

$$\langle(\varepsilon^{f_{13}}_{f_{14}^*(\boldsymbol{G}^{[f_{24},f_{25}]})}\times_{D(1)}id_{D(0)})^{f_{13}f_{01}} : ((H^{f_{13}}\times_{D(3)}D(1))\times_{D(1)}D(0))^{f_{13}f_{01}} \to (H\times_{D(1)}D(0))^{f_{13}f_{01}}, id_{D(3)}\rangle \tag{2.5.3}$$

Suppose that each rectangles of the following diagrams are cartesian.

$$\begin{tikzcd}
((G\times_{D(5)}D(2))^{f_{24}}\times_{D(4)}D(1))\times_{D(1)}D(0) \arrow[r, "(f_{01})_{(\pi^{f_{24}}_{f_{25}})_{f_{14}}}"] \arrow[d, "((\pi^{f_{24}}_{f_{25}})_{f_{14}})_{f_{01}}"] & (G\times_{D(5)}D(2))^{f_{24}}\times_{D(4)}D(1) \arrow[r, "(f_{14})_{\pi^{f_{24}}_{f_{25}}}"] \arrow[d, "(\pi^{f_{24}}_{f_{25}})_{f_{14}}"] & (G\times_{D(5)}D(2))^{f_{24}} \arrow[d, "\pi^{f_{24}}_{f_{25}}"] \\
D(0) \arrow[r, "f_{01}"] & D(1) \arrow[r, "f_{14}"] & D(4)
\end{tikzcd}$$

$$\begin{tikzcd}
((G\times_{D(5)}D(2))^{f_{24}}\times_{D(4)}D(2))\times_{D(2)}D(0) \arrow[r, "(f_{02})_{(\pi^{f_{24}}_{f_{25}})_{f_{24}}}"] \arrow[d, "((\pi^{f_{24}}_{f_{25}})_{f_{24}})_{f_{02}}"] & (G\times_{D(5)}D(2))^{f_{24}}\times_{D(4)}D(2) \arrow[r, "(f_{24})_{\pi^{f_{24}}_{f_{25}}}"] \arrow[d, "(\pi^{f_{24}}_{f_{25}})_{f_{24}}"] & (G\times_{D(5)}D(2))^{f_{24}} \arrow[d, "\pi^{f_{24}}_{f_{25}}"] \\
D(0) \arrow[r, "f_{02}"] & D(2) \arrow[r, "f_{24}"] & D(4)
\end{tikzcd}$$

Then, we have

$$(f_{13}f_{01})_!(f_{01}^*(f_{14}^*(\boldsymbol{G}^{[f_{24},f_{25}]}))) = \Big((((G\times_{D(5)}D(2))^{f_{24}}\times_{D(4)}D(1))\times_{D(1)}D(0))^{f_{13}f_{01}} \xrightarrow{((\pi^{f_{24}}_{f_{25}})_{f_{14}})^{f_{13}f_{01}}_{f_{01}}} D(3)\Big)$$
$$(f_{13}f_{01})_!(f_{02}^*(f_{24}^*(\boldsymbol{G}^{[f_{24},f_{25}]}))) = \Big((((G\times_{D(5)}D(2))^{f_{24}}\times_{D(4)}D(2))\times_{D(2)}D(0))^{f_{13}f_{01}} \xrightarrow{((\pi^{f_{24}}_{f_{25}})_{f_{24}})^{f_{13}f_{01}}_{f_{02}}} D(3)\Big)$$

and it follows from (2.4.6) that an isomorphism $(f_{13}f_{01})_!(c_{f_{24},f_{02}}((f_{24})_!(f_{25}^*(\boldsymbol{G})))^{-1}c_{f_{14},f_{01}}((f_{24})_!(f_{25}^*(\boldsymbol{G}))))$ from $(f_{13}f_{01})_!(f_{01}^*(f_{14}^*(\boldsymbol{G}^{[f_{24},f_{25}]})))$ to $(f_{13}f_{01})_!(f_{02}^*(f_{24}^*(\boldsymbol{G}^{[f_{24},f_{25}]})))$ is given by

$$\Big\langle\Big(\Big((f_{24})_{\pi^{f_{24}}_{f_{25}}}(f_{02})_{(\pi^{f_{24}}_{f_{25}})_{f_{24}}}, f_{02}\big((\pi^{f_{24}}_{f_{25}})_{f_{14}}\big)_{f_{01}}\Big), \big((\pi^{f_{24}}_{f_{25}})_{f_{14}}\big)_{f_{01}}\Big)^{f_{13}f_{01}}, id_{D(3)}\Big\rangle. \tag{2.5.4}$$

Suppose that the following diagrams are cartesian.

$$\begin{tikzcd}
(G\times_{D(5)}D(2))\times_{D(2)}D(0) \arrow[r, "(f_{02})_{\pi_{f_{25}}}"] \arrow[d, "(\pi_{f_{25}})_{f_{02}}"] & G\times_{D(5)}D(2) \arrow[d, "\pi_{f_{25}}"] \\
D(0) \arrow[r, "f_{02}"] & D(2)
\end{tikzcd}
\qquad
\begin{tikzcd}
G\times_{D(5)}D(0) \arrow[r, "(f_{25}f_{02})_\pi"] \arrow[d, "\pi_{f_{25}f_{02}}"] & G \arrow[d, "\pi"] \\
D(0) \arrow[r, "f_{25}f_{02}"] & D(5)
\end{tikzcd}$$

Then, we have the following.

$$(f_{13}f_{01})_!(f_{02}^*(f_{25}^*(\boldsymbol{G}))) = \Big(((G\times_{D(5)}D(2))\times_{D(2)}D(0))^{f_{13}f_{01}} \xrightarrow{(\pi_{f_{25}})^{f_{13}f_{01}}_{f_{02}}} D(3)\Big)$$
$$(f_{13}f_{01})_!((f_{25}f_{02})^*(\boldsymbol{G})) = \Big((G\times_{D(5)}D(0))^{f_{13}f_{01}} \xrightarrow{\pi^{f_{13}f_{01}}_{f_{25}f_{02}}} D(3)\Big)$$

We note that $(f_{13}f_{01})_!(f_{02}^*(f_{24}^*(\boldsymbol{G}^{[f_{24},f_{25}]}))) = (f_{13}f_{01})_!(f_{02}^*(f_{24}^*((f_{24})_!(f_{25}^*(\boldsymbol{G})))))$ and that

$$(f_{13}f_{01})_!\big(f_{02}^*\big(\boldsymbol{\varepsilon}^{f_{24}}_{f_{25}^*(\boldsymbol{G})}\big)\big) : (f_{13}f_{01})_!(f_{02}^*(f_{24}^*((f_{24})_!(f_{25}^*(\boldsymbol{G}))))) \to (f_{13}f_{01})_!(f_{02}^*(f_{25}^*(\boldsymbol{G})))$$
$$(f_{13}f_{01})_!(c_{f_{25},f_{02}}(\boldsymbol{G})) : (f_{13}f_{01})_!(f_{02}^*(f_{25}^*(\boldsymbol{G}))) \to (f_{13}f_{01})_!((f_{25}f_{02})^*(\boldsymbol{G}))$$

are given as follows, respectively.

$$(f_{13}f_{01})_!\big(f_{02}^*\big(\boldsymbol{\varepsilon}^{f_{24}}_{f_{25}^*(\boldsymbol{G})}\big)\big) = \big\langle\big(\varepsilon^{f_{24}}_{f_{25}^*(\boldsymbol{G})} \times_{D(2)} id_{D(0)}\big)^{f_{13}f_{01}}, id_{D(3)}\big\rangle \tag{2.5.5}$$

$$(f_{13}f_{01})_!(c_{f_{25},f_{02}}(\boldsymbol{G})) = \langle((f_{02})_{\pi_{f_{25}}}(f_{25})_\pi, (\pi_{f_{25}})_{f_{02}})^{f_{13}f_{01}}, id_{D(3)}\rangle \tag{2.5.6}$$

Here, the sources and the targets of $\big(\varepsilon^{f_{24}}_{f_{25}^*(\boldsymbol{G})} \times_{D(2)} id_{D(0)}\big)^{f_{13}f_{01}}$ and $(f_{02})_{\pi_{f_{25}}}(f_{25})_\pi, (\pi_{f_{25}})_{f_{02}})^{f_{13}f_{01}}$ are given as follows.

$(\varepsilon^{f_{24}}_{f_{25}^*(\boldsymbol{G})} \times_{D(2)} id_{D(0)})^{f_{13}f_{01}} : (((G \times_{D(5)} D(2))^{f_{24}} \times_{D(4)} D(2)) \times_{D(2)} D(0))^{f_{13}f_{01}} \to ((G \times_{D(5)} D(2)) \times_{D(2)} D(0))^{f_{13}f_{01}}$

$(f_{02})_{\pi_{f_{25}}}(f_{25})_\pi, (\pi_{f_{25}})_{f_{02}})^{f_{13}f_{01}} : ((G \times_{D(5)} D(2)) \times_{D(2)} D(0))^{f_{13}f_{01}} \to (G \times_{D(5)} D(0))^{f_{13}f_{01}}$

3 Representations of internal categories

3.1 Definitions and basic properties of representations of internal categories

We first recall the notions of internal categories and internal functors.

Definition 3.1.1 *Let $\mathcal{E}$ be a category with finite limits. An internal category $\boldsymbol{C}$ in $\mathcal{E}$ consists of the following objects and morphisms.*

(1) A pair of objects C_0 (the object-of-objects) and C_1 (the object-of-morphisms) of $\mathcal{E}$.

(2) Four morphisms $\sigma : C_1 \to C_0$ (source), $\tau : C_1 \to C_0$ (target), $\varepsilon : C_0 \to C_1$ (identity), $\mu : C_1 \times_{C_0} C_1 \to C_1$ (composition), where $C_1 \xleftarrow{\mathrm{pr}_1} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_2} C_1$ is a limit of diagram $C_1 \xrightarrow{\tau} C_0 \xleftarrow{\sigma} C_1$, such that $\sigma\varepsilon = \tau\varepsilon = id_{C_0}$ and the following diagrams commute.

$$\begin{tikzcd}
C_1 \arrow[d, "\sigma"'] & C_1 \times_{C_0} C_1 \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mu"] & C_1 \arrow[d, "\tau"] \\
C_0 & C_1 \arrow[l, "\sigma"] \arrow[r, "\tau"'] & C_0
\end{tikzcd}
\quad
\begin{tikzcd}
C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "\mu \times_{C_0} id_{C_1}"] \arrow[d, "id_{C_1} \times_{C_0} \mu"] & C_1 \times_{C_0} C_1 \arrow[d, "\mu"] \\
C_1 \times_{C_0} C_1 \arrow[r, "\mu"] & C_1
\end{tikzcd}
\quad
\begin{tikzcd}
C_1 \times_{C_0} C_1 \arrow[dr, "\mu"] & C_1 \arrow[l, "(\varepsilon\sigma, id_{C_1})"'] \arrow[d, "id_{C_1}"] \\
C_1 \arrow[u, "(id_{C_1}, \varepsilon\tau)"] \arrow[r, "id_{C_1}"'] & C_1
\end{tikzcd}$$

Here, $C_1 \times_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_i} C_1$ $(i = 1, 2, 3)$ is a limit of diagram $C_1 \xrightarrow{\tau} C_0 \xleftarrow{\sigma} C_1 \xrightarrow{\tau} C_0 \xleftarrow{\sigma} C_1$. We denote by $(C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ an internal category $\boldsymbol{C}$ whose object-of-objects and object-of-morphisms are C_0 and C_1, respectively, with structure morphisms σ, τ, ε, μ.

A morphism $\boldsymbol{f} : \boldsymbol{C} \to \boldsymbol{D}$ of internal categories (internal functor) is a pair (f_0, f_1) of two morphisms $f_0 : C_0 \to D_0$ and $f_1 : C_1 \to D_1$ of $\mathcal{E}$ such that the following diagrams commute if $\boldsymbol{D} = (D_0, D_1; \sigma', \tau', \varepsilon', \mu')$.

$$\begin{tikzcd}
C_0 \arrow[d, "f_0"] & C_1 \arrow[l, "\sigma"'] \arrow[r, "\tau"] \arrow[d, "f_1"] & C_0 \arrow[d, "f_0"] \\
D_0 & D_1 \arrow[l, "\sigma'"] \arrow[r, "\tau'"'] & D_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \times_{C_0} C_1 \arrow[r, "\mu"] \arrow[d, "f_1 \times_{C_0} f_1"] & C_1 \arrow[d, "f_1"] & C_0 \arrow[l, "\varepsilon"'] \arrow[d, "f_0"] \\
D_1 \times_{D_0} D_1 \arrow[r, "\mu'"'] & D_1 & D_0 \arrow[l, "\varepsilon'"]
\end{tikzcd}$$

If both f_0 and f_1 are monomorphisms, $\boldsymbol{D}$ is called an internal subcategory of $\boldsymbol{C}$.

An internal natural transformation $\varphi : \boldsymbol{f} \to \boldsymbol{g}$ from an internal functor $\boldsymbol{f} = (f_0, f_1) : \boldsymbol{C} \to \boldsymbol{D}$ to an internal functor $\boldsymbol{g} = (g_0, g_1) : \boldsymbol{C} \to \boldsymbol{D}$ is a morphism $\varphi : C_0 \to D_1$ in $\mathcal{E}$ making the following diagrams commute.

$$\begin{tikzcd}
D_0 & C_0 \arrow[l, "f_0"'] \arrow[r, "g_0"] \arrow[d, "\varphi"] & D_0 \\
& D_1 \arrow[ul, "\sigma'"] \arrow[ur, "\tau'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[r, "(f_1, \varphi\tau)"] \arrow[d, "(\varphi\sigma, g_1)"] & D_1 \times_{D_0} D_1 \arrow[d, "\mu'"] \\
D_1 \times_{D_0} D_1 \arrow[r, "\mu'"] & D_1
\end{tikzcd}$$

We denote by $\boldsymbol{cat}(\mathcal{E})$ the category of internal categories in $\mathcal{E}$.

Let $p : \mathcal{F} \to \mathcal{E}$ be a normalized cloven fibered category over $\mathcal{E}$ and $f : X \to Y$, $g : X \to Z$, $k : V \to X$ morphisms in $\mathcal{E}$. For objects M of $\mathcal{F}_Y$, N of $\mathcal{F}_Z$ and a morphism $\xi : f^*(M) \to g^*(N)$ of $\mathcal{F}_X$, we denote $k^\sharp_{M,N}(\xi)$ by ξ_k for short. That is, ξ_k is the following composition.

$$(fk)^*(M) \xrightarrow{c_{f,k}(M)^{-1}} k^*f^*(M) \xrightarrow{k^*(\xi)} k^*g^*(N) \xrightarrow{c_{g,k}(N)} (gk)^*(N)$$

Definition 3.1.2 *Suppose that $p : \mathcal{F} \to \mathcal{E}$ be a normalized cloven fibered category and that $\mathcal{E}$ is a category with finite limits. Let $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathcal{E}$. A pair (M, ξ) of an object M of $\mathcal{F}_{C_0}$ and a morphism $\xi : \sigma^*(M) \to \tau^*(M)$ of $\mathcal{F}_{C_1}$ is called a representation of $\boldsymbol{C}$ on M if the following conditions are satisfied.*

(A) Let $C_1 \xleftarrow{\mathrm{pr}_1} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_2} C_1$ be a limit of diagram $C_1 \xrightarrow{\tau} C_0 \xleftarrow{\sigma} C_1$. $\xi_\mu : (\sigma\mu)^*(M) \to (\tau\mu)^*(M)$ coincides with the following composition.

$$(\sigma\mu)^*(M) = (\sigma\mathrm{pr}_1)^*(M) \xrightarrow{\xi_{\mathrm{pr}_1}} (\tau\mathrm{pr}_1)^*(M) = (\sigma\mathrm{pr}_2)^*(M) \xrightarrow{\xi_{\mathrm{pr}_2}} (\tau\mathrm{pr}_2)^*(M) = (\tau\mu)^*(M)$$

(U) $\xi_\varepsilon : M = (\sigma\varepsilon)^*(M) \to (\tau\varepsilon)^*(M) = M$ coincides with the identity morphism of M.

Let (M,ξ) and (N,ζ) be representations of $\boldsymbol{C}$ on M and N, respectively. A morphism $\varphi : M \to N$ in $\mathcal{F}_{C_0}$ is called a morphism in representations of $\boldsymbol{C}$ if φ makes the following diagram commute.

$$\begin{array}{ccc} \sigma^*(M) & \xrightarrow{\quad \xi \quad} & \tau^*(M) \\ \downarrow{\scriptstyle \sigma^*(\varphi)} & & \downarrow{\scriptstyle \tau^*(\varphi)} \\ \sigma^*(N) & \xrightarrow{\quad \zeta \quad} & \tau^*(N) \end{array}$$

We denote by $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ *the category of the representations of* $\boldsymbol{C}$*.*

We denote by $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$ the forgetful functor which assigns $(M,\xi) \in \mathrm{Ob}\,\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ to $M \in \mathrm{Ob}\,\mathcal{F}_{C_0}$ and $(\varphi : (M,\xi) \to (N,\zeta)) \in \mathrm{Mor}\,\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ to $\varphi : M \to N$.

Definition 3.1.3 *Let $\varphi : (M,\xi) \to (N,\zeta)$ be a morphism in* $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$*.*
(1) *If $\mathscr{F}_{\boldsymbol{C}}(\varphi) : M \to N$ is a monomorphism in $\mathcal{F}_{C_0}$, we call (M,ξ) a subrepresentation of (N,ζ).*
(2) *If $\mathscr{F}_{\boldsymbol{C}}(\varphi) : M \to N$ is an epimorphism in $\mathcal{F}_{C_0}$, we call (N,ζ) a quotient representation of (M,ξ).*

Proposition 3.1.4 *Let $\varphi : (M,\xi) \to (N,\zeta)$ be a morphism of representations of an internal category $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ in $\mathcal{E}$.*
(1) *Suppose that $\mathscr{F}_{\boldsymbol{C}}(\varphi) : M \to N$ is a monomorphism in $\mathcal{F}_{C_1}$. For a representation (M,ξ') of $\boldsymbol{C}$ and a morphism $\varphi' : (M,\xi') \to (N,\zeta)$ of representations such that $\mathscr{F}_{\boldsymbol{C}}(\varphi) = \mathscr{F}_{\boldsymbol{C}}(\varphi')$, if one of the following conditions is satisfied, we have $\xi' = \xi$.*

(i) $\tau^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_1}$ preserves monomorphisms. *(ii) (σ,τ) is a left fibered representable pair with respect to M.*

(2) *Suppose that $\mathscr{F}_{\boldsymbol{C}}(\varphi) : M \to N$ is an epimorphism in $\mathcal{F}_{C_1}$. For a representation (N,ζ') of $\mathcal{C}$ and a morphism $\varphi' : (M,\xi) \to (N,\zeta')$ of representations such that $\mathscr{F}_{\boldsymbol{C}}(\varphi) = \mathscr{F}_{\boldsymbol{C}}(\varphi')$, if one of the following conditions is satisfied, we have $\zeta' = \zeta$.*

(i) $\sigma^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_1}$ preserves epimorphisms. *(ii) (σ,τ) is a right fibered representable pair with respect to N.*

Proof. (1) Since $\tau^*(\varphi)\xi' = \zeta\sigma^*(\varphi) = \tau^*(\varphi)\xi$ by the assumption, it suffices to show that

$$\tau^*(\varphi)_* : \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(M)) \to \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(N))$$

is injective. If (i) is satisfied, then $\tau^*(\varphi)$ is a monomorphism, hence $\tau^*(\varphi)_*$ is injective.
Suppose that (ii) is satisfied. Then the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(M)) & \xrightarrow{P_{\sigma,\tau}(M)_M} & \mathcal{F}_{C_0}(M_{[\sigma,\tau]}, M) \\ \downarrow{\scriptstyle \tau^*(\varphi)_*} & & \downarrow{\scriptstyle \varphi_*} \\ \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(N)) & \xrightarrow{P_{\sigma,\tau}(M)_N} & \mathcal{F}_{C_0}(M_{[\sigma,\tau]}, N) \end{array}$$

Since both φ_* and $P_{\sigma,\tau}(M)_M$ are injective, so is $\tau^*(\varphi)_*$.
(2) Since $\zeta'\sigma^*(\varphi) = \tau^*(\varphi)\xi = \zeta\sigma^*(\varphi)$ by the assumption, it suffices to show that

$$\sigma^*(\varphi)^* : \mathcal{F}_{C_1}(\sigma^*(N), \tau^*(N)) \to \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(N))$$

is injective. If (i) is satisfied, then $\sigma^*(\varphi)$ is an epimorphism, hence $\sigma^*(\varphi)_*$ is injective.
Suppose that (ii) is satisfied. Then the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{F}_{C_1}(\sigma^*(N), \tau^*(N)) & \xrightarrow{E_{\sigma,\tau}(N)_N} & \mathcal{F}_{C_0}(N, N^{[\sigma,\tau]}) \\ \downarrow{\scriptstyle \sigma^*(\varphi)^*} & & \downarrow{\scriptstyle \varphi^*} \\ \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(N)) & \xrightarrow{E_{\sigma,\tau}(N)_M} & \mathcal{F}_{C_0}(M, N^{[\sigma,\tau]}) \end{array}$$

Since both φ^* and $E_{\sigma,\tau}(N)_N$ are injective, so is $\sigma^*(\varphi)^*$. $\square$

Proposition 3.1.5 *Let M, N be objects of $\mathcal{F}_{C_0}$ and $\xi : \sigma^*(M) \to \tau^*(M)$, $\zeta : \sigma^*(N) \to \tau^*(N)$ morphisms in $\mathcal{F}_{C_1}$. We assume that a morphism $\varphi : M \to N$ of $\mathcal{F}_{C_0}$ makes the following diagram commute.*

$$\begin{array}{ccc}\sigma^*(M) & \xrightarrow{\xi} & \tau^*(M)\\ \downarrow{\scriptstyle \sigma^*(\varphi)} & & \downarrow{\scriptstyle \tau^*(\varphi)}\\ \sigma^*(N) & \xrightarrow{\zeta} & \tau^*(N)\end{array}$$

(1) *Suppose that* (N,ζ) *is a representation of* $\boldsymbol{C}$ *on* N *and that* $\varphi : M \to N$ *is an monomorphism. If*

$$(\tau\mu)^*(\varphi)_* : \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M),(\tau\mu)^*(M)) \to \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M),(\tau\mu)^*(N))$$

is injective, ξ *is a representation of* $\boldsymbol{C}$ *on* M*.*

(2) *Suppose that* (M,ξ) *is a representation of* $\boldsymbol{C}$ *on* M *and that* $\varphi : M \to N$ *is an epimorphism. If*

$$(\sigma\mu)^*(\varphi)^* : \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(N),(\tau\mu)^*(N)) \to \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M),(\tau\mu)^*(N))$$

is injective, ζ *is a representation of* $\boldsymbol{C}$ *on* N*.*

Proof. The following diagrams commute by the assumption and (1.1.15).

$$\begin{array}{ccccccccccc}(\sigma\mu)^*(M) & = & (\sigma\mathrm{pr}_1)^*(M) & \xrightarrow{\xi_{\mathrm{pr}_1}} & (\tau\mathrm{pr}_1)^*(M) & = & (\sigma\mathrm{pr}_2)^*(M) & \xrightarrow{\xi_{\mathrm{pr}_2}} & (\tau\mathrm{pr}_2)^*(M) & = & (\tau\mu)^*(M)\\ \downarrow{\scriptstyle (\sigma\mu)^*(\varphi)} & & \downarrow{\scriptstyle (\sigma\mathrm{pr}_1)^*(\varphi)} & & \downarrow{\scriptstyle (\tau\mathrm{pr}_1)^*(\varphi)} & & \downarrow{\scriptstyle (\sigma\mathrm{pr}_2)^*(\varphi)} & & \downarrow{\scriptstyle (\tau\mathrm{pr}_2)^*(\varphi)} & & \downarrow{\scriptstyle (\tau\mu)^*(\varphi)}\\ (\sigma\mu)^*(N) & = & (\sigma\mathrm{pr}_1)^*(N) & \xrightarrow{\zeta_{\mathrm{pr}_1}} & (\tau\mathrm{pr}_1)^*(N) & = & (\sigma\mathrm{pr}_2)^*(N) & \xrightarrow{\zeta_{\mathrm{pr}_2}} & (\tau\mathrm{pr}_2)^*(N) & = & (\tau\mu)^*(N)\end{array}$$

$$\begin{array}{ccc}(\sigma\mu)^*(M) & \xrightarrow{\xi_\mu} & (\tau\mu)^*(M)\\ \downarrow{\scriptstyle (\sigma\mu)^*(\varphi)} & & \downarrow{\scriptstyle (\tau\mu)^*(\varphi)}\\ (\sigma\mu)^*(N) & \xrightarrow{\zeta_\mu} & (\tau\mu)^*(N)\end{array} \qquad \begin{array}{ccc}(\sigma\varepsilon)^*(M) & \xrightarrow{\xi_\varepsilon} & (\tau\varepsilon)^*(M)\\ \downarrow{\scriptstyle (\sigma\varepsilon)^*(\varphi)=\varphi} & & \downarrow{\scriptstyle (\tau\varepsilon)^*(\varphi)=\varphi}\\ (\sigma\varepsilon)^*(N) & \xrightarrow{\zeta_\varepsilon} & (\tau\varepsilon)^*(N)\end{array}$$

(1) It follows from the commutativity of the above diagrams that we have

$$(\tau\mu)^*(\varphi)\xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1} = \zeta_{\mathrm{pr}_1}\zeta_{\mathrm{pr}_2}(\sigma\mu)^*(\varphi) = \zeta_\mu(\sigma\mu)^*(\varphi) = (\tau\mu)^*(\varphi)\xi_\mu \ \text{ and } \ \varphi\xi_\varepsilon = \zeta_\varepsilon\varphi = \varphi.$$

Hence we have $\xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1} = \xi_\mu$ and $\xi_\varepsilon = id_M$ by the assumption.

(2) It follows from the commutativity of the above diagrams that we have

$$\zeta_{\mathrm{pr}_2}\zeta_{\mathrm{pr}_1}(\sigma\mu)^*(\varphi) = (\tau\mathrm{pr}_2)^*(\varphi)\xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1} = (\tau\mu)^*(\varphi)\xi_\mu = \zeta_\mu(\sigma\mu)^*(\varphi) \ \text{ and } \ \zeta_\varepsilon\varphi = \varphi\xi_\varepsilon = \varphi.$$

Hence we have $\zeta_{\mathrm{pr}_2}\zeta_{\mathrm{pr}_1} = \zeta_\mu$ and $\zeta_\varepsilon = id_N$ by the assumption. □

Proposition 3.1.6 *Let* $\varphi : M \to N$ *be a morphism in* $\mathcal{F}_{C_0}$*.*

(1) If φ *is a monomorphism and one of the following conditions is satisfied, the condition of (1) of (3.1.5) is satisfied.*

(i) $(\tau\mu)^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_1\times_{C_0}C_1}$ *preserves monomorphisms.*
(ii) $(\sigma\mu,\tau\mu)$ *is a left fibered representable pair with respect to* M*.*
(iii) $(\sigma\mu,\tau\mu)$ *is a right fibered representable pair with respect to* M*,* N *and the following map is injective.*

$$\varphi_*^{[\sigma\mu,\tau\mu]} : \mathcal{F}_{C_0}(M, M^{[\sigma\mu,\tau\mu]}) \to \mathcal{F}_{C_0}(M, N^{[\sigma\mu,\tau\mu]})$$

(2) If $\varphi : M \to N$ *is an epimorphism and one of the following conditions is satisfied, the condition of (2) of (3.1.5) is satisfied.*

(i) $(\sigma\mu)^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_1\times_{C_0}C_1}$ *preserves epimorphisms.*
(ii) $(\sigma\mu,\tau\mu)$ *is a right fibered representable pair with respect to* N*.*
(iii) $(\sigma\mu,\tau\mu)$ *is a left fibered representable pair with respect to* M*,* N *and the following map is injective.*

$$\varphi^*_{[\sigma\mu,\tau\mu]} : \mathcal{F}_{C_0}(N_{[\sigma\mu,\tau\mu]}, N) \to \mathcal{F}_{C_0}(M_{[\sigma\mu,\tau\mu]}, N)$$

Proof. (1) If (i) is satisfied, $(\tau\mu)^*(\varphi)$ is a monomorphism. Assume that (ii) is satisfied. Then, we have the following commutative diagram by the assumption.

$$\begin{array}{ccc}
\mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M),(\tau\mu)^*(M)) & \xrightarrow{P_{\sigma\mu,\tau\mu}(M)_M} & \mathcal{F}_{C_0}(M_{[\sigma\mu,\tau\mu]},M) \\
\Big\downarrow{\scriptstyle (\tau\mu)^*(\varphi)_*} & & \Big\downarrow{\scriptstyle \varphi_*} \\
\mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M),(\tau\mu)^*(N)) & \xrightarrow{P_{\sigma\mu,\tau\mu}(M)_N} & \mathcal{F}_{C_0}(M_{[\sigma\mu,\tau\mu]},N)
\end{array}$$

Since both φ_* and $P_{\sigma\mu,\tau\mu}(M)_M$ are injective, so is $(\tau\mu)^*(\varphi)_*$. Assume that (iii) is satisfied. The following diagram is commutative by (1.4.4),

$$\begin{array}{ccc}
\mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M),(\tau\mu)^*(M)) & \xrightarrow{E_{\sigma\mu,\tau\mu}(M)_M} & \mathcal{F}_{C_0}(M,M^{[\sigma\mu,\tau\mu]}) \\
\Big\downarrow{\scriptstyle (\tau\mu)^*(\varphi)_*} & & \Big\downarrow{\scriptstyle \varphi_*^{[\sigma\mu,\tau\mu]}} \\
\mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M),(\tau\mu)^*(N)) & \xrightarrow{E_{\sigma\mu,\tau\mu}(N)_M} & \mathcal{F}_{C_0}(M,N^{[\sigma\mu,\tau\mu]})
\end{array}$$

Since both $\varphi_*^{[\sigma\mu,\tau\mu]}$ and $E_{\sigma\mu,\tau\mu}(M)_M$ are injective, so is $(\tau\mu)^*(\varphi)^*$.

(2) If (i) is satisfied, $(\sigma\mu)^*(\varphi)$ is an epimorphism. Assume that (ii) is satisfied. Then, we have the following commutative diagram by the assumption.

$$\begin{array}{ccc}
\mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(N),(\tau\mu)^*(N)) & \xrightarrow{E_{\sigma\mu,\tau\mu}(N)_N} & \mathcal{F}_{C_0}(N,N^{[\sigma\mu,\tau\mu]}) \\
\Big\downarrow{\scriptstyle (\sigma\mu)^*(\varphi)^*} & & \Big\downarrow{\scriptstyle \varphi^*} \\
\mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M),(\tau\mu)^*(N)) & \xrightarrow{E_{\sigma\mu,\tau\mu}(N)_M} & \mathcal{F}_{C_0}(M,N^{[\sigma\mu,\tau\mu]})
\end{array}$$

Since both φ^* and $E_{\sigma\mu,\tau\mu}(N)_N$ are injective, so is $(\sigma\mu)^*(\varphi)^*$. Assume that (iii) is satisfied. The following diagram is commutative by (1.3.4),

$$\begin{array}{ccc}
\mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(N),(\tau\mu)^*(N)) & \xrightarrow{P_{\sigma\mu,\tau\mu}(N)_N} & \mathcal{F}_{C_0}(N_{[\sigma\mu,\tau\mu]},N) \\
\Big\downarrow{\scriptstyle (\sigma\mu)^*(\varphi)^*} & & \Big\downarrow{\scriptstyle \varphi^*_{[\sigma\mu,\tau\mu]}} \\
\mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M),(\tau\mu)^*(N)) & \xrightarrow{P_{\sigma\mu,\tau\mu}(M)_N} & \mathcal{F}_{C_0}(M_{[\sigma\mu,\tau\mu]},N)
\end{array}$$

Since both $\varphi^*_{[\sigma\mu,\tau\mu]}$ and $P_{\sigma\mu,\tau\mu}(N)_N$ are injective, so is $(\sigma\mu)^*(\varphi)^*$. □

Proposition 3.1.7 *Let $D:\mathcal{D}\to \mathrm{Rep}(\boldsymbol{C};\mathcal{F})$ be a functor.*

(1) *Let $(\pi_i : M\to \mathscr{F}_{\boldsymbol{C}}D(i))_{i\in \mathrm{Ob}\,\mathcal{D}}$ be a limiting cone of $\mathscr{F}_{\boldsymbol{C}}D : D\to \mathcal{F}_{C_0}$. Assume that*

$$\left(\tau^*(\pi_i)_* : \mathcal{F}_{C_1}(\sigma^*(M),\tau^*(M))\to \mathcal{F}_{C_1}(\sigma^*(M),\tau^*\mathscr{F}_{\boldsymbol{C}}D(i))\right)_{i\in \mathrm{Ob}\,\mathcal{D}}$$

is a limiting cone of a functor $\mathcal{D}\to \mathcal{S}et$ which assigns $i\in \mathrm{Ob}\,\mathcal{D}$ to $\mathcal{F}_{C_1}(\sigma^*(M),\tau^*\mathscr{F}_{\boldsymbol{C}}D(i))$ and $\alpha\in \mathcal{D}(i,j)$ to $\tau^*\mathscr{F}_{\boldsymbol{C}}D(\alpha)_* : \mathcal{F}_{C_1}(\sigma^*(M),\tau^*\mathscr{F}_{\boldsymbol{C}}D(i)))\to \mathcal{F}_{C_1}(\sigma^*(M),\tau^*\mathscr{F}_{\boldsymbol{C}}D(j)))$. We also assume that

$$\left((\tau\mu)^*(\pi_i)_* : \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M),(\tau\mu)^*(M))\to \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M),(\tau\mu)^*\mathscr{F}_{\boldsymbol{C}}D(i))\right)_{i\in \mathrm{Ob}\,\mathcal{D}}$$

is a monomorphic family. Then, there exists a unique morphism $\xi:\sigma^*(M)\to\tau^*(M)$ such that (M,ξ) is a representation of $\boldsymbol{C}$ on M and $(\pi_i : (M,\xi)\to D(i))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of D.

(2) *Let $(\iota_i : \mathscr{F}_{\boldsymbol{C}}D(i)\to M)_{i\in \mathrm{Ob}\,\mathcal{D}}$ be a colimiting cone of $\mathscr{F}_{\boldsymbol{C}}D : D\to \mathcal{F}_{C_0}$. Asuume that*

$$\left(\sigma^*(\iota_i)^* : \mathcal{F}_{C_1}(\sigma^*(M),\tau^*(M))\to \mathcal{F}_{C_1}(\sigma^*\mathscr{F}_{\boldsymbol{C}}D(i),\tau^*(M))\right)_{i\in \mathrm{Ob}\,\mathcal{D}}$$

is a limiting cone of a functor $\mathcal{D}^{op}\to \mathcal{S}et$ which assigns $i\in \mathrm{Ob}\,\mathcal{D}$ to $\mathcal{F}_{C_1}(\sigma^*\mathscr{F}_{\boldsymbol{C}}D(i),\tau^*(M))$ and $\alpha\in\mathcal{D}(i,j)$ to $\tau^*\mathscr{F}_{\boldsymbol{C}}D(\alpha)^* : \mathcal{F}_{C_1}(\sigma^*\mathscr{F}_{\boldsymbol{C}}D(j),\tau^*(M))\to \mathcal{F}_{C_1}(\sigma^*\mathscr{F}_{\boldsymbol{C}}D(i),\tau^*(M))$. We also assume that

$$\left((\sigma\mu)^*(\iota_i)^* : \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M),(\tau\mu)^*(M))\to \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*\mathscr{F}_{\boldsymbol{C}}D(i),(\tau\mu)^*(M))\right)_{i\in \mathrm{Ob}\,\mathcal{D}}$$

is a monomorphic family. Then, there exists a unique morphism $\xi:\sigma^*(M)\to\tau^*(M)$ such that (M,ξ) is a representation of $\boldsymbol{C}$ on M and $(\iota_i : D(i)\to (M,\xi))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of D.

Proof. For $i \in \operatorname{Ob}\mathcal{D}$, we denote by $\xi_i : \sigma^*\mathscr{F}_{\boldsymbol{C}}D(i) \to \tau^*\mathscr{F}_{\boldsymbol{C}}D(i)$ the structure morphism in the representation of $\boldsymbol{C}$ on $\mathscr{F}_{\boldsymbol{C}}D(i)$.

(1) Since $\xi_j\sigma^*D(\alpha) = \tau^*D(\alpha)\xi_i$ for any morphism $\alpha : i \to j$ of $\mathcal{D}$,

$$\left(\xi_{i*}\sigma^*(\pi_i)_* : \mathcal{F}_{C_1}(\sigma^*(M), \sigma^*(M)) \to \mathcal{F}_{C_1}(\sigma^*(M), \tau^*\mathscr{F}_{\boldsymbol{C}}D(i))\right)_{i\in\operatorname{Ob}\mathcal{D}}$$

is a cone of a functor $\mathcal{D} \to \mathcal{S}et$ which assigns $i \in \operatorname{Ob}\mathcal{D}$ to $\mathcal{F}_{C_1}(\sigma^*(M), \sigma^*\mathscr{F}_{\boldsymbol{C}}D(i))$. Hence there exists a unique map $\chi : \mathcal{F}_{C_1}(\sigma^*(M), \sigma^*(M)) \to \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(M))$ satisfying $\tau^*(\pi_i)_*\chi = \xi_{i*}\sigma^*(\pi_i)_*$ for every $i \in \operatorname{Ob}\mathcal{D}$. Put $\xi = \chi(id_{\sigma^*(M)})$, then we have $\tau^*(\pi_i)\xi = \xi_i\sigma^*(\pi_i)$ and

$$\begin{aligned} f^\sharp_{\sigma^*\mathscr{F}_{\boldsymbol{C}}D(i),\tau^*\mathscr{F}_{\boldsymbol{C}}D(i)}(\xi_i)f^\sharp_{\sigma^*(M),\sigma^*\mathscr{F}_{\boldsymbol{C}}D(i)}(\sigma^*(\pi_i)) &= f^\sharp_{\sigma^*(M),\tau^*\mathscr{F}_{\boldsymbol{C}}D(i)}(\xi_i\sigma^*(\pi_i)) = f^\sharp_{\sigma^*(M),\tau^*\mathscr{F}_{\boldsymbol{C}}D(i)}(\tau^*(\pi_i)\xi) \\ &= f^\sharp_{\tau^*(M),\tau^*\mathscr{F}_{\boldsymbol{C}}D(i)}(\tau^*(\pi_i))f^\sharp_{\sigma^*(M),\tau^*(M)}(\xi) \end{aligned}$$

for $f = \mathrm{pr}_1, \mathrm{pr}_2, \mu : C_1 \times_{C_0} C_1 \to C_1$. We note that $\mu^\sharp(\tau^*(\pi_i)) = (\tau\mu)^*(\pi_i) = (\tau\mathrm{pr}_2)^*(\pi_i) = \mathrm{pr}_2^\sharp(\tau^*(\pi_i))$, $\mathrm{pr}_1^\sharp(\tau^*(\pi_i)) = (\tau\mathrm{pr}_1)^*(\pi_i) = (\sigma\mathrm{pr}_2)^*(\pi_i)) = \mathrm{pr}_2^\sharp(\sigma^*(\pi_i))$ and $\mu^\sharp(\sigma^*(\pi_i)) = (\sigma\mu)^*(\pi_i) = (\sigma\mathrm{pr}_1)^*(\pi_i) = \mathrm{pr}_1^\sharp(\sigma^*(\pi_i))$. Since ξ_i satisfies (A) of (3.1.2), we have

$$\begin{aligned} \mu^\sharp(\tau^*(\pi_i))\mu^\sharp(\xi) &= \mu^\sharp(\xi_i)\mu^\sharp(\sigma^*(\pi_i)) = \mathrm{pr}_2^\sharp(\xi_i)\mathrm{pr}_1^\sharp(\xi_i)\mathrm{pr}_1^\sharp(\sigma^*(\pi_i)) = \mathrm{pr}_2^\sharp(\xi_i)\mathrm{pr}_1^\sharp(\xi_i\sigma^*(\pi_i)) = \mathrm{pr}_2^\sharp(\xi_i)\mathrm{pr}_1^\sharp(\tau^*(\pi_i)\xi) \\ &= \mathrm{pr}_2^\sharp(\xi_i)\mathrm{pr}_1^\sharp(\tau^*(\pi_i))\mathrm{pr}_1^\sharp(\xi) = \mathrm{pr}_2^\sharp(\xi_i)\mathrm{pr}_2^\sharp(\sigma^*(\pi_i))\mathrm{pr}_1^\sharp(\xi) = \mathrm{pr}_2^\sharp(\xi_i\sigma^*(\pi_i))\mathrm{pr}_1^\sharp(\xi) \\ &= \mathrm{pr}_2^\sharp(\tau^*(\pi_i)\xi)\mathrm{pr}_1^\sharp(\xi) = \mathrm{pr}_2^\sharp(\tau^*(\pi_i))\mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\xi) = \mu^\sharp(\tau^*(\pi_i))\mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\xi) \end{aligned}$$

for any $i \in \operatorname{Ob}\mathcal{D}$. Since $\mu^\sharp(\xi), \mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\xi) \in \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M), (\tau\mu)^*(M))$, the second assumption implies that ξ satisfies (A) of (3.1.2). Since $\varepsilon^\sharp(\xi_i)$ is the identity morphism of $\mathscr{F}_{\boldsymbol{C}}D(i)$, we have

$$\begin{aligned} \pi_i\varepsilon^\sharp(\xi) &= (\tau\varepsilon)^*(\pi_i)\varepsilon^\sharp(\xi) = \varepsilon^\sharp(\tau^*(\pi_i))\varepsilon^\sharp(\xi) = \varepsilon^\sharp(\tau^*(\pi_i)\xi) = \varepsilon^\sharp(\xi_i\sigma^*(\pi_i)) \\ &= \varepsilon^\sharp(\xi_i)\varepsilon^\sharp(\sigma^*(\pi_i)) = \varepsilon^\sharp(\sigma^*(\pi_i)) = (\sigma\varepsilon)^*(\pi_i) = \pi_i \end{aligned}$$

for any $i \in \operatorname{Ob}\mathcal{D}$. Since $(\pi_i : M \to \mathscr{F}_{\boldsymbol{C}}D(i))_{i\in\operatorname{Ob}\mathcal{D}}$ is a monomorphic family, ξ satisfies (U) of (3.1.2).

(2) Since $\xi_j\sigma^*D(\alpha) = \tau^*D(\alpha)\xi_i$ for any morphism $\alpha : i \to j$ of $\mathcal{D}$,

$$\left(\xi_i^*\tau^*(\iota_i)^* : \mathcal{F}_{C_1}(\tau^*(M), \tau^*(M)) \to \mathcal{F}_{C_1}(\sigma^*\mathscr{F}_{\boldsymbol{C}}D(i), \tau^*(M))\right)_{i\in\operatorname{Ob}\mathcal{D}}$$

is a cone of a functor $\mathcal{D}^{op} \to \mathcal{S}et$ which assigns $i \in \operatorname{Ob}\mathcal{D}$ to $\mathcal{F}_{C_1}(\sigma^*\mathscr{F}_{\boldsymbol{C}}D(i), \tau^*(M))$. Hence there exists a unique map $\chi : \mathcal{F}_{C_1}(\tau^*(M), \tau^*(M)) \to \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(M))$ satisfying $\sigma^*(\iota_i)^*\chi = \xi_i^*\tau^*(\iota_i)^*$ for every $i \in \operatorname{Ob}\mathcal{D}$. Put $\xi = \chi(id_{\tau^*(M)})$, then we have $\xi\sigma^*(\iota_i) = \tau^*(\iota_i)\xi_i$ and

$$\begin{aligned} f^\sharp_{\tau^*\mathscr{F}_{\boldsymbol{C}}D(i),\tau^*(M)}(\tau^*(\iota_i))f^\sharp_{\sigma^*\mathscr{F}_{\boldsymbol{C}}D(i),\tau^*\mathscr{F}_{\boldsymbol{C}}D(i)}(\xi_i) &= f^\sharp_{\sigma^*\mathscr{F}_{\boldsymbol{C}}D(i),\tau^*(M)}(\tau^*(\iota_i)\xi_i) = f^\sharp_{\sigma^*\mathscr{F}_{\boldsymbol{C}}D(i),\tau^*(M)}(\xi\sigma^*(\iota_i)) \\ &= f^\sharp_{\sigma^*(M),\tau^*(M)}(\xi)f^\sharp_{\sigma^*\mathscr{F}_{\boldsymbol{C}}D(i),\sigma^*(M)}(\sigma^*(\iota_i)) \end{aligned}$$

for $f = \mathrm{pr}_1, \mathrm{pr}_2, \mu : C_1 \times_{C_0} C_1 \to C_1$. We note that $\mu^\sharp(\tau^*(\iota_i)) = (\tau\mu)^*(\iota_i) = (\tau\mathrm{pr}_2)^*(\iota_i) = \mathrm{pr}_2^\sharp(\tau^*(\iota_i))$, $\mathrm{pr}_2^\sharp(\sigma^*(\iota_i)) = (\sigma\mathrm{pr}_2)^*(\iota_i)) = (\tau\mathrm{pr}_1)^*(\iota_i) = \mathrm{pr}_1^\sharp(\tau^*(\iota_i))$ and $\mathrm{pr}_1^\sharp(\sigma^*(\iota_i)) = (\sigma\mathrm{pr}_1)^*(\iota_i) = (\sigma\mu)^*(\iota_i) = \mu^\sharp(\sigma^*(\iota_i))$. Since ξ_i satisfies (A) of (3.1.2), we have

$$\begin{aligned} \mu^\sharp(\xi)\mu^\sharp(\sigma^*(\iota_i)) &= \mu^\sharp(\tau^*(\iota_i))\mu^\sharp(\xi_i) = \mathrm{pr}_2^\sharp(\tau^*(\iota_i))\mathrm{pr}_2^\sharp(\xi_i)\mathrm{pr}_1^\sharp(\xi_i) = \mathrm{pr}_2^\sharp(\tau^*(\iota_i)\xi_i)\mathrm{pr}_1^\sharp(\xi_i) = \mathrm{pr}_2^\sharp(\xi\sigma^*(\iota_i))\mathrm{pr}_1^\sharp(\xi_i) \\ &= \mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_2^\sharp(\sigma^*(\iota_i))\mathrm{pr}_1^\sharp(\xi_i) = \mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\tau^*(\iota_i))\mathrm{pr}_1^\sharp(\xi_i) = \mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\tau^*(\iota_i)\xi_i) \\ &= \mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\xi\sigma^*(\iota_i)) = \mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\xi)\mathrm{pr}_1^\sharp(\sigma^*(\iota_i)) = \mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\xi)\mu^\sharp(\sigma^*(\iota_i)) \end{aligned}$$

for any $i \in \operatorname{Ob}\mathcal{D}$. Since $\mu^\sharp(\xi), \mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\xi) \in \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M), (\tau\mu)^*(M))$, the second assumption implies that ξ satisfies (A) of (3.1.2). Since $\varepsilon^\sharp(\xi_i)$ is the identity morphism of $\mathscr{F}_{\boldsymbol{C}}D(i)$, we have

$$\begin{aligned} \varepsilon^\sharp(\xi)\iota_i &= \varepsilon^\sharp(\xi)(\sigma\varepsilon)^*(\iota_i) = \varepsilon^\sharp(\xi)\varepsilon^\sharp(\sigma^*(\iota_i)) = \varepsilon^\sharp(\xi\sigma^*(\iota_i)) = \varepsilon^\sharp(\tau^*(\iota_i)\xi_i) \\ &= \varepsilon^\sharp(\tau^*(\iota_i))\varepsilon^\sharp(\xi_i) = \varepsilon^\sharp(\tau^*(\iota_i)) = (\tau\varepsilon)^*(\iota_i) = \iota_i \end{aligned}$$

for any $i \in \operatorname{Ob}\mathcal{D}$. Since $(\iota_i : \mathscr{F}_{\boldsymbol{C}}D(i) \to M)_{i\in\operatorname{Ob}\mathcal{D}}$ is an epimorphic family, ξ satisfies (U) of (3.1.2). □

Remark 3.1.8 (1) *If $\tau^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_1}$ preserves limits and $\mu^* : \mathcal{F}_{C_1} \to \mathcal{F}_{C_1 \times_{C_0} C_1}$ preserves monomorphic families, the assumptions of* (1) *of* (3.1.7) *are satisfied for any functor $D : \mathcal{D} \to \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ such that $\mathscr{F}_{\boldsymbol{C}}D : \mathcal{D} \to \mathcal{F}_{C_0}$ has a limit. This case, $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$ creates limits in the sense of Mac Lane ([11], chapter V). In particular, if $p : \mathcal{F} \to \mathcal{E}$ is a bifibered category, $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$ creates limits.*

(2) *If $\sigma^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_1}$ preserves colimits and $\mu^* : \mathcal{F}_{C_1} \to \mathcal{F}_{C_1 \times_{C_0} C_1}$ preserves epimorphic families, the assumptions of* (2) *of* (3.1.7) *are satisfied for any functor $D : \mathcal{D} \to \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ such that $\mathscr{F}_{\boldsymbol{C}}D : \mathcal{D} \to \mathcal{F}_{C_0}$ has a colimit. This case, $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$ creates colimits.*

(3) *If (σ, τ) is a left fibered representable pair with respect to M, then the first assumption of* (1) *of* (3.1.7) *is satisfied. In fact, $\big(\pi_{i*} : \mathcal{F}_{C_0}(M_{[\sigma,\tau]}, M) \to \mathcal{F}_{C_0}(M_{[\sigma,\tau]}, \mathscr{F}_{\boldsymbol{C}}D(i))\big)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of a functor $\mathcal{D} \to \mathcal{S}et$ which assigns $i \in \mathrm{Ob}\,\mathcal{D}$ to $\mathcal{F}_{C_0}(M_{[\sigma,\tau]}, \mathscr{F}_{\boldsymbol{C}}D(i))$, $\alpha \in \mathcal{D}(i,j)$ to $\mathscr{F}_{\boldsymbol{C}}D(\alpha)_*$ and the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{F}_{C_1}(\sigma^*(M), \tau^*(M)) \arrow[r, "\tau^*(\pi_i)_*"] \arrow[d, "P_{\sigma,\tau}(M)_M"] & \mathcal{F}_{C_1}(\sigma^*(M), \tau^*\mathscr{F}_{\boldsymbol{C}}D(i)) \arrow[d, "P_{\sigma,\tau}(M)_{\mathscr{F}_{\boldsymbol{C}}D(i)}"] \\
\mathcal{F}_{C_0}(M_{[\sigma,\tau]}, M) \arrow[r, "\pi_{i*}"] & \mathcal{F}_{C_0}(M_{[\sigma,\tau]}, \mathscr{F}_{\boldsymbol{C}}D(i))
\end{tikzcd}$$

Similarly, if $(\sigma\mu, \tau\mu)$ is a left fibered representable pair with respect to M, then the second assumption of (1) *of* (3.1.7) *is satisfied. In fact, $(\pi_{i*})_{i\in \mathrm{Ob}\,\mathcal{D}} : \mathcal{F}_{C_0}(M_{[\sigma\mu,\tau\mu]}, M) \to \prod_{i\in\mathrm{Ob}\,\mathcal{D}} \mathcal{F}_{C_0}(M_{[\sigma\mu,\tau\mu]}, \mathscr{F}_{\boldsymbol{C}}D(i))$ is injective and the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M), (\tau\mu)^*(M)) \arrow[r, "((\tau\mu)^*(\pi_i)_*)_{i\in\mathrm{Ob}\,\mathcal{D}}"] \arrow[d, "P_{\sigma\mu,\tau\mu}(M)_M"] & \prod_{i\in\mathrm{Ob}\,\mathcal{D}} \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M), (\tau\mu)^*\mathscr{F}_{\boldsymbol{C}}D(i)) \arrow[d, "\prod_{i\in\mathrm{Ob}\,\mathcal{D}} P_{\sigma\mu,\tau\mu}(M)_{\mathscr{F}_{\boldsymbol{C}}D(i)}"] \\
\mathcal{F}_{C_0}(M_{[\sigma\mu,\tau\mu]}, M) \arrow[r, "(\pi_{i*})_{i\in\mathrm{Ob}\,\mathcal{D}}"] & \prod_{i\in\mathrm{Ob}\,\mathcal{D}} \mathcal{F}_{C_0}(M_{[\sigma\mu,\tau\mu]}, \mathscr{F}_{\boldsymbol{C}}D(i))
\end{tikzcd}$$

(4) *If (σ, τ) is a right fibered representable pair with respect to M, then the first assumption of* (2) *of* (3.1.7) *is satisfied. In fact, $\big(\iota_i^* : \mathcal{F}_{C_0}(M, M^{[\sigma,\tau]}) \to \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}D(i), M^{[\sigma,\tau]})\big)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of a functor $\mathcal{D}^{op} \to \mathcal{S}et$ which assigns $i \in \mathrm{Ob}\,\mathcal{D}$ to $\mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}D(i), M^{[\sigma,\tau]})$, $\alpha \in \mathcal{D}(i,j)$ to $\mathscr{F}_{\boldsymbol{C}}D(\alpha)^*$ and the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{F}_{C_1}(\sigma^*(M), \tau^*(M)) \arrow[r, "\sigma^*(\iota_i)^*"] \arrow[d, "E_{\sigma,\tau}(M)_M"] & \mathcal{F}_{C_1}(\sigma^*\mathscr{F}_{\boldsymbol{C}}D(i), \tau^*(M)) \arrow[d, "E_{\sigma,\tau}(M)_{\mathscr{F}_{\boldsymbol{C}}D(i)}"] \\
\mathcal{F}_{C_0}(M, M^{[\sigma,\tau]}) \arrow[r, "\iota_i^*"] & \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}D(i), M^{[\sigma,\tau]})
\end{tikzcd}$$

Similarly, if $(\sigma\mu, \tau\mu)$ is a right fibered representable pair with respect to M, then the second assumption of (2) *of* (3.1.7) *is satisfied. In fact, $(\iota_i^*)_{i\in\mathrm{Ob}\,\mathcal{D}} : \mathcal{F}_{C_0}(M, M^{[\sigma\mu,\tau\mu]}) \to \prod_{i\in\mathrm{Ob}\,\mathcal{D}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}D(i), M^{[\sigma\mu,\tau\mu]})$ is injective and the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M), (\tau\mu)^*(M)) \arrow[r, "((\sigma\mu)^*(\iota_i)^*)_{i\in\mathrm{Ob}\,\mathcal{D}}"] \arrow[d, "E_{\sigma\mu,\tau\mu}(M)_M"] & \prod_{i\in\mathrm{Ob}\,\mathcal{D}} \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*\mathscr{F}_{\boldsymbol{C}}D(i), (\tau\mu)^*(M)) \arrow[d, "\prod_{i\in\mathrm{Ob}\,\mathcal{D}} E_{\sigma\mu,\tau\mu}(M)_{\mathscr{F}_{\boldsymbol{C}}D(i)}"] \\
\mathcal{F}_{C_0}(M, M^{[\sigma\mu,\tau\mu]}) \arrow[r, "(\iota_i^*)_{i\in\mathrm{Ob}\,\mathcal{D}}"] & \prod_{i\in\mathrm{Ob}\,\mathcal{D}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}D(i), M^{[\sigma\mu,\tau\mu]})
\end{tikzcd}$$

Proposition 3.1.9 *The forgetful functor $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$ reflects isomorphisms.*

Proof. Let $\varphi : \xi \to \zeta$ be a morphism in $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ such that $\mathscr{F}_{\boldsymbol{C}}(\varphi)$ is an isomorphism. Since $\tau^*(\varphi^{-1})\zeta = \tau^*(\varphi^{-1})\zeta\sigma^*(\varphi)\sigma^*(\varphi^{-1}) = \tau^*(\varphi^{-1})\tau^*(\varphi)\xi\sigma^*(\varphi^{-1}) = \xi\sigma^*(\varphi^{-1})$, φ^{-1} is also a morphism in $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$. Hence φ is an isomorphism in $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$. □

Proposition 3.1.10 *Let $\xi : \sigma^*(M) \to \tau^*(M)$ be a morphism in $\mathcal{F}_{C_1}$.*

(1) *If ξ is a monomorphism or epimorphism which satisfies* (A) *of* (3.1.2)*, then ξ satisfies* (U) *of* (3.1.2).

(2) *If $\boldsymbol{C}$ is an internal groupoid in $\mathcal{E}$ and ξ satisfies* (A) *and* (U) *of* (3.1.2)*, then ξ is an isomorphism.*

Proof. (1) We put $\varepsilon_1=(id_{C_1},\varepsilon\tau),\varepsilon_2=(\varepsilon\sigma,id_{C_1}):C_1\to C_1\times_{C_0}C_1$. Since $\mu\varepsilon_1=\mu\varepsilon_2=id_{C_1}$, we have maps

$$\varepsilon_i^\sharp:\mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M),(\tau\mu)^*(M))\to\mathcal{F}_{C_1}(\sigma^*(M),\tau^*(M))$$

for $i=1,2$. Then, we have the following by (1.1.15) and (1.1.16).

$$\xi=(\mu\varepsilon_i)^\sharp(\xi)=\varepsilon_i^\sharp(\mu^\sharp(\xi))=\varepsilon_i^\sharp(\mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\xi))=\varepsilon_i^\sharp(\mathrm{pr}_2^\sharp(\xi))\varepsilon_i^\sharp(\mathrm{pr}_1^\sharp(\xi))=(\mathrm{pr}_2\varepsilon_i)^\sharp(\xi)(\mathrm{pr}_1\varepsilon_i)^\sharp(\xi)=\begin{cases}(\varepsilon\tau)^\sharp(\xi)\xi & i=1\\ \xi(\varepsilon\sigma)^\sharp(\xi) & i=2\end{cases}$$

Hence $(\varepsilon\tau)^\sharp(\xi)\xi=\xi(\varepsilon\sigma)^\sharp(\xi)=\xi$ which implies $(\varepsilon\tau)^\sharp(\xi)=id_{\tau^*(M)}$ if ξ is an epimorphism, $(\varepsilon\sigma)^\sharp(\xi)=id_{\sigma^*(M)}$ if ξ is a monomorphism. In the former case, since $\varepsilon^\sharp:\mathcal{F}_{C_1}(\tau^*(M),\tau^*(M))\to\mathcal{F}_{C_0}(M,M)$ maps $id_{\tau^*(M)}$ and $(\varepsilon\tau)^\sharp(\xi)$ to id_M and $(\varepsilon\tau\varepsilon)^\sharp(\xi)=\varepsilon^\sharp(\xi)=\xi_\varepsilon$ respectively, ξ satisfies (U) of (3.1.2). In the latter case, since $\varepsilon^\sharp:\mathcal{F}_{C_1}(\sigma^*(M),\sigma^*(M))\to\mathcal{F}_{C_0}(M,M)$ maps $id_{\sigma^*(M)}$ and $(\varepsilon\sigma)^\sharp(\xi)$ to id_M and $(\varepsilon\sigma\varepsilon)^\sharp(\xi)=\varepsilon^\sharp(\xi)=\xi_\varepsilon$ respectively, ξ satisfies (U) of (3.1.2).

(2) Let us denote by $\iota:C_1\to C_1$ the inverse of $\boldsymbol{C}$. Since $\sigma\iota=\tau$ and $\tau\iota=\sigma$, we have a morphism $\xi_\iota=\iota^\sharp(\xi):\tau^*(M)\to\sigma^*(M)$ $\mathcal{F}_{C_1}$ and morphisms $\iota_1=(id_{C_1},\iota),\iota_2=(\iota,id_{C_1}):C_1\to C_1\times_{C_0}C_1$ of $\mathcal{E}$. Since $(\mathrm{pr}_2\iota_i)^\sharp(\xi)(\mathrm{pr}_1\iota_i)^\sharp(\xi)=\iota_i^\sharp(\mathrm{pr}_2^\sharp(\xi))\iota_i^\sharp(\mathrm{pr}_1^\sharp(\xi))=\iota_i^\sharp(\mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\xi))=\iota_i^\sharp(\mu^\sharp(\xi))=(\mu\iota_i)^\sharp(\xi)$ for $i=1,2$ and $\mu\iota_1=\varepsilon\sigma$, $\mu\iota_2=\varepsilon\tau$, we have $\xi_\iota\xi=\iota^\sharp(\xi)\xi=(\mathrm{pr}_2\iota_1)^\sharp(\xi)(\mathrm{pr}_1\iota_1)^\sharp(\xi)=(\mu\iota_1)^\sharp(\xi)=(\varepsilon\sigma)^\sharp(\xi)=\sigma^\sharp(\varepsilon^\sharp(\xi))=\sigma^\sharp(id_M)=id_{\sigma^*(M)}$ and $\xi\xi_\iota=\xi\iota^\sharp(\xi)=(\mathrm{pr}_2\iota_2)^\sharp(\xi)(\mathrm{pr}_1\iota_2)^\sharp(\xi)=(\mu\iota_2)^\sharp(\xi)=(\varepsilon\tau)^\sharp(\xi)=\tau^\sharp(\varepsilon^\sharp(\xi))=\tau^\sharp(id_M)=id_{\tau^*(M)}$. □

Proposition 3.1.11 *Let $\boldsymbol{C}=(C_0,C_1;\sigma,\tau,\varepsilon,\mu)$ be an internal category in $\mathcal{E}$ and $s:\mathcal{E}\to\mathcal{F}$ a cartesian section. Then, $s_{\sigma,\tau}:\sigma^*s(C_0)\to\tau^*s(C_0)$ defined in (1.1.23) is a representation of $\boldsymbol{C}$ on $s(C_0)$.*

Proof. By (3.1.10), we only have to verify the condition (A) of (3.1.2). Since we assumed that $\mathcal{E}$ has finite limits, we may assume that $s=s_T$ for some $T\in\operatorname{Ob}\mathcal{F}_1$ by (1.1.22), here o_{C_0} denotes the unique morphism $C_0\to 1$. Then, $s_\sigma=c_{o_{C_0},\sigma}(T)^{-1}$, $s_\tau=c_{o_{C_0},\tau}(T)^{-1}$ and we have the following equalities by (1.1.12) for $f=\mu,\mathrm{pr}_1,\mathrm{pr}_2$.

$$c_{\tau,f}(s(C_0))f^*(s_\tau)=c_{\tau,f}(o_{C_0}^*(T))f^*(c_{o_{C_0},\tau}(T)^{-1})=c_{o_{C_0},\tau f}(T)^{-1}c_{o_{C_0}\tau,f}(T)=c_{o_{C_0},\tau f}(T)^{-1}c_{o_{C_1},f}(T)$$
$$f^*(s_\sigma^{-1})c_{\sigma,f}(s(C_0))^{-1}=f^*(c_{o_{C_0},\sigma}(T))c_{\sigma,f}(o_{C_0}^*(T))^{-1}=c_{o_{C_0}\sigma,f}(T)^{-1}c_{o_{C_0},\sigma f}(T)=c_{o_{C_1},f}(T)^{-1}c_{o_{C_0},\sigma f}(T)$$

Hence we have $f^\sharp(s_{\sigma,\tau})=c_{\tau,f}(s(C_0))f^*(s_\tau)f^*(s_\sigma^{-1})c_{\sigma,f}(s(C_0))^{-1}=c_{o_{C_0},\tau f}(T)^{-1}c_{o_{C_0},\sigma f}(T)$. Since $\tau\mathrm{pr}_2=\tau\mu$, $\sigma\mathrm{pr}_2=\tau\mathrm{pr}_1$ and $\sigma\mathrm{pr}_1=\sigma\mu$, above equality implies

$$\mathrm{pr}_2^\sharp(s_{\sigma,\tau})\mathrm{pr}_1^\sharp(s_{\sigma,\tau})=c_{o_{C_0},\tau\mathrm{pr}_2}(T)^{-1}c_{o_{C_0},\sigma\mathrm{pr}_2}(T)c_{o_{C_0},\tau\mathrm{pr}_1}(T)^{-1}c_{o_{C_0},\sigma\mathrm{pr}_1}(T)=c_{o_{C_0},\tau\mu}(T)^{-1}c_{o_{C_0},\sigma\mu}(T)=\mu^\sharp(s_{\sigma,\tau}).$$

Thus $s_{\sigma,\tau}$ satisfies the condition (A) of (3.1.2). □

Definition 3.1.12 *Let $\boldsymbol{C}=(C_0,C_1;\sigma,\tau,\varepsilon,\mu)$ be an internal category in $\mathcal{E}$ and $s:\mathcal{E}\to\mathcal{F}$ a cartesian section.*

(1) We set $s_{\boldsymbol{C}}=s_{\sigma,\tau}$ and call $(s(C_0),s_{\boldsymbol{C}})$ the trivial representation associated with s. In the case $s=s_T$ for some $T\in\operatorname{Ob}\mathcal{F}_1$, we also call $(s_T(C_0),(s_T)_{\boldsymbol{C}})$ the trivial representation associated with T.

(2) Let $\xi:\sigma^*(M)\to\tau^*(M)$ be a representation of $\boldsymbol{C}$ on M and T an object of $\mathcal{F}_1$. We call a morphism $\varphi:(M,\xi)\to(s(C_0),(s_T)_{\boldsymbol{C}})$ a primitive element of (M,ξ) with respect to T.

Let $p:\mathcal{F}\to\mathcal{E}$, $q:\mathcal{G}\to\mathcal{C}$ be normalized cloven fibered categories and $\boldsymbol{C}=(C_0,C_1;\sigma,\tau,\varepsilon,\mu)$ an internal category in $\mathcal{E}$. Suppose that functors $F:\mathcal{E}\to\mathcal{C}$ and $\Phi:\mathcal{F}\to\mathcal{G}$ are given such that $q\Phi=Fp$ and Φ preserves cartesian morphisms. We assume that $F(C_1)\xleftarrow{F(\mathrm{pr}_1)}F(C_1\times_{C_0}C_1)\xrightarrow{F(\mathrm{pr}_2)}F(C_1)$ is a limit of $F(C_1)\xrightarrow{F(\tau)}F(C_0)\xleftarrow{F(\sigma)}F(C_1)$. Then, $(F(C_0),F(C_1);F(\sigma),F(\tau),F(\varepsilon),F(\mu))$ is an internal category in $\mathcal{C}$. We denote this internal category by $F(\boldsymbol{C})$.

Proposition 3.1.13 *Let M be an object of $\mathcal{F}_{C_0}$ and $\xi:\sigma^*(M)\to\tau^*(M)$ a morphism in $\mathcal{F}_{C_1}$.*

(1) If (M,ξ) is a representation of $\boldsymbol{C}$ on M, $(\Phi(M),\Phi_{M,M}^{\sigma,\tau}(\xi))$ is a representation of $F(\boldsymbol{C})$ on $\Phi(M)$.

(2) If Φ is faithful and $(\Phi(M),\Phi_{M,M}^{\sigma,\tau}(\xi))$ is a representation of $F(\boldsymbol{C})$ on $\Phi(M)$, (M,ξ) is a representation of $\boldsymbol{C}$ on M.

Proof. (1) It follows from (1.1.19) and (1.1.17) that we have the following equality.

$$\Phi_{M,M}^{\sigma,\tau}(\xi)_{F(\mathrm{pr}_2)}\Phi_{M,M}^{\sigma,\tau}(\xi)_{F(\mathrm{pr}_1)}=\Phi_{M,M}^{\sigma\mathrm{pr}_2,\tau\mathrm{pr}_2}(\xi_{\mathrm{pr}_2})\Phi_{M,M}^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_1}(\xi_{\mathrm{pr}_1})=\Phi_{M,M}^{\sigma\mathrm{pr}_2,\tau\mu}(\xi_{\mathrm{pr}_2})\Phi_{M,M}^{\sigma\mu,\sigma\mathrm{pr}_2}(\xi_{\mathrm{pr}_1})=\Phi_{M,M}^{\sigma\mu,\tau\mu}(\xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1})$$

Thus $\Phi^{\sigma,\tau}_{M,M}(\xi)_{F(\mathrm{pr}_2)}\Phi^{\sigma,\tau}_{M,M}(\xi)_{F(\mathrm{pr}_1)} = \Phi^{\sigma\mu,\tau\mu}_{M,M}(\xi_\mu) = \Phi^{\sigma,\tau}_{M,M}(\xi)_{F(\mu)}$ by the assumption and (1.1.19). We also have $\Phi^{\sigma,\tau}_{M,M}(\xi)_{F(\varepsilon)} = \Phi^{\sigma\varepsilon,\tau\varepsilon}_{M,M}(\xi_\varepsilon) = \Phi^{id_{C_0},id_{C_0}}_{M,M}(id_M) = id_{\Phi(M)}$ by (1.1.19) and the assumption. Hence $(\Phi(M),\Phi^{\sigma,\tau}_{M,M}(\xi))$ is a representation of $F(\boldsymbol{C})$ on $\Phi(M)$.

(2) By (1.1.19), the assumption and the equality of (1) above, we have

$$\Phi^{\sigma\mu,\tau\mu}_{M,M}(\xi_\mu) = \Phi^{\sigma,\tau}_{M,M}(\xi)_{F(\mu)} = \Phi^{\sigma,\tau}_{M,M}(\xi)_{F(\mathrm{pr}_2)}\Phi^{\sigma,\tau}_{M,M}(\xi)_{F(\mathrm{pr}_1)} = \Phi^{\sigma\mu,\tau\mu}_{M,M}(\xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1})$$
$$\Phi^{id_{C_0},id_{C_0}}_{M,M}(\xi_\varepsilon) = \Phi^{\sigma\varepsilon,\tau\varepsilon}_{M,M}(\xi_\varepsilon) = \Phi^{\sigma,\tau}_{M,M}(\xi)_{F(\varepsilon)} = id_{\Phi(M)} = \Phi^{id_{C_0},id_{C_0}}_{M,M}(id_M)$$

Since Φ is faithful, $\Phi^{\sigma\mu,\tau\mu}_{M,M} : \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M),(\tau\mu)^*(M)) \to \mathcal{G}_{F(C_1\times_{C_0}C_1)}(F(\sigma\mu)^*(\Phi(M)),F(\tau\mu)^*(\Phi(M)))$ and $\Phi^{id_{C_0},id_{C_0}}_{M,M} : \mathcal{F}_{C_0}(id^*_{C_0}(M),id^*_{C_0}(M)) \to \mathcal{G}_{F(C_0)}(id^*_{F(C_0)}(\Phi(M)),id^*_{F(C_0)}(\Phi(M)))$ are injective, which implies $\xi_\mu = \xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1}$ and $\xi_\varepsilon = id_M$. □

Proposition 3.1.14 *Let $\varphi : M \to N$ be a morphism in $\mathcal{F}_{C_0}$ and (M,ξ), (N,ζ) representations of $\boldsymbol{C}$.*

(1) If $\varphi : (M,\xi) \to (N,\zeta)$ is a morphism representations of $\boldsymbol{C}$, $\Phi(\varphi) : (\Phi(M),\Phi^{\sigma,\tau}_{M,M}(\xi)) \to (\Phi(N),\Phi^{\sigma,\tau}_{N,N}(\zeta))$ is a morphism representations of $F(\boldsymbol{C})$.

(2) If Φ is faithful and $\Phi(\varphi) : (\Phi(M),\Phi^{\sigma,\tau}_{M,M}(\xi)) \to (\Phi(N),\Phi^{\sigma,\tau}_{N,N}(\zeta))$ is a morphism representations of $F(\boldsymbol{C})$, $\varphi : (M,\xi) \to (N,\zeta)$ is a morphism representations of $\boldsymbol{C}$.

Proof. It follows from (1.1.13) that the left and the right rectangles of the following diagram $(*)$ are commutative.

$$\begin{tikzcd}
F(\sigma)^*(\Phi(M)) \arrow[r, "c_{\sigma,\Phi}(M)^{-1}"] \arrow[d, "F(\sigma)^*(\Phi(\varphi))"] & \Phi(\sigma^*(M)) \arrow[r, "\Phi(\xi)"] \arrow[d, "\Phi(\sigma^*(\varphi))"] & \Phi(\tau^*(M)) \arrow[r, "c_{\tau,\Phi}(M)"] \arrow[d, "\Phi(\tau^*(\varphi))"] & F(\tau)^*(\Phi(M)) \arrow[d, "F(\tau)^*(\Phi(\varphi))"] \\
F(\sigma)^*(\Phi(N)) \arrow[r, "c_{\sigma,\Phi}(N)^{-1}"] & \Phi(\sigma^*(N)) \arrow[r, "\Phi(\zeta)"] & \Phi(\tau^*(N)) \arrow[r, "c_{\tau,\Phi}(N)"] & F(\tau)^*(\Phi(N))
\end{tikzcd} \qquad (*)$$

(1) Since $\Phi^{\sigma,\tau}_{M,M}(\xi) = c_{\tau,\Phi}(M)\Phi(\xi)c_{\sigma,\Phi}(M)^{-1}$, $\Phi^{\sigma,\tau}_{N,N}(\zeta) = c_{\tau,\Phi}(N)\Phi(\zeta)c_{\sigma,\Phi}(N)^{-1}$ and the middle rectangle of $(*)$ is commutative, the assertion follows.

(2) Since the outer rectangle of $(*)$ is commutative, we have

$$c_{\tau,\Phi}(N)\Phi(\tau^*(\varphi)\xi)c_{\sigma,\Phi}(M)^{-1} = c_{\tau,\Phi}(N)\Phi(\zeta\sigma^*(\varphi))c_{\sigma,\Phi}(M)^{-1}.$$

Thus $\Phi(\tau^*(\varphi)\xi) = \Phi(\zeta\sigma^*(\varphi))$ which implies $\tau^*(\varphi)\xi = \zeta\sigma^*(\varphi)$ by the assumption. □

Under the above situation, we can define a functor $\Phi_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C};\mathcal{F}) \to \mathrm{Rep}(F(\boldsymbol{C});\mathcal{G})$ by $\Phi_{\boldsymbol{C}}(M,\xi) = (\Phi(M),\Phi^{\sigma,\tau}_{M,M}(\xi))$ and $\Phi_{\boldsymbol{C}}(\varphi) = \Phi(\varphi)$. It follows from (3.1.14) that $\Phi_{\boldsymbol{C}}$ is fully faithful if Φ is so.

3.2 Restrictions, regular representations

Let $\boldsymbol{C} = (C_0,C_1;\sigma,\tau,\varepsilon,\mu)$ and $\boldsymbol{D} = (D_0,D_1;\sigma',\tau',\varepsilon',\mu')$ be internal categories in $\mathcal{E}$, $\boldsymbol{f} = (f_0,f_1) : \boldsymbol{D} \to \boldsymbol{C}$ an internal functor and $p : \mathcal{F} \to \mathcal{E}$ a cloven fibered category. Suppose that a representation (M,ξ) of $\boldsymbol{C}$ on $M \in \mathrm{Ob}\,\mathcal{F}_{C_0}$ is given. We denote by $\xi_{\boldsymbol{f}} : \sigma'^*(f_0^*(M)) \to \tau'^*(f_0^*(M))$ the following composition.

$$\sigma'^*(f_0^*(M)) \xrightarrow{c_{f_0,\sigma'}(M)} (f_0\sigma')^*(M) = (\sigma f_1)^*(M) \xrightarrow{(f_1)^\sharp_{M,M}(\xi)} (\tau f_1)^*(M) = (f_0\tau')^*(M) \xrightarrow{c_{f_0,\tau'}(M)^{-1}} \tau'^*(f_0^*(M))$$

Proposition 3.2.1 *$(f_0^*(M),\xi_{\boldsymbol{f}})$ is a representation of $\boldsymbol{D}$ on $f_0^*(M) \in \mathrm{Ob}\,\mathcal{F}_{D_0}$.*

Proof. $(\mathrm{pr}_i^\sharp)_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}})$ is the following composition for $i = 1,2$.

$$(\sigma'\mathrm{pr}_i)^*(f_0^*(M)) \xrightarrow{c_{\sigma',\mathrm{pr}_i}(f_0^*(M))^{-1}} \mathrm{pr}_i^*\sigma'^*(f_0^*(M)) \xrightarrow{\mathrm{pr}_i^*(c_{f_0,\sigma'}(M))} \mathrm{pr}_i^*(f_0\sigma')^*(M) = \mathrm{pr}_i^*(\sigma f_1)^*(M) \xrightarrow{\mathrm{pr}_i^*((f_1)^\sharp_{M,M}(\xi))}$$
$$\mathrm{pr}_i^*(\tau f_1)^*(M) = \mathrm{pr}_i^*(f_0\tau')^*(M) \xrightarrow{\mathrm{pr}_i^*(c_{f_0,\tau'}(M)^{-1})} \mathrm{pr}_i^*\tau'^*(f_0^*(M)) \xrightarrow{c_{\tau',\mathrm{pr}_i}(f_0^*(M))} (\tau'\mathrm{pr}_i)^*(f_0^*(M))$$

It follows from (1.1.12) and $f_0\sigma' = \sigma f_1$, $f_0\tau' = \tau f_1$ that $(\mathrm{pr}_i^\sharp)_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}})$ is the following composition.

$$(\sigma'\mathrm{pr}_i)^*(f_0^*(M)) \xrightarrow{c_{f_0,\sigma'\mathrm{pr}_i}(M)} (f_0\sigma'\mathrm{pr}_i)^*(M) = (\sigma f_1\mathrm{pr}_i)^*(M) \xrightarrow{(\mathrm{pr}_i^\sharp)_{M,M}((f_1)^\sharp_{M,M}(\xi))} (\tau f_1\mathrm{pr}_i)^*(M)$$
$$= (f_0\tau'\mathrm{pr}_i)^*(M) \xrightarrow{c_{f_0,\tau'\mathrm{pr}_i}(M)^{-1}} (\tau'\mathrm{pr}_i)^*(f_0^*(M))$$

Moreover, since $(\mathrm{pr}_i^\sharp)_{M,M}((f_1)^\sharp_{M,M}(\xi))=(f_1\mathrm{pr}_i)^\sharp_{M,M}(\xi)=(\mathrm{pr}_i(f_1\times_{C_0}f_1))^\sharp_{M,M}(\xi)=(f_1\times_{C_0}f_1)^\sharp_{M,M}((\mathrm{pr}_i)^\sharp_{M,M}(\xi))$ by (1.1.16), $(\mathrm{pr}_i^\sharp)_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}})$ is the following composition.

$$(\sigma'\mathrm{pr}_i)^*(f_0^*(M)) \xrightarrow{c_{f_0,\sigma'\mathrm{pr}_i}(M)} (f_0\sigma'\mathrm{pr}_i)^*(M) = (\sigma\mathrm{pr}_i(f_1\times_{C_0}f_1))^*(M) \xrightarrow{(f_1\times_{C_0}f_1)^\sharp_{M,M}((\mathrm{pr}_i)^\sharp_{M,M}(\xi))} (\tau\mathrm{pr}_i(f_1\times_{C_0}f_1))^*(M) = (f_0\tau'\mathrm{pr}_i)^*(M) \xrightarrow{c_{f_0,\tau'\mathrm{pr}_i}(M)^{-1}} (\tau'\mathrm{pr}_i)^*(f_0^*(M))$$

Hence the composition

$$(\sigma'\mu')^*(f_0^*(M)) = (\sigma'\mathrm{pr}_1)^*(f_0^*(M)) \xrightarrow{(\mathrm{pr}_1^\sharp)_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}})} (\tau'\mathrm{pr}_1)^*(f_0^*(M)) = (\sigma'\mathrm{pr}_2)^*(f_0^*(M)) \xrightarrow{(\mathrm{pr}_2^\sharp)_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}})} (\tau'\mathrm{pr}_2)^*(f_0^*(M)) = (\tau'\mu')^*(f_0^*(M)) \quad \cdots (*)$$

coincides with the following composition since $\sigma'\mathrm{pr}_1=\sigma'\mu'$, $\tau'\mathrm{pr}_2=\tau'\mu'$.

$$(\sigma'\mu')^*(f_0^*(M)) \xrightarrow{c_{f_0,\sigma'\mu'}(M)} (f_0\sigma'\mu')^*(M) = (\sigma\mathrm{pr}_1(f_1\times_{C_0}f_1))^*(M) \xrightarrow{(f_1\times_{C_0}f_1)^\sharp_{M,M}((\mathrm{pr}_1)^\sharp_{M,M}(\xi))} (\tau\mathrm{pr}_1(f_1\times_{C_0}f_1))^*(M) = (\sigma\mathrm{pr}_2(f_1\times_{C_0}f_1))^*(M) \xrightarrow{(f_1\times_{C_0}f_1)^\sharp_{M,M}((\mathrm{pr}_2)^\sharp_{M,M}(\xi))} (\tau\mathrm{pr}_2(f_1\times_{C_0}f_1))^*(M) = (f_0\tau'\mu')^*(M) \xrightarrow{c_{f_0,\tau'\mu'}(M)^{-1}} (\tau'\mu')^*(f_0^*(M))$$

Since ξ satisfies (A) of (3.1.2), it follows from (1.1.15) that we have

$$\begin{aligned}(f_1\times_{C_0}f_1)^\sharp_{M,M}((\mathrm{pr}_2)^\sharp_{M,M}(\xi))(f_1\times_{C_0}f_1)^\sharp_{M,M}((\mathrm{pr}_1)^\sharp_{M,M}(\xi)) &= (f_1\times_{C_0}f_1)^\sharp_{M,M}((\mathrm{pr}_2)^\sharp_{M,M}(\xi)(\mathrm{pr}_1)^\sharp_{M,M}(\xi))\\ &= (f_1\times_{C_0}f_1)^\sharp_{M,M}(\mu^\sharp_{M,M}(\xi)).\end{aligned}$$

Therefore the above composition $(*)$ coincides with the following composition.

$$(\sigma'\mu')^*(f_0^*(M)) \xrightarrow{c_{f_0,\sigma'\mu'}(M)} (f_0\sigma'\mu')^*(M) = (\sigma\mu(f_1\times_{C_0}f_1))^*(M) \xrightarrow{(f_1\times_{C_0}f_1)^\sharp_{M,M}(\mu^\sharp_{M,M}(\xi))} (\tau\mu(f_1\times_{C_0}f_1))^*(M) = (f_0\tau'\mu')^*(M) \xrightarrow{c_{f_0,\tau'\mu'}(M)^{-1}} (\tau'\mu')^*(f_0^*(M))$$

On the other hand, ${\mu'}^\sharp_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}})$ is the following composition.

$$(\sigma'\mu')^*(f_0^*(M)) \xrightarrow{c_{\sigma',\mu'}(f_0^*(M))^{-1}} \mu'^*\sigma'^*(f_0^*(M)) \xrightarrow{\mu'^*(c_{f_0,\sigma'}(M))} \mu'^*(f_0\sigma')^*(M) = \mu'^*(\sigma f_1)^*(M) \xrightarrow{\mu'^*((f_1)^\sharp_{M,M}(\xi))} \mu'^*(\tau f_1)^*(M) = \mu'^*(f_0\tau')^*(M) \xrightarrow{\mu'^*(c_{f_0,\tau'}(M)^{-1})} \mu'^*\tau'^*(f_0^*(M)) \xrightarrow{c_{\tau',\mu'}(f_0^*(M))} (\tau'\mu')^*(f_0^*(M))$$

It follows from (1.1.12) and $f_0\sigma'=\sigma f_1$, $f_0\tau'=\tau f_1$ that ${\mu'}^\sharp_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}})$ is the following composition.

$$(\sigma'\mu')^*(f_0^*(M)) \xrightarrow{c_{f_0,\sigma'\mu'}(M)} (f_0\sigma'\mu')^*(M) = (\sigma f_1\mu')^*(M) \xrightarrow{{\mu'}^\sharp_{M,M}((f_1)^\sharp_{M,M}(\xi))} (\tau f_1\mu')(M) = (f_0\tau'\mu')^*(M) \xrightarrow{c_{f_0,\tau'\mu'}(M)^{-1}} (\tau'\mu')^*(f_0^*(M))$$

By (1.1.16), ${\mu'}^\sharp_{M,M}((f_1)^\sharp_{M,M}(\xi)) : (\sigma\mu(f_1\times_{C_0}f_1))^*(M) = (\sigma f_1\mu')^*(M) \to (\tau f_1\mu')^*(M) = (\tau\mu(f_1\times_{C_0}f_1))^*(M)$ coincides with

$$(f_1\mu')^\sharp_{M,M}(\xi)=(\mu(f_1\times_{C_0}f_1))^\sharp_{M,M}(\xi)=(f_1\times_{C_0}f_1)^\sharp_{M,M}(\mu^\sharp_{M,M}(\xi)) : (\sigma\mu(f_1\times_{C_0}f_1))^*(M) \to (\tau\mu(f_1\times_{C_0}f_1))^*(M).$$

Thus we have verified that $\xi_{\boldsymbol{f}}$ satisfies (A) of (3.1.2).

${\varepsilon'}^\sharp_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}}) : f_0^*(M) = (\sigma'\varepsilon')^*(f_0^*(M)) \to (\tau'\varepsilon')^*(f_0^*(M)) = f_0^*(M)$ is the following composition.

$$(\sigma'\varepsilon')^*(f_0^*(M)) \xrightarrow{c_{\sigma',\varepsilon'}(f_0^*(M))^{-1}} \varepsilon'^*\sigma'^*(f_0^*(M)) \xrightarrow{\varepsilon'^*(c_{f_0,\sigma'}(M))} \varepsilon'^*(f_0\sigma')^*(M) = \varepsilon'^*(\sigma f_1)^*(M) \xrightarrow{\varepsilon'^*((f_1)^\sharp_{M,M}(\xi))} \varepsilon'^*(\tau f_1)^*(M) = \varepsilon'^*(f_0\tau')^*(M) \xrightarrow{\varepsilon'^*(c_{f_0,\tau'}(M)^{-1})} \varepsilon'^*\tau'^*(f_0^*(M)) \xrightarrow{c_{\tau',\varepsilon'}(f_0^*(M))} (\tau'\varepsilon')^*(f_0^*(M))$$

It follows from (1.1.12) and $f_0\sigma' = \sigma f_1$, $f_0\tau' = \tau f_1$ that ${\varepsilon'}^{\sharp}_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}})$ is the following composition.

$$(\sigma'\varepsilon')^*(f_0^*(M)) \xrightarrow{c_{f_0,\sigma'\varepsilon'}(M)} (f_0\sigma'\varepsilon')^*(M) = (\sigma f_1\varepsilon')^*(M) \xrightarrow{{\varepsilon'}^{\sharp}_{M,M}((f_1)^{\sharp}_{M,M}(\xi))} (\tau f_1\varepsilon')^*(M) = (f_0\tau'\varepsilon')^*(M)$$
$$\xrightarrow{c_{f_0,\tau'\varepsilon'}(M)^{-1}} (\tau'\varepsilon')^*(f_0^*(M))$$

Since ${\varepsilon'}^{\sharp}_{M,M}((f_1)^{\sharp}_{M,M}(\xi)) = (f_1\varepsilon')^{\sharp}_{M,M}(\xi) = (\varepsilon f_0)^{\sharp}_{M,M}(\xi) = (f_0^{\sharp})_{M,M}(\varepsilon^{\sharp}_{M,M}(\xi)) = (f_0^{\sharp})_{M,M}(id_M) = id_{f_0^*(M)}$ by (1.1.15) and (1.1.16), the above composition is the identity morphism of $f_0^*(M)$. □

Proposition 3.2.2 *Let (M,ξ) and (N,ζ) be representations of $\boldsymbol{C}$ and $\boldsymbol{f} : \boldsymbol{D} \to \boldsymbol{C}$ an internal functor. For a morphism of representations $\varphi : (M,\xi) \to (N,\zeta)$ of $\boldsymbol{C}$, $f_0^*(\varphi) : f_0^*(M) \to f_0^*(N)$ defines a morphism $f_0^*(\varphi) : (f_0^*(M), \xi_{\boldsymbol{f}}) \to (f_0^*(N), \zeta_{\boldsymbol{f}})$ of representations.*

Proof. By the naturality of $f_1^{\sharp}$, we have $(\tau f_1)^*(\varphi)f_1^{\sharp}(\xi) = f_1^{\sharp}(\tau^*(\varphi)\xi) = f_1^{\sharp}(\zeta\sigma^*(\varphi)) = f_1^{\sharp}(\zeta)(\sigma f_1)^*(\varphi)$. Then, the following diagram commute.

$$\begin{array}{ccccccccccc}
\sigma'^*f_0^*(M) & \xrightarrow{c_{f_0,\sigma'}(M)} & (f_0\sigma')^*(M) & = & (\sigma f_1)^*(M) & \xrightarrow{f_1^{\sharp}(\xi)} & (\tau f_1)^*(M) & = & (f_0\tau')^*(M) & \xrightarrow{c_{f_0,\tau'}(M)^{-1}} & \tau'^*f_0^*(M) \\
\downarrow{\scriptstyle \sigma'^*f_0^*(\varphi)} & & \downarrow{\scriptstyle (f_0\sigma')^*(\varphi)} & & \downarrow{\scriptstyle (\sigma f_1)^*(\varphi)} & & \downarrow{\scriptstyle (\tau f_1)^*(\varphi)} & & \downarrow{\scriptstyle (f_0\tau')^*(\varphi)} & & \downarrow{\scriptstyle \tau'^*f_0^*(\varphi)} \\
\sigma'^*f_0^*(N) & \xrightarrow{c_{f_0,\sigma'}(N)} & (f_0\sigma')^*(N) & = & (\sigma f_1)^*(N) & \xrightarrow{f_1^{\sharp}(\zeta)} & (\tau f_1)^*(N) & = & (f_0\tau')^*(N) & \xrightarrow{c_{f_0,\tau'}(N)^{-1}} & \tau'^*f_0^*(N)
\end{array}$$

Hence $f_0^*(\varphi) : f_0^*(M) \to f_0^*(N)$ defines a morphism $f_0^*(\varphi) : (f_0^*(M), \xi_{\boldsymbol{f}}) \to (f_0^*(N), \zeta_{\boldsymbol{f}})$ of representations. □

Definition 3.2.3 *We call $(f_0^*(M), \xi_{\boldsymbol{f}})$ the restriction of (M,ξ) along $\boldsymbol{f}$. It follows that we have a functor $\boldsymbol{f}^{\bullet} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})$ given by $\boldsymbol{f}^{\bullet}(M,\xi) = (f_0^*(M), \xi_{\boldsymbol{f}})$ for an object (M,ξ) of $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ and $\boldsymbol{f}^{\bullet}(\varphi) = f_0^*(\varphi)$ for a morphism φ of $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$.*

Let $p : \mathcal{F} \to \mathcal{E}$, $q : \mathcal{G} \to \mathcal{C}$ be normalized cloven fibered categories and $F : \mathcal{E} \to \mathcal{C}$, $\Phi : \mathcal{F} \to \mathcal{G}$ functors such that $q\Phi = Fp$ and Φ preserves cartesian morphisms. For internal categories $\boldsymbol{C}$ and $\boldsymbol{D}$ of $\mathcal{E}$, we assume that $F(C_1) \xleftarrow{F(\mathrm{pr}_1)} F(C_1 \times_{C_0} C_1) \xrightarrow{F(\mathrm{pr}_2)} F(C_1)$ is a limit of $F(C_1) \xrightarrow{F(\tau)} F(C_0) \xleftarrow{F(\sigma)} F(C_1)$ and that $F(D_1) \xleftarrow{F(\mathrm{pr}_1)} F(D_1 \times_{D_0} D_1) \xrightarrow{F(\mathrm{pr}_2)} F(D_1)$ is a limit of $F(D_1) \xrightarrow{F(\tau')} F(D_0) \xleftarrow{F(\sigma')} F(D_1)$ Then, $(F(C_0), F(C_1); F(\sigma), F(\tau), F(\varepsilon), F(\mu))$ and $(F(D_0), F(D_1); F(\sigma'), F(\tau'), F(\varepsilon'), F(\mu'))$ are internal categories in $\mathcal{C}$. We denote these internal categories by $F(\boldsymbol{C})$ and $F(\boldsymbol{D})$, respectively. For an internal functor $\boldsymbol{f} : \boldsymbol{C} \to \boldsymbol{D}$, $(F(f_0), F(f_1)) : F(\boldsymbol{D}) \to F(\boldsymbol{C})$ is an internal functor and we denote this by $F(\boldsymbol{f})$.

Proposition 3.2.4 *For a representation (M,ξ) of $\boldsymbol{C}$, the isomorphism $c_{f_0,\Phi}(M) : \Phi(f_0^*(M)) \to F(f_0)^*(\Phi(M))$ defines an isomorphism $\left(\Phi(f_0^*(M)), \Phi^{\sigma',\tau'}_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}})\right) \longrightarrow \left(F(f_0)^*(\Phi(M)), \Phi^{\sigma,\tau}_{M,M}(\xi)_{F(\boldsymbol{f})}\right)$ of representations of $F(\boldsymbol{D})$. Thus we have a natural equivalence $\Phi_{\boldsymbol{D}}\boldsymbol{f}^{\bullet} \to F(\boldsymbol{f})^{\bullet}\Phi_{\boldsymbol{C}}$.*

Proof. The upper and lower rectangles of the following diagram is commutative by (1.1.14). The left middle rectangle is commutative by the definition of $\xi_{\boldsymbol{f}}$ and the right middle rectangle is commutative by (1.1.19).

$$\begin{array}{ccccc}
F(\sigma')^*(\Phi(f_0^*(M))) & & \xrightarrow{F(\sigma')^*(c_{f_0,\Phi}(M))} & & F(\sigma')^*(F(f_0)^*(\Phi(M))) \\
\downarrow{\scriptstyle c_{\sigma',\Phi}(f_0^*(M))^{-1}} & & & & \downarrow{\scriptstyle c_{F(f_0),F(\sigma')}(\Phi(M))} \\
\Phi(\sigma'^*(f_0^*(M))) & \xrightarrow{\Phi(c_{f_0,\sigma'}(M))} & \Phi((f_0\sigma')^*(M)) & \xrightarrow{c_{f_0\sigma',\Phi}(M)} & F(f_0\sigma')^*(\Phi(M)) \\
\downarrow{\scriptstyle \Phi(\xi_{\boldsymbol{f}})} & & \downarrow{\scriptstyle \Phi(f_1^{\sharp}(\xi))} & & \downarrow{\scriptstyle F(f_1)^{\sharp}(\Phi^{\sigma,\tau}_{M,M}(\xi))} \\
\Phi(\tau'^*(f_0^*(M))) & \xrightarrow{\Phi(c_{f_0,\tau'}(M))} & \Phi((f_0\tau')^*(M)) & \xrightarrow{c_{f_0\tau',\Phi}(M)} & F(f_0\tau')^*(\Phi(M)) \\
\downarrow{\scriptstyle c_{\tau',\Phi}(f_0^*(M))} & & & & \downarrow{\scriptstyle c_{F(f_0),F(\tau')}(\Phi(M))^{-1}} \\
F(\tau')^*(\Phi(f_0^*(M))) & & \xrightarrow{F(\tau')^*(c_{f_0,\Phi}(M))} & & F(\tau')^*(F(f_0)^*(\Phi(M)))
\end{array}$$

Since the left vertical composition of the above diagram is $\Phi^{\sigma',\tau'}_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}})$ and the right vertical composition is $\Phi^{\sigma,\tau}_{M,M}(\xi)_{F(\boldsymbol{f})}$, the assertion follows. □

If $g = (g_0, g_1) : \boldsymbol{D} \to \boldsymbol{C}$ is an internal functor and χ is an internal natural transformation from f to g, let us define a morphism $\chi^{\bullet}_{(M,\xi)} : f_0^*(M) \to g_0^*(M)$ in $\mathcal{F}_{D_0}$ to be $\chi^{\sharp}_{M,M}(\xi) : f_0^*(M) = (\sigma\chi)^*(M) \to (\tau\chi)^*(M) = g_0^*(M)$.

Proposition 3.2.5 *$\chi^\bullet_{(M,\xi)}$ is a morphism of representations from $(f_0^*(M),\xi_{\boldsymbol{f}})$ to $(g_0^*(M),\xi_{\boldsymbol{g}})$ and the following diagram in $\mathrm{Rep}(\boldsymbol{D};\mathcal{F})$ commutes for a morphism $\varphi:(M,\xi)\to(N,\zeta)$ of representations of $\boldsymbol{C}$.*

$$\begin{tikzcd}
(f_0^*(M),\xi_{\boldsymbol{f}}) \arrow[r, "f^*(\varphi)"] \arrow[d, "\chi^\sharp_{M,M}(\xi)"] & (f_0^*(N),\zeta_{\boldsymbol{f}}) \arrow[d, "\chi^\sharp_{N,N}(\xi)"] \\
(g_0^*(M),\xi_{\boldsymbol{g}}) \arrow[r, "g^*(\varphi)"] & (g_0^*(N),\zeta_{\boldsymbol{g}})
\end{tikzcd}$$

Thus we have a natural transformation $\chi^\bullet:\boldsymbol{f}^\bullet\to\boldsymbol{g}^\bullet$

Proof. Since ξ satisfies the condition (A) of (3.1.2), it follows from (1.1.15) and (1.1.16) that we have

$$\begin{aligned}
(\chi\tau')^\sharp(\xi)(f_1)^\sharp(\xi) &= (\mathrm{pr}_2(f_1,\chi\tau'))^\sharp(\xi)(\mathrm{pr}_1(f_1,\chi\tau'))^\sharp(\xi) = (f_1,\chi\tau')^\sharp((\mathrm{pr}_2)^\sharp(\xi))(f_1,\chi\tau')^\sharp((\mathrm{pr}_1)^\sharp(\xi)) \\
&= (f_1,\chi\tau')^\sharp((\mathrm{pr}_2)^\sharp(\xi)(\mathrm{pr}_1)^\sharp(\xi)) = (f_1,\chi\tau')^\sharp(\mu^\sharp(\xi)) = (\mu(f_1,\chi\tau'))^\sharp(\xi) = (\mu(\chi\sigma',g_1))^\sharp(\xi) \\
&= (\chi\sigma',g_1)^\sharp(\mu^\sharp(\xi)) = (\chi\sigma',g_1)^\sharp((\mathrm{pr}_2)^\sharp(\xi)(\mathrm{pr}_1)^\sharp(\xi)) = (\chi\sigma',g_1)^\sharp((\mathrm{pr}_2)^\sharp(\xi))(\chi\sigma',g_1)^\sharp((\mathrm{pr}_1)^\sharp(\xi)) \\
&= (\mathrm{pr}_2(\chi\sigma',g_1))^\sharp(\xi)(\mathrm{pr}_1(\chi\sigma',g_1))^\sharp(\xi) = (g_1)^\sharp(\xi)(\chi\sigma')^\sharp(\xi).
\end{aligned}$$

Hence the middle rectangle of the following diagram is commutative.

$$\begin{tikzcd}
\sigma'^*f_0^*(M) \arrow[r, equal] \arrow[d, "c_{f_0,\sigma'}(M)"] & \sigma'^*(\sigma\chi)^*(M) \arrow[r, "\sigma'^*(\chi^\sharp_{M,M}(\xi))"] \arrow[d, "c_{\sigma\chi,\sigma'}(M)"] & \sigma'^*(\tau\chi)^*(M) \arrow[r, equal] \arrow[d, "c_{\tau\chi,\sigma'}(M)"] & \sigma'^*g_0^*(M) \arrow[d, "c_{g_0,\sigma'}(M)"] \\
(f_0\sigma')^*(M) \arrow[r, equal] \arrow[d, equal] & (\sigma\chi\sigma')^*(M) \arrow[r, "(\chi\sigma')^\sharp_{M,M}(\xi)"] & (\tau\chi\sigma')^*(M) \arrow[r, equal] & (g_0\sigma')^*(M) \arrow[d, equal] \\
(\sigma f_1)^*(M) \arrow[d, "(f_1)^\sharp_{M,M}(\xi)"] & & & (\sigma g_1)^*(M) \arrow[d, "(g_1)^\sharp_{M,M}(\xi)"] \\
(\tau f_1)^*(M) \arrow[d, equal] & & & (\tau g_1)^*(M) \arrow[d, equal] \\
(f_0\tau')^*(M) \arrow[r, equal] \arrow[d, "c_{f_0,\tau'}(M)^{-1}"] & (\sigma\chi\tau')^*(M) \arrow[r, "(\chi\tau')^\sharp_{M,M}(\xi)"] \arrow[d, "c_{\sigma\chi,\tau'}(M)^{-1}"] & (\tau\chi\tau')^*(M) \arrow[r, equal] \arrow[d, "c_{\tau\chi,\tau'}(M)^{-1}"] & (g_0\tau')^*(M) \arrow[d, "c_{g_0,\tau'}(M)^{-1}"] \\
\tau'^*f_0^*(M) \arrow[r, equal] & \tau'^*(\sigma\chi)^*(M) \arrow[r, "\tau'^*(\chi^\sharp_{M,M}(\xi))"] & \tau'^*(\tau\chi)^*(M) \arrow[r, equal] & \tau'^*g_0^*(M)
\end{tikzcd}$$

Since the upper and lower middle small rectangles of the above diagram also commutes by (1.1.16) the outer rectangle of the above diagram is commutative. Since the left (resp. right) vertical composition of the above is $\xi_{\boldsymbol{f}}$ (resp. $\xi_{\boldsymbol{g}}$), we see that $\chi^\bullet_{(M,\xi)}$ is a morphism of representations from $(f_0^*(M),\xi_{\boldsymbol{f}})$ to $(g_0^*(M),\xi_{\boldsymbol{g}})$.

The following diagram commutes by by (1.1.11) and (1.1.12).

$$\begin{tikzcd}
\sigma'^*f_0^*(M) \arrow[r, "c_{f_0,\sigma'}(M)"] \arrow[d, equal] & (f_0\sigma')^*(M) \arrow[r, "(f_0\sigma')^*(\varphi)"] \arrow[d, equal] & (f_0\sigma')^*(N) \arrow[r, "c_{f_0,\sigma'}(N)^{-1}"] \arrow[d, equal] & \sigma'^*f_0^*(N) \arrow[d, equal] \\
\sigma'^*(\sigma\chi)^*(M) \arrow[r, "c_{\sigma\chi,\sigma'}(M)"] \arrow[d, "\sigma'^*(c_{\sigma,\chi}(M))^{-1}"] & (\sigma\chi\sigma')^*(M) \arrow[r, "(\sigma\chi\sigma')^*(\varphi)"] \arrow[d, "c_{\sigma,\chi\sigma'}(M)^{-1}"] & (\sigma\chi\sigma')^*(N) \arrow[r, "c_{\sigma\chi,\sigma'}(N)^{-1}"] \arrow[d, "c_{\sigma,\chi\sigma'}(N)^{-1}"] & \sigma'^*(\sigma\chi)^*(N) \arrow[d, "\sigma'^*(c_{\sigma,\chi}(N))^{-1}"] \\
\sigma'^*\chi^*\sigma^*(M) \arrow[r, "c_{\chi,\sigma'}(\sigma^*(M))"] \arrow[d, "\sigma'^*\chi^*(\xi)"] & (\chi\sigma')^*\sigma^*(M) \arrow[r, "(\chi\sigma')^*\sigma^*(\varphi)"] \arrow[d, "(\chi\sigma')^*(\xi)"] & (\chi\sigma')^*\sigma^*(N) \arrow[r, "c_{\chi,\sigma'}(\sigma^*(N))^{-1}"] \arrow[d, "(\chi\sigma')^*(\zeta)"] & \sigma'^*\chi^*\sigma^*(N) \arrow[d, "\sigma'^*\chi^*(\zeta)"] \\
\sigma'^*\chi^*\tau^*(M) \arrow[r, "c_{\chi,\sigma'}(\tau^*(M))"] \arrow[d, "\sigma'^*(c_{\tau,\chi}(M))"] & (\chi\sigma')^*\tau^*(M) \arrow[r, "(\chi\sigma')^*\tau^*(\varphi)"] \arrow[d, "c_{\tau,\chi\sigma'}(M)"] & (\chi\sigma')^*\tau^*(N) \arrow[r, "c_{\chi,\sigma'}(\tau^*(N))^{-1}"] \arrow[d, "c_{\tau,\chi\sigma'}(N)"] & \sigma'^*\chi^*\tau^*(N) \arrow[d, "\sigma'^*(c_{\tau,\chi}(N))"] \\
\sigma'^*(\tau\chi)^*(M) \arrow[r, "c_{\tau\chi,\sigma'}(M)"] \arrow[d, equal] & (\tau\chi\sigma')^*(M) \arrow[r, "(\tau\chi\sigma')^*(\varphi)"] \arrow[d, equal] & (\tau\chi\sigma')^*(N) \arrow[r, "c_{\tau\chi,\sigma'}(N)^{-1}"] \arrow[d, equal] & \sigma'^*(\tau\chi)^*(N) \arrow[d, equal] \\
\sigma'^*g_0^*(M) \arrow[r, "c_{g_0,\sigma'}(M)"] & (g_0\sigma')^*(M) \arrow[r, "(g_0\sigma')^*(\varphi)"] & (g_0\sigma')^*(N) \arrow[r, "c_{g_0,\sigma'}(N)^{-1}"] & \sigma'^*g_0^*(N)
\end{tikzcd}$$

The composition of the left (resp. right) vertical morphisms in the above diagram is $\chi^{\bullet}_{(M,\xi)}$ (resp. $\chi^{\bullet}_{(N,\zeta)}$) and the composition of the upper (resp. lower) horizontal morphisms is $f_0^*(\varphi)$ (resp. $g_0^*(\varphi)$). Thus the second assertion follows. □

Define a functor Res : $\boldsymbol{cat}(\mathcal{E})(\boldsymbol{D},\boldsymbol{C})\times \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})$ by $\mathrm{Res}(\boldsymbol{f},\xi)=\xi_{\boldsymbol{f}}$ for $\boldsymbol{f}\in \mathrm{Ob}\,\boldsymbol{cat}(\mathcal{E})(\boldsymbol{D},\boldsymbol{C})$, $(M,\xi)\in \mathrm{Ob}\,\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ and $\mathrm{Res}(\chi,\varphi)=g^*(\varphi)\chi^{\bullet}_{(M,\xi)}=\chi^{\bullet}_{(N,\zeta)}f^*(\varphi)$ for $\chi\in \boldsymbol{cat}(\mathcal{E})(\boldsymbol{D},\boldsymbol{C})(\boldsymbol{f},\boldsymbol{g})$ and $\varphi\in \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\xi),(N,\zeta))$. If $\mathcal{F}=\mathcal{F}(\boldsymbol{G})$ for an internal category $\boldsymbol{G}$, we remark that Res is identified with the composition of internal functors by the isomorphism in Theorem 3.17 of [19], that is, the following diagram commutes.

$$\begin{array}{ccc}
\boldsymbol{cat}(\mathcal{E})(\boldsymbol{D},\boldsymbol{C})\times \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C},\boldsymbol{G}) & \xrightarrow{\text{composition}} & \boldsymbol{cat}(\mathcal{E})(\boldsymbol{D},\boldsymbol{G}) \\
\Big\downarrow{\scriptstyle id\times F} & & \Big\downarrow{\scriptstyle F} \\
\boldsymbol{cat}(\mathcal{E})(\boldsymbol{D},\boldsymbol{C})\times \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}(\boldsymbol{G})) & \xrightarrow{\mathrm{Res}} & \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F}(\boldsymbol{G}))
\end{array}$$

Definition 3.2.6 *Let (M,ρ) be a representation of $\boldsymbol{C}$ on $M\in \mathrm{Ob}\,\mathcal{F}_{C_0}$.*

(1) (M,ρ) is called a left regular representation if there exist an object L of $\mathcal{F}_{C_0}$ and a bijection

$$\mathscr{A}^l_{(N,\xi)}:\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\rho),(N,\xi))\to \mathcal{F}_{C_0}(L,\mathscr{F}_{\boldsymbol{C}}(N,\xi))$$

for each $(N,\xi)\in \mathrm{Ob}\,\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ which is natural in (N,ξ).

(2) (M,ρ) is called a right regular representation if there exist an object R of $\mathcal{F}_{C_0}$ and a bijection

$$\mathscr{A}^r_{(N,\xi)}:\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((N,\xi),(M,\rho))\to \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(N,\xi),R)$$

for each $(N,\xi)\in \mathrm{Ob}\,\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ which is natural in (N,ξ).

Proposition 3.2.7 *Let (M,ρ) be a representation of $\boldsymbol{C}$ on $M\in \mathcal{F}_{C_0}$.*

(1) (M,ρ) is a left regular representation if and only if there exists a morphism $\eta:L\to \mathscr{F}_{\boldsymbol{C}}(M,\rho)$ of $\mathcal{F}_{C_0}$ such that, for any $(N,\xi)\in \mathrm{Ob}\,\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$, the following composition is bijective.

$$\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\rho),(N,\xi))\xrightarrow{\mathscr{F}_{\boldsymbol{C}}}\mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(M,\rho),\mathscr{F}_{\boldsymbol{C}}(N,\xi))\xrightarrow{\eta^*}\mathcal{F}_{C_0}(L,\mathscr{F}_{\boldsymbol{C}}(N,\xi))$$

(2) (M,ρ) is a right regular representation if and only if there exists a morphism $\varepsilon:\mathscr{F}_{\boldsymbol{C}}(M,\rho)\to R$ of $\mathcal{F}_{C_0}$ such that, for any $(N,\xi)\in \mathrm{Ob}\,\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$, the following composition is bijective.

$$\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((N,\xi),(M,\rho))\xrightarrow{\mathscr{F}_{\boldsymbol{C}}}\mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(N,\xi),\mathscr{F}_{\boldsymbol{C}}(M,\rho))\xrightarrow{\varepsilon_*}\mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(N,\xi),R)$$

Proof. (1) Suppose that (M,ρ) is a left regular representation. We take $L\in \mathrm{Ob}\,\mathcal{F}_{C_0}$ and a natural bijection $\mathscr{A}^l_{(N,\xi)}$ as in (1) of (3.2.6). Put $\eta=\mathscr{A}^l_{(M,\rho)}(id_{(M,\rho)}):L\to \mathscr{F}_{\boldsymbol{C}}(M,\rho)$. For $f\in \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\rho),(N,\xi))$, the naturality of $\mathscr{A}^l$ implies $\mathscr{F}_{\boldsymbol{C}}(f)\eta=\mathscr{F}_{\boldsymbol{C}}(f)\mathscr{A}^l_{(M,\rho)}(id_{(M,\rho)})=\mathscr{A}^l_{(N,\xi)}(f)$. Hence the composition $\eta^*\mathscr{F}_{\boldsymbol{C}}:\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\rho),(N,\xi))\to \mathcal{F}_{C_0}(L,\mathscr{F}_{\boldsymbol{C}}(N,\xi))$ coincides with $\mathscr{A}^l_{(N,\xi)}$. The converse is obvious.

(2) Suppose that (M,ρ) is a right regular representation. We take $R\in \mathrm{Ob}\,\mathcal{F}_{C_0}$ and a natural bijection $\mathscr{A}^r_{(N,\xi)}$ as in (2) of (3.2.6). Put $\varepsilon=\mathscr{A}^r_{(M,\rho)}(id_{(M,\rho)}):\mathscr{F}_{\boldsymbol{C}}(M,\rho)\to R$. For $f\in \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((N,\xi),(M,\rho))$, the naturality of $\mathscr{A}^r$ implies $\varepsilon\mathscr{F}_{\boldsymbol{C}}(f)=\mathscr{A}^r_{(M,\rho)}(id_{(M,\rho)})\mathscr{F}_{\boldsymbol{C}}(f)=\mathscr{A}^r_{(N,\xi)}(f)$. Hence the composition $\varepsilon_*\mathscr{F}_{\boldsymbol{C}}:\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((N,\xi),(M,\rho))\to \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(N,\xi),R)$ coincides with $\mathscr{A}^r_{(N,\xi)}$. The converse is obvious. □

By the above result and Theorem 3.17 of [19], we have the following.

Corollary 3.2.8 *Let $\boldsymbol{C}=(C_0,C_1;\sigma,\tau,\varepsilon,\mu)$ and $\boldsymbol{G}=(G_0,G_1;\sigma',\tau',\varepsilon',\mu')$ be internal categories in $\mathcal{E}$. Consider the fibered category $p_{\boldsymbol{C}}:\mathcal{F}(\boldsymbol{C})\to\mathcal{E}$ represented by $\boldsymbol{C}$ given in Example 2.18 of [19].*

(1) A representation $((G_0,\rho_0),(id_{G_1},\rho_1))$ of $\boldsymbol{G}$ on (G_0,ρ_0) is a left regular representation if and only if there exists a morphism $(id_{G_0},\eta):(G_0,u)\to(G_0,\rho_0)$ of $\mathcal{F}(\boldsymbol{C})_{G_0}$ such that, for any internal functor $(f_0,f_1):\boldsymbol{G}\to\boldsymbol{C}$, a map $\boldsymbol{cat}(\mathcal{E})(\boldsymbol{G},\boldsymbol{C})((\rho_0,\rho_1),(f_0,f_1))\to\Gamma_{\boldsymbol{C}}(G_0)(u,f_0)=\{\varphi\in\mathcal{E}(G_0,C_1)\,|\,\sigma\varphi=u,\ \tau\varphi=f_0\}$ given by $\varphi\mapsto\mu(\eta,\varphi)$ is bijective.

(2) A representation $((G_0,\rho_0),(id_{G_1},\rho_1))$ of $\boldsymbol{G}$ on (G_0,ρ_0) is a right regular representation if and only if there exists a morphism $(id_{G_0},\varepsilon):(G_0,\rho_0)\to(G_0,v)$ of $\mathcal{F}(\boldsymbol{C})_{G_0}$ such that, for any internal functor $(f_0,f_1):\boldsymbol{G}\to\boldsymbol{C}$, a map $\boldsymbol{cat}(\mathcal{E})(\boldsymbol{G},\boldsymbol{C})((f_0,f_1),(\rho_0,\rho_1))\to\Gamma_{\boldsymbol{C}}(G_0)(f_0,v)=\{\varphi\in\mathcal{E}(G_0,C_1)\,|\,\sigma\varphi=f_0,\ \tau\varphi=v\}$ given by $\varphi\mapsto\mu(\varphi,\varepsilon)$ is bijective.

Proof. (1) It follows from (1) of (3.2.7) and Theorem 3.17 of [19] that (G_0, ρ_0) is a left regular representation if and only if there exists a morphism $(id_{G_0}, \eta) : (G_0, u) \to (G_0, \rho_0)$ of $\mathcal{F}(\boldsymbol{C})_{G_0}$ such that, for any internal functor $(f_0, f_1) : \boldsymbol{G} \to \boldsymbol{C}$, the following composition is bijective.

$$\boldsymbol{cat}(\mathcal{E})(\boldsymbol{G}, \boldsymbol{C})((\rho_0, \rho_1), (f_0, f_1)) \xrightarrow{\mathscr{F}_{\boldsymbol{C}}F} \mathcal{F}(\boldsymbol{C})_{G_0}((G_0, \rho_0), (G_0, f_0)) \xrightarrow{(id_{G_0}, \eta)^*} \mathcal{F}(\boldsymbol{C})_{G_0}((G_0, u), (G_0, f_0))$$

The above composition maps $\varphi \in \boldsymbol{cat}(\mathcal{E})(\boldsymbol{G}, \boldsymbol{C})((\rho_0, \rho_1), (f_0, f_1))$ to a composition $G_0 \xrightarrow{(\eta, \varphi)} C_1 \times_{C_0} C_1 \xrightarrow{\mu} C_1$.

(2) It follows from (2) of (3.2.7) and Theorem 3.17 of [19] that (G_0, ρ_0) is a right regular representation if and only if there exists a morphism $(id_{G_0}, \varepsilon) : (G_0, \rho_0) \to (G_0, v)$ of $\mathcal{F}(\boldsymbol{C})_{G_0}$ such that, for any internal functor $(f_0, f_1) : \boldsymbol{G} \to \boldsymbol{C}$, the following composition is bijective.

$$\boldsymbol{cat}(\mathcal{E})(\boldsymbol{G}, \boldsymbol{C})((f_0, f_1), (\rho_0, \rho_1)) \xrightarrow{\mathscr{F}_{\boldsymbol{C}}F} \mathcal{F}(\boldsymbol{C})_{G_0}((G_0, f_0), (G_0, \rho_0)) \xrightarrow{(id_{G_0}, \varepsilon)_*} \mathcal{F}(\boldsymbol{C})_{G_0}((G_0, f_0), (G_0, v))$$

The above composition maps $\varphi : (f_0, f_1) \to (\rho_0, \rho_1)$ to a composition $G_0 \xrightarrow{(\varphi, \varepsilon)} C_1 \times_{C_0} C_1 \xrightarrow{\mu} C_1$. □

Proposition 3.2.9 *The following assertions hold.*

(1) *The forgetful functor $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,; \mathcal{F}) \to \mathcal{F}_{C_0}$ has a left adjoint if and only if, for every $L \in \mathrm{Ob}\,\mathcal{F}_{C_0}$, there exist a representation (M_L, ρ_L) of $\boldsymbol{C}$ and a morphism $\eta_L : L \to \mathscr{F}_{\boldsymbol{C}}(M_L, \rho_L)$ of $\mathcal{F}_{C_0}$ such that, for any $(N, \xi) \in \mathrm{Ob}\,\mathrm{Rep}(\boldsymbol{C}\,; \mathcal{F})$, the following composition is bijective.*

$$\mathrm{Rep}(\boldsymbol{C}\,; \mathcal{F})((M_L, \rho_L), (N, \xi)) \xrightarrow{\mathscr{F}_{\boldsymbol{C}}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(M_L, \rho_L), \mathscr{F}_{\boldsymbol{C}}(N, \xi)) \xrightarrow{\eta_L^*} \mathcal{F}_{C_0}(L, \mathscr{F}_{\boldsymbol{C}}(N, \xi))$$

(2) *The forgetful functor $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,; \mathcal{F}) \to \mathcal{F}_{C_0}$ has a right adjoint if and only if, for every $R \in \mathrm{Ob}\,\mathcal{F}_{C_0}$, there exist a representation (M_R, ρ_R) of $\boldsymbol{C}$ and a morphism $\varepsilon_R : \mathscr{F}_{\boldsymbol{C}}(M_R, \rho_R) \to R$ of $\mathcal{F}_{C_0}$ such that, for any $(N, \xi) \in \mathrm{Ob}\,\mathrm{Rep}(\boldsymbol{C}\,; \mathcal{F})$, the following composition is bijective.*

$$\mathrm{Rep}(\boldsymbol{C}\,; \mathcal{F})((N, \xi), (M_R, \rho_R)) \xrightarrow{\mathscr{F}_{\boldsymbol{C}}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(N, \xi), \mathscr{F}_{\boldsymbol{C}}(M_R, \rho_R)) \xrightarrow{\varepsilon_{R*}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(N, \xi), R)$$

Proof. (1) Suppose that $\mathscr{F}_{\boldsymbol{C}}$ has a left adjoint $\mathscr{L}_{\boldsymbol{C}} : \mathcal{F}_{C_0} \to \mathrm{Rep}(\boldsymbol{C}\,; \mathcal{F})$. Let $\eta : id_{\mathcal{F}_{C_0}} \to \mathscr{F}_{\boldsymbol{C}}\mathscr{L}_{\boldsymbol{C}}$ be the unit of this adjunction. For $L \in \mathrm{Ob}\,\mathcal{F}_{C_0}$, a representation $\mathscr{L}_{\boldsymbol{C}}(L)$ and a morphism $\eta_L : L \to \mathscr{F}_{\boldsymbol{C}}\mathscr{L}_{\boldsymbol{C}}(L)$ satisfies the condition. In fact, for $(N, \xi) \in \mathrm{Ob}\,\mathrm{Rep}(\boldsymbol{C}\,; \mathcal{F})$, the composition

$$\mathrm{Rep}(\boldsymbol{C}\,; \mathcal{F})(\mathscr{L}_{\boldsymbol{C}}(L), (N, \xi)) \xrightarrow{\mathscr{F}_{\boldsymbol{C}}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}\mathscr{L}_{\boldsymbol{C}}(L), \mathscr{F}_{\boldsymbol{C}}(N, \xi)) \xrightarrow{\eta_L^*} \mathcal{F}_{C_0}(L, \mathscr{F}_{\boldsymbol{C}}(N, \xi))$$

is the adjoint bijection. We show the converse. Define a functor $\mathscr{L}_{\boldsymbol{C}} : \mathcal{F}_{C_0} \to \mathrm{Rep}(\boldsymbol{C}\,; \mathcal{F})$ as follows. For an object L of $\mathcal{F}_{C_0}$, put $\mathscr{L}_{\boldsymbol{C}}(L) = (M_L, \rho_L)$. For a morphism $\varphi : L \to K$ of $\mathcal{F}_{C_0}$, let $\mathscr{L}_{\boldsymbol{C}}(\varphi) : (M_L, \rho_L) \to (M_K, \rho_K)$ be the morphism in $\mathrm{Rep}(\boldsymbol{C}\,; \mathcal{F})$ which maps to $\eta_K\varphi$ by the composition

$$\mathrm{Rep}(\boldsymbol{C}\,; \mathcal{F})((M_L, \rho_L), (M_K, \rho_K)) \xrightarrow{\mathscr{F}_{\boldsymbol{C}}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(M_L, \rho_L), \mathscr{F}_{\boldsymbol{C}}(M_K, \rho_K)) \xrightarrow{\eta_L^*} \mathcal{F}_{C_0}(L, \mathscr{F}_{\boldsymbol{C}}(M_K, \rho_K)).$$

It is easy to verify that $\mathscr{L}_{\boldsymbol{C}}$ is a functor and that it is a left adjoint of $\mathscr{F}_{\boldsymbol{C}}$.

(2) Suppose that $\mathscr{F}_{\boldsymbol{C}}$ has right adjoint $\mathscr{R}_{\boldsymbol{C}} : \mathcal{F}_{C_0} \to \mathrm{Rep}(\boldsymbol{C}\,; \mathcal{F})$. Let $\varepsilon : \mathscr{F}_{\boldsymbol{C}}\mathscr{R}_{\boldsymbol{C}} \to id_{\mathcal{F}_{C_0}}$ be the counit of this adjunction. For $R \in \mathrm{Ob}\,\mathcal{F}_{C_0}$, a representation $\mathscr{R}_{\boldsymbol{C}}(R)$ and a morphism $\varepsilon_R : \mathscr{F}_{\boldsymbol{C}}\mathscr{R}_{\boldsymbol{C}}(R) \to R$ satisfies the condition. In fact, for $(N, \xi) \in \mathrm{Ob}\,\mathrm{Rep}(\boldsymbol{C}\,; \mathcal{F})$, the composition

$$\mathrm{Rep}(\boldsymbol{C}\,; \mathcal{F})((N, \xi), \mathscr{R}_{\boldsymbol{C}}(R)) \xrightarrow{\mathscr{F}_{\boldsymbol{C}}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(N, \xi), \mathscr{F}_{\boldsymbol{C}}\mathscr{R}_{\boldsymbol{C}}(R)) \xrightarrow{\varepsilon_{R*}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(N, \xi), R)$$

is the adjoint bijection. We show the converse. Define a functor $\mathscr{R}_{\boldsymbol{C}} : \mathcal{F}_{C_0} \to \mathrm{Rep}(\boldsymbol{C}\,; \mathcal{F})$ as follows. For an object R of $\mathcal{F}_{C_0}$, put $\mathscr{R}_{\boldsymbol{C}}(R) = (M_R, \rho_R)$. For a morphism $\varphi : Q \to R$ of $\mathcal{F}_{C_0}$, let $\mathscr{R}_{\boldsymbol{C}}(\varphi) : (M_Q, \rho_Q) \to (M_R, \rho_R)$ be the morphism in $\mathrm{Rep}(\boldsymbol{C}\,; \mathcal{F})$ which maps to $\varphi\varepsilon_Q$ by the composition

$$\mathrm{Rep}(\boldsymbol{C}\,; \mathcal{F})((M_Q, \rho_Q), (M_R, \rho_R)) \xrightarrow{\mathscr{F}_{\boldsymbol{C}}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(M_Q, \rho_Q), \mathscr{F}_{\boldsymbol{C}}(M_R, \rho_R)) \xrightarrow{\varepsilon_{R*}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(M_Q, \rho_Q), R).$$

It is easy to verify that $\mathscr{R}_{\boldsymbol{C}}$ is a functor and that it is a right adjoint of $\mathscr{F}_{\boldsymbol{C}}$. □

Proposition 3.2.10 *The following assertions hold.*

(1) *Suppose that $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$ has a left adjoint $\mathscr{L}_{\boldsymbol{C}}$. Let us denote by η and ε the unit and the counit of this adjunction. Put $T = \mathscr{F}_{\boldsymbol{C}}\mathscr{L}_{\boldsymbol{C}}$ and consider the monad $\boldsymbol{T} = (T, \eta, \mathscr{F}_{\boldsymbol{C}}(\epsilon_{\mathscr{L}_{\boldsymbol{C}}}))$ associated with this adjunction. Then, the comparision functor $K : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}^{\boldsymbol{T}}$ given by $K(M,\xi) = \langle M, \mathscr{F}_{\boldsymbol{C}}(\varepsilon_{(M,\xi)})\rangle$ is an isomorphism in categories.*

(2) *Suppose that $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$ has a right adjoint $\mathscr{R}_{\boldsymbol{C}}$. Let us denote by η and ε the unit and the counit of this adjunction. Put $T = \mathscr{F}_{\boldsymbol{C}}\mathscr{R}_{\boldsymbol{C}}$ and consider the comonad $\boldsymbol{T} = (T, \varepsilon, \mathscr{F}_{\boldsymbol{C}}(\epsilon_L))$ associated with this adjunction. Then, the comparision functor $K : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}^{\boldsymbol{T}}$ given by $K(M,\xi) = \langle M, \mathscr{F}_{\boldsymbol{C}}(\eta_{(M,\xi)})\rangle$ is an isomorphism in categories.*

Proof. (1) Let $(M,\xi) \overset{\varphi}{\underset{\psi}{\rightrightarrows}} (N,\zeta)$ be parallel arrows in $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ such that $\mathscr{F}_{\boldsymbol{C}}(M,\xi) \overset{\mathscr{F}_{\boldsymbol{C}}(\varphi)}{\underset{\mathscr{F}_{\boldsymbol{C}}(\psi)}{\rightrightarrows}} \mathscr{F}_{\boldsymbol{C}}(N,\zeta)$ has a split coequalizer in $\mathcal{F}_{C_0}$. Since σ^* preserves split coequalizers and μ^* preserves split epimorphism, $\mathscr{F}_{\boldsymbol{C}}$ creates the coequalizer of $\mathscr{F}_{\boldsymbol{C}}(M,\xi) \overset{\mathscr{F}_{\boldsymbol{C}}(\varphi)}{\underset{\mathscr{F}_{\boldsymbol{C}}(\psi)}{\rightrightarrows}} \mathscr{F}_{\boldsymbol{C}}(N,\zeta)$ by (2) of (3.1.7). Hence, by the theorem of Beck ([11], p.151) the assertion follows.

(2) Let $(M,\xi) \overset{\varphi}{\underset{\psi}{\rightrightarrows}} (N,\zeta)$ be parallel arrows in $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ such that $\mathscr{F}_{\boldsymbol{C}}(M,\xi) \overset{\mathscr{F}_{\boldsymbol{C}}(\varphi)}{\underset{\mathscr{F}_{\boldsymbol{C}}(\psi)}{\rightrightarrows}} \mathscr{F}_{\boldsymbol{C}}(N,\zeta)$ has a split equalizer in $\mathcal{F}_{C_0}$. Since τ^* preserves split equalizers and μ^* preserves split epimorphism, $\mathscr{F}_{\boldsymbol{C}}$ creates the equalizer of $\mathscr{F}_{\boldsymbol{C}}(M,\xi) \overset{\mathscr{F}_{\boldsymbol{C}}(\varphi)}{\underset{\mathscr{F}_{\boldsymbol{C}}(\psi)}{\rightrightarrows}} \mathscr{F}_{\boldsymbol{C}}(N,\zeta)$ by (1) of (3.1.7). Hence, by the theorem of Beck ([11], p.151) the assertion follows. □

3.3 Representations of left fibered representable internal categories

Let $p : \mathcal{F} \to \mathcal{E}$ be a normalized cloven fibered category.

Definition 3.3.1 *Let $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathcal{E}$. We call $\boldsymbol{C}$ a left fibered representable internal category if (σ,τ) and $(\sigma\mathrm{pr}_1, \tau\mathrm{pr}_2)$ are left fibered representable pairs.*

We assume that all internal categories in this subsection are left fibered representable internal categories. We also assume that, for morphisms $f : X \to Y$, $g : X \to Z$ and an object M of $\mathcal{F}_Y$, (f,g) is a left fibered representable pair with respect to M if necessary.

Proposition 3.3.2 *For $M \in \mathrm{Ob}\,\mathcal{F}_{C_0}$ and $\xi \in \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(M))$, we put $\hat{\xi} = P_{\sigma,\tau}(M)_M(\xi) : M_{[\sigma,\tau]} \to M$. ξ satisfies condition (A) of (3.1.2) if and only if the following diagram commutes.*

$$\begin{tikzcd}
M_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[r, "\theta_{\sigma,\tau,\sigma,\tau}(M)"] \arrow[d, equal] & (M_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\hat{\xi}_{[\sigma,\tau]}"] & M_{[\sigma,\tau]} \arrow[d, "\hat{\xi}"] \\
M_{[\sigma\mu,\tau\mu]} \arrow[r, "M_\mu"] & M_{[\sigma,\tau]} \arrow[r, "\hat{\xi}"] & M
\end{tikzcd}$$

ξ satisfies condition (U) of (3.1.2) if and only if a composition $M = M_{[\sigma\varepsilon,\tau\varepsilon]} \xrightarrow{M_\varepsilon} M_{[\sigma,\tau]} \xrightarrow{\hat{\xi}} M$ coincides with the identity morphism of M.

Proof. We have $P_{\sigma\mu,\tau\mu}(M)_M(\xi_\mu) = \hat{\xi}M_\mu$ and $P_{\sigma\mathrm{pr}_i,\tau\mathrm{pr}_i}(M)_M(\xi_{\mathrm{pr}_i}) = \hat{\xi}M_{\mathrm{pr}_i}$ for $i = 1,2$ by (1) of (1.3.7). Hence (1.3.4), (1.3.7), (1.3.9), (1.3.16) imply

$$\begin{aligned}
P_{\sigma\mu,\tau\mu}(M)_M(\xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1}) &= P_{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M)_M(\xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1}) = \hat{\xi}M_{\mathrm{pr}_2}(\hat{\xi}M_{\mathrm{pr}_1})_{[\sigma\mathrm{pr}_2,\tau\mathrm{pr}_2]}\delta_{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_1,\tau\mathrm{pr}_2,M} \\
&= \hat{\xi}\hat{\xi}_{[\sigma,\tau]}(M_{[\sigma,\tau]})_{\mathrm{pr}_2}(M_{\mathrm{pr}_1})_{[\sigma\mathrm{pr}_2,\tau\mathrm{pr}_2]}\delta_{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_1,\tau\mathrm{pr}_2,M} = \hat{\xi}\hat{\xi}_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M) \\
\xi_\varepsilon &= P_{id_{C_0},id_{C_0}}(M)_M(\xi_\varepsilon) = P_{\sigma\varepsilon,\tau\varepsilon}(M)_M(\xi_\varepsilon) = \hat{\xi}M_\varepsilon
\end{aligned}$$

Thus $\xi_\mu = \xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1}$ and $\xi_\varepsilon = id_M$ are equivalent to $\hat{\xi}\hat{\xi}_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M) = \hat{\xi}M_\mu$ and $\hat{\xi}M_\varepsilon = id_M$, respectively. □

Remark 3.3.3 *If we denote $M_{[\sigma,\tau]}$ by $M\times\boldsymbol{C}$ and $M = M_{[id_{C_0},id_{C_0}]}$ by $M\times 1$, $\hat{\xi} : M\times\boldsymbol{C}\to M$ can be regarded as a right action of $\boldsymbol{C}$ on M and $M_\varepsilon : M\times 1\to M\times\boldsymbol{C}$ which is denoted by $M\times\varepsilon$ can be regarded as the unital morphism. Then the equality $\hat{\xi}(M\times\varepsilon) = id_M$ means that the right action $\hat{\xi}$ is untary. Moreover, if we denote $M\times\mu : M\times(\boldsymbol{C}\times\boldsymbol{C})\to M\times\boldsymbol{C}$ instead of $M_\mu : M_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}\to M_{[\sigma,\tau]}$ and denote $\hat{\xi}\times id_{\boldsymbol{C}} : (M\times\boldsymbol{C})\times\boldsymbol{C}\to M\times\boldsymbol{C}$ instead of $\hat{\xi}_{[\sigma,\tau]} : (M_{[\sigma,\tau]})_{[\sigma,\tau]}\to M_{[\sigma,\tau]}$, the fact that the following diagram commutes means that the right action $\hat{\xi} : M\times\boldsymbol{C}\to M$ of $\boldsymbol{C}$ is associative.*

$$\begin{array}{ccccc}
M\times(\boldsymbol{C}\times\boldsymbol{C}) & \xrightarrow{\theta_{\sigma,\tau,\sigma,\tau}(M)} & (M\times\boldsymbol{C})\times\boldsymbol{C} & \xrightarrow{\hat{\xi}\times id_{\boldsymbol{C}}} & M\times\boldsymbol{C} \\
\downarrow{\scriptstyle M\times\mu} & & & & \downarrow{\scriptstyle\hat{\xi}} \\
M\times\boldsymbol{C} & & \xrightarrow{\qquad\qquad\hat{\xi}\qquad\qquad} & & M
\end{array}$$

For morphisms $f : X\to Y$, $g : X\to Z$ of $\mathcal{E}$, we define a functor $D_{f,g} : \mathcal{Q}\to\mathcal{E}$ by $D_{f,g}(0) = X$, $D_{f,g}(1) = Y$, $D_{f,g}(2) = Z$, $D_{f,g}(\tau_{01}) = f$, $D_{f,g}(\tau_{02}) = g$. If $h : Y\to V$, $i : Z\to W$ are morphisms in $\mathcal{E}$, we define a natural transformation $\omega(f,g\,;h,i) : D_{f,g}\to D_{hf,ig}$ by $\omega(f,g\,;h,i)_0 = id_X$, $\omega(f,g\,;h,i)_1 = h$, $\omega(f,g\,;h,i)_2 = i$.

Proposition 3.3.4 *Let $(s(C_0), s_{\boldsymbol{C}})$ be the trivial representation associated with a cartesian section $s : \mathcal{E}\to\mathcal{F}$. Put $T = s(1)$. The image of $s_{\boldsymbol{C}}\in\mathcal{F}_{C_1}(\sigma^*s(C_0),\tau^*s(C_0))$ by $P_{\sigma,\tau}(s(C_0))_{s(C_0)} : \mathcal{F}_{C_1}(\sigma^*s(C_0),\tau^*s(C_0))\to\mathcal{F}_{C_0}(s(C_0)_{[\sigma,\tau]}, s(C_0))$ is $o_{C_0}^*(P_{o_{C_1},o_{C_1}}(T)_T(id_{s(C_1)}))\omega(\sigma,\tau\,;o_{C_0},o_{C_0})_T$.*

Proof. It follows from (1.1.22) and the definition of $s_{\boldsymbol{C}}$ that we have $s_{\boldsymbol{C}} = c_{o_{C_0},\tau}(T)^{-1}c_{o_{C_0},\sigma}(T)$. We note that $o_{C_0}\sigma = o_{C_0}\tau = o_{C_1}$ and $s(C_i) = o_{C_i}^*(T)$ for $i = 0,1$. The following diagram is commutative by (1.3.30).

$$\begin{array}{ccccc}
\mathcal{F}_{C_1}(s(C_1), s(C_1)) & \xrightarrow{c_{o_{C_0},\tau}(T)_*^{-1}} & \mathcal{F}_{C_1}(s(C_1),\tau^*(s(C_0))) & \xrightarrow{c_{o_{C_0},\sigma}(T)^*} & \mathcal{F}_{C_1}(\sigma^*(s(C_0)),\tau^*(s(C_0))) \\
\downarrow{\scriptstyle P_{o_{C_1},o_{C_1}}(T)_T} & & & & \downarrow{\scriptstyle P_{\sigma,\tau}(s(C_0))_{s(C_0)}} \\
\mathcal{F}_1(T_{[o_{C_1},o_{C_1}]}, T) & \xrightarrow{o_{C_0}^*} & \mathcal{F}_{C_0}(o_{C_0}^*(T_{[o_{C_1},o_{C_1}]}), s(C_0)) & \xrightarrow{\omega(\sigma,\tau\,;o_{C_0},o_{C_0})_T^*} & \mathcal{F}_{C_0}(s(C_0)_{[\sigma,\tau]}, s(C_0))
\end{array}$$

Hence we have $P_{\sigma,\tau}(s(C_0))_{s(C_0)}(s_{\boldsymbol{C}}) = o_{C_0}^*(P_{o_{C_1},o_{C_1}}(T)_T(id_{s(C_1)}))\omega(\sigma,\tau\,;o_{C_0},o_{C_0})_T$. □

Proposition 3.3.5 *Let $\boldsymbol{f} = (f_0,f_1) : \boldsymbol{D}\to\boldsymbol{C}$ be an internal functor and (M,ξ) a representation of $\boldsymbol{C}$. We denote by $\sigma',\tau' : D_1\to D_0$ the source and target of $\boldsymbol{D}$, respectively. Then, the following equality holds.*

$$P_{\sigma',\tau'}(f_0^*(M))_{f_0^*(M)}(\xi_{\boldsymbol{f}}) = f_0^*(\hat{\xi}M_{f_1})\,\omega(\sigma',\tau';f_0,f_0)_M$$

Proof. The upper rectangle of the following diagram is commutative by (1) of (1.3.7) and the lower one is commutative (1.3.30).

$$\begin{array}{ccc}
\mathcal{F}_{C_1}(\sigma^*(M),\tau^*(M)) & \xrightarrow{P_{\sigma,\tau}(M)_M} & \mathcal{F}_{C_0}(M_{[\sigma,\tau]}, M) \\
\downarrow{\scriptstyle f_1^\sharp} & & \downarrow{\scriptstyle M_{f_1}^*} \\
\mathcal{F}_{D_1}((f_0\sigma')^*(M),(f_0\tau')^*(M)) & \xrightarrow{P_{f_0\sigma',f_0\tau'}(M)_M} & \mathcal{F}_{C_0}(M_{[f_0\sigma',f_0\tau']}, M) \\
\downarrow{\scriptstyle c_{f_0,\tau'}(M)_*^{-1}} & & \downarrow{\scriptstyle f_0^*} \\
\mathcal{F}_{D_1}((f_0\sigma')^*(M),\tau'(f_0^*(M))) & & \mathcal{F}_{D_0}(f_0^*(M_{[f_0\sigma',f_0\tau']}), f_0^*(M)) \\
\downarrow{\scriptstyle c_{f_0,\sigma'}(M)^*} & & \downarrow{\scriptstyle \omega(\sigma',\tau';f_0,f_0)_M^*} \\
\mathcal{F}_{D_1}(\sigma'^*(f_0^*(M)),\tau'^*(f_0^*(M))) & \xrightarrow{P_{\sigma',\tau'}(f_0^*(M))_{f_0^*(M)}} & \mathcal{F}_{D_0}(f_0^*(M)_{[\sigma',\tau']}, f_0^*(M))
\end{array}$$

The assertion follows from the above diagram and the definition of $\xi_{\boldsymbol{f}}$. □

The following fact is a direct consequence of (1.3.6).

Proposition 3.3.6 *Let (M,ξ) and (N,ζ) be representations of $\boldsymbol{C}$ and $\varphi : M\to N$ a morphism in $\mathcal{F}_{C_0}$. We put $\hat{\xi} = P_{\sigma,\tau}(M)_M(\xi)$ and $\hat{\zeta} = P_{\sigma,\tau}(N)_N(\zeta)$. Then, φ is a morphism of representations if and only if the following diagram is commutative.*

$$\begin{tikzcd}
M_{[\sigma,\tau]} \arrow[r, "\hat{\xi}"] \arrow[d, "\varphi_{[\sigma,\tau]}"] & M \arrow[d, "\varphi"] \\
N_{[\sigma,\tau]} \arrow[r, "\hat{\zeta}"] & N
\end{tikzcd}$$

Let $(\pi : X \to C_0,\ \alpha : X \times^{\sigma}_{C_0} C_1 \to X)$ be an internal diagram on $\boldsymbol{C}$. Let $X \times^{\sigma}_{C_0} C_1 \xleftarrow{\tilde{\mathrm{pr}}_{12}} X \times^{\sigma}_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{\tilde{\mathrm{pr}}_{23}} C_1 \times_{C_0} C_1$ be a limit of $X \times^{\sigma}_{C_0} C_1 \xrightarrow{\pi_\sigma} C_1 \xleftarrow{\mathrm{pr}_1} C_1 \times_{C_0} C_1$. Then, $X \xleftarrow{\sigma_\pi \tilde{\mathrm{pr}}_{12}} X \times^{\sigma}_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{\tilde{\mathrm{pr}}_{23}} C_1 \times_{C_0} C_1$ is a limit of $X \xrightarrow{\pi} C_0 \xleftarrow{\sigma \mathrm{pr}_1} C_1 \times_{C_0} C_1$. We also note that $X \times^{\sigma}_{C_0} C_1 \xleftarrow{\tilde{\mathrm{pr}}_{12}} X \times^{\sigma}_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_2 \tilde{\mathrm{pr}}_{23}} C_1$ is a limit of $X \times^{\sigma}_{C_0} C_1 \xrightarrow{\tau\pi_\sigma} C_0 \xleftarrow{\sigma} C_1$.

$$\begin{tikzcd}
X \times^{\sigma}_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "\tilde{\mathrm{pr}}_{23}"] \arrow[d, "\tilde{\mathrm{pr}}_{12}"] & C_1 \times_{C_0} C_1 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & C_1 \arrow[d, "\sigma"] \\
X \times^{\sigma}_{C_0} C_1 \arrow[r, "\pi_\sigma"] \arrow[d, "\sigma_\pi"] & C_1 \arrow[r, "\tau"] \arrow[d, "\sigma"] & C_0 \\
X \arrow[r, "\pi"] & C_0 &
\end{tikzcd}$$

Define a functor $D_\alpha : \mathcal{P} \to \mathcal{E}$ by $D_\alpha(0) = X \times^{\sigma}_{C_0} C_1$, $D_\alpha(1) = C_1$, $D_\alpha(2) = X$, $D_\alpha(3) = D_\alpha(4) = D_\alpha(5) = C_0$ and $D_\alpha(\tau_{01}) = \pi_\sigma$, $D_\alpha(\tau_{02}) = \alpha$, $D_\alpha(\tau_{13}) = \sigma$, $D_\alpha(\tau_{14}) = \tau$, $D_\alpha(\tau_{24}) = D_\alpha(\tau_{25}) = \pi$. For a representation (M, ξ) of $\boldsymbol{C}$, we put $\hat{\xi} = P_{\sigma,\tau}(M)_M(\xi)$. Assume that $\theta_{\pi,\pi,\sigma,\tau}(M) : M_{[\pi\sigma_\pi,\, \tau\pi_\sigma]} \to (M_{[\pi,\pi]})_{[\sigma,\tau]}$ is an isomorphism and define a morphism $\hat{\xi}_\alpha : (M_{[\pi,\pi]})_{[\sigma,\tau]} \to M_{[\pi,\pi]}$ to be the following composition.

$$(M_{[\pi,\pi]})_{[\sigma,\tau]} \xrightarrow{\theta_{\pi,\pi,\sigma,\tau}(M)^{-1}} M_{[\pi\sigma_\pi,\, \tau\pi_\sigma]} = M_{[\sigma\pi_\sigma,\, \pi\alpha]} \xrightarrow{\theta_{D_\alpha}(M)} (M_{[\sigma,\tau]})_{[\pi,\pi]} \xrightarrow{\hat{\xi}_{[\pi,\pi]}} M_{[\pi,\pi]}$$

Proposition 3.3.7 *Assume that $\theta_{\pi,\pi,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M) : M_{[\pi\sigma_\pi\tilde{\mathrm{pr}}_{12},\, \tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \to (M_{[\pi,\pi]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$ is an epimorphism. Put $P_{\sigma,\tau}(M_{[\pi,\pi]})^{-1}_{M_{[\pi,\pi]}}(\hat{\xi}_\alpha) = \xi_\alpha$. Then, $(M_{[\pi,\pi]}, \xi_\alpha)$ is a representation of $\boldsymbol{C}$ and $M_\pi : (M_{[\pi,\pi]}, \xi_\alpha) \to (M, \xi)$ is a morphism of representations.*

Proof. The left rectangle of the following diagram is commutative by (1.3.25) and the right rectangle is commutative by (1.3.21).

$$\begin{tikzcd}
(M_{[\pi\sigma_\pi,\, \tau\pi_\sigma]})_{[\sigma,\tau]} \arrow[d, "\theta_{\pi,\pi,\sigma,\tau}(M)_{[\sigma,\tau]}"] & M_{[\pi\sigma_\pi\tilde{\mathrm{pr}}_{12},\, \tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[l, "\theta_{\pi\sigma_\pi,\, \tau\pi_\sigma,\sigma,\tau}(M)"'] \arrow[r, "M_{id_X \times_{C_0} \mu}"] \arrow[d, "\theta_{\pi,\pi,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M)"] & M_{[\pi\sigma_\pi,\, \tau\pi_\sigma]} \arrow[d, "\theta_{\pi,\pi,\sigma,\tau}(M)"] \\
((M_{[\pi,\pi]})_{[\sigma,\tau]})_{[\sigma,\tau]} & (M_{[\pi,\pi]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[l, "\theta_{\sigma,\tau,\sigma,\tau}(M_{[\pi,\pi]})"'] \arrow[r, "(M_{[\pi,\pi]})_\mu"] & (M_{[\pi,\pi]})_{[\sigma,\tau]}
\end{tikzcd}$$

Since $\pi\alpha = \tau\pi_\sigma$, $\pi_\sigma(\alpha \times_{C_0} id_{C_1}) = \mathrm{pr}_2\tilde{\mathrm{pr}}_{23}$ and $\alpha(\alpha \times_{C_0} id_{C_1}) = \alpha(id_X \times_{C_0} \mu)$, we can define functors $E, F : \mathcal{P} \to \mathcal{E}$ and a natural transformation $\lambda : E \to D_\alpha$ by $E(0) = F(0) = X \times^{\sigma}_{C_0} C_1 \times_{C_0} C_1$, $E(1) = C_1 \times_{C_0} C_1$, $F(1) = C_1$, $E(2) = X$, $F(2) = X \times^{\sigma}_{C_0} C_1$, $E(i) = F(i) = C_0$ for $i = 3, 4, 5$, $E(\tau_{01}) = \tilde{\mathrm{pr}}_{23}$, $F(\tau_{01}) = \mathrm{pr}_1\tilde{\mathrm{pr}}_{23}$, $E(\tau_{02}) = \alpha(\alpha \times_{C_0} id_{C_1})$, $F(\tau_{02}) = \alpha \times_{C_0} id_{C_1}$, $E(\tau_{13}) = \sigma\mathrm{pr}_1$, $F(\tau_{13}) = \sigma$, $E(\tau_{14}) = \tau\mathrm{pr}_2$, $F(\tau_{14}) = \tau$, $E(\tau_{24}) = \pi$, $F(\tau_{24}) = \sigma\pi_\sigma$, $E(\tau_{25}) = \pi$, $F(\tau_{25}) = \pi\alpha$ and $\lambda_0 = id_X \times_{C_0} \mu$, $\lambda_1 = \mu$, $\lambda_2 = id_X$, $\lambda_3 = \lambda_4 = \lambda_5 = id_{C_0}$. We also note that $\mathrm{pr}_1\tilde{\mathrm{pr}}_{23} = \pi_\sigma\tilde{\mathrm{pr}}_{12}$. Then, the following diagram commutes by (1.3.24)

$$\begin{tikzcd}
(M_{[\sigma\pi_\sigma,\, \pi\alpha]})_{[\sigma,\tau]} \arrow[d, "\theta_{D_\alpha}(M)_{[\sigma,\tau]}"] & M_{[\sigma\pi_\sigma\tilde{\mathrm{pr}}_{12},\, \tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[l, "\theta_{\sigma\pi_\sigma,\pi\alpha,\sigma,\tau}(M)"'] \arrow[r, "\theta_E(M)"] \arrow[d, "\theta_F(M)"] & (M_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]})_{[\pi,\pi]} \arrow[d, "\theta_{\sigma,\tau,\sigma,\tau}(M)_{[\pi,\pi]}"] \\
((M_{[\sigma,\tau]})_{[\pi,\pi]})_{[\sigma,\tau]} & (M_{[\sigma,\tau]})_{[\pi\sigma_\pi,\, \tau\pi_\sigma)} \arrow[l, "\theta_{\pi,\pi,\sigma,\tau}(M_{[\sigma,\tau]})"'] \arrow[r, "\theta_{D_\alpha}(M_{[\sigma,\tau]})"] & ((M_{[\sigma,\tau]})_{[\sigma,\tau]})_{[\pi,\pi]}
\end{tikzcd}$$

and the following diagram commutes by (1.3.20).

$$\begin{tikzcd}
M_{[\sigma\pi_\sigma\tilde{\mathrm{pr}}_{12},\, \tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "M_{id_X \times_{C_0} \mu}"] \arrow[d, "\theta_E(M)"] & M_{[\sigma\pi_\sigma,\, \pi\alpha]} \arrow[d, "\theta_{D_\alpha}(M)"] \\
(M_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]})_{[\pi,\pi]} \arrow[r, "(M_\mu)_{[\pi,\, \pi]}"] & (M_{[\sigma,\, \tau]})_{[\pi,\, \pi]}
\end{tikzcd}$$

It follows from the above facts and (1.3.19), (1.3.21), (3.3.2) that the following diagram is commutative

$$
\begin{tikzcd}
((M_{[\pi,\pi]})_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[d, "\theta_{\pi,\pi,\sigma,\tau}(M)^{-1}_{[\sigma,\tau]}"] & & (M_{[\pi,\pi]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[ll, "\theta_{\sigma,\tau,\sigma,\tau}(M_{[\pi,\pi]})"'] \arrow[rr, "(M_{[\pi,\pi]})_{\mu}"] & & (M_{[\pi,\pi]})_{[\sigma,\tau]} \arrow[d, "\theta_{\pi,\pi,\sigma,\tau}(M)^{-1}"'] \\
(M_{[\pi\sigma_\pi,\,\tau\pi_\sigma]})_{[\sigma,\tau]} \arrow[d, equal] & & M_{[\pi\sigma_\pi\tilde{\mathrm{pr}}_{12},\,\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[u, "\theta_{\pi,\pi,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M)"'] \arrow[ll, "\theta_{\pi\sigma_\pi,\,\tau\pi_\sigma,\sigma,\tau}(M)"'] \arrow[rr, "M_{id_X\times_{C_0}\mu}"] \arrow[d, equal] & & M_{[\pi\sigma_\pi,\,\tau\pi_\sigma]} \arrow[d, equal] \\
(M_{[\sigma\pi_\sigma,\,\pi\alpha]})_{[\sigma,\tau]} \arrow[d, "\theta_{D_\alpha}(M)_{[\sigma,\tau]}"] & & M_{[\sigma\pi_\sigma\tilde{\mathrm{pr}}_{12},\,\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[ll, "\theta_{\sigma\pi_\sigma,\pi\alpha,\sigma,\tau}(M)"'] \arrow[rr, "M_{id_X\times_{C_0}\mu}"] \arrow[d, "\theta_E(M)"] \arrow[dddl, "\theta_F(M)"'] & & M_{[\sigma\pi_\sigma,\,\pi\alpha]} \arrow[d, "\theta_{D_\alpha}(M)"] \\
((M_{[\sigma,\tau]})_{[\pi,\pi]})_{[\sigma,\tau]} \arrow[dd, "(\hat{\xi}_{[\pi,\pi]})_{[\sigma,\tau]}"] & & (M_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]})_{[\pi,\pi]} \arrow[rr, "(M_\mu)_{[\pi,\pi]}"] \arrow[d, "\theta_{\sigma,\tau,\sigma,\tau}(M)_{[\pi,\pi]}"] & & (M_{[\sigma,\tau]})_{[\pi,\pi]} \arrow[d, "\hat{\xi}_{[\pi,\pi]}"] \\
 & & ((M_{[\sigma,\tau]})_{[\sigma,\tau]})_{[\pi,\pi]} \arrow[dr, "(\hat{\xi}_{[\sigma,\tau]})_{[\pi,\pi]}"] & & M_{[\pi,\pi]} \\
(M_{[\pi,\pi]})_{[\sigma,\tau]} \arrow[dr, "\theta_{\pi,\pi,\sigma,\tau}(M)^{-1}"'] & (M_{[\sigma,\tau]})_{[\sigma\pi_\sigma,\,\pi\alpha]} \arrow[uul, "\theta_{\pi,\pi,\sigma,\tau}(M_{[\sigma,\tau]})"'] \arrow[ur, "\theta_{D_\alpha}(M_{[\sigma,\tau]})"'] \arrow[dr, "\hat{\xi}_{[\sigma\pi_\sigma,\,\pi\alpha]}"] & & (M_{[\sigma,\tau]})_{[\pi,\pi]} \arrow[ur, "\hat{\xi}_{[\pi,\pi]}"] & \\
 & M_{[\pi\sigma_\pi,\,\tau\pi_\sigma]} \arrow[r, equal] & M_{[\sigma\pi_\sigma,\,\pi\alpha]} \arrow[ur, "\theta_{D_\alpha}(M)"] & &
\end{tikzcd}
$$

Hence $\hat{\xi}_\alpha$ make the diagram of (3.3.2) commute.

Since functors $D_{\pi,\pi,id_{C_0},id_{C_0}}, D_{id_{C_0},id_{C_0},\pi,\pi} : \mathcal{P} \to \mathcal{E}$ are given by

$$
\begin{aligned}
D_{\pi,\pi,id_{C_0},id_{C_0}}(i) &= D_{id_{C_0},id_{C_0},\pi,\pi}(j) = X \quad (i = 0,1,\ j = 0,2), \\
D_{\pi,\pi,id_{C_0},id_{C_0}}(i) &= D_{id_{C_0},id_{C_0},\pi,\pi}(j) = C_0 \quad (i = 2,3,4,5,\ j = 1,3,4,5), \\
D_{\pi,\pi,id_{C_0},id_{C_0}}(\tau_{01}) &= D_{id_{C_0},id_{C_0},\pi,\pi}(\tau_{02}) = id_X, \\
D_{\pi,\pi,id_{C_0},id_{C_0}}(\tau_{ij}) &= D_{id_{C_0},id_{C_0},\pi,\pi}(\tau_{kl}) = \pi \quad ((i,j) = (0,2),(1,3),(1,4),(k,l) = (0,1),(1,3),(1,4)), \\
D_{\pi,\pi,id_{C_0},id_{C_0}}(\tau_{2j}) &= D_{id_{C_0},id_{C_0},\pi,\pi}(\tau_{2j}) = id_{C_0} \quad (j = 3,4,5),
\end{aligned}
$$

we define natural transformations $\nu : D_{\pi,\pi,id_{C_0},id_{C_0}} \to D_{\pi,\pi,\sigma,\tau}$ and $\kappa : D_{id_{C_0},id_{C_0},\pi,\pi} \to D_\alpha$ by $\nu_0 = \kappa_0 = (id_X, \varepsilon\pi) : X \to X \times^\sigma_{C_0} C_1$, $\nu_1 = \kappa_2 = id_X$, $\nu_2 = \kappa_1 = \varepsilon$, $\nu_i = \kappa_i = id_{C_0}$ $(i = 3,4,5)$. Then, the following diagram is commutative by (1.3.19), (1.3.21).

$$
\begin{tikzcd}
(M_{[\pi,\pi]})_{[\sigma\varepsilon,\tau\varepsilon]} \arrow[r, "\theta_{\pi,\pi,id_{C_0},id_{C_0}}(M)^{-1}"] \arrow[d, "(M_{[\pi,\pi]})_\varepsilon"] & M_{[\pi id_X,\,\tau\varepsilon\pi]} \arrow[r, equal] \arrow[d, "M_{(id_X,\varepsilon\pi)}"] & M_{[id_{C_0}\pi,\,\pi id_X]} \arrow[r, "\theta_{id_{C_0},id_{C_0},\pi,\pi}(M)"] & (M_{[id_{C_0},id_{C_0}]})_{[\pi,\,\pi]} \arrow[d, "(M_\varepsilon)_{[\pi,\,\pi]}"] \\
(M_{[\pi,\pi]})_{[\sigma,\tau]} \arrow[r, "\theta_{\pi,\pi,\sigma,\tau}(M)^{-1}"] & M_{[\pi\sigma_\pi,\,\tau\pi_\sigma]} \arrow[r, equal] & M_{[\sigma\pi_\sigma,\,\pi\alpha]} \arrow[r, "\theta_{D_\alpha}(M)"] & (M_{[\sigma,\tau]})_{[\pi,\pi]}
\end{tikzcd}
$$

The upper row of the above diagram is identified with the identity morphism of $M_{[\pi,\pi]}$. Since $\hat{\xi}M_\varepsilon$ is the identity morphism of M by (3.3.2), $\hat{\xi}_{[\pi,\pi]}(M_\varepsilon)_{[\pi,\,\pi]}$ is the identity morphism of $M_{[\pi,\,\pi]}$. It follows from the above facts and the definition of $\hat{\xi}_\alpha$ that $M_{[\pi,\pi]} = (M_{[\pi,\pi]})_{[\sigma\varepsilon,\tau\varepsilon]} \xrightarrow{(M_{[\pi,\pi]})_\varepsilon} (M_{[\pi,\pi]})_{[\sigma,\tau]} \xrightarrow{\hat{\xi}_\alpha} M_{[\pi,\pi]}$ coincides with the identity morphism of $M_{[\pi,\pi]}$.

By (1.3.9) and (1.3.19), (1.3.21), the following diagram is commutative.

$$
\begin{tikzcd}
(M_{[\pi,\pi]})_{[\sigma,\tau]} \arrow[r, "\theta_{\pi,\pi,\sigma,\tau}(M)^{-1}"] \arrow[d, "(M_\pi)_{[\sigma,\tau]}"] & M_{[\pi\sigma_\pi,\tau\pi_\sigma]} = M_{[\sigma\pi_\sigma,\pi\alpha]} \arrow[r, "\theta_{D_\alpha}(M)"] \arrow[d, "M_{\pi\sigma}"] & (M_{[\sigma,\tau]})_{[\pi,\pi]} \arrow[r, "\hat{\xi}_{[\pi,\pi]}"] \arrow[d, "(M_{[\sigma,\tau]})_\pi"] & M_{[\pi,\pi]} \arrow[d, "M_\pi"] \\
(M_{[id_{C_0},id_{C_0}]})_{[\sigma,\tau]} \arrow[r, "\theta_{id_{C_0},id_{C_0},\sigma,\tau}(M)^{-1}"] & M_{[id_{C_0}\sigma,\tau id_{C_1}]} = M_{[\sigma id_{C_1},id_{C_0}\tau]} \arrow[r, "\theta_{\sigma,\tau,id_{C_0},id_{C_0}}(M)"] & (M_{[\sigma,\tau]})_{[id_{C_0},id_{C_0}]} \arrow[r, "\hat{\xi}"] & M
\end{tikzcd}
$$

Therefore $M_\pi : (M_{[\pi,\pi]}, \xi_\alpha) \to (M, \xi)$ is a morphism in representations by (3.3.6). $\square$

Proposition 3.3.8 *Let $\varphi : (M, \xi) \to (N, \zeta)$ be a morphism of representations of $\boldsymbol{C}$. Assume that the following left morphism is an isomorphism for $L = M, N$ and that the right morphism is an epimorphisms for $L = M, N$*

$$\theta_{\pi,\pi,\sigma,\tau}(L) : L_{[\pi\sigma_\pi,\, \tau\pi_\sigma]} \to (L_{[\pi,\pi]})_{[\sigma,\tau]}, \quad \theta_{\pi,\pi,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(L) : L_{[\pi\sigma_\pi\tilde{\mathrm{pr}}_{12},\, \tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \to (L_{[\pi,\pi]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$$

Then, $\varphi_{[\pi,\pi]} : M_{[\pi,\pi]} \to N_{[\pi,\pi]}$ gives a morphism in representations from $(M_{[\pi,\pi]}, \xi_\alpha)$ to $(N_{[\pi,\pi]}, \zeta_\alpha)$.

Proof. The following diagram is commutative by (1.3.4) and (1.3.19).

$$\begin{tikzcd}
(M_{[\pi,\pi]})_{[\sigma,\tau]} \arrow[r, "\theta_{\pi,\pi,\sigma,\tau}(M)^{-1}"] \arrow[d, "(\varphi_{[\pi,\pi]})_{[\sigma,\tau]}"] & M_{[\pi\sigma_\pi,\, \tau\pi_\sigma]} = M_{[\sigma\pi_\sigma,\, \pi\alpha]} \arrow[r, "\theta_{D_\alpha}(M)"] \arrow[d, "\varphi_{[\pi\sigma_\pi,\, \tau\pi_\sigma]}"] & (M_{[\sigma,\tau]})_{[\pi,\pi]} \arrow[r, "\hat{\xi}_{[\pi,\pi]}"] \arrow[d, "(\varphi_{[\sigma,\tau]})_{[\pi,\pi]}"] & M_{[\pi,\pi]} \arrow[d, "\varphi_{[\pi,\pi]}"] \\
(N_{[\pi,\pi]})_{[\sigma,\tau]} \arrow[r, "\theta_{\pi,\pi,\sigma,\tau}(N)^{-1}"] & N_{[\pi\sigma_\pi,\, \tau\pi_\sigma]} = N_{[\sigma\pi_\sigma,\, \pi\alpha]} \arrow[r, "\theta_{D_\alpha}(N)"] & (N_{[\sigma,\tau]})_{[\pi,\pi]} \arrow[r, "\hat{\zeta}_{[\pi,\pi]}"] & N_{[\pi,\pi]}
\end{tikzcd}$$

Hence the assertion follows. □

Proposition 3.3.9 *Let $(\pi : X \to C_0,\ \alpha : X \times^\sigma_{C_0} C_1 \to X)$ and $(\rho : Y \to C_0,\ \beta : Y \times^\sigma_{C_0} C_1 \to Y)$ be internal diagrams on $\boldsymbol{C}$ and (M, ξ) a representation of $\boldsymbol{C}$. Assume that the following left morphism is an isomorphism for $\chi = \pi, \rho$ and that the right morphism is an epimorphism for $\chi = \pi, \rho$.*

$$\theta_{\chi,\chi,\sigma,\tau}(M) : M_{[\chi\sigma_\chi,\, \tau\chi_\sigma]} \to (M_{[\chi,\chi]})_{[\sigma,\tau]}, \quad \theta_{\chi,\chi,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M) : M_{[\chi\sigma_\chi\tilde{\mathrm{pr}}_{12},\, \tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \to (M_{[\chi,\chi]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$$

If a morphism $f : X \to Y$ of $\mathcal{E}$ defines a morphism in internal diagrams from $(\pi : X \to C_0, \alpha)$ to $(\rho : Y \to C_0, \beta)$, $M_f : M_{[\pi,\pi]} \to M_{[\rho,\rho]}$ is a morphism of representations from $(M_{[\pi,\pi]}, \xi_\alpha)$ to $(M_{[\rho,\rho]}, \xi_\beta)$.

Proof. Define a natural transformation $\lambda : D_\alpha \to D_\beta$ by $\lambda_0 = f \times_{C_0} id_{C_1}$, $\lambda_1 = id_{C_1}$, $\lambda_2 = f$, $\lambda_i = id_{C_0}$ $(i = 3, 4, 5)$. The following diagram is commutative by (1.3.7) and (1.3.20).

$$\begin{tikzcd}
(M_{[\pi,\pi]})_{[\sigma,\tau]} \arrow[r, "\theta_{\pi,\pi,\sigma,\tau}(M)^{-1}"] \arrow[d, "(M_f)_{[\sigma,\tau]}"] & M_{[\pi\sigma_\pi,\, \tau\pi_\sigma]} = M_{[\sigma\pi_\sigma,\, \pi\alpha]} \arrow[r, "\theta_{D_\alpha}(M)"] \arrow[d, "M_{f\times_{C_0} id_{C_1}}"] & (M_{[\sigma,\tau]})_{[\pi,\pi]} \arrow[r, "\hat{\xi}_{[\pi,\pi]}"] \arrow[d, "(M_{[\sigma,\tau]})_f"] & M_{[\pi,\pi]} \arrow[d, "M_f"] \\
(M_{[\rho,\rho]})_{[\sigma,\tau]} \arrow[r, "\theta_{\rho,\rho,\sigma,\tau}(M)^{-1}"] & M_{[\rho\sigma_\rho,\, \tau\rho_\sigma]} = M_{[\sigma\rho_\sigma,\, \rho\beta]} \arrow[r, "\theta_{D_\beta}(M)"] & (M_{[\sigma,\tau]})_{[\rho,\rho]} \arrow[r, "\hat{\zeta}_{[\rho,\rho]}"] & M_{[\rho,\rho]}
\end{tikzcd}$$

Hence the assertion follows. □

For an object M of $\mathcal{F}_{C_0}$, we define a morphism $\hat{\mu}_M : (M_{[\sigma,\tau]})_{[\sigma,\tau]} \to M_{[\sigma,\tau]}$ to be the following composition assuming that $\theta_{\sigma,\tau,\sigma,\tau}(M) : M_{[\sigma\mathrm{pr}_1,\, \tau\mathrm{pr}_2]} \to (M_{[\sigma,\tau]})_{[\sigma,\tau]}$ is an isomorphism.

$$(M_{[\sigma,\tau]})_{[\sigma,\tau]} \xrightarrow{\theta_{\sigma,\tau,\sigma,\tau}(M)^{-1}} M_{[\sigma\mathrm{pr}_1,\, \tau\mathrm{pr}_2]} = M_{[\sigma\mu,\, \tau\mu]} \xrightarrow{M_\mu} M_{[\sigma,\tau]}$$

Let $C_1 \times_{C_0} C_1 \xleftarrow{\mathrm{pr}_{12}} C_1 \times_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_{23}} C_1 \times_{C_0} C_1$ be a limit of a diagram $C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_2} C_1 \xleftarrow{\mathrm{pr}_1} C_1 \times_{C_0} C_1$.

Proposition 3.3.10 *We assume that $\theta_{\sigma,\tau,\sigma,\tau}(M) : M_{[\sigma\mathrm{pr}_1,\, \tau\mathrm{pr}_2]} \to (M_{[\sigma,\tau]})_{[\sigma,\tau]}$ is an isomorphism and that $\theta_{\sigma,\tau,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M) : M_{[\sigma\mathrm{pr}_1\mathrm{pr}_{12},\, \tau\mathrm{pr}_2\mathrm{pr}_{23}]} \to (M_{[\sigma,\tau]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$ is an epimorphism. Let us denote by μ^l_M a morphism $P_{\sigma,\tau}(M_{[\sigma,\tau]})^{-1}_{M_{[\sigma,\tau]}}(\hat{\mu}_M)$ in $\mathcal{F}_{C_1}$. Then $(M_{[\sigma,\tau]}, \mu^l_M)$ is a representation of $\boldsymbol{C}$. Moreover, if $\xi : \sigma^*(M) \to \tau^*(M)$ is a morphism in $\mathcal{F}_{C_1}$ such that (M, ξ) is a representation of $\boldsymbol{C}$, then $\hat{\xi} = P_{\sigma,\tau}(M)_M(\xi) : M_{[\sigma,\tau]} \to M$ defines a morphism of representations from $(M_{[\sigma,\tau]}, \mu^l_M)$ to (M, ξ).*

Proof. The following diagram is commutative by (1.3.21) and (1.3.25).

$$\begin{tikzcd}
((M_{[\sigma,\tau]})_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\theta_{\sigma,\tau,\sigma,\tau}(M)^{-1}_{[\sigma,\tau]}"] & (M_{[\sigma\mathrm{pr}_1,\, \tau\mathrm{pr}_2]})_{[\sigma,\tau]} = (M_{[\sigma\mu,\, \tau\mu]})_{[\sigma,\tau]} \arrow[r, "(M_\mu)_{[\sigma,\tau]}"] & (M_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[d, "\theta_{\sigma,\tau,\sigma,\tau}(M)^{-1}"'] \\
(M_{[\sigma,\tau]})_{[\sigma\mathrm{pr}_1,\, \tau\mathrm{pr}_2]} \arrow[u, "\theta_{\sigma,\tau,\sigma,\tau}(M_{[\sigma,\tau]})"] \arrow[d, equal] & M_{[\sigma\mathrm{pr}_1\mathrm{pr}_{12},\, \tau\mathrm{pr}_2\mathrm{pr}_{23}]} \arrow[l, "\theta_{\sigma,\tau,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M)"'] \arrow[r, "M_{\mu\times_{C_0} id_{C_1}}"] \arrow[u, "\theta_{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\sigma,\tau}(M)"] \arrow[d, equal] & M_{[\sigma\mathrm{pr}_1,\, \tau\mathrm{pr}_2]} \arrow[d, equal] \\
(M_{[\sigma,\tau]})_{[\sigma\mu,\, \tau\mu]} \arrow[d, "(M_{[\sigma,\tau]})_\mu"] & M_{[\sigma\mu\mathrm{pr}_{12},\, \tau\mu\mathrm{pr}_{23}]} \arrow[l, "\theta_{\sigma,\tau,\sigma\mu,\tau\mu}(M)"'] \arrow[r, "M_{\mu\times_{C_0} id_{C_1}}"] \arrow[d, "M_{id_{C_1}\times_{C_0}\mu}"] & M_{[\sigma\mu,\, \tau\mu]} \arrow[d, "M_\mu"'] \\
(M_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\theta_{\sigma,\tau,\sigma,\tau}(M)^{-1}"] & M_{[\sigma\mathrm{pr}_1,\, \tau\mathrm{pr}_2]} = M_{[\sigma\mu,\, \tau\mu]} \arrow[r, "M_\mu"] & M_{[\sigma,\, \tau]}
\end{tikzcd}$$

Since the functor $D_{\sigma,\tau,id_{C_0},id_{C_0}} : \mathcal{P} \to \mathcal{E}$ are given by

$$\begin{array}{ll} D_{\sigma,\tau,id_{C_0},id_{C_0}}(i) = C_1 \quad (i = 0,1), & D_{\sigma,\tau,id_{C_0},id_{C_0}}(i) = C_0 \quad (i = 2,3,4,5), \\ D_{\sigma,\tau,id_{C_0},id_{C_0}}(\tau_{01}) = id_{C_1}, & D_{\sigma,\tau,id_{C_0},id_{C_0}}(\tau_{13}) = \sigma, \\ D_{\sigma,\tau,id_{C_0},id_{C_0}}(\tau_{02}) = D_{\sigma,\tau,id_{C_0},id_{C_0}}(\tau_{14}) = \tau, & D_{\sigma,\tau,id_{C_0},id_{C_0}}(\tau_{23}) = D_{\sigma,\tau,id_{C_0},id_{C_0}}(\tau_{24}) = id_{C_0}, \end{array}$$

we define a natural transformations $\nu : D_{\sigma,\tau,id_{C_0},id_{C_0}} \to D_{\sigma,\tau,\sigma,\tau}$ by $\nu_0 = (id_{C_1}, \varepsilon\tau) : C_1 \to C_1 \times_{C_0} C_1$, $\nu_1 = id_{C_1}$, $\nu_2 = \varepsilon$, $\nu_i = \kappa_i = id_{C_0}$ $(i = 3,4,5)$. Then, the following diagram is commutative by (1.3.19), (1.3.7).

$$\begin{tikzcd} (M_{[\sigma,\tau]})_{[\sigma\varepsilon,\tau\varepsilon]} \arrow[r, "\theta_{\sigma,\tau,id_{C_0},id_{C_0}}(M)^{-1}"] \arrow[d, "(M_{[\sigma,\tau]})_\varepsilon"] & M_{[\sigma id_{C_1},\, id_{C_0}\tau]} \arrow[r, equal] \arrow[d, "M_{[id_{C_1},\varepsilon\tau]}"] & M_{[\sigma id_{C_1},\, \tau id_{C_1}]} \arrow[r, "M_{id_{C_1}}"] & M_{[\sigma,\tau]} \arrow[d, "id_{M_{[\sigma,\tau]}}"] \\ (M_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\theta_{\sigma,\tau,\sigma,\tau}(M)^{-1}"] & M_{[\sigma \mathrm{pr}_1,\, \tau \mathrm{pr}_2]} \arrow[r, equal] & M_{[\sigma\mu,\tau\mu]} \arrow[r, "M_\mu"] & M_{[\sigma,\tau]} \end{tikzcd}$$

The upper row of the above diagram is identified with the identity morphism of $M_{[\sigma,\tau]}$ which implies that $\hat{\mu}_M(M_{[\sigma,\tau]})_\varepsilon$ is the identity morphism of $M_{[\sigma,\tau]}$. Thus $(M_{[\sigma,\tau]}, \mu^l_M)$ is a representation of $\boldsymbol{C}$ by (3.3.2).

If (M,ξ) is a representation of $\boldsymbol{C}$, then, $\hat{\xi}\hat{\xi}_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M) = \hat{\xi}M_\mu$ by (3.3.2). Hence $\hat{\xi}\hat{\xi}_{[\sigma,\tau]} = \hat{\xi}\hat{\mu}_M$ by the definition of $\hat{\mu}_M$ and it follows from (3.3.6) that $\hat{\xi}$ defines a morphism of representations from $(M_{[\sigma,\tau]}, \mu^l_M)$ to (M,ξ). □

Proposition 3.3.11 *Assume that $\theta_{\sigma,\tau,\sigma,\tau}(L) : L_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \to (L_{[\sigma,\tau]})_{[\sigma,\tau]}$ is an isomorphism for $L = M,N$ and that $\theta_{\sigma,\tau,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(L) : L_{[\sigma\mathrm{pr}_1\mathrm{pr}_{12},\,\tau\mathrm{pr}_2\mathrm{pr}_{23}]} \to (L_{[\sigma,\tau]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$ is an epimorphisms for $L = M,N$. For a morphism $\varphi : M \to N$, $\varphi_{[\sigma,\tau]} : M_{[\sigma,\tau]} \to N_{[\sigma,\tau]}$ defines a morphism of representations from $(M_{[\sigma,\tau]}, \mu^l_M)$ to $(N_{[\sigma,\tau]}, \mu^l_N)$.*

Proof. The following diagram is commtative by (1.3.9) and (1.3.21).

$$\begin{tikzcd} (M_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\theta_{\sigma,\tau,\sigma,\tau}(M)^{-1}"] \arrow[d, "(\varphi_{[\sigma,\tau]})_{[\sigma,\tau]}"] & M_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \arrow[r, equal] \arrow[d, "\varphi_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]}"] & M_{[\sigma\mu,\tau\mu]} \arrow[r, "M_\mu"] & M_{[\sigma,\tau]} \arrow[d, "\varphi_{[\sigma,\tau]}"] \\ (N_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\theta_{\sigma,\tau,\sigma,\tau}(N)^{-1}"] & N_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \arrow[r, equal] & N_{[\sigma\mu,\tau\mu]} \arrow[r, "N_\mu"] & N_{[\sigma,\tau]} \end{tikzcd}$$

Hence the assertion follows from (3.3.6). □

Remark 3.3.12 *If $\varphi : (M,\xi) \to (N,\zeta)$ is a morphism of representations of $\boldsymbol{C}$, we have the following commutative diagram in* $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$.

$$\begin{tikzcd} (M_{[\sigma,\tau]}, \mu^l_M) \arrow[r, "\hat{\xi}"] \arrow[d, "\varphi_{[\sigma,\tau]}"] & (M,\xi) \arrow[d, "\varphi"] \\ (N_{[\sigma,\tau]}, \mu^l_N) \arrow[r, "\hat{\zeta}"] & (N,\zeta) \end{tikzcd}$$

Theorem 3.3.13 *Let M be an object of $\mathcal{F}_{C_0}$ and (N,ζ) a representation of $\boldsymbol{C}$. Assume that $\theta_{\sigma,\tau,\sigma,\tau}(L) : L_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \to (L_{[\sigma,\tau]})_{[\sigma,\tau]}$ is an isomorphism for $L = M,N$ and that $\theta_{\sigma,\tau,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(L) : L_{[\sigma\mathrm{pr}_1\mathrm{pr}_{12},\,\tau\mathrm{pr}_2\mathrm{pr}_{23}]} \to (L_{[\sigma,\tau]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$ is an epimorphism for $L = M,N$. Then, a map*

$$\Phi : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M_{[\sigma,\tau]}, \mu^l_M), (N,\zeta)) \to \mathcal{F}_{C_0}(M,N)$$

defined by $\Phi(\varphi) = \varphi M_\varepsilon$ is bijective. Hence, if $\theta_{\sigma,\tau,\sigma,\tau}(L)$ is an isomorphism and $\theta_{\sigma,\tau,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(L)$ is an epimorphisms for all $L \in \mathrm{Ob}\,\mathcal{F}_{C_0}$, a functor $\mathscr{L}_{\boldsymbol{C}} : \mathcal{F}_{C_0} \to \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ defined by $\mathscr{L}_{\boldsymbol{C}}(M) = (M_{[\sigma,\tau]}, \mu^l_M)$ for $M \in \mathrm{Ob}\,\mathcal{F}_{C_0}$ and $\mathscr{L}_{\boldsymbol{C}}(\varphi) = \varphi_{[\sigma,\tau]}$ for $\varphi \in \mathrm{Mor}\mathcal{F}_{C_0}$ is a left adjoint of the forgetful functor $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$.

Proof. We put $\hat{\zeta} = P_{\sigma,\tau}(N)_N(\zeta) : N_{[\sigma,\tau]} \to N$. For $\psi \in \mathcal{F}_{C_0}(M,N)$, it follows from (3.3.11) that we have a morphism $\psi_{[\sigma,\tau]} : (M_{[\sigma,\tau]}, \mu^l_M) \to (N_{[\sigma,\tau]}, \mu^l_N)$ of representations. Since $\hat{\zeta} : (N_{[\sigma,\tau]}, \mu^l_N) \to (N,\zeta)$ is a morphism of representations by (3.3.10), $\hat{\zeta}\psi_{[\sigma,\tau]} : (M_{[\sigma,\tau]}, \mu^l_M) \to (N,\zeta)$ is a morphism of representations. It follows from (1.3.9) and (3.3.2) that we have $\Phi(\hat{\zeta}\psi_{[\sigma,\tau]}) = \hat{\zeta}\psi_{[\sigma,\tau]}M_\varepsilon = \hat{\zeta}N_\varepsilon\psi = \psi$. On the other hand, for $\varphi \in \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M_{[\sigma,\tau]}, \mu^l_M)), (N,\zeta))$, since $\hat{\zeta}\varphi_{[\sigma,\tau]} = \varphi\hat{\mu}_M = \varphi M_\mu \theta_{\sigma,\tau,\sigma,\tau}(M)^{-1}$ by (3.3.6) and the following diagram commutes by (1.3.7) and (1.3.21),

$$\begin{tikzcd}
(M_{[id_{C_0}, id_{C_0}]})_{[\sigma,\tau]} \arrow[d, "(M_\varepsilon)_{[\sigma,\tau]}"] & M_{[id_{C_0}\sigma, \tau id_{C_1}]} \arrow[l, "\theta_{id_{C_0},id_{C_0},\sigma,\tau}(M)"'] \arrow[r, "id_{M_{[\sigma,\tau]}}"] \arrow[d, "M_{(\varepsilon\sigma, id_{C_1})}"] & M_{[\sigma,\tau]} \\
(M_{[\sigma,\tau]})_{[\sigma,\tau]} & M_{[\sigma\mathrm{pr}_1, \tau\mathrm{pr}_2]} \arrow[l, "\theta_{\sigma,\tau,\sigma,\tau}(M)"'] \arrow[r, equal] & M_{[\sigma\mu, \tau\mu]} \arrow[u, "M_\mu"']
\end{tikzcd}$$

we have $\hat{\zeta}(\varphi M_\varepsilon)_{[\sigma,\tau]} = \hat{\zeta}\varphi_{[\sigma,\tau]}(M_\varepsilon)_{[\sigma,\tau]} = \varphi M_\mu \theta_{\sigma,\tau,\sigma,\tau}(M)^{-1}(M_\varepsilon)_{[\sigma,\tau]} = \varphi$ by (1.3.4) and (1.3.26). Therefore a correspondence $\psi \mapsto \hat{\zeta}\psi_{[\sigma,\tau]}$ gives the inverse map of Φ. □

For morphisms $f : X \to Y$ and $g : X \to Z$ of $\mathcal{E}$, we denote by $[f,g]_* : \mathcal{F}_Y \to \mathcal{F}_Z$ the functor defined by $[f,g]_*(M) = M_{[f,g]}$ for $M \in \mathrm{Ob}\,\mathcal{F}_Y$ and $[f,g]_*(\varphi) = \varphi_{[f,g]}$ for $\varphi \in \mathrm{Mor}\,\mathcal{F}_Y$.

Proposition 3.3.14 *Let (M,ξ) and (M,ζ) be representations of $\boldsymbol{C}$ on $M \in \mathrm{Ob}\,\mathcal{F}_{C_0}$. We put $\hat{\xi} = P_{\sigma,\tau}(M)_M(\xi)$ and $\hat{\zeta} = P_{\sigma,\tau}(M)_M(\zeta)$. Assume that $[\sigma,\tau]_* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_0}$ preserves coequalizers ((σ,τ) is a right fibered representable pair, for example. See (1.5.2).) and that $\theta_{\sigma,\tau,\sigma,\tau}(M)$ is an epimorphism. Let $\pi_{\xi,\zeta} : M \to M_{(\xi:\zeta)}$ be a coequalizer of $\hat{\xi}, \hat{\zeta} : M_{[\sigma,\tau]} \to M$.*

(1) There exists unique morphism $\hat{\lambda} : (M_{(\xi:\zeta)})_{[\sigma,\tau]} \to M_{(\xi:\zeta)}$ that makes the following diagram commute.

$$\begin{tikzcd}
M_{[\sigma,\tau]} \arrow[r, "(\pi_{\xi,\zeta})_{[\sigma,\tau]}"] \arrow[d, "\hat{\xi}"] & (M_{(\xi:\zeta)})_{[\sigma,\tau]} \arrow[d, "\hat{\lambda}"] & M_{[\sigma,\tau]} \arrow[l, "(\pi_{\xi,\zeta})_{[\sigma,\tau]}"'] \arrow[d, "\hat{\zeta}"] \\
M \arrow[r, "\pi_{\xi,\zeta}"] & M_{(\xi:\zeta)} & M \arrow[l, "\pi_{\xi,\zeta}"']
\end{tikzcd}$$

(2) Moreover, we assume that $[\sigma\mathrm{pr}_1, \tau\mathrm{pr}_2]_* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_0}$ maps coequalizers to epimorphisms ($(\sigma\mathrm{pr}_1, \tau\mathrm{pr}_2)$ is a right fibered representable pair, for example. See (1.5.2).). Put $\lambda = P_{\sigma,\tau}(M_{(\xi:\zeta)})^{-1}_{M_{(\xi:\zeta)}}(\hat{\lambda})$. Then, $(M_{(\xi:\zeta)}, \lambda)$ is a representation of $\boldsymbol{C}$ and $\pi_{\xi,\zeta}$ defines morphisms of representations $(M,\xi) \to (M_{(\xi:\zeta)}, \lambda)$ and $(M,\zeta) \to (M_{(\xi:\zeta)}, \lambda)$.

(3) Let (N,ν) be a representation of $\boldsymbol{C}$. Suppose that a morphism $\varphi : M \to N$ of $\mathcal{F}_{C_0}$ gives morphisms $(M,\xi) \to (N,\nu)$ and $(M,\zeta) \to (N,\nu)$ of $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$. Then, there exists unique morphism $\tilde{\varphi} : (M_{(\xi:\zeta)}, \lambda) \to (N,\nu)$ of $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ that satisfies $\tilde{\varphi}\pi_{\xi,\zeta} = \varphi$.

Proof. (1) Put $\chi = \pi_{\xi,\zeta}\hat{\xi} = \pi_{\xi,\zeta}\hat{\zeta} : M_{[\sigma,\tau]} \to M_{(\xi:\zeta)}$. Then, it follows from (3.3.2) that

$$\chi\hat{\xi}_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M) = \pi_{\xi,\zeta}\hat{\xi}\hat{\xi}_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M) = \pi_{\xi,\zeta}\hat{\xi}M_\mu = \pi_{\xi,\zeta}\hat{\zeta}M_\mu = \pi_{\xi,\zeta}\hat{\zeta}\hat{\zeta}_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M) = \chi\hat{\zeta}_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M),$$

which implies $\chi\hat{\xi}_{[\sigma,\tau]} = \chi\hat{\zeta}_{[\sigma,\tau]}$ since $\theta_{\sigma,\tau,\sigma,\tau}(M)$ is an epimorphism. Since $(\pi_{\xi,\zeta})_{[\sigma,\tau]} : M_{[\sigma,\tau]} \to (M_{(\xi:\zeta)})_{[\sigma,\tau]}$ is a coequalizer of $\hat{\xi}_{[\sigma,\tau]}, \hat{\zeta}_{[\sigma,\tau]} : (M_{[\sigma,\tau]})_{[\sigma,\tau]} \to M_{[\sigma,\tau]}$ by the assumption, there exists unique morphism $\hat{\lambda} : (M_{(\xi:\zeta)})_{[\sigma,\tau]} \to M_{(\xi:\zeta)}$ that satisfies $\hat{\lambda}(\pi_{\xi,\zeta})_{[\sigma,\tau]} = \chi$.

(2) By (1.3.4), (1.3.7), (1.3.21) and (3.3.2), the following diagrams are commutative.

$$\begin{tikzcd}
M_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[r, "\theta_{\sigma,\tau,\sigma,\tau}(M)"] \arrow[d, "(\pi_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}"] & (M_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\hat{\xi}_{[\sigma,\tau]}"] \arrow[d, "((\pi_{\xi,\zeta})_{[\sigma,\tau]})_{[\sigma,\tau]}"] & M_{[\sigma,\tau]} \arrow[r, "\hat{\xi}"] \arrow[d, "(\pi_{\xi,\zeta})_{[\sigma,\tau]}"] & M \arrow[d, "\pi_{\xi,\zeta}"] \\
(M_{(\xi:\zeta)})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[r, "\theta_{\sigma,\tau,\sigma,\tau}(M_{(\xi:\zeta)})"] & ((M_{(\xi:\zeta)})_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\hat{\lambda}_{[\sigma,\tau]}"] & (M_{(\xi:\zeta)})_{[\sigma,\tau]} \arrow[r, "\hat{\lambda}"] & M_{(\xi:\zeta)}
\end{tikzcd}$$

$$\begin{tikzcd}
M_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[r, equal] \arrow[d, "(\pi_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}"] & M_{[\sigma\mu,\tau\mu]} \arrow[r, "M_\mu"] \arrow[d, "(\pi_{\xi,\zeta})_{[\sigma\mu,\tau\mu]}"] & M_{[\sigma,\tau]} \arrow[r, "\hat{\xi}"] \arrow[d, "(\pi_{\xi,\zeta})_{[\sigma,\tau]}"] & M \arrow[d, "\pi_{\xi,\zeta}"] \\
(M_{(\xi:\zeta)})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[r, equal] & (M_{(\xi:\zeta)})_{[\sigma\mu,\tau\mu]} \arrow[r, "(M_{(\xi:\zeta)})_\mu"] & (M_{(\xi:\zeta)})_{[\sigma,\tau]} \arrow[r, "\hat{\lambda}"] & M_{(\xi:\zeta)}
\end{tikzcd}$$

$$\begin{tikzcd}
M \arrow[r, equal] \arrow[d, "\pi_{\xi,\zeta}"] & M_{[\sigma\varepsilon,\tau\varepsilon]} \arrow[r, "M_\varepsilon"] \arrow[d, "(\pi_{\xi,\zeta})_{[\sigma\varepsilon,\tau\varepsilon]}"] & M_{[\sigma,\tau]} \arrow[r, "\hat{\xi}"] \arrow[d, "(\pi_{\xi,\zeta})_{[\sigma,\tau]}"] & M \arrow[d, "\pi_{\xi,\zeta}"] \\
M_{(\xi:\zeta)} \arrow[r, equal] & (M_{(\xi:\zeta)})_{[\sigma\varepsilon,\tau\varepsilon]} \arrow[r, "(M_{(\xi:\zeta)})_\varepsilon"] & (M_{(\xi:\zeta)})_{[\sigma,\tau]} \arrow[r, "\hat{\lambda}"] & M_{(\xi:\zeta)}
\end{tikzcd}$$

It follows from (3.3.2) that we have

$$\hat{\lambda}\hat{\lambda}_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M_{(\xi:\zeta)})(\pi_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} = \pi_{\xi,\zeta}\hat{\xi}\hat{\xi}_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M) = \pi_{\xi,\zeta}\hat{\xi}M_{\mu} = \hat{\lambda}(M_{(\xi:\zeta)})_{\mu}(\pi_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$$
$$\hat{\lambda}(M_{(\xi:\zeta)})_{\varepsilon}\pi_{\xi,\zeta} = \pi_{\xi,\zeta}\hat{\xi}M_{\varepsilon} = \pi_{\xi,\zeta}$$

Since $\pi_{\xi,\zeta}$ and $(\pi_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$ are epimorphisms, it follows that $\hat{\lambda}(\hat{\lambda}_{[\sigma,\tau]})\theta_{\sigma,\tau,\sigma,\tau}(M_{(\xi:\zeta)}) = \hat{\lambda}(M_{(\xi:\zeta)})_{\mu}$ and $\hat{\lambda}(M_{(\xi:\zeta)})_{\varepsilon} = id_{M_{(\xi:\zeta)}}$. Therefore λ is a representation of $\boldsymbol{C}$ on $M_{(\xi:\zeta)}$ by (3.3.2). $\pi_{\xi,\zeta} : (M,\xi) \to (M_{(\xi:\zeta)},\lambda)$ and $\pi_{\xi,\zeta} : (M,\zeta) \to (M_{(\xi:\zeta)},\lambda)$ are morphisms of representations by the first assertion and (1.3.6).

(3) Put $\hat{\nu} = P_{\sigma,\tau}(N)_N(\nu)$. Since $\varphi\hat{\xi} = \hat{\nu}\varphi_{[\sigma,\tau]} = \varphi\hat{\zeta}$ by (3.3.6), there exists unique morphism $\tilde{\varphi} : M_{(\xi:\zeta)} \to N$ that satisfies $\tilde{\varphi}\pi_{\xi,\zeta} = \varphi$. Then, we have $\tilde{\varphi}\hat{\lambda}(\pi_{\xi,\zeta})_{[\sigma,\tau]} = \tilde{\varphi}\pi_{\xi,\zeta}\hat{\xi} = \varphi\hat{\xi} = \hat{\nu}\varphi_{[\sigma,\tau]} = \hat{\nu}\tilde{\varphi}_{[\sigma,\tau]}(\pi_{\xi,\zeta})_{[\sigma,\tau]}$. Since $(\pi_{\xi,\zeta})_{[\sigma,\tau]}$ is an epimorphism, it follows $\tilde{\varphi}\hat{\lambda} = \hat{\nu}\tilde{\varphi}_{[\sigma,\tau]}$, which implies that $\tilde{\varphi}$ gives a morphism $(M_{(\xi:\zeta)},\lambda) \to (N,\nu)$ of representations of $\boldsymbol{C}$. □

Remark 3.3.15 *Assume that one of the following conditions.*

(i) $[\sigma,\tau]_* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_0}$ *preserves epimorphisms.*
(ii) $\sigma^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_1}$ *preserves epimorphisms.*
(iii) (σ,τ) *is a right fibered representable pair with respect to* $N \in \mathrm{Ob}\,\mathcal{F}_{C_0}$.

For representations (M,ξ), (N,ζ) *and* (N,ζ') *of* $\boldsymbol{C}$, *suppose that there exists an epimorphism* $\varphi : M \to N$ *of* $\mathcal{F}_{C_0}$ *such that* $\varphi : (M,\xi) \to (N,\zeta)$ *and* $\varphi : (M,\xi) \to (N,\zeta')$ *are morphisms in* $\mathrm{Rep}(\boldsymbol{C};\mathcal{F})$. *Then,* $\sigma^*(\varphi)^* : \mathcal{F}_{C_1}(\sigma^*(N),\tau^*(N)) \to \mathcal{F}_{C_1}(\sigma^*(M),\tau^*(N))$ *is injective by the assumption. Hence* $\zeta\sigma^*(\varphi) = \tau^*(\varphi)\xi = \zeta'\sigma^*(\varphi)$ *implies* $\zeta = \zeta'$.

Proposition 3.3.16 *Let* (M,ξ), (N,ξ'), (M,ζ) *and* (N,ζ') *be objects of* $\mathrm{Rep}(\boldsymbol{C};\mathcal{F})$. *Put* $\hat{\xi} = P_{\sigma,\tau}(M)_M(\xi)$, $\hat{\xi}' = P_{\sigma,\tau}(N)_N(\xi')$, $\hat{\zeta} = P_{\sigma,\tau}(M)_M(\zeta)$ *and* $\hat{\zeta}' = P_{\sigma,\tau}(N)_N(\zeta')$. *Assume that* $[\sigma,\tau]_* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_0}$ *preserves coequalizers and that* $[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]_* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_0}$ *maps coequalizers to epimorphisms (e.g.,* (σ,τ) *and* $(\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2)$ *are right fibered representable pairs. See (1.5.2)). Suppose that* $\pi_{\xi,\zeta} : M \to M_{(\xi:\zeta)}$ *is a coequalizer of* $\hat{\xi},\hat{\zeta} : M_{[\sigma,\tau]} \to M$ *and that* $\pi_{\xi',\zeta'} : N \to N_{(\xi':\zeta')}$ *is a coequalizer of* $\hat{\xi}',\hat{\zeta}' : N_{[\sigma,\tau]} \to N$. *We denote by* $(M_{(\xi:\zeta)},\lambda)$ *and* $(N_{(\xi':\zeta')},\lambda')$ *the representations of* $\boldsymbol{C}$ *given in (3.3.14). If a morphism* $\varphi : M \to N$ *defines morphisms of representations* $(M,\xi) \to (N,\xi')$ *and* $(M,\zeta) \to (N,\zeta')$, *then there exists unique morphism* $\tilde{\varphi} : (M_{(\xi:\zeta)},\lambda) \to (N_{(\xi':\zeta')},\lambda')$ *of representations of* $\boldsymbol{C}$ *that satisfies* $\tilde{\varphi}\pi_{\xi,\zeta} = \pi_{\xi',\zeta'}\varphi$.

Proof. Since $\pi_{\xi',\zeta'} : N \to N_{(\xi':\zeta')}$ defines morphisms $(N,\xi') \to (N_{(\xi':\zeta')},\lambda')$, $(N,\zeta') \to (N_{(\xi':\zeta')},\lambda')$ of representations of $\boldsymbol{C}$, $\pi_{\xi',\zeta'}\varphi : M \to N_{(\xi':\zeta')}$ defines morphisms $(M,\xi) \to (N_{(\xi':\zeta')},\lambda')$, $(M,\zeta) \to (N_{(\xi':\zeta')},\lambda')$ of representations of $\boldsymbol{C}$. Hence it follows from (3) of (3.3.16) that there exists unique morphism $\tilde{\varphi} : M_{(\xi:\zeta)} \to N_{(\xi':\zeta')}$ that satisfies $\tilde{\varphi}\pi_{\xi,\zeta} = \pi_{\xi',\zeta'}\varphi$ and gives a morphism $(M_{(\xi:\zeta)},\lambda) \to (N_{(\xi':\zeta')},\lambda')$ of representations of $\boldsymbol{C}$. □

3.4 Representations of right fibered representable internal categories

Let $p : \mathcal{F} \to \mathcal{E}$ be a normalized cloven fibered category with exponents and $\boldsymbol{C} = (C_0,C_1;\sigma,\tau,\varepsilon,\mu)$ an internal category in $\mathcal{E}$.

Definition 3.4.1 *Let* $\boldsymbol{C} = (C_0,C_1;\sigma,\tau,\varepsilon,\mu)$ *be an internal category in* $\mathcal{E}$. *We call* $\boldsymbol{C}$ *a right fibered representable internal category if* (σ,τ) *and* $(\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2)$ *are right fibered representable pairs.*

We assume that all internal categories in this subsection are right fibered representable internal categories. We also assume that, for morphisms $f : X \to Y$, $g : X \to Z$ and an object N of $\mathcal{F}_Z$, (f,g) is a right fibered representable pair with respect to N if necessary.

Proposition 3.4.2 *For* $M \in \mathrm{Ob}\,\mathcal{F}_{C_0}$ *and* $\xi \in \mathcal{F}_{C_1}(\sigma^*(M),\tau^*(M))$, *we put* $\check{\xi} = E_{\sigma,\tau}(M)_M(\xi) : M \to M^{[\sigma,\tau]}$. ξ *satisfies condition (A) of (3.1.2) if and only if the following diagram commutes.*

$$\begin{tikzcd}
M \arrow[r, "\check{\xi}"] \arrow[d, "\check{\xi}"'] & M^{[\sigma,\tau]} \arrow[r, "\check{\xi}^{[\sigma,\tau]}"] & (M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[d, "\theta^{\sigma,\tau,\sigma,\tau}(M)"] \\
M^{[\sigma,\tau]} \arrow[r, "M^{\mu}"] & M^{[\sigma\mu,\tau\mu]} \arrow[r, equal] & M^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}
\end{tikzcd}$$

ξ satisfies condition (U) of (3.1.2) if and only if a composition $M \xrightarrow{\check{\xi}} M^{[\sigma,\tau]} \xrightarrow{M^{\varepsilon}} M^{[\sigma\varepsilon,\tau\varepsilon]} = M$ coincides with the identity morphism of M.

Proof. We have $E_{\sigma\mu,\tau\mu}(M)_M(\xi_\mu) = M^\mu\check{\xi}$ and $E_{\sigma\mu,\tau\mu}(M)_M(\xi_{\mathrm{pr}_i}) = M^{\mathrm{pr}_i}\check{\xi}$ for $i = 1, 2$ by (1.4.7). Hence (1.4.4), (1.4.7), (1.4.9), (1.4.16) imply

$$\begin{aligned} E_{\sigma\mu,\tau\mu}(M)_M(\xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1}) &= E_{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M)_M(\xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1}) = \epsilon_M^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_1,\tau\mathrm{pr}_2}(M^{\mathrm{pr}_2}\check{\xi})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_1]}M^{\mathrm{pr}_1}\check{\xi} \\ &= \epsilon_M^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_1,\tau\mathrm{pr}_2}(M^{\mathrm{pr}_2})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_1]}(M^{[\sigma,\tau]})^{\mathrm{pr}_1}\check{\xi}^{[\sigma,\tau]}\check{\xi} = \theta^{\sigma,\tau,\sigma,\tau}(M)\check{\xi}^{[\sigma,\tau]}\check{\xi} \end{aligned}$$

Thus $\xi_\mu = \xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1}$ and $\xi_\varepsilon = id_M$ are equivalent to $\theta^{\sigma,\tau,\sigma,\tau}(M)\check{\xi}^{[\sigma,\tau]}\check{\xi} = M^\mu\check{\xi}$ and $M^\varepsilon\check{\xi} = id_M$, respectively. □

Proposition 3.4.3 *Let $(s(C_0), s_{\boldsymbol{C}})$ be the trivial representation associated with a cartesian section $s : \mathcal{E} \to \mathcal{F}$. Put $T = s(1)$. The image of $s_{\boldsymbol{C}} \in \mathcal{F}_{C_1}(\sigma^*s(C_0), \tau^*s(C_0))$ by $E_{\sigma,\tau}(s(C_0))_{s(C_0)} : \mathcal{F}_{C_1}(\sigma^*s(C_0), \tau^*s(C_0)) \to \mathcal{F}_{C_0}(s(C_0), s(C_0)^{[\sigma,\tau]})$ is $\omega(\sigma,\tau; o_{C_0}, o_{C_0})^T o_{C_0}^*(E_{o_{C_1},o_{C_1}}(T)_T(id_{s(C_1)}))$.*

Proof. It follows from (1.1.22) and the definition of $s_{\boldsymbol{C}}$ that we have $s_{\boldsymbol{C}} = c_{o_{C_0},\tau}(T)^{-1}c_{o_{C_0},\sigma}(T)$. We note that $o_{C_0}\sigma = o_{C_0}\tau = o_{C_1}$ and $s(C_i) = o_{C_i}^*(T)$ for $i = 0, 1$. The following diagram is commutative by (1.4.30).

$$\begin{tikzcd} \mathcal{F}_{C_1}(s(C_1), s(C_1)) \arrow[r, "c_{o_{C_0},\tau}(T)_*^{-1}"] \arrow[d, "E_{o_{C_1},o_{C_1}}(T)_T"] & \mathcal{F}_{C_1}(s(C_1), \tau^*(s(C_0))) \arrow[r, "c_{o_{C_0},\sigma}(T)^*"] & \mathcal{F}_{C_1}(\sigma^*(s(C_0)), \tau^*(s(C_0))) \arrow[d, "E_{\sigma,\tau}(s(C_0))_{s(C_0)}"] \\ \mathcal{F}_1(T, T^{[o_{C_1},o_{C_1}]}) \arrow[r, "o_{C_0}^*"] & \mathcal{F}_{C_0}(s(C_0), o_{C_0}^*(T^{[o_{C_1},o_{C_1}]})) \arrow[r, "\omega(\sigma,\tau;o_{C_0},o_{C_0})_*^T"] & \mathcal{F}_{C_0}(s(C_0), s(C_0)^{[\sigma,\tau]}) \end{tikzcd}$$

Hence we have $E_{\sigma,\tau}(s(C_0))_{s(C_0)}(s_{\boldsymbol{C}}) = \omega(\sigma,\tau; o_{C_0}, o_{C_0})^T o_{C_0}^*(E_{o_{C_1},o_{C_1}}(T)_T(id_{s(C_1)}))$. □

Proposition 3.4.4 *Let $\boldsymbol{f} = (f_0, f_1) : \boldsymbol{D} \to \boldsymbol{C}$ be an internal functor and (M, ξ) a representation of $\boldsymbol{C}$. Then,*

$$E_{\sigma',\tau'}(f_0^*(M))_{f_0^*(M)}(\xi_{\boldsymbol{f}}) = \omega(\sigma',\tau'; f_0, f_0)^M f_0^*(M^{f_1}\check{\xi}).$$

Proof. The upper rectangle of the following diagram is commutative by (1) of (1.4.7) and the lower one is commutative (1.4.30).

$$\begin{tikzcd} \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(M)) \arrow[r, "E_{\sigma,\tau}(M)_M"] \arrow[d, "f_1^\sharp"] & \mathcal{F}_{C_0}(M, M^{[\sigma,\tau]}) \arrow[d, "M_*^{f_1}"] \\ \mathcal{F}_{D_1}((f_0\sigma')^*(M), (f_0\tau')^*(M)) \arrow[r, "E_{f_0\sigma',f_0\tau'}(M)_M"] \arrow[d, "c_{f_0,\tau'}(M)_*^{-1}"] & \mathcal{F}_{C_0}(M, M^{[f_0\sigma',f_0\tau']}) \arrow[d, "f_0^*"] \\ \mathcal{F}_{D_1}((f_0\sigma')^*(M), \tau'(f_0^*(M))) \arrow[d, "c_{f_0,\sigma'}(M)^*"] & \mathcal{F}_{D_0}(f_0^*(M), f_0^*(M^{[f_0\sigma',f_0\tau']})) \arrow[d, "\omega(\sigma',\tau';f_0,f_0)_*^M"] \\ \mathcal{F}_{D_1}(\sigma'^*(f_0^*(M)), \tau'^*(f_0^*(M))) \arrow[r, "E_{\sigma',\tau'}(f_0^*(M))_{f_0^*(M)}"] & \mathcal{F}_{D_0}(f_0^*(M), f_0^*(M)^{[\sigma',\tau']}) \end{tikzcd}$$

The assertion follows from the above diagram and the definition of $\xi_{\boldsymbol{f}}$. □

The following fact is a direct consequence of (1.4.6).

Proposition 3.4.5 *Let (M, ξ) and (N, ζ) be representations of $\boldsymbol{C}$ and $\varphi : M \to N$ a morphism in $\mathcal{F}_{C_0}$. We put $\check{\xi} = E_{\sigma,\tau}(M)_M(\xi)$ and $\check{\zeta} = E_{\sigma,\tau}(N)_N(\zeta)$. Then, φ is a morphism of representations if and only if the following diagram is commutative.*

$$\begin{tikzcd} M \arrow[r, "\check{\xi}"] \arrow[d, "\varphi"] & M^{[\sigma,\tau]} \arrow[d, "\varphi^{[\sigma,\tau]}"] \\ N \arrow[r, "\check{\zeta}"] & N^{[\sigma,\tau]} \end{tikzcd}$$

For a morphism $\pi : X \to C_0$ of $\mathcal{E}$, we consider a limit $C_1 \xleftarrow{\pi_\tau} C_1 \times^\tau_{C_0} X \xrightarrow{\tau_\pi} X$ of a diagram $C_1 \xrightarrow{\tau} C_0 \xleftarrow{\pi} X$. Let $(\pi : X \to C_0,\ \alpha : C_1 \times^\tau_{C_0} X \to X)$ be an internal presheaf on $\boldsymbol{C}$. That is, the following diagrams are commutative.

$$\begin{tikzcd}
C_1 \times^\tau_{C_0} X \arrow[r, "\alpha"] \arrow[d, "\pi_\tau"] & X \arrow[d, "\pi"] \\
C_1 \arrow[r, "\sigma"] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \times_{C_0} C_1 \times^\tau_{C_0} X \arrow[r, "id_{C_1}\times\alpha"] \arrow[d, "\mu\times id_X"] & C_1 \times^\tau_{C_0} X \arrow[d, "\alpha"] \\
C_1 \times^\tau_{C_0} X \arrow[r, "\alpha"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \times^\tau_{C_0} X \arrow[r, "\alpha"] & X \\
C_0 \times^\tau_{C_0} X \arrow[u, "\varepsilon\times id_X"] \arrow[ur, "\mathrm{pr}_2"'] &
\end{tikzcd}$$

Let $C_1 \times^\tau_{C_0} X \xleftarrow{\bar{\mathrm{pr}}_{23}} C_1 \times_{C_0} C_1 \times^\tau_{C_0} X \xrightarrow{\bar{\mathrm{pr}}_{12}} C_1 \times_{C_0} C_1$ be a limit of $C_1 \times^\tau_{C_0} X \xrightarrow{\pi_\tau} C_1 \xleftarrow{\mathrm{pr}_2} C_1 \times_{C_0} C_1$. Then, $X \xleftarrow{\tau_\pi \bar{\mathrm{pr}}_{23}} C_1 \times_{C_0} C_1 \times^\tau_{C_0} X \xrightarrow{\bar{\mathrm{pr}}_{12}} C_1 \times_{C_0} C_1$ is a limit of $X \xrightarrow{\pi} C_0 \xleftarrow{\tau\mathrm{pr}_2} C_1 \times_{C_0} C_1$. We also note that $C_1 \times^\tau_{C_0} X \xleftarrow{\bar{\mathrm{pr}}_{23}} C_1 \times_{C_0} C_1 \times^\tau_{C_0} X \xrightarrow{\mathrm{pr}_1\bar{\mathrm{pr}}_{12}} C_1$ is a limit of $C_1 \times^\tau_{C_0} X \xrightarrow{\sigma\pi_\tau} C_0 \xleftarrow{\tau} C_1$.

$$\begin{tikzcd}
C_1 \times_{C_0} C_1 \times^\tau_{C_0} X \arrow[r, "\bar{\mathrm{pr}}_{23}"] \arrow[d, "\bar{\mathrm{pr}}_{12}"] & C_1 \times^\tau_{C_0} X \arrow[r, "\tau_\pi"] \arrow[d, "\pi_\tau"] & X \arrow[d, "\pi"] \\
C_1 \times_{C_0} C_1 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & C_1 \arrow[r, "\tau"] \arrow[d, "\sigma"] & C_0 \\
C_1 \arrow[r, "\tau"] & C_0 &
\end{tikzcd}$$

Define a functor $D^\alpha : \mathcal{P} \to \mathcal{E}$ by $D^\alpha(0) = C_1 \times^\tau_{C_0} X$, $D^\alpha(1) = X$, $D^\alpha(2) = C_1$, $D^\alpha(3) = D^\alpha(4) = D^\alpha(5) = C_0$ and $D^\alpha(\tau_{01}) = \alpha$, $D^\alpha(\tau_{02}) = \pi_\tau$, $D^\alpha(\tau_{13}) = D^\alpha(\tau_{14}) = \pi$, $D^\alpha(\tau_{24}) = \sigma$, $D^\alpha(\tau_{25}) = \tau$. For a representation (M, ξ) of $\boldsymbol{C}$, we put $\check{\xi} = E_{\sigma,\tau}(M)_M(\xi)$. Assume that $\theta^{\sigma,\tau,\pi,\pi}(M) : (M^{[\pi,\pi]})^{[\sigma,\tau]} \to M^{[\sigma\pi_\tau,\,\pi\tau_\pi]}$ is an isomorphism and define a morphism $\check{\xi}^\alpha : M^{[\pi,\pi]} \to (M^{[\pi,\pi]})^{[\sigma,\tau]}$ to be the following composition.

$$M^{[\pi,\pi]} \xrightarrow{\check{\xi}^{[\pi,\pi]}} (M^{[\sigma,\tau]})^{[\pi,\pi]} \xrightarrow{\theta^{D^\alpha}(M)} M^{[\pi\alpha,\,\tau\pi_\tau]} = M^{[\sigma\pi_\tau,\,\pi\tau_\pi]} \xrightarrow{\theta^{\sigma,\tau,\pi,\pi}(M)^{-1}} (M^{[\pi,\pi]})^{[\sigma,\tau]}$$

Proposition 3.4.6 *Assume that $\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\pi,\pi}(M) : (M^{[\pi,\pi]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \to M^{[\sigma\mathrm{pr}_1\bar{\mathrm{pr}}_{12},\,\pi\tau_\pi\bar{\mathrm{pr}}_{23}]}$ is a monomorphism. Put $E_{\sigma,\tau}(M^{[\pi,\pi]})^{-1}_{M^{[\pi,\pi]}}(\check{\xi}^\alpha) = \xi^\alpha$. Then, $(M^{[\pi,\pi]}, \xi^\alpha)$ is a representation of $\boldsymbol{C}$ and $M^\pi : (M, \xi) \to (M^{[\pi,\pi]}, \xi^\alpha)$ is a morphism of representations.*

Proof. The left rectangle of the following diagram is commutative by (1.4.21) and the right rectangle is commutative by (1.4.25).

$$\begin{tikzcd}
(M^{[\pi,\pi]})^{[\sigma,\tau]} \arrow[r, "(M^{[\pi,\pi]})^\mu"] \arrow[d, "\theta^{\sigma,\tau,\pi,\pi}(M)"] & (M^{[\pi,\pi]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[d, "\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\pi,\pi}(M)"] & ((M^{[\pi,\pi]})^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[l, "\theta^{\sigma,\tau,\sigma,\tau}(M^{[\pi,\pi]})"'] \arrow[d, "(\theta^{\sigma,\tau,\pi,\pi}(M)^{[\sigma,\tau]})"] \\
M^{[\sigma\pi_\tau,\,\pi\tau_\pi]} \arrow[r, "M^{\mu\times_{C_0} id_X}"] & M^{[\sigma\mathrm{pr}_1\bar{\mathrm{pr}}_{12},\,\pi\tau_\pi\bar{\mathrm{pr}}_{23}]} & (M^{[\sigma\pi_\tau,\,\pi\tau_\pi]})^{[\sigma,\tau]} \arrow[l, "\theta^{\sigma,\tau,\sigma\pi_\tau,\,\pi\tau_\pi}(M)"']
\end{tikzcd}$$

Since $\pi\alpha = \sigma\pi_\tau$, $\pi_\tau(id_{C_1} \times_{C_0} \alpha) = \mathrm{pr}_1\bar{\mathrm{pr}}_{12}$ and $\alpha(id_{C_1} \times_{C_0} \alpha) = \alpha(\mu \times_{C_0} id_X)$, we can define functors $E, F : \mathcal{P} \to \mathcal{E}$ and a natural transformation $\lambda : E \to D^\alpha$ by $E(0) = F(0) = C_1 \times_{C_0} C_1 \times^\tau_{C_0} X$, $E(1) = X$, $F(1) = C_1 \times^\tau_{C_0} X$, $E(2) = C_1 \times_{C_0} C_1$, $F(2) = C_1$, $E(i) = F(i) = C_0$ for $i = 3, 4, 5$, $E(\tau_{01}) = \alpha(id_{C_1} \times_{C_0} \alpha)$, $F(\tau_{01}) = id_{C_1} \times_{C_0} \alpha$, $E(\tau_{02}) = \bar{\mathrm{pr}}_{12}$, $F(\tau_{02}) = \pi_\tau\bar{\mathrm{pr}}_{23}$, $E(\tau_{13}) = \pi$, $F(\tau_{13}) = \sigma\pi_\tau$, $E(\tau_{14}) = \pi$, $F(\tau_{14}) = \pi\tau_\pi$, $E(\tau_{24}) = \sigma\mathrm{pr}_1$, $F(\tau_{24}) = \sigma\pi_\tau$, $E(\tau_{25}) = \tau\mathrm{pr}_2$, $F(\tau_{25}) = \pi\alpha$ and $\lambda_0 = \mu \times_{C_0} id_X$, $\lambda_1 = id_X$, $\lambda_2 = \mu$, $\lambda_3 = \lambda_4 = \lambda_5 = id_{C_0}$. We also note that $\mathrm{pr}_2\bar{\mathrm{pr}}_{12} = \pi_\tau\bar{\mathrm{pr}}_{23}$. Then, the following diagram commutes by (1.4.24)

$$\begin{tikzcd}
((M^{[\sigma,\tau]})^{[\pi,\pi]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\pi,\pi}(M^{[\sigma,\tau]})"] \arrow[d, "\theta^{D^\alpha}(M)^{[\sigma,\tau]}"] & (M^{[\sigma,\tau]})^{[\sigma\pi_\tau,\,\pi\tau_\pi]} \arrow[d, "\theta^F(M)"] & ((M^{[\sigma,\tau]})^{[\sigma,\tau]})^{[\pi,\pi]} \arrow[l, "\theta^{D^\alpha}(M^{[\sigma,\tau]})"'] \arrow[d, "\theta^{\sigma,\tau,\sigma,\tau}(M)^{[\pi,\pi]}"] \\
(M^{[\pi\alpha,\,\tau\pi_\tau]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\pi\alpha,\,\tau\pi_\tau}(M)"] & M^{[\sigma\mathrm{pr}_1\bar{\mathrm{pr}}_{12},\,\tau\pi_\tau\bar{\mathrm{pr}}_{23}]} & (M^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]})^{[\pi,\pi]} \arrow[l, "\theta^E(M)"']
\end{tikzcd}$$

and the following diagram commutes by (1.4.20).

$$\begin{tikzcd}
(M^{[\sigma,\tau]})^{[\pi,\,\pi]} \arrow[r, "(M^\mu)^{[\pi,\pi]}"] \arrow[d, "\theta^{D^\alpha}(M)"] & (M^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]})^{[\pi,\pi]} \arrow[d, "\theta^E(M)"] \\
M^{[\pi\alpha,\,\tau\pi_\tau]} \arrow[r, "M^{\mu\times_{C_0} id_X}"] & M^{[\sigma\mathrm{pr}_1\bar{\mathrm{pr}}_{12},\,\tau\pi_\tau\bar{\mathrm{pr}}_{23}]}
\end{tikzcd}$$

It follows from the above facts and (1.4.19), (1.4.21), (3.4.2) that the following diagram is commutative

$$
\begin{tikzcd}
 & & M^{[\pi\alpha,\,\tau\pi_\tau]} \arrow[r, equal] \arrow[dr, "\check{\xi}^{[\pi\alpha,\,\tau\pi_\tau]}"'] & M^{[\sigma\pi_\tau,\,\pi\tau_\pi]} \arrow[dr, "\theta^{\sigma,\tau,\pi,\pi}(M)^{-1}"] & \\
 & (M^{[\sigma,\tau]})^{[\pi,\pi]} \arrow[ur, "\theta^{D^\alpha}(M)"] \arrow[dr, "(\check{\xi}^{[\sigma,\tau]})^{[\pi,\pi]}"] & & (M^{[\sigma,\tau]})^{[\pi\alpha,\,\tau\pi_\tau]} \arrow[dddl, "\theta^F(M)"'] & (M^{[\pi,\pi]})^{[\sigma,\tau]} \arrow[dd] \arrow[dd, "(\check{\xi}^{[\pi,\pi]})^{[\sigma,\tau]}"'] \\
M^{[\pi,\pi]} \arrow[ur, "\check{\xi}^{[\pi,\pi]}"] \arrow[d, "\check{\xi}^{[\pi,\pi]}"] & & ((M^{[\sigma,\tau]})^{[\sigma,\tau]})^{[\pi,\pi]} \arrow[ur, "\theta^{D^\alpha}(M^{[\sigma,\tau]})"] \arrow[d, "\theta^{\sigma,\tau,\sigma,\tau}(M)^{[\pi,\pi]}"] & & \\
(M^{[\sigma,\tau]})^{[\pi,\pi]} \arrow[rr, "(M^\mu)^{[\pi,\pi]}"] \arrow[d, "\theta^{D^\alpha}(M)"] & & (M^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]})^{[\pi,\pi]} \arrow[d, "\theta^E(M)"'] & & ((M^{[\sigma,\tau]})^{[\pi,\pi]})^{[\sigma,\tau]} \arrow[uul, "\theta^{\sigma,\tau,\pi,\pi}(M^{[\sigma,\tau]})"'] \arrow[d, "\theta^{D^\alpha}(M)^{[\sigma,\tau]}"'] \\
M^{[\pi\alpha,\,\tau\pi_\tau]} \arrow[rr, "M^{\mu\times_{C_0}id_X}"] \arrow[d, equal] & & M^{[\sigma\mathrm{pr}_1\bar{\mathrm{pr}}_{12},\,\tau\pi_\tau\bar{\mathrm{pr}}_{23}]} \arrow[d, equal] & & (M^{[\pi\alpha,\,\tau\pi_\tau]})^{[\sigma,\tau]} \arrow[ll, "\theta^{\sigma,\tau,\pi\alpha,\,\tau\pi_\tau}(M)"'] \arrow[d, equal] \\
M^{[\sigma\pi_\tau,\,\pi\tau_\pi]} \arrow[rr, "M^{\mu\times_{C_0}id_X}"] \arrow[d, "\theta^{\sigma,\tau,\pi,\pi}(M)^{-1}"] & & M^{[\sigma\mathrm{pr}_1\bar{\mathrm{pr}}_{12},\,\pi\tau_\pi\bar{\mathrm{pr}}_{23}]} & & (M^{[\sigma\pi_\tau,\,\pi\tau_\pi]})^{[\sigma,\tau]} \arrow[ll, "\theta^{\sigma,\tau,\sigma\pi_\tau,\,\pi\tau_\pi}(M)"'] \arrow[d, "(\theta^{\sigma,\tau,\pi,\pi}(M)^{[\sigma,\tau]})^{-1}"'] \\
(M^{[\pi,\pi]})^{[\sigma,\tau]} \arrow[rr, "(M^{[\pi,\pi]})^\mu"] & & (M^{[\pi,\pi]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[u, "\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\pi,\pi}(M)"] & & ((M^{[\pi,\pi]})^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[ll, "\theta^{\sigma,\tau,\sigma,\tau}(M^{[\pi,\pi]})"']
\end{tikzcd}
$$

Hence $\check{\xi}^\alpha$ make the diagram of (3.4.2) commute.

Since functors $D_{\pi,\pi,id_{C_0},id_{C_0}}, D_{id_{C_0},id_{C_0},\pi,\pi} : \mathcal{P} \to \mathcal{E}$ are given by

$$
\begin{aligned}
D_{\pi,\pi,id_{C_0},id_{C_0}}(i) &= D_{id_{C_0},id_{C_0},\pi,\pi}(j) = X \quad (i = 0,1,\ j = 0,2),\\
D_{\pi,\pi,id_{C_0},id_{C_0}}(i) &= D_{id_{C_0},id_{C_0},\pi,\pi}(j) = C_0 \quad (i = 2,3,4,5,\ j = 1,3,4,5),\\
D_{\pi,\pi,id_{C_0},id_{C_0}}(\tau_{01}) &= D_{id_{C_0},id_{C_0},\pi,\pi}(\tau_{02}) = id_X,\\
D_{\pi,\pi,id_{C_0},id_{C_0}}(\tau_{ij}) &= D_{id_{C_0},id_{C_0},\pi,\pi}(\tau_{kl}) = \pi \quad ((i,j) = (0,2),(1,3),(1,4),(k,l) = (0,1),(1,3),(1,4)),\\
D_{\pi,\pi,id_{C_0},id_{C_0}}(\tau_{2j}) &= D_{id_{C_0},id_{C_0},\pi,\pi}(\tau_{2j}) = id_{C_0} \quad (j = 3,4,5),
\end{aligned}
$$

we define natural transformations $\nu : D_{id_{C_0},id_{C_0},\pi,\pi} \to D_{\sigma,\tau,\pi,\pi}$ and $\kappa : D_{\pi,\pi,id_{C_0},id_{C_0}} \to D^\alpha$ by $\nu_0 = \kappa_0 = (\varepsilon\pi, id_X) : X \to C_1 \times^\tau_{C_0} X$, $\nu_1 = \kappa_2 = \varepsilon$, $\nu_2 = \kappa_1 = id_X$, $\nu_i = \kappa_i = id_{C_0}$ $(i = 3,4,5)$. Then, the following diagram is commutative by (1.4.19), (1.4.21).

$$
\begin{tikzcd}
(M^{[\sigma,\tau]})^{[\pi,\pi]} \arrow[r, "\theta^{D^\alpha}(M)"] \arrow[d, "(M^\varepsilon)^{[\pi,\,\pi]}"] & M^{[\pi\alpha,\,\tau\pi_\tau]} = M^{[\sigma\pi_\tau,\,\pi\tau_\pi]} \arrow[r, "\theta^{\sigma,\tau,\pi,\pi}(M)^{-1}"] \arrow[d, "M^{(id_X,\,\varepsilon\pi)}"] & (M^{[\pi,\pi]})^{[\sigma,\tau]} \arrow[d, "(M^{[\pi,\pi]})^\varepsilon"] \\
(M^{[id_{C_0},id_{C_0}]})^{[\pi,\,\pi]} \arrow[r, "\theta^{\pi,\pi,id_{C_0},id_{C_0}}(M)"] & M^{[id_{C_0}\pi,\,\pi id_X]} = M^{[\pi id_X,\,\tau\varepsilon\pi]} \arrow[r, "\theta^{id_{C_0},id_{C_0},\pi,\pi}(M)^{-1}"] & (M^{[\pi,\pi]})^{[\sigma\varepsilon,\tau\varepsilon]}
\end{tikzcd}
$$

The lower row of the above diagram is identified with the identity morphism of $M^{[\pi,\pi]}$. Since $\check{\xi}M^\varepsilon$ is the identity morphism of M by (3.4.2), $\check{\xi}^{[\pi,\pi]}(M^\varepsilon)^{[\pi,\,\pi]}$ is the identity morphism of $M^{[\pi,\,\pi]}$. It follows from the above facts and the definition of $\check{\xi}^\alpha$ that $M^{[\pi,\pi]} = (M^{[\pi,\pi]})^{[\sigma\varepsilon,\tau\varepsilon]} \xrightarrow{(M^{[\pi,\pi]})^\varepsilon} (M^{[\pi,\pi]})^{[\sigma,\tau]} \xrightarrow{\check{\xi}^\alpha} M^{[\pi,\pi]}$ coincides with the identity morphism of $M^{[\pi,\pi]}$.

By (1.4.9) and (1.4.19), (1.4.21), the following diagram is commutative.

$$
\begin{tikzcd}
M \arrow[r, "\check{\xi}"] \arrow[d, "M^\pi"] & (M^{[\sigma,\tau]})^{[id_{C_0},id_{C_0}]} \arrow[r, "\theta^{id_{C_0},id_{C_0},\sigma,\tau}(M)"] \arrow[d, "(M^{[\sigma,\tau]})^\pi"] & M^{[id_{C_0}\sigma,\tau id_{C_1}]} = M^{[\sigma id_{C_1},id_{C_0}\tau]} \arrow[r, "\theta^{id_{C_0},id_{C_0},\sigma,\tau}(M)^{-1}"] \arrow[d, "M^{\pi_\tau}"] & (M^{[id_{C_0},id_{C_0}]})^{[\sigma,\tau]} \arrow[d, "(M^\pi)^{[\sigma,\tau]}"] \\
M^{[\pi,\pi]} \arrow[r, "\check{\xi}^{[\pi,\pi]}"] & (M^{[\sigma,\tau]})^{[\pi,\pi]} \arrow[r, "\theta^{D^\alpha}(M)"] & M^{[\pi\alpha,\,\tau\pi_\tau]} = M^{[\sigma\pi_\tau,\,\pi\tau_\pi]} \arrow[r, "\theta^{\sigma,\tau,\pi,\pi}(M)^{-1}"] & (M^{[\pi,\pi]})^{[\sigma,\tau]}
\end{tikzcd}
$$

Therefore $M^\pi : (M,\xi) \to (M^{[\pi,\pi]}, \xi^\alpha)$ is a morphism in representations by (3.4.5). $\square$

Proposition 3.4.7 *Let $\varphi : (M, \xi) \to (N, \zeta)$ be a morphism of representations of $\boldsymbol{C}$. Assume that the following left morphism is an isomorphism for $L = M, N$ and that the right morphism is a monoomorphism for $L = M, N$.*

$$\theta^{\sigma,\tau,\pi,\pi}(L) : (L^{[\pi,\pi]})^{[\sigma,\tau]} \to L^{[\sigma\pi_\tau,\, \pi\tau_\pi]}, \quad \theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\pi,\pi}(L) : (L^{[\pi,\pi]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \to L^{[\sigma\mathrm{pr}_1\bar{\mathrm{pr}}_{12},\, \pi\tau_\pi\bar{\mathrm{pr}}_{23}]}$$

Then, $\varphi^{[\pi,\pi]} : M^{[\pi,\pi]} \to N^{[\pi,\pi]}$ gives a morphism in representations from $(M^{[\pi,\pi]}, \xi^\alpha)$ to $(N^{[\pi,\pi]}, \zeta^\alpha)$.

Proof. The following diagram is commutative by (1.4.4) and (1.4.19).

$$\begin{array}{ccccccc}
M^{[\pi,\pi]} & \xrightarrow{\check{\xi}^{[\pi,\pi]}} & (M^{[\sigma,\tau]})^{[\pi,\pi]} & \xrightarrow{\theta^{D^\alpha}(M)} & M^{[\pi\alpha,\, \tau\pi_\tau]} = M^{[\sigma\pi_\tau,\, \pi\tau_\pi]} & \xrightarrow{\theta^{\sigma,\tau,\pi,\pi}(M)^{-1}} & (M^{[\pi,\pi]})^{[\sigma,\tau]} \\
\downarrow{\scriptstyle \varphi^{[\pi,\pi]}} & & \downarrow{\scriptstyle (\varphi^{[\sigma,\tau]})^{[\pi,\pi]}} & & \downarrow{\scriptstyle \varphi^{[\pi\sigma_\pi,\, \tau\pi_\sigma]}} & & \downarrow{\scriptstyle (\varphi^{[\pi,\pi]})^{[\sigma,\tau]}} \\
N^{[\pi,\pi]} & \xrightarrow{\check{\xi}^{[\pi,\pi]}} & (N^{[\sigma,\tau]})^{[\pi,\pi]} & \xrightarrow{\theta^{D^\alpha}(N)} & N^{[\pi\alpha,\, \tau\pi_\tau]} = N^{[\sigma\pi_\tau,\, \pi\tau_\pi]} & \xrightarrow{\theta^{\sigma,\tau,\pi,\pi}(N)^{-1}} & (N^{[\pi,\pi]})^{[\sigma,\tau]}
\end{array}$$

Hence the assertion follows. □

Proposition 3.4.8 *Let $(\pi : X \to C_0,\ \alpha : C_1 \times^\tau_{C_0} X \to X)$ and $(\rho : Y \to C_0,\ \beta : C_1 \times^\tau_{C_0} Y \to Y)$ be internal presheaves on $\boldsymbol{C}$ and (M, ξ) a representation of $\boldsymbol{C}$. Assume that the following left morphism is an isomorphism for $\chi = \pi, \rho$ and that the right morphism is a monomorphism for $\chi = \pi, \rho$.*

$$\theta^{\sigma,\tau,\chi,\chi}(M) : (M^{[\chi,\chi]})^{[\sigma,\tau]} \to M^{[\sigma\chi_\tau,\, \chi\tau_\chi]}, \quad \theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\chi,\chi}(M) : (M^{[\chi,\chi]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \to M^{[\sigma\mathrm{pr}_1\bar{\mathrm{pr}}_{12},\, \chi\tau_\chi\bar{\mathrm{pr}}_{23}]}$$

If a morphism $f : X \to Y$ of $\mathcal{E}$ defines a morphism in internal presheaves from $(\pi : X \to C_0, \alpha)$ to $(\rho : Y \to C_0, \beta)$, $M^f : M^{[\rho,\rho]} \to M^{[\pi,\pi]}$ is a morphism of representations from $(M^{[\rho,\rho]}, \xi^\beta)$ to $(M^{[\pi,\pi]}, \xi^\alpha)$.

Proof. Define a natural transformation $\lambda : D^\alpha \to D^\beta$ by $\lambda_0 = id_{C_1} \times_{C_0} f$, $\lambda_1 = f$, $\lambda_2 = id_{C_1}$, $\lambda_i = id_{C_0}$ $(i = 3, 4, 5)$. The following diagram is commutative by (1.4.7) and (1.4.20).

$$\begin{array}{ccccccc}
(M^{[\rho,\rho]})^{[\sigma,\tau]} & \xrightarrow{\theta^{\sigma,\tau,\rho,\rho}(M)^{-1}} & M^{[\rho\sigma_\rho,\, \tau\rho_\sigma]} = M^{[\sigma\rho_\sigma,\, \rho\beta]} & \xrightarrow{\theta^{D^\beta}(M)} & (M^{[\sigma,\tau]})^{[\rho,\rho]} & \xrightarrow{\hat{\xi}^{[\rho,\rho]}} & M^{[\rho,\rho]} \\
\downarrow{\scriptstyle (M^f)^{[\sigma,\tau]}} & & \downarrow{\scriptstyle M^{id_{C_1} \times_{C_0} f}} & & \downarrow{\scriptstyle (M^{[\sigma,\tau]})^f} & & \downarrow{\scriptstyle M^f} \\
(M^{[\pi,\pi]})^{[\sigma,\tau]} & \xrightarrow{\theta^{\sigma,\tau,\pi,\pi}(M)^{-1}} & M^{[\pi\sigma_\pi,\, \tau\pi_\sigma]} = M^{[\sigma\pi_\sigma,\, \pi\alpha]} & \xrightarrow{\theta^{D^\alpha}(M)} & (M^{[\sigma,\tau]})^{[\pi,\pi]} & \xrightarrow{\hat{\zeta}^{[\pi,\pi]}} & M^{[\pi,\pi]}
\end{array}$$

Hence the assertion follows. □

For an object M of $\mathcal{F}_{C_0}$, we define a morphism $\check{\mu}_M : M^{[\sigma,\tau]} \to (M^{[\sigma,\tau]})^{[\sigma,\tau]}$ to be the following composition assuming that $\theta^{\sigma,\tau,\sigma,\tau}(M) : (M^{[\sigma,\tau]})^{[\sigma,\tau]} \to M^{[\sigma\mathrm{pr}_1,\, \tau\mathrm{pr}_2]}$ is an isomorphism.

$$M^{[\sigma,\tau]} \xrightarrow{M^\mu} M^{[\sigma\mu,\, \tau\mu]} = M^{[\sigma\mathrm{pr}_1,\, \tau\mathrm{pr}_2]} \xrightarrow{\theta^{\sigma,\tau,\sigma,\tau}(M)^{-1}} (M^{[\sigma,\tau]})^{[\sigma,\tau]}$$

Let $C_1 \times_{C_0} C_1 \xleftarrow{\mathrm{pr}_{12}} C_1 \times_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_{23}} C_1 \times_{C_0} C_1$ be a limit of a diagram $C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_2} C_1 \xleftarrow{\mathrm{pr}_1} C_1 \times_{C_0} C_1$.

Proposition 3.4.9 *We assume that $\theta^{\sigma,\tau,\sigma,\tau}(M) : (M^{[\sigma,\tau]})^{[\sigma,\tau]} \to M^{[\sigma\mathrm{pr}_1,\, \tau\mathrm{pr}_2]}$ is an isomorphism and that $\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\sigma,\tau}(M) : (M^{[\sigma,\tau]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \to M^{[\sigma\mathrm{pr}_1\mathrm{pr}_{12},\, \tau\mathrm{pr}_2\mathrm{pr}_{23}]}$ is a monomorphism. Let us denote by μ^r_M a morphism $E_{\sigma,\tau}(M^{[\sigma,\tau]})^{-1}_{M^{[\sigma,\tau]}}(\check{\mu}_M)$ of $\mathcal{F}_{C_1}$. Then, $(M^{[\sigma,\tau]}, \mu^r_M)$ is a representation of $\boldsymbol{C}$. Moreover, if $\xi : \sigma^*(M) \to \tau^*(M)$ is a morphism in $\mathcal{F}_{C_1}$ such that (M, ξ) is a representation of $\boldsymbol{C}$, then $\check{\xi} = E_{\sigma,\tau}(M)_M(\xi) : M \to M^{[\sigma,\tau]}$ defines a morphism of representations from (M, ξ) to $(M^{[\sigma,\tau]}, \mu^r_M)$.*

Proof. The following diagram is commutative by (1.4.21) and (1.4.25).

$$\begin{array}{ccccc}
M^{[\sigma,\tau]} & \xrightarrow{M^\mu} & M^{[\sigma\mu,\, \tau\mu]} = M^{[\sigma\mathrm{pr}_1,\, \tau\mathrm{pr}_2]} & \xrightarrow{\theta^{\sigma,\tau,\sigma,\tau}(M)^{-1}} & (M^{[\sigma,\tau]})^{[\sigma,\tau]} \\
{\scriptstyle M^\mu}\downarrow & & \downarrow{\scriptstyle M^{\mu \times_{C_0} id_{C_1}}} & & \downarrow{\scriptstyle (M^{[\sigma,\tau]})^\mu} \\
M^{[\sigma\mu,\, \tau\mu]} & \xrightarrow{M^{id_{C_1} \times_{C_0} \mu}} & M^{[\sigma\mu\mathrm{pr}_{12},\, \tau\mu\mathrm{pr}_{23}]} & \xrightarrow{\theta^{\sigma\mu,\tau\mu,\sigma,\tau}(M)^{-1}} & (M^{[\sigma,\tau]})^{[\sigma\mu,\, \tau\mu]} \\
\| & & \| & & \| \\
M^{[\sigma\mathrm{pr}_1,\, \tau\mathrm{pr}_2]} & \xrightarrow{M^{id_{C_1} \times_{C_0} \mu}} & M^{[\sigma\mathrm{pr}_1\mathrm{pr}_{12},\, \tau\mathrm{pr}_2\mathrm{pr}_{23}]} & \xrightarrow{\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\sigma,\tau}(M)^{-1}} & (M^{[\sigma,\tau]})^{[\sigma\mathrm{pr}_1,\, \tau\mathrm{pr}_2]} \\
\downarrow{\scriptstyle \theta^{\sigma,\tau,\sigma,\tau}(M)^{-1}} & & \uparrow{\scriptstyle \theta^{\sigma,\tau,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M)} & & \uparrow{\scriptstyle \theta^{\sigma,\tau,\sigma,\tau}(M^{[\sigma,\tau]})} \\
(M^{[\sigma,\tau]})^{[\sigma,\tau]} & \xrightarrow{(M^\mu)^{[\sigma,\tau]}} & (M^{[\sigma\mu,\, \tau\mu]})^{[\sigma,\tau]} = (M^{[\sigma\mathrm{pr}_1,\, \tau\mathrm{pr}_2]})^{[\sigma,\tau]} & \xrightarrow{(\theta^{\sigma,\tau,\sigma,\tau}(M)^{[\sigma,\tau]})^{-1}} & ((M^{[\sigma,\tau]})^{[\sigma,\tau]})^{[\sigma,\tau]}
\end{array}$$

Since the functor $D_{id_{C_0},id_{C_0},\sigma,\tau} : \mathcal{P} \to \mathcal{E}$ are given by

$$\begin{array}{ll} D_{id_{C_0},id_{C_0},\sigma,\tau}(i) = C_1 \quad (i = 0, 2), & D_{id_{C_0},id_{C_0},\sigma,\tau}(i) = C_0 \quad (i = 1, 3, 4, 5), \\ D_{id_{C_0},id_{C_0},\sigma,\tau}(\tau_{01}) = D_{id_{C_0},id_{C_0},\sigma,\tau}(\tau_{24}) = \sigma, & D_{id_{C_0},id_{C_0},\sigma,\tau}(\tau_{02}) = id_{C_1} \\ D_{id_{C_0},id_{C_0},\sigma,\tau}(\tau_{13}) = D_{id_{C_0},id_{C_0},\sigma,\tau}(\tau_{14}) = id_{C_0}, & D_{id_{C_0},id_{C_0},\sigma,\tau}(\tau_{25}) = \tau, \end{array}$$

we define a natural transformations $\nu : D_{id_{C_0},id_{C_0},\sigma,\tau} \to D_{\sigma,\tau,\sigma,\tau}$ by $\nu_0 = (\varepsilon\sigma, id_{C_1}) : C_1 \to C_1 \times_{C_0} C_1$, $\nu_1 = \varepsilon$, $\nu_2 = id_{C_1}$, $\nu_i = \kappa_i = id_{C_0}$ $(i = 3, 4, 5)$. Then, the following diagram is commutative by (1.4.19), (1.4.7).

$$\begin{array}{ccccccc} M^{[\sigma,\tau]} & \xrightarrow{M^{\mu}} & M^{[\sigma\mu,\tau\mu]} & = & M^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} & \xrightarrow{\theta^{\sigma,\tau,\sigma,\tau}(M)^{-1}} & (M^{[\sigma,\tau]})^{[\sigma,\tau]} \\ \downarrow{\scriptstyle id_{M^{[\sigma,\tau]}}} & & & & \downarrow{\scriptstyle M^{[\varepsilon\sigma, id_{C_1}]}} & & \downarrow{\scriptstyle (M^{[\sigma,\tau]})^{\varepsilon}} \\ M^{[\sigma,\tau]} & \xrightarrow{M^{id_{C_1}}} & M^{[\sigma id_{C_1},\,\tau id_{C_1}]} & = & M^{[id_{C_0}\sigma,\,\tau id_{C_1}]} & \xrightarrow{\theta^{id_{C_0},id_{C_0},\sigma,\tau}(M)^{-1}} & (M^{[\sigma,\tau]})^{[\sigma\varepsilon,\tau\varepsilon]} \end{array}$$

The lower row of the above diagram is identified with the identity morphism of $M^{[\sigma,\tau]}$ which implies that $\check{\mu}_M(M^{[\sigma,\tau]})^{\varepsilon}$ is the identity morphism of $M^{[\sigma,\tau]}$. Thus $(M^{[\sigma,\tau]}, \mu^r_M)$ is a representation of $\boldsymbol{C}$ by (3.4.2).

If (M, ξ) is a representation of $\boldsymbol{C}$, then, $\theta^{\sigma,\tau,\sigma,\tau}(M)\check{\xi}^{[\sigma,\tau]}\check{\xi} = M^{\mu}\check{\xi}$ by (3.4.2). Hence $\check{\xi}^{[\sigma,\tau]}\check{\xi} = \check{\mu}_M\check{\xi}$ by the definition of $\check{\mu}_M$ and it follows from (3.4.5) that $\check{\xi}$ defines a morphism in representations from (M, ξ) to $(M^{[\sigma,\tau]}, \mu^r_M)$. $\square$

Proposition 3.4.10 *Assume that $\theta^{\sigma,\tau,\sigma,\tau}(L) : (L^{[\sigma,\tau]})^{[\sigma,\tau]} \to L^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]}$ is an isomorphism for $L = M, N$ and that $\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\sigma,\tau}(L) : (L^{[\sigma,\tau]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \to L^{[\sigma\mathrm{pr}_1\mathrm{pr}_{12},\,\tau\mathrm{pr}_2\mathrm{pr}_{23}]}$ is a monomorphism for $L = M, N$. For a morphism $\varphi : M \to N$, $\varphi^{[\sigma,\tau]} : M^{[\sigma,\tau]} \to N^{[\sigma,\tau]}$ defines a morphism of representations from $(M^{[\sigma,\tau]}, \mu^r_M)$ to $(N^{[\sigma,\tau]}, \mu^r_N)$.*

Proof. The following diagram is commtative by (1.4.9) and (1.4.21).

$$\begin{array}{ccccccc} M^{[\sigma,\tau]} & \xrightarrow{M^{\mu}} & M^{[\sigma\mu,\tau\mu]} & = & M^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} & \xrightarrow{\theta^{\sigma,\tau,\sigma,\tau}(M)^{-1}} & (M^{[\sigma,\tau]})^{[\sigma,\tau]} \\ \downarrow{\scriptstyle \varphi^{[\sigma,\tau]}} & & & & \downarrow{\scriptstyle \varphi^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]}} & & \downarrow{\scriptstyle (\varphi^{[\sigma,\tau]})^{[\sigma,\tau]}} \\ N^{[\sigma,\tau]} & \xrightarrow{N^{\mu}} & N^{[\sigma\mu,\tau\mu]} & = & N^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} & \xrightarrow{\theta^{\sigma,\tau,\sigma,\tau}(N)^{-1}} & (N^{[\sigma,\tau]})^{[\sigma,\tau]} \end{array}$$

Hence the assertion follows from (3.4.5). $\square$

Remark 3.4.11 *If $\varphi : (M, \xi) \to (N, \zeta)$ is a morphism of representations of $\boldsymbol{C}$, we have the following commutative diagram in $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$.*

$$\begin{array}{ccc} (M, \xi) & \xrightarrow{\hat{\xi}} & (M^{[\sigma,\tau]}, \mu^r_M) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi^{[\sigma,\tau]}} \\ (N, \zeta) & \xrightarrow{\hat{\zeta}} & (N^{[\sigma,\tau]}, \mu^r_N) \end{array}$$

Theorem 3.4.12 *Let M be an object of $\mathcal{F}_{C_0}$ and (N, ζ) a representation of $\boldsymbol{C}$. Assume that $\theta^{\sigma,\tau,\sigma,\tau}(L) : (L^{[\sigma,\tau]})^{[\sigma,\tau]} \to L^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]}$ is an isomorphism for $L = M, N$ and that $\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\sigma,\tau}(L) : (L^{[\sigma,\tau]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \to L^{[\sigma\mathrm{pr}_1\mathrm{pr}_{12},\,\tau\mathrm{pr}_2\mathrm{pr}_{23}]}$ is a monomorphism for $L = M, N$. Then, a map*

$$\Phi : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M, \xi), (N^{[\sigma,\tau]}, \mu^r_N)) \to \mathcal{F}_{C_0}(M, N)$$

defined by $\Phi(\varphi) = N^{\varepsilon}\varphi$ is bijective. Hence, if $\theta^{\sigma,\tau,\sigma,\tau}(L)$ an isomorphism and $\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\sigma,\tau}(L)$ is a monomorphism for all $L \in \mathrm{Ob}\,\mathcal{F}_{C_0}$, a functor $\mathscr{R}_{\boldsymbol{C}} : \mathcal{F}_{C_0} \to \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ defined by $\mathscr{R}_{\boldsymbol{C}}(N) = (N^{[\sigma,\tau]}, \mu^r_N)$ for $N \in \mathrm{Ob}\,\mathcal{F}_{C_0}$ and $\mathscr{R}_{\boldsymbol{C}}(\varphi) = \varphi^{[\sigma,\tau]}$ for $\varphi \in \mathrm{Mor}\mathcal{F}_{C_0}$ is a right adjoint of the forgetful functor $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$.

Proof. We put $\check{\xi} = E_{\sigma,\tau}(M)_M(\xi) : M \to M^{[\sigma,\tau]}$. For $\psi \in \mathcal{F}_{C_0}(M, N)$, it follows from (3.4.10) that we have a morphism $\psi^{[\sigma,\tau]} : (M^{[\sigma,\tau]}, \mu^r_M) \to (N^{[\sigma,\tau]}, \mu^r_N)$ of representations. Since $\check{\xi} : (M, \xi) \to (M^{[\sigma,\tau]}, \mu^r_M)$ is a morphism of representations by (3.4.9), $\psi^{[\sigma,\tau]}\check{\xi} : (M, \xi) \to (N^{[\sigma,\tau]}, \mu^r_N)$ is a morphism of representations. It follows from (1.4.9) and (3.4.2) that we have $\Phi(\psi^{[\sigma,\tau]}\check{\xi}) = N^{\varepsilon}\psi^{[\sigma,\tau]}\check{\xi} = \psi M^{\varepsilon}\check{\xi} = \psi$. On the other hand, for $\varphi \in \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M, \xi), (N^{[\sigma,\tau]}, \mu^r_N))$, since $\varphi^{[\sigma,\tau]}\check{\xi} = \check{\mu}_N\varphi = N^{\mu}\theta^{\sigma,\tau,\sigma,\tau}(N)^{-1}\varphi$ by (3.4.5) and the following diagram commutes by (1.4.7) and (1.4.21),

$$\begin{tikzcd}
(N^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\sigma,\tau}(N)"] \arrow[d, "(N^{\varepsilon})^{[\sigma,\tau]}"] & N^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \arrow[r, equal] \arrow[d, "N^{(id_{C_1},\,\varepsilon\tau)}"] & N^{[\sigma\mu,\,\tau\mu]} \\
(N^{[id_{C_0},\,id_{C_0}]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,id_{C_0},id_{C_0}}(N)"] & N^{[id_{C_0}\sigma,\,\tau id_{C_1}]} & N^{[\sigma,\tau]} \arrow[l, "id_{N^{[\sigma,\tau]}}"'] \arrow[u, "N^{\mu}"']
\end{tikzcd}$$

we have $(N^{\varepsilon}\varphi)^{[\sigma,\tau]}\check{\xi} = (N^{\varepsilon})^{[\sigma,\tau]}\varphi^{[\sigma,\tau]}\check{\xi} = (N^{\varepsilon})^{[\sigma,\tau]}\theta^{\sigma,\tau,\sigma,\tau}(N)^{-1}N^{\mu}\varphi = \varphi$ by (1.4.4) and (1.4.26). Therefore a correspondence $\psi \mapsto \psi^{[\sigma,\tau]}\check{\xi}$ gives the inverse map of Φ. $\square$

For morphisms $f : X \to Y$ and $g : X \to Z$ of $\mathcal{E}$, we denote by $[f,g]^* : \mathcal{F}_Z \to \mathcal{F}_Y$ the functor defined by $[f,g]^*(N) = N^{[f,g]}$ for $N \in \mathrm{Ob}\,\mathcal{F}_Z$ and $[f,g]^*(\varphi) = \varphi^{[f,g]}$ for $\varphi \in \mathrm{Mor}\,\mathcal{F}_Z$.

Proposition 3.4.13 *Let (N,ξ) and (N,ζ) be representations of $\boldsymbol{C}$ on $N \in \mathrm{Ob}\,\mathcal{F}_{C_0}$. We put $\check{\xi} = E_{\sigma,\tau}(N)_N(\xi)$ and $\check{\zeta} = E_{\sigma,\tau}(N)_N(\zeta)$. Assume that $[\sigma,\tau]^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_0}$ preserves equalizers ((σ,τ) is a left fibered representable pair, for example. See (1.5.2).) and that $\theta^{\sigma,\tau,\sigma,\tau}(N)$ is a monomorphism. Let $\iota_{\xi,\zeta} : N^{(\xi:\zeta)} \to N$ be an equalizer of $\check{\xi},\check{\zeta} : N \to N^{[\sigma,\tau]}$.*

(1) There exists unique morphism $\check{\lambda} : (N^{(\xi:\zeta)})^{[\sigma,\tau]} \to N^{(\xi:\zeta)}$ that makes the following diagram commute.

$$\begin{tikzcd}
N \arrow[d, "\check{\xi}"] & N^{(\xi:\zeta)} \arrow[l, "\iota_{\xi,\zeta}"'] \arrow[r, "\iota_{\xi,\zeta}"] \arrow[d, "\check{\lambda}"] & N \arrow[d, "\check{\zeta}"] \\
N^{[\sigma,\tau]} & (N^{(\xi:\zeta)})^{[\sigma,\tau]} \arrow[l, "(\iota_{\xi,\zeta})^{[\sigma,\tau]}"'] \arrow[r, "(\iota_{\xi,\zeta})^{[\sigma,\tau]}"] & N^{[\sigma,\tau]}
\end{tikzcd}$$

(2) Moreover, we assume that $[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_0}$ maps equalizers to monomorphisms ($(\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2)$ is a left fibered representable pair, for example. See (1.5.2).). Put $\lambda = E_{\sigma,\tau}(N^{(\xi:\zeta)})^{-1}_{N^{(\xi:\zeta)}}(\check{\lambda})$. Then, $(N^{(\xi:\zeta)},\lambda)$ is a representation of $\boldsymbol{C}$ and $\iota_{\xi,\zeta}$ defines morphisms of representations $(N^{(\xi:\zeta)},\lambda) \to (N,\xi)$ and $(N^{(\xi:\zeta)},\lambda) \to (N,\zeta)$. Hence $(N^{(\xi:\zeta)},\lambda)$ is a subrepresentation of both (N,ξ) and (N,ζ).

(3) Let (M,ν) be a representation of $\boldsymbol{C}$. Suppose that a morphism $\varphi : M \to N$ of $\mathcal{F}_{C_0}$ gives morphisms $(M,\nu) \to (N,\xi)$ and $(M,\nu) \to (N,\zeta)$ of $\mathrm{Rep}(\boldsymbol{C};\mathcal{F})$. Then, there exists unique morphism $\tilde{\varphi} : (M,\nu) \to (N_{(\xi:\zeta)},\lambda)$ of $\mathrm{Rep}(\boldsymbol{C};\mathcal{F})$ that satisfies $\iota_{\xi,\zeta}\tilde{\varphi} = \varphi$.

Proof. (1) Put $\chi = \check{\xi}\iota_{\xi,\zeta} = \check{\zeta}\iota_{\xi,\zeta} : N^{(\xi:\zeta)} \to N^{[\sigma,\tau]}$. Then, it follows from (3.4.2) that

$$\theta^{\sigma,\tau,\sigma,\tau}(N)\check{\xi}^{[\sigma,\tau]}\chi = \theta^{\sigma,\tau,\sigma,\tau}(N)\check{\xi}^{[\sigma,\tau]}\check{\xi}\iota_{\xi,\zeta} = N^{\mu}\check{\xi}\iota_{\xi,\zeta} = N^{\mu}\check{\zeta}\iota_{\xi,\zeta} = \theta^{\sigma,\tau,\sigma,\tau}(N)\check{\xi}^{[\sigma,\tau]}\check{\zeta}\iota_{\xi,\zeta} = \theta^{\sigma,\tau,\sigma,\tau}(N)\check{\xi}^{[\sigma,\tau]}\chi,$$

which implies $\check{\xi}^{[\sigma,\tau]}\chi = \check{\zeta}^{[\sigma,\tau]}\chi$ since $\theta^{\sigma,\tau,\sigma,\tau}(N)$ is a monomorphism. Since $(\iota_{\xi,\zeta})^{[\sigma,\tau]} : (N^{(\xi:\zeta)})^{[\sigma,\tau]} \to N^{[\sigma,\tau]}$ is an equalizer of $\check{\xi}^{[\sigma,\tau]},\check{\zeta}^{[\sigma,\tau]} : N^{[\sigma,\tau]} \to (N^{[\sigma,\tau]})^{[\sigma,\tau]}$ by the assumption, there exists unique morphism $\check{\lambda} : N^{(\xi:\zeta)} \to (N^{(\xi:\zeta)})^{[\sigma,\tau]}$ that satisfies $(\iota_{\xi,\zeta})^{[\sigma,\tau]}\check{\lambda} = \chi$.

(2) By (1.4.4), (1.4.7), (1.4.21) and (3.4.2), the following diagrams are commutative.

$$\begin{tikzcd}
N^{(\xi:\zeta)} \arrow[r, "\check{\lambda}"] \arrow[d, "\iota_{\xi,\zeta}"] & (N^{(\xi:\zeta)})^{[\sigma,\tau]} \arrow[r, "\check{\lambda}^{[\sigma,\tau]}"] \arrow[d, "(\iota_{\xi,\zeta})^{[\sigma,\tau]}"] & ((N^{(\xi:\zeta)})^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\sigma,\tau}(N^{(\xi:\zeta)})"] \arrow[d, "((\iota_{\xi,\zeta})^{[\sigma,\tau]})^{[\sigma,\tau]}"] & (N^{(\xi:\zeta)})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[d, "(\iota_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}"] \\
N \arrow[r, "\check{\xi}"] & N^{[\sigma,\tau]} \arrow[r, "\check{\xi}^{[\sigma,\tau]}"] & (N^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\sigma,\tau}(N)"] & N_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}
\end{tikzcd}$$

$$\begin{tikzcd}
N^{(\xi:\zeta)} \arrow[r, "\check{\lambda}"] \arrow[d, "\iota_{\xi,\zeta}"] & (N^{(\xi:\zeta)})^{[\sigma,\tau]} \arrow[r, "(N^{(\xi:\zeta)})^{\mu}"] \arrow[d, "(\iota_{\xi,\zeta})^{[\sigma,\tau]}"] & (N^{(\xi:\zeta)})^{[\sigma\mu,\tau\mu]} \arrow[r, equal] \arrow[d, "(\iota_{\xi,\zeta})^{[\sigma\mu,\tau\mu]}"] & (N^{(\xi:\zeta)})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[d, "(\iota_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}"] \\
N \arrow[r, "\check{\xi}"] & N^{[\sigma,\tau]} \arrow[r, "N^{\mu}"] & N^{[\sigma\mu,\tau\mu]} \arrow[r, equal] & N_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}
\end{tikzcd}$$

$$\begin{tikzcd}
N^{(\xi:\zeta)} \arrow[r, "\check{\lambda}"] \arrow[d, "\iota_{\xi,\zeta}"] & (N^{(\xi:\zeta)})^{[\sigma,\tau]} \arrow[r, "(N^{(\xi:\zeta)})^{\varepsilon}"] \arrow[d, "(\iota_{\xi,\zeta})^{[\sigma,\tau]}"] & (N^{(\xi:\zeta)})^{[\sigma\varepsilon,\tau\varepsilon]} \arrow[r, equal] \arrow[d, "(\iota_{\xi,\zeta})^{[\sigma\varepsilon,\tau\varepsilon]}"] & N^{(\xi:\zeta)} \arrow[d, "\iota_{\xi,\zeta}"] \\
N \arrow[r, "\check{\xi}"] & N^{[\sigma,\tau]} \arrow[r, "N^{\varepsilon}"] & N^{[\sigma\varepsilon,\tau\varepsilon]} \arrow[r, equal] & N
\end{tikzcd}$$

It follows from (3.4.2) that we have

$$(\iota_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}\theta^{\sigma,\tau,\sigma,\tau}(N^{(\xi:\zeta)})\check{\lambda}^{[\sigma,\tau]}\check{\lambda} = \theta^{\sigma,\tau,\sigma,\tau}(N)\check{\xi}^{[\sigma,\tau]}\check{\xi}\iota_{\xi,\zeta} = N^{\mu}\check{\xi}\iota_{\xi,\zeta} = (\iota_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}(N^{(\xi:\zeta)})^{\mu}\check{\lambda}$$
$$\iota_{\xi,\zeta}(N^{(\xi:\zeta)})^{\varepsilon}\check{\lambda} = N^{\varepsilon}\check{\xi}\iota_{\xi,\zeta} = \iota_{\xi,\zeta}$$

Since $\iota_{\xi,\zeta}$ and $(\iota_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$ are monomorphisms, it follows that $\theta^{\sigma,\tau,\sigma,\tau}(N^{(\xi:\zeta)})\check{\lambda}^{[\sigma,\tau]}\check{\lambda} = (N^{(\xi:\zeta)})^{\mu}\check{\lambda}$ and $N^{\varepsilon}\check{\xi}\iota_{\xi,\zeta} = id_{N^{(\xi:\zeta)}}$. Therefore λ is a representation of $\boldsymbol{C}$ on $N^{(\xi:\zeta)}$ by (3.4.2). $\iota_{\xi,\zeta} : (N^{(\xi:\zeta)},\lambda) \to (N,\xi)$ and $\iota_{\xi,\zeta} : (N^{(\xi:\zeta)},\lambda) \to (N,\zeta)$ are morphisms of representations by the first assertion and (1.4.6).

(3) Put $\check{\nu} = E_{\sigma,\tau}(N)_N(\nu)$. Since $\varphi\check{\xi} = \check{\nu}\varphi^{[\sigma,\tau]} = \varphi\check{\zeta}$ by (3.4.5), there exists unique morphism $\tilde{\varphi} : M \to N^{(\xi:\zeta)}$ that satisfies $\iota_{\xi,\zeta}\tilde{\varphi} = \varphi$. Then, we have $(\iota_{\xi,\zeta})^{[\sigma,\tau]}\check{\lambda}\tilde{\varphi} = \check{\xi}\iota_{\xi,\zeta}\tilde{\varphi} = \check{\xi}\varphi = \varphi^{[\sigma,\tau]}\check{\nu} = (\iota_{\xi,\zeta})^{[\sigma,\tau]}\tilde{\varphi}^{[\sigma,\tau]}\check{\nu}$. Since $(\iota_{\xi,\zeta})^{[\sigma,\tau]}$ is a monomorphism, it follows $\check{\lambda}\tilde{\varphi} = \tilde{\varphi}^{[\sigma,\tau]}\check{\nu}$, which implies that $\tilde{\varphi}$ gives a morphism $(M,\nu) \to (N^{(\xi:\zeta)},\lambda)$ of representations of $\boldsymbol{C}$. □

Remark 3.4.14 *Assume that one of the following conditions.*

(i) $[\sigma,\tau]^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_0}$ *preserves monomorphisms.*
(ii) $\sigma^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_1}$ *preserves monomorphisms.*
(iii) (σ,τ) *is a left fibered representable pair with respect to* $M \in \operatorname{Ob}\mathcal{F}_{C_0}$.

For representations (M,ξ), (M,ξ') *and* (N,ζ) *of* $\boldsymbol{C}$, *suppose that there exists a monomorphism* $\varphi : M \to N$ *of* $\mathcal{F}_{C_0}$ *such that* $\varphi : (M,\xi) \to (N,\zeta)$ *and* $\varphi : (M,\xi') \to (N,\zeta)$ *are morphisms in* $\mathrm{Rep}(\boldsymbol{C};\mathcal{F})$. *Then,* $\tau^*(\varphi)_* : \mathcal{F}_{C_1}(\sigma^*(M),\tau^*(M)) \to \mathcal{F}_{C_1}(\sigma^*(M),\tau^*(N))$ *is injective by the assumption. Hence* $\tau_*(\varphi)\xi = \zeta\sigma^*(\varphi) = \tau^*(\varphi)\xi'$ *implies* $\xi = \xi'$.

Proposition 3.4.15 *Let* (M,ξ), (N,ξ'), (M,ζ) *and* (N,ζ') *be objects of* $\mathrm{Rep}(\boldsymbol{C};\mathcal{F})$. *Put* $\check{\xi} = E_{\sigma,\tau}(M)_M(\xi)$, $\check{\xi}' = E_{\sigma,\tau}(N)_N(\xi')$, $\check{\zeta} = E_{\sigma,\tau}(M)_M(\zeta)$ *and* $\check{\zeta}' = E_{\sigma,\tau}(N)_N(\zeta')$. *Assume that* $[\sigma,\tau]^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_0}$ *preserves equalizers and that* $[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_0}$ *map equalizers to monomorphisms (e.g.,* (σ,τ) *and* $(\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2)$ *are left fibered representable pairs. See (1.5.2)). Suppose that* $\iota_{\xi,\zeta} : M^{(\xi:\zeta)} \to M$ *is an equalizer of* $\check{\xi},\check{\zeta} : M \to M^{[\sigma,\tau]}$ *and that* $\iota_{\xi',\zeta'} : N^{(\xi':\zeta')} \to N$ *is an equalizer of* $\check{\xi}',\check{\zeta}' : N \to N^{[\sigma,\tau]}$. *We denote by* $(M^{(\xi:\zeta)},\lambda)$ *and* $(N^{(\xi':\zeta')},\lambda')$ *the representations of* $\boldsymbol{C}$ *given in (3.4.13). If a morphism* $\varphi : M \to N$ *defines morphisms of representations* $(M,\xi) \to (N,\xi')$ *and* $(M,\zeta) \to (N,\zeta')$, *then there exists unique morphism* $\tilde{\varphi} : (M^{(\xi:\zeta)},\lambda) \to (N^{(\xi':\zeta')},\lambda')$ *of representations that satisfies* $\iota_{\xi',\zeta'}\tilde{\varphi} = \varphi\iota_{\xi,\zeta}$.

Proof. Since $\iota_{\xi,\zeta} : M^{(\xi:\zeta)} \to M$ defines morphisms $(M^{(\xi:\zeta)},\lambda) \to (M,\xi)$, $(M^{(\xi:\zeta)},\lambda) \to (M,\zeta)$ of representations of $\boldsymbol{C}$, $\varphi\iota_{\xi,\zeta} : M^{(\xi:\zeta)} \to N$ defines morphisms $(M^{(\xi:\zeta)},\lambda) \to (N,\xi')$, $(M^{(\xi:\zeta)},\lambda) \to (N,\zeta')$ of representations of $\boldsymbol{C}$. Hence it follows from (3) of (3.4.15) that there exists unique morphism $\tilde{\varphi} : M^{(\xi:\zeta)} \to N^{(\xi':\zeta')}$ that satisfies $\iota_{\xi',\zeta'}\tilde{\varphi} = \varphi\iota_{\xi,\zeta}$ and gives a morphism $(M^{(\xi:\zeta)},\lambda) \to (N^{(\xi':\zeta')},\lambda')$ of representations of $\boldsymbol{C}$. □

3.5 Construction of left induced representations

Let $p : \mathcal{F} \to \mathcal{E}$ be a normalized cloven fibered category. For morphisms $f : X \to Y$, $g : X \to Z$ of $\mathcal{E}$ and an object M of $\mathcal{F}_Y$, we assume that (f,g) is a left fibered representable pair with respect to M if necessary.

Let $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ and $\boldsymbol{D} = (D_0, D_1; \sigma', \tau', \varepsilon', \mu')$ be internal categories in $\mathcal{E}$. For an internal functor $\boldsymbol{f} = (f_0, f_1) : \boldsymbol{D} \to \boldsymbol{C}$ in $\mathcal{E}$, we consider the following diagram whose rectangles are all cartesian.

$$\begin{tikzcd}
D_0 \times_{C_0} C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "id_{D_0}\times_{C_0}(\mathrm{pr}_1,\mathrm{pr}_2)"] \arrow[d, "(f_0)_{\sigma\mathrm{pr}_1(\mathrm{pr}_1,\mathrm{pr}_2)}"] & D_0 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "id_{D_0}\times_{C_0}\mathrm{pr}_1"] \arrow[d, "(f_0)_{\sigma\mathrm{pr}_1}"] & D_0 \times_{C_0} C_1 \arrow[r, "\sigma_{f_0}"] \arrow[d, "(f_0)_{\sigma}"] & D_0 \arrow[d, "f_0"] \\
C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "(\mathrm{pr}_1,\mathrm{pr}_2)"] & C_1 \times_{C_0} C_1 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & C_1 \arrow[r, "\sigma"] \arrow[d, "\tau"] & C_0 \\
& C_1 \arrow[r, "\sigma"] & C_0 &
\end{tikzcd}$$

Diagram 3.5.1

For simplicity, we set $\tilde{\mathrm{pr}}_{123} = id_{D_0} \times_{C_0} (\mathrm{pr}_1,\mathrm{pr}_2)$, $\tilde{\mathrm{pr}}_{12} = id_{D_0} \times_{C_0} \mathrm{pr}_1$, $\tilde{\mathrm{pr}}_{234} = (f_0)_{\sigma\mathrm{pr}_1(\mathrm{pr}_1,\mathrm{pr}_2)}$, $\tilde{\mathrm{pr}}_{23} = (f_0)_{\sigma\mathrm{pr}_1}$ and $\mathrm{pr}_{12} = (\mathrm{pr}_1,\mathrm{pr}_2)$. Since $id_{D_0} \times_{C_0} \mu = (\sigma_{f_0}\tilde{\mathrm{pr}}_{12}, \mu\tilde{\mathrm{pr}}_{23})$ holds, we have $\sigma_{f_0}\tilde{\mathrm{pr}}_{12} = \sigma_{f_0}(id_{D_0} \times_{C_0} \mu)$ and $\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{12} = \tau\mu\tilde{\mathrm{pr}}_{23} = \tau(f_0)_\sigma(id_{D_0} \times_{C_0} \mu)$. Let M be an object of $\mathcal{F}_{D_0}$. If

$$\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma,\tau}(M) : M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \to (M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]}$$

is an isomorphism, we define a morphism $\hat{\mu}_{\boldsymbol{f}}(M) : (M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \to M_{[\sigma_{f_0},\tau(f_0)_\sigma]}$ to be the following composition.

$$(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \xrightarrow{\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma,\tau}(M)^{-1}} M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} = M_{[\sigma_{f_0}(id_{D_0}\times_{C_0}\mu),\tau(f_0)_\sigma(id_{D_0}\times_{C_0}\mu)]} \xrightarrow{M_{id_{D_0}\times_{C_0}\mu}} M_{[\sigma_{f_0},\tau(f_0)_\sigma]}$$

We consider the following commutative diagram.

$$
\begin{tikzcd}[column sep=small]
 & & & & D_0\times_{C_0}C_1\times_{C_0}C_1\times_{C_0}C_1 \arrow[dll, "\tilde{\mathrm{pr}}_{123}"'] \arrow[drr, "\tilde{\mathrm{pr}}_{234}"] & & & & & \\
 & & D_0\times_{C_0}C_1\times_{C_0}C_1 \arrow[dl, "\tilde{\mathrm{pr}}_{12}"'] \arrow[drr, "\tilde{\mathrm{pr}}_{23}"] & & & & C_1\times_{C_0}C_1\times_{C_0}C_1 \arrow[dll, "\mathrm{pr}_{12}"'] \arrow[dr, "\mathrm{pr}_{23}"] & & & \\
 & D_0\times_{C_0}C_1 \arrow[dl, "\sigma_{f_0}"'] \arrow[dr, "(f_0)_\sigma"] & & & C_1\times_{C_0}C_1 \arrow[dll, "\mathrm{pr}_1"'] \arrow[drr, "\mathrm{pr}_2"] & & & C_1\times_{C_0}C_1 \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & & \\
D_0 & & C_1 \arrow[drr, "\tau"] & & & & C_1 \arrow[dll, "\sigma"'] \arrow[dr, "\tau"] & & C_1 \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\
 & & & & C_0 & & & C_0 & & C_0
\end{tikzcd}
$$

Diagram 3.5.2

Proposition 3.5.1 *Assume that that $\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma,\tau}(M) : M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \to (M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]}$ is an isomorphism and that $\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M) : M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12}\tilde{\mathrm{pr}}_{123},\tau\mathrm{pr}_2\mathrm{pr}_{23}\tilde{\mathrm{pr}}_{234}]} \to (M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$ is an epimorphism. We put*

$$\mu^l_{\boldsymbol{f}}(M) = P_{\sigma,\tau}(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})^{-1}_{M_{[\sigma_{f_0},\tau(f_0)_\sigma]}}(\hat{\mu}_{\boldsymbol{f}}(M)) : \sigma^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]}) \to \tau^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]}).$$

Then, $(M_{[\sigma_{f_0},\tau(f_0)_\sigma]},\mu^l_{\boldsymbol{f}}(M))$ is a representation of $\boldsymbol{C}$.

Proof. It follows from (1.3.21) that the following diagram is commutative.

$$
\begin{tikzcd}
(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma\varepsilon,\tau\varepsilon]} \arrow[r, "\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma\varepsilon,\tau\varepsilon}(M)^{-1}"] \arrow[d, "(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_\varepsilon"] & M_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[d, "M_{id_{D_0}\times_{C_0}C_1\times_{C_0}\varepsilon}"] \arrow[dr, bend left, "id_{M_{[\sigma_{f_0},\tau(f_0)_\sigma]}}"] & \\
(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \arrow[r, "\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma,\tau}(M)^{-1}"] & M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "M_{id_{D_0}\times_{C_0}\mu}"] & M_{[\sigma_{f_0},\tau(f_0)_\sigma]}
\end{tikzcd}
$$

Hence a composition $M_{[\sigma_{f_0},\tau(f_0)_\sigma]} = (M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma\varepsilon,\tau\varepsilon]} \xrightarrow{(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_\varepsilon} (M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \xrightarrow{\hat{\mu}_{\boldsymbol{f}}(M)} M_{[\sigma_{f_0},\tau(f_0)_\sigma]}$ coincides with the identity morphism of $M_{[\sigma_{f_0},\tau(f_0)_\sigma]}$.
Note that we have the following equalities.

$$
\begin{aligned}
\sigma_{f_0}\tilde{\mathrm{pr}}_{12}\tilde{\mathrm{pr}}_{123} &= \sigma_{f_0}\tilde{\mathrm{pr}}_{12}(id_{D_0}\times_{C_0} id_{C_0}\times_{C_0}\mu) = \sigma_{f_0}\tilde{\mathrm{pr}}_{12}(id_{D_0}\times_{C_0}\mu\times_{C_0} id_{C_0})\\
\tau\mathrm{pr}_2\mathrm{pr}_{23}\tilde{\mathrm{pr}}_{234} &= \tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}(id_{D_0}\times_{C_0} id_{C_0}\times_{C_0}\mu) = \tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}(id_{D_0}\times_{C_0}\mu\times_{C_0} id_{C_0})\\
\sigma_{f_0}\tilde{\mathrm{pr}}_{12} &= \sigma_{f_0}(id_{D_0}\times_{C_0}\mu)\\
\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23} &= \tau(f_0)_\sigma(id_{D_0}\times_{C_0}\mu)
\end{aligned}
$$

It follows from (2) of (1.3.7), (1.3.21) and (1.3.25) that the following diagram commutes.

$$
\begin{tikzcd}[column sep=small]
(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} & M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[l, "\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma,\tau}(M)"'] \arrow[r, "M_{id_{D_0}\times_{C_0}\mu}"] & M_{[\sigma_{f_0},\tau(f_0)_\sigma]} \\
(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[u, "(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_\mu"] \arrow[d, "\theta_{\sigma,\tau,\sigma,\tau}(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})"] & M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12}\tilde{\mathrm{pr}}_{123},\tau\mathrm{pr}_2\mathrm{pr}_{23}\tilde{\mathrm{pr}}_{234}]} \arrow[l, "\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M)"'] \arrow[u, "M_{id_{D_0}\times_{C_0}id_{C_0}\times_{C_0}\mu}"] \arrow[r, "M_{id_{D_0}\times_{C_0}\mu\times_{C_0}id_{C_0}}"] \arrow[d, "\theta_{\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23},\sigma,\tau}(M)"] & M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[u, "M_{id_{D_0}\times_{C_0}\mu}"'] \arrow[d, "\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma,\tau}(M)"'] \\
((M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]})_{[\sigma,\tau]} & (M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]})_{[\sigma,\tau]} \arrow[l, "\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma,\tau}(M)_{[\sigma,\tau]}"'] \arrow[r, "(M_{id_{D_0}\times_{C_0}\mu})_{[\sigma,\tau]}"] & (M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]}
\end{tikzcd}
$$

Thus the following diagram commutes.

$$
\begin{tikzcd}
(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[r, "(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_\mu"] \arrow[d, "\theta_{\sigma,\tau,\sigma,\tau}(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})"] & (M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \arrow[r, "\hat{\mu}_{\boldsymbol{f}}(M)"] & M_{[\sigma_{f_0},\tau(f_0)_\sigma]} \\
((M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "(\hat{\mu}_{\boldsymbol{f}}(M))_{[\sigma,\tau]}"] & (M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \arrow[ur, "\hat{\mu}_{\boldsymbol{f}}(M)"'] &
\end{tikzcd}
$$

and $\hat{\mu}_{\boldsymbol{f}}(M)$ satisfies the conditions of (3.3.2). □

Proposition 3.5.2 *Let $\varphi : M \to N$ be a morphisms in $\mathcal{F}_{D_0}$. Assume that that the following upper morphism is an isomorphism and that the lower morphism is an epimorphism for $L = M, N$.*

$$\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma,\tau}(L) : L_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \longrightarrow (L_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]}$$
$$\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(L) : L_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12}\tilde{\mathrm{pr}}_{123},\tau\mathrm{pr}_2\mathrm{pr}_{23}\tilde{\mathrm{pr}}_{234}]} \longrightarrow (L_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$$

Then, $\varphi_{[\sigma_{f_0},\tau(f_0)_\sigma]} : (M_{[\sigma_{f_0},\tau(f_0)_\sigma]}, \mu^l_{\boldsymbol{f}}(M)) \to (N_{[\sigma_{f_0},\tau(f_0)_\sigma]}, \mu^l_{\boldsymbol{f}}(N))$ is a morphism of representations of $\boldsymbol{C}$.

Proof. The following diagram is commutative by (1.3.9) and (1.3.21).

$$\begin{tikzcd}
(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \arrow[r, "\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma,\tau}(M)^{-1}"] \arrow[d, "(\varphi_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]}"] & M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "M_{id_{D_0}\times_{C_0}\mu}"] \arrow[d, "\varphi_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]}"] & M_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[d, "\varphi_{[\sigma_{f_0},\tau(f_0)_\sigma]}"] \\
(N_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \arrow[r, "\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma,\tau}(N)^{-1}"] & N_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "N_{id_{D_0}\times_{C_0}\mu}"] & N_{[\sigma_{f_0},\tau(f_0)_\sigma]}
\end{tikzcd}$$

Hence the assertion follows from (3.3.7). □

We consider the following cartesian square.

$$\begin{tikzcd}
D_1 \times_{C_0} C_1 \arrow[r, "\tilde{\mathrm{pr}}_2"] \arrow[d, "\tilde{\mathrm{pr}}_1"] & C_1 \arrow[d, "\sigma"] \\
D_1 \arrow[r, "f_0\tau'"] & C_0
\end{tikzcd}$$

There exist unique morphisms $\tau' \times_{C_0} id_{C_1} : D_1 \times_{C_0} C_1 \to D_0 \times_{C_0} C_1$ and $f_1 \times_{C_0} id_{C_1} : D_1 \times_{C_0} C_1 \to C_1 \times_{C_0} C_1$ that satisfy $\sigma_{f_0}(\tau' \times_{C_0} id_{C_1}) = \tau'\tilde{\mathrm{pr}}_1$, $(f_0)_\sigma(\tau' \times_{C_0} id_{C_1}) = \tilde{\mathrm{pr}}_2$ and $\mathrm{pr}_1(f_1 \times_{C_0} id_{C_1}) = f_1\tilde{\mathrm{pr}}_1$, $\mathrm{pr}_2(f_1 \times_{C_0} id_{C_1}) = \tilde{\mathrm{pr}}_2$.

$$\begin{tikzcd}
D_1 \times_{C_0} C_1 \arrow[dr, dashed, "\tau' \times_{C_0} id_{C_1}"] \arrow[rr, bend left, "\tilde{\mathrm{pr}}_2"] \arrow[dd, "\tilde{\mathrm{pr}}_1"] & & \\
& D_0 \times_{C_0} C_1 \arrow[r, "(f_0)_\sigma"] \arrow[d, "\sigma_{f_0}"] & C_1 \arrow[d, "\sigma"] \\
D_1 \arrow[r, "\tau'"] & D_0 \arrow[r, "f_0"] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
D_1 \times_{C_0} C_1 \arrow[dr, dashed, "f_1 \times_{C_0} id_{C_1}"] \arrow[rr, bend left, "\tilde{\mathrm{pr}}_2"] \arrow[dd, "\tilde{\mathrm{pr}}_1"] & & \\
& C_1 \times_{C_0} C_1 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & C_1 \arrow[d, "\sigma"] \\
D_1 \arrow[r, "f_1"] & C_1 \arrow[r, "\tau"] & C_0
\end{tikzcd}$$

We note that the following diagrams are cartesian.

$$\begin{tikzcd}
D_1 \times_{C_0} C_1 \arrow[r, "\tau' \times_{C_0} id_{C_1}"] \arrow[d, "\tilde{\mathrm{pr}}_1"] & D_0 \times_{C_0} C_1 \arrow[d, "\sigma_{f_0}"] \\
D_1 \arrow[r, "\tau'"] & D_0
\end{tikzcd}
\qquad
\begin{tikzcd}
D_1 \times_{C_0} C_1 \arrow[r, "f_1 \times_{C_0} id_{C_1}"] \arrow[d, "\tilde{\mathrm{pr}}_1"] & C_1 \times_{C_0} C_1 \arrow[d, "\mathrm{pr}_1"] \\
D_1 \arrow[r, "f_1"] & C_1
\end{tikzcd}$$

Since $\sigma\mu(f_1 \times_{C_0} id_{C_1}) = \sigma\mathrm{pr}_1(f_1 \times_{C_0} id_{C_1}) = \sigma f_1\tilde{\mathrm{pr}}_1 = f_0\sigma'\tilde{\mathrm{pr}}_1$, there exists unique morphism

$$(\sigma'\tilde{\mathrm{pr}}_1, \mu(f_1 \times_{C_0} id_{C_1})) : D_1 \times_{C_0} C_1 \to D_0 \times_{C_0} C_1$$

that makes the following diagram commutes.

$$\begin{tikzcd}
D_1 \times_{C_0} C_1 \arrow[rr, "f_1 \times_{C_0} id_{C_1}"] \arrow[dr, dashed, "(\sigma'\tilde{\mathrm{pr}}_1, \mu(f_1 \times_{C_0} id_{C_1}))"] \arrow[dd, "\tilde{\mathrm{pr}}_1"] & & C_1 \times_{C_0} C_1 \arrow[d, "\mu"] \\
& D_0 \times_{C_0} C_1 \arrow[r, "(f_0)_\sigma"] \arrow[d, "\sigma_{f_0}"] & C_1 \arrow[d, "\sigma"] \\
D_1 \arrow[r, "\sigma'"] & D_0 \arrow[r, "f_0"] & C_0
\end{tikzcd}$$

Hence we have $\tau(f_0)_\sigma(\sigma'\tilde{\mathrm{pr}}_1, \mu(f_1 \times_{C_0} id_{C_1})) = \tau\mu(f_1 \times_{C_0} id_{C_1}) = \tau\mathrm{pr}_2(f_1 \times_{C_0} id_{C_1}) = \tau\tilde{\mathrm{pr}}_2 = \tau(f_0)_\sigma(\tau' \times_{C_0} id_{C_1})$ which shows the following result.

Lemma 3.5.3 *The following equalities holds.*

$$\sigma'\tilde{\mathrm{pr}}_1 = \sigma_{f_0}(\sigma'\tilde{\mathrm{pr}}_1,\, \mu(f_1 \times_{C_0} id_{C_1}))$$
$$\tau(f_0)_\sigma(\tau' \times_{C_0} id_{C_1}) = \tau(f_0)_\sigma(\sigma'\tilde{\mathrm{pr}}_1,\, \mu(f_1 \times_{C_0} id_{C_1}))$$

We also consider the following cartesian square.

$$\begin{tikzcd}
D_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "\tilde{\mathrm{pr}}_3"] \arrow[d, "\tilde{\mathrm{pr}}_{12}"] & C_1 \arrow[d, "\sigma"] \\
D_1 \times_{C_0} C_1 \arrow[r, "\tau\tilde{\mathrm{pr}}_2"] & C_0
\end{tikzcd}$$

Assumption 3.5.4 *For a representation (M,ξ) of $\boldsymbol{D}$, we put $\hat{\xi} = P_{\sigma',\tau'}(M)_M : M_{[\sigma',\tau']} \to M$. We assume the following.*

(i) A coequalizer of the following morphisms in $\mathcal{F}_{C_0}$ exists.

$$M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0} id_{C_1})]} \xrightarrow{\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M)} (M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]} \xrightarrow{\hat{\xi}_{[\sigma_{f_0},\tau(f_0)_\sigma]}} M_{[\sigma_{f_0},\tau(f_0)_\sigma]}$$

$$M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0} id_{C_1})]} = M_{[\sigma_{f_0}(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1})),\,\tau(f_0)_\sigma(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1}))]} \xrightarrow{M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1}))}} M_{[\sigma_{f_0},\tau(f_0)_\sigma]}$$

(ii) Let us denote by $P^{\boldsymbol{f}}_{(M,\xi)} : M_{[\sigma_{f_0},\tau(f_0)_\sigma]} \to (M,\xi)_{\boldsymbol{f}}$ a coequalizer of the above morphisms. Then

$$\big(P^{\boldsymbol{f}}_{(M,\xi)}\big)_{[\sigma,\tau]} : (M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \to ((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]}$$

is a coequalizer of the following morphisms.

$$\big(M_{[\sigma'\tilde{\mathrm{pr}}_1,\tau(f_0)_\sigma(\tau'\times_{C_0} id_{C_1})]}\big)_{[\sigma,\tau]} \xrightarrow{\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M)_{[\sigma,\tau]}} \big((M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]}\big)_{[\sigma,\tau]} \xrightarrow{(\hat{\xi}_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]}} \big(M_{[\sigma_{f_0},\tau(f_0)_\sigma]}\big)_{[\sigma,\tau]}$$

$$\big(M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0} id_{C_1})]}\big)_{[\sigma,\tau]} \xrightarrow{(M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1}))})_{[\sigma,\tau]}} \big(M_{[\sigma_{f_0},\,\tau(f_0)_\sigma]}\big)_{[\sigma,\tau]}$$

(iii) The following map is injective.

$$(\sigma\mu)^*\big(P^{\boldsymbol{f}}_{(M,\xi)}\big)^* : \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*((M,\xi)_{\boldsymbol{f}}),(\tau\mu)^*((M,\xi)_{\boldsymbol{f}})) \to \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]}),(\tau\mu)^*((M,\xi)_{\boldsymbol{f}}))$$

(iv) $\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma,\tau}(M) : M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \to \big(M_{[\sigma_{f_0},\tau(f_0)_\sigma]}\big)_{[\sigma,\tau]}$ is an isomorphism.
(v) The following morphisms are epimorphisms.

$$\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M) : M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12}\tilde{\mathrm{pr}}_{123},\tau\mathrm{pr}_2\mathrm{pr}_{23}\tilde{\mathrm{pr}}_{234}]} \longrightarrow \big(M_{[\sigma_{f_0},\tau(f_0)_\sigma]}\big)_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$$
$$\theta_{\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0} id_{C_1}),\sigma,\tau}(M) : M_{[\sigma'\tilde{\mathrm{pr}}_1\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}(\tau'\times_{C_0} id_{C_1}\times_{C_0} id_{C_1})]} \longrightarrow \big(M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0} id_{C_1})]}\big)_{[\sigma,\tau]}$$

The following diagram commutes.

$$\begin{tikzcd}
D_0 \times_{C_0} C_1 \arrow[d, "\sigma_{f_0}"] & D_0 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[l, "\tilde{\mathrm{pr}}_{12}"'] \arrow[r, "\tilde{\mathrm{pr}}_{23}"] \arrow[d, "id_{D_0}\times_{C_0}\mu"] & C_1 \times_{C_0} C_1 \arrow[d, "\mu"] \\
D_0 & D_0 \times_{C_0} C_1 \arrow[l, "\sigma_{f_0}"'] \arrow[r, "(f_0)_\sigma"] & C_1
\end{tikzcd}$$

Hence we have $\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23} = \tau\mu\tilde{\mathrm{pr}}_{23} = \tau(f_0)_\sigma(id_{D_0}\times_{C_0}\mu)$ and $\sigma_{f_0}\tilde{\mathrm{pr}}_{12} = \sigma_{f_0}(id_{D_0}\times_{C_0}\mu)$.

Consider the following diagram whose rhombuses are all cartesian.

$$\begin{tikzcd}
 & & & & D_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[dll, "\tilde{\mathrm{pr}}_{12}"'] \arrow[drr, "\tau'\times_{C_0} id_{C_1}\times_{C_0} id_{C_1}"] & & & & \\
 & & D_1 \times_{C_0} C_1 \arrow[dl, "\tilde{\mathrm{pr}}_1"'] \arrow[drr, "\tau'\times_{C_0} id_{C_1}"] & & & & D_0 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[dll, "\tilde{\mathrm{pr}}_{12}"'] \arrow[dr, "\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}"] & & \\
 & D_1 \arrow[dl, "\sigma'"'] \arrow[dr, "\tau'"] & & & D_0 \times_{C_0} C_1 \arrow[dll, "\sigma_{f_0}"'] \arrow[drr, "\tau(f_0)_\sigma"] & & & C_1 \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\
D_0 & & D_0 & & & & C_0 & & C_0
\end{tikzcd}$$

It follows from (1.3.25) that

$$\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}(\tau'\times_{C_0}id_{C_1}\times_{C_0}id_{C_1})]} \arrow[r, "\theta_{\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1}),\sigma,\tau}(M)"] \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}}(M)"] & \big(M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]}\big)_{[\sigma,\tau]} \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M)_{[\sigma,\tau]}"] \\
(M_{[\sigma',\tau']})_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "\theta_{\sigma_{f_0},\,\tau(f_0)_\sigma,\sigma,\tau}(M_{[\sigma',\tau']})"] & \big((M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]}\big)_{[\sigma,\tau]}
\end{tikzcd}$$

is commutative. The following diagrams are commutative by (1.3.21), (1.3.19), (1.3.9), respectively.

$$\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}(\tau'\times_{C_0}id_{C_1}\times_{C_0}id_{C_1})]} \arrow[r, "M_{id_{D_1}\times_{C_0}\mu}"] \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}}(M)"] & M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]} \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M)"] \\
(M_{[\sigma',\tau']})_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "(M_{[\sigma',\tau']})_{id_{D_0}\times_{C_0}\mu}"] & (M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]}
\end{tikzcd}$$

$$\begin{tikzcd}
(M_{[\sigma',\tau']})_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "\theta_{\sigma_{f_0},\,\tau(f_0)_\sigma,\sigma,\tau}(M_{[\sigma',\tau']})"] \arrow[d, "\hat{\xi}_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]}"] & \big((M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]}\big)_{[\sigma,\tau]} \arrow[d, "(\hat{\xi}_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]}"] \\
M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "\theta_{\sigma_{f_0},\,\tau(f_0)_\sigma,\sigma,\tau}(M)"] & (M_{[\sigma_{f_0},\,\tau(f_0)_\sigma]})_{[\sigma,\tau]}
\end{tikzcd}$$

$$\begin{tikzcd}
(M_{[\sigma',\tau']})_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "(M_{[\sigma',\tau']})_{id_{D_0}\times_{C_0}\mu}"] \arrow[d, "\hat{\xi}_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]}"] & (M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[d, "\hat{\xi}_{[\sigma_{f_0},\tau(f_0)_\sigma]}"] \\
M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "M_{id_{D_0}\times_{C_0}\mu}"] & M_{[\sigma_{f_0},\,\tau(f_0)_\sigma]}
\end{tikzcd}$$

The associativity of μ implies that a diagram

$$\begin{tikzcd}
D_1\times_{C_0}C_1\times_{C_0}C_1 \arrow[r, "id_{D_1}\times_{C_0}\mu"] \arrow[d, "(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))\times_{C_0}id_{C_1}"] & D_1\times_{C_0}C_1 \arrow[d, "(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))"] \\
D_0\times_{C_0}C_1\times_{C_0}C_1 \arrow[r, "id_{D_0}\times_{C_0}\mu"] & D_0\times_{C_0}C_1
\end{tikzcd}$$

is commutative. Hence the following diagram is commutative by (1.3.7).

$$\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1\tilde{\mathrm{pr}}_{12},\,\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}(\tau'\times_{C_0}id_{C_1}\times_{C_0}id_{C_1})]} \arrow[r, "M_{id_{D_1}\times_{C_0}\mu}"] \arrow[d, "M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))\times_{C_0}id_{C_1}}"] & M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]} \arrow[d, "M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))}"] \\
M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "M_{id_{D_0}\times_{C_0}\mu}"] & M_{[\sigma_{f_0},\,\tau(f_0)_\sigma]}
\end{tikzcd}$$

Moreover, it follows from (1.3.21) that the following diagram commutes.

$$\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1\tilde{\mathrm{pr}}_{12},\,\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}(\tau'\times_{C_0}id_{C_1}\times_{C_0}id_{C_1})]} \arrow[r, "\theta_{\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1}),\sigma,\tau}(M)"] \arrow[d, "M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))\times_{C_0}id_{C_1}}"] & \big(M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]}\big)_{[\sigma,\tau]} \arrow[d, "(M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))})_{[\sigma,\tau]}"'] \\
M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "\theta_{\sigma_{f_0},\,\tau(f_0)_\sigma,\sigma,\tau}(M)"] & \big(M_{[\sigma_{f_0},\,\tau(f_0)_\sigma]}\big)_{[\sigma,\tau]}
\end{tikzcd}$$

Since $P^{\boldsymbol{f}}_{(M,\xi)}$ is a coequalizer of $\hat{\xi}_{[\sigma_{f_0},\tau(f_0)_\sigma]}\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M)$ and $M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))}$, we have

$$\begin{aligned}
&P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M)(\hat{\xi}_{[\sigma_{f_0},\tau(f_0)_\sigma]}\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M))_{[\sigma,\tau]}\theta_{\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1}),\sigma,\tau}(M)\\
&\quad= P^{\boldsymbol{f}}_{(M,\xi)}M_{id_{D_0}\times_{C_0}\mu}\theta_{\sigma_{f_0},\,\tau(f_0)_\sigma,\sigma,\tau}(M)^{-1}(\hat{\xi}_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]}\theta_{\sigma_{f_0},\,\tau(f_0)_\sigma,\sigma,\tau}(M_{[\sigma',\tau']})\theta_{\sigma',\tau',\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}}(M)\\
&\quad= P^{\boldsymbol{f}}_{(M,\xi)}M_{id_{D_0}\times_{C_0}\mu}\hat{\xi}_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]}\theta_{\sigma',\tau',\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}}(M)\\
&\quad= P^{\boldsymbol{f}}_{(M,\xi)}\hat{\xi}_{[\sigma_{f_0},\tau(f_0)_\sigma]}(M_{[\sigma',\tau']})_{id_{D_0}\times_{C_0}\mu}\theta_{\sigma',\tau',\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}}(M)\\
&\quad= P^{\boldsymbol{f}}_{(M,\xi)}\hat{\xi}_{[\sigma_{f_0},\tau(f_0)_\sigma]}\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M)M_{id_{D_1}\times_{C_0}\mu} = P^{\boldsymbol{f}}_{(M,\xi)}M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))}M_{id_{D_1}\times_{C_0}\mu}\\
&\quad= P^{\boldsymbol{f}}_{(M,\xi)}M_{id_{D_0}\times_{C_0}\mu}M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))\times_{C_0}id_{C_1}}\\
&\quad= P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M)\theta_{\sigma_{f_0},\,\tau(f_0)_\sigma,\sigma,\tau}(M)M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))\times_{C_0}id_{C_1}}\\
&\quad= P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M)(M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))})_{[\sigma,\tau]}\theta_{\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1}),\sigma,\tau}(M).
\end{aligned}$$

Therefore, it follows from the assumption (v) of (3.5.4) that we have

$$P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M)(\hat{\xi}_{[\sigma_{f_0},\tau(f_0)_\sigma]}\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M))_{[\sigma,\tau]} = P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M)(M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))})_{[\sigma,\tau]}.$$

Hence (ii) of (3.5.4) implies that there exists unique morphism $\hat{\xi}_{\boldsymbol{f}} : ((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]} \to (M,\xi)_{\boldsymbol{f}}$ that satisfies $\hat{\xi}_{\boldsymbol{f}}(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]} = P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M)$. We put $\xi^l_{\boldsymbol{f}} = P_{\sigma,\tau}((M,\xi)_{\boldsymbol{f}})^{-1}_{(M,\xi)_{\boldsymbol{f}}}(\hat{\xi}_{\boldsymbol{f}}) : \sigma^*((M,\xi)_{\boldsymbol{f}}) \to \tau^*((M,\xi)_{\boldsymbol{f}})$.

Proposition 3.5.5 *$((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}})$ is a representation of $\boldsymbol{C}$ and $P^{\boldsymbol{f}}_{(M,\xi)} : (M_{[\sigma_{f_0},\,\tau(f_0)_\sigma]},\mu^l_{\boldsymbol{f}}(M)) \to ((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}})$ is a morphism of representations of $\boldsymbol{C}$.*

Proof. It follows from (3.3.6) that $\hat{\xi}_{\boldsymbol{f}}(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]} = P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M)$ implies the commutativity of the following diagram.

$$\begin{tikzcd}
\sigma^*(M_{[\sigma_{f_0},\,\tau(f_0)_\sigma]}) \arrow[r, "\mu^l_{\boldsymbol{f}}(M)"] \arrow[d, "\sigma^*\left(P^{\boldsymbol{f}}_{(M,\xi)}\right)"] & \tau^*(M_{[\sigma_{f_0},\,\tau(f_0)_\sigma]}) \arrow[d, "\tau^*\left(P^{\boldsymbol{f}}_{(M,\xi)}\right)"] \\
\sigma^*((M,\xi)_{\boldsymbol{f}}) \arrow[r, "\xi^l_{\boldsymbol{f}}"] & \tau^*((M,\xi)_{\boldsymbol{f}})
\end{tikzcd}$$

Hence the assertion follows from (iii) of (3.5.4) and (2) of (3.1.5). □

We assume (3.5.4) also for a representation (N,ζ) of $\boldsymbol{D}$. Let $\varphi : (M,\xi) \to (N,\zeta)$ be a morphism of representations of $\boldsymbol{D}$. The following diagrams are commutative by (1.3.21), (1.3.4) and (1.3.9).

$$\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]} \arrow[r, "\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M)"] \arrow[d, "\varphi_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]}"] & (M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[r, "\hat{\xi}_{[\sigma_{f_0},\tau(f_0)_\sigma]}"] \arrow[d, "(\varphi_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]}"] & M_{[\sigma_{f_0},\,\tau(f_0)_\sigma]} \arrow[d, "\varphi_{[\sigma_{f_0},\,\tau(f_0)_\sigma]}"] \\
N_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]} \arrow[r, "\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(N)"] & (N_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[r, "\hat{\zeta}_{[\sigma_{f_0},\tau(f_0)_\sigma]}"] & N_{[\sigma_{f_0},\,\tau(f_0)_\sigma]}
\end{tikzcd}$$

$$\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]} \arrow[r, "M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))}"] \arrow[d, "\varphi_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]}"] & M_{[\sigma_{f_0},\,\tau(f_0)_\sigma]} \arrow[d, "\varphi_{[\sigma_{f_0},\,\tau(f_0)_\sigma]}"] \\
N_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]} \arrow[r, "N_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))}"] & N_{[\sigma_{f_0},\,\tau(f_0)_\sigma]}
\end{tikzcd}$$

Hence there exists unique morphism $\varphi_{\boldsymbol{f}} : (M,\xi)_{\boldsymbol{f}} \to (N,\zeta)_{\boldsymbol{f}}$ that satisfies $\varphi_{\boldsymbol{f}}P^{\boldsymbol{f}}_{(M,\xi)} = P^{\boldsymbol{f}}_{(N,\zeta)}\varphi_{[\sigma_{f_0},\,\tau(f_0)_\sigma]}$.

Proposition 3.5.6 *$\varphi_{\boldsymbol{f}} : ((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}}) \to ((N,\zeta)_{\boldsymbol{f}},\zeta^l_{\boldsymbol{f}})$ is a morphism of representations of $\boldsymbol{C}$.*

Proof. It follows from (3.5.2) that the outer rectangle of the following diagram is commutative.

$$\begin{tikzcd}
(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \arrow[rrr, "\hat{\mu}_{\boldsymbol{f}}(M)"] \arrow[dr, "(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]}"] \arrow[ddd, "(\varphi_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]}"'] & & & M_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[dl, "P^{\boldsymbol{f}}_{(M,\xi)}"'] \arrow[ddd, "\varphi_{[\sigma_{f_0},\tau(f_0)_\sigma]}"] \\
& ((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]} \arrow[r, "\hat{\xi}_{\boldsymbol{f}}"] \arrow[d, "(\varphi_{\boldsymbol{f}})_{[\sigma,\tau]}"] & (M,\xi)_{\boldsymbol{f}} \arrow[d, "\varphi_{\boldsymbol{f}}"] & \\
& ((N,\zeta)_{\boldsymbol{f}})_{[\sigma,\tau]} \arrow[r, "\hat{\zeta}_{\boldsymbol{f}}"] & (N,\zeta)_{\boldsymbol{f}} & \\
(N_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \arrow[rrr, "\hat{\mu}_{\boldsymbol{f}}(N)"] \arrow[ur, "(P^{\boldsymbol{f}}_{(N,\zeta)})_{[\sigma,\tau]}"] & & & N_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[ul, "P^{\boldsymbol{f}}_{(N,\zeta)}"']
\end{tikzcd}$$

Then, by the definitions of $\hat{\xi}_{\boldsymbol{f}}$, $\hat{\zeta}_{\boldsymbol{f}}$ and $\varphi_{\boldsymbol{f}}$, we have

$$\begin{aligned}
\varphi_{\boldsymbol{f}}\hat{\xi}_{\boldsymbol{f}}(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]} &= \varphi_{\boldsymbol{f}}P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M) = P^{\boldsymbol{f}}_{(N,\zeta)}\varphi_{[\sigma_{f_0},\tau(f_0)_\sigma]}\hat{\mu}_{\boldsymbol{f}}(M) = P^{\boldsymbol{f}}_{(N,\zeta)}\hat{\mu}_{\boldsymbol{f}}(N)(\varphi_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \\
&= \hat{\zeta}_{\boldsymbol{f}}(P^{\boldsymbol{f}}_{(N,\zeta)})_{[\sigma,\tau]}(\varphi_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} = \hat{\zeta}_{\boldsymbol{f}}(\varphi_{\boldsymbol{f}})_{[\sigma,\tau]}(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]}.
\end{aligned}$$

Since $(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]}$ is an epimorphism by (ii) of (3.5.4), the above equality implies $\varphi_{\boldsymbol{f}}\hat{\xi}_{\boldsymbol{f}} = \hat{\zeta}_{\boldsymbol{f}}(\varphi_{\boldsymbol{f}})_{[\sigma,\tau]}$. Therefore $\varphi_{\boldsymbol{f}}$ is a morphism of representations of $\boldsymbol{D}$ by (3.3.6). □

Define functors $S,T,U:\mathcal{P}\to\mathcal{E}$ and natural transformations $\alpha:S\to T$, $\beta:T\to U$ as follows.

$S(0)=D_1$	$S(1)=D_1$	$S(2)=D_0$	$S(3)=D_0$	$S(4)=D_0$	$S(5)=D_0$
$S(\tau_{01})=id_{D_1}$	$S(\tau_{02})=\tau'$	$S(\tau_{13})=\sigma'$	$S(\tau_{14})=\tau'$	$S(\tau_{24})=id_{D_0}$	$S(\tau_{25})=id_{D_0}$
$T(0)=D_1\times_{C_0}C_1$	$T(1)=D_1$	$T(2)=D_0\times_{C_0}C_1$	$T(3)=D_0$	$T(4)=D_0$	$T(5)=C_0$
$T(\tau_{01})=\tilde{\mathrm{pr}}_1$	$T(\tau_{02})=\tau'\times_{C_0}id_{C_1}$	$T(\tau_{13})=\sigma'$	$T(\tau_{14})=\tau'$	$T(\tau_{24})=\sigma_{f_0}$	$T(\tau_{25})=\tau(f_0)_\sigma$
$U(0)=D_0\times_{C_0}C_1\times_{C_0}C_1$	$U(1)=D_0\times_{C_0}C_1$	$U(2)=C_1$	$U(3)=D_0$	$U(4)=C_0$	$U(5)=C_0$
$U(\tau_{01})=\tilde{\mathrm{pr}}_{12}$	$U(\tau_{02})=\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}$	$U(\tau_{13})=\sigma_{f_0}$	$U(\tau_{14})=\tau(f_0)_\sigma$	$U(\tau_{24})=\sigma$	$U(\tau_{25})=\tau$
$\alpha_0=(id_{D_1},f_1\varepsilon'\tau')$	$\alpha_1=id_{D_1}$	$\alpha_2=(id_{D_0},f_1\varepsilon')$	$\alpha_3=id_{D_0}$	$\alpha_4=id_{D_0}$	$\alpha_5=f_0$
$\beta_0=(\sigma'\tilde{\mathrm{pr}}_1,f_1\tilde{\mathrm{pr}}_1,\tilde{\mathrm{pr}}_2)$	$\beta_1=(\sigma',f_1)$	$\beta_2=(f_0)_\sigma$	$\beta_3=id_{D_0}$	$\beta_4=f_0$	$\beta_5=id_{C_0}$

Hence if we define functors $S_i,T_i,U_i:\mathcal{Q}\to\mathcal{E}$ for $i=0,1,2$ by

$S_0(0)=S(0)$	$S_0(1)=S(3)$	$S_0(2)=S(5)$	$S_0(\tau_{01})=S(\tau_{13}\tau_{01})$	$S_0(\tau_{02})=S(\tau_{25}\tau_{02})$
$T_0(0)=T(0)$	$T_0(1)=T(3)$	$T_0(2)=T(5)$	$T_0(\tau_{01})=T(\tau_{13}\tau_{01})$	$T_0(\tau_{02})=T(\tau_{25}\tau_{02})$
$U_0(0)=U(0)$	$U_0(1)=U(3)$	$U_0(2)=U(5)$	$U_0(\tau_{01})=U(\tau_{13}\tau_{01})$	$U_0(\tau_{02})=U(\tau_{25}\tau_{02})$
$S_1(0)=S(1)$	$S_1(1)=S(3)$	$S_1(2)=S(4)$	$S_1(\tau_{01})=S(\tau_{13})$	$S_1(\tau_{02})=S(\tau_{14})$
$T_1(0)=T(1)$	$T_1(1)=T(3)$	$T_1(2)=T(4)$	$T_1(\tau_{01})=T(\tau_{13})$	$T_1(\tau_{02})=T(\tau_{14})$
$U_1(0)=U(1)$	$U_1(1)=U(3)$	$U_1(2)=U(4)$	$U_1(\tau_{01})=U(\tau_{13})$	$U_1(\tau_{02})=U(\tau_{14})$
$S_2(0)=S(2)$	$S_2(1)=S(4)$	$S_2(2)=S(5)$	$S_2(\tau_{01})=S(\tau_{24})$	$S_2(\tau_{02})=S(\tau_{25})$
$T_2(0)=T(2)$	$T_2(1)=T(4)$	$T_2(2)=T(5)$	$T_2(\tau_{01})=T(\tau_{24})$	$T_2(\tau_{02})=T(\tau_{25})$
$U_2(0)=U(2)$	$U_2(1)=U(4)$	$U_2(2)=U(5)$	$U_2(\tau_{01})=U(\tau_{24})$	$U_2(\tau_{02})=U(\tau_{25})$

and natural transformations $\alpha^i:S_i\to T_i$, $\beta^i:T_i\to U_i$ for $i=0,1,2$ by

$$\begin{matrix}
\alpha^0_0=\alpha_0 & \alpha^0_1=\alpha_3 & \alpha^0_2=\alpha_5 & \alpha^1_0=\alpha_1 & \alpha^1_1=\alpha_3 & \alpha^1_2=\alpha_4 & \alpha^2_0=\alpha_2 & \alpha^2_1=\alpha_4 & \alpha^2_2=\alpha_5, \\
\beta^0_0=\beta_0 & \beta^0_1=\beta_3 & \beta^0_2=\beta_5 & \beta^1_0=\beta_1 & \beta^1_1=\beta_3 & \beta^1_2=\beta_4 & \beta^2_0=\beta_2 & \beta^2_1=\beta_4 & \beta^2_2=\beta_5,
\end{matrix}$$

then we have $S_0=S_1=T_1$, $U_1=T_2$.

For morphisms $f:X\to Y$, $g:X\to Z$ and $k:W\to X$ of $\mathcal{E}$, we denote by $\omega(k\,;f,g):D_{fk,gk}\to D_{f,g}$ a natural transformation given by $\omega(k\,;f,g)_0=k$, $\omega(k\,;f,g)_1=id_Y$, $\omega(k\,;f,g)_2=id_Z$. We note that $\omega(k\,;f,g)_M=M_k:M_{[fk,gk]}\to M_{[f,g]}$ for $M\in\mathrm{Ob}\,\mathcal{F}_Y$ by (1.3.29).

Lemma 3.5.7 *For a representation (M,ξ) of $\boldsymbol{D}$, the following diagram is commutative.*

$$\begin{tikzcd}
M \arrow[d, equal] & M_{[\sigma',\tau']} \arrow[l, "\hat{\xi}"'] \arrow[rr, "\beta^1_M"] & & f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]}) \arrow[d, "f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})"] \\
M_{[id_{D_0},id_{D_0}]} \arrow[r, "\alpha^2_M"] & f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]}) \arrow[rr, "f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})"] & & f_0^*((M,\xi)_{\boldsymbol{f}})
\end{tikzcd}$$

Proof. The following diagram is commutative by the definition of $P^{\boldsymbol{f}}_{(M,\xi)}$.

$$\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1,\, \tau(f_0)_\sigma(\tau'\times_{C_0} id_{C_1})]} \arrow[r, "\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M)"] \arrow[d, "M_{(\sigma'\tilde{\mathrm{pr}}_1,\, \mu(f_1\times_{C_0} id_{C_1}))}"] & (M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[r, "\hat{\xi}_{[\sigma_{f_0},\tau(f_0)_\sigma]}"] & M_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[d, "P^{\boldsymbol{f}}_{(M,\xi)}"] \\
M_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[rr, "P^{\boldsymbol{f}}_{(M,\xi)}"] & & (M,\xi)_{\boldsymbol{f}}
\end{tikzcd}$$

It follows from (1.3.34) that the following diagram is commutative.

$$\begin{tikzcd}
M_{[\sigma' id_{D_1}, id_{D_0}\tau']} \arrow[rr, "\alpha^0_M"] \arrow[d, "\theta_{\sigma',\tau',id_{D_0},id_{D_0}}(M)"] & & f_0^*(M_{[\sigma'\tilde{\mathrm{pr}}_1,\tau(f_0)_\sigma(\tau'\times_{C_0} id_{C_1})]}) \arrow[d, "f_0^*(\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M))"] \\
(M_{[\sigma',\tau']})_{[id_{D_0},id_{D_0}]} \arrow[r, "(\alpha^1_M)_{[id_{D_0},id_{D_0}]}"] & (M_{[\sigma',\tau']})_{[id_{D_0},id_{D_0}]} \arrow[r, "\alpha^2_{M_{[\sigma',\tau']}}"] & f_0^*((M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]})
\end{tikzcd}$$

We note that $\theta_{\sigma',\tau',id_{D_0},id_{D_0}}(M)$ and $(\alpha^1_M)_{[id_{D_0},id_{D_0}]}$ are the identity morphism of $M_{[\sigma',\tau']}$ by (1.3.26) and the definition of α^1_M. Therefore the following diagram commutes by the commutativity of the above diagrams and (1.3.31).

$$\begin{tikzcd}
M_{[\sigma',\tau']} \arrow[r, "\theta_{\sigma',\tau',id_{D_0},id_{D_0}}(M)=id_{M_{[\sigma',\tau']}}"] \arrow[d, "\alpha^0_M"] & M_{[\sigma',\tau']} \arrow[r, "\hat{\xi}"] \arrow[d, "\alpha^2_{M_{[\sigma',\tau']}}"] & M \arrow[d, "\alpha^2_M"] \\
f_0^*(M_{[\sigma'\tilde{\mathrm{pr}}_1,\, \tau(f_0)_\sigma(\tau'\times_{C_0} id_{C_1})]}) \arrow[r, "f_0^*(\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M))"] \arrow[d, "f_0^*(M_{(\sigma'\tilde{\mathrm{pr}}_1,\, \mu(f_1\times_{C_0} id_{C_1}))})"] & f_0^*((M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]}) \arrow[r, "f_0^*(\hat{\xi}_{[\sigma_{f_0},\tau(f_0)_\sigma]})"] & f_0^*(M_{[\sigma_{f_0},\, \tau(f_0)_\sigma]}) \arrow[d, "f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})"] \\
f_0^*(M_{[\sigma_{f_0},\, \tau(f_0)_\sigma]}) \arrow[rr, "f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})"] & & f_0^*((M,\xi)_{\boldsymbol{f}})
\end{tikzcd}$$

We put $\bar{\beta} = \omega((\sigma'\tilde{\mathrm{pr}}_1, \mu(f_1 \times_{C_0} id_{C_1})\,; \sigma_{f_0}, \tau(f_0)_\sigma) : T_0 \to T_2$. Then, $\beta^1 = \bar{\beta}\alpha^0$ holds. It follows from (1.3.33) that the following diagram is commutative.

$$\begin{tikzcd}
M_{[\sigma',\tau']} \arrow[r, "\alpha^0_M"] \arrow[d, "c_{id_{D_0},id_{D_0}}(M)_{[\sigma',\tau']}=id_{M_{[\sigma',\tau']}}"] & f_0^*(M_{[\sigma'\tilde{\mathrm{pr}}_1,\, \tau(f_0)_\sigma(\tau'\times_{C_0} id_{C_1})]}) \arrow[r, "f_0^*(\bar{\beta}_M)"] & f_0^*(M_{[\sigma_{f_0},\, \tau(f_0)_\sigma]}) \arrow[d, "c_{id_{C_0},f_0}(M_{[\sigma_{f_0},\, \tau(f_0)_\sigma]})=id_{f_0^*(M_{[\sigma_{f_0},\, \tau(f_0)_\sigma]})}"'] \\
M_{[\sigma',\tau']} \arrow[rr, "\beta^1_M=(\bar{\beta}\alpha^0)_M"] & & f_0^*(M_{[\sigma_{f_0},\, \tau(f_0)_\sigma]})
\end{tikzcd}$$

Since $\bar{\beta}_M = \omega((\sigma'\tilde{\mathrm{pr}}_1, \mu(f_1 \times_{C_0} id_{C_1})\,; \sigma_{f_0}, \tau(f_0)_\sigma)_M = M_{(\sigma'\tilde{\mathrm{pr}}_1,\, \mu(f_1\times_{C_0} id_{C_1}))}$ by (1.3.29), we have

$$f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})\alpha^2_M\hat{\xi} = f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})f_0^*(M_{(\sigma'\tilde{\mathrm{pr}}_1,\, \mu(f_1\times_{C_0} id_{C_1}))})\alpha^0_M = f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})f_0^*(\bar{\beta}_M)\alpha^0_M = f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})\beta^1_M$$

□

Proposition 3.5.8 *A composition*

$$M = M_{[id_{D_0},\, id_{D_0}]} \xrightarrow{\alpha^2_M} f_0^*(M_{[\sigma_{f_0},\, \tau(f_0)_\sigma]}) \xrightarrow{f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})} f_0^*((M,\xi)_{\boldsymbol{f}})$$

defines a morphism $(M,\xi) \to (f_0^*((M,\xi)_{\boldsymbol{f}}), (\xi^l_{\boldsymbol{f}})_{\boldsymbol{f}})$ *of representations of* $\boldsymbol{D}$.

Proof. By applying (1.3.34) to $\beta : \mathcal{P} \to \mathcal{E}$, we see that the following diagram (i) is commutative.

$$\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1,\tau(f_0)_\sigma(\tau'\times_{C_0} id_{C_1})]} \arrow[rr, "\beta^0_M=M_{(\sigma'\tilde{\mathrm{pr}}_1,\, f_1\tilde{\mathrm{pr}}_1,\, \tilde{\mathrm{pr}}_2)}"] \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M)"] & & M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[d, "\theta_{\sigma_{f_0},\, \tau(f_0)_\sigma,\sigma,\tau}(M)"'] \\
(M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[r, "(\beta^1_M)_{[\sigma_{f_0},\tau(f_0)_\sigma]}"] & (f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]}))_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[r, "\beta^2_{M_{[\sigma_{f_0},\tau(f_0)_\sigma]}}"] & (M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]}
\end{tikzcd}$$

diagram (i)

Let $D_0 \xleftarrow{\hat{\mathrm{pr}}_1} D_0 \times_{C_0} D_1 \xrightarrow{\hat{\mathrm{pr}}_2} D_1$ be a limit of a diagram $D_0 \xrightarrow{f_0} C_0 \xleftarrow{\sigma f_1} D_1$. Define a natural transformation $\bar{\beta}^2 : D_{\hat{\mathrm{pr}}_1,\tau f_1 \hat{\mathrm{pr}}_2} \to D_{\sigma f_1,\tau f_1}$ by $\bar{\beta}^2_0 = \hat{\mathrm{pr}}_2$, $\bar{\beta}^2_1 = f_0$, $\bar{\beta}^2_2 = id_{C_0}$. We also consider natural transformations $\omega(id_{D_0} \times_{C_0} f_1\,; \sigma_{f_0}, \tau(f_0)_\sigma) : D_{\hat{\mathrm{pr}}_1,\tau f_1 \hat{\mathrm{pr}}_2} \to D_{\sigma_{f_0},\tau(f_0)_\sigma} = T_2$ and $\omega(f_1\,;\sigma,\tau) : D_{\sigma f_1,\tau f_1} \to D_{\sigma,\tau} = U_2$. Then, we have $\omega(f_1\,;\sigma,\tau)\bar{\beta}^2 = \beta^2\omega(id_{D_0} \times_{C_0} f_1\,;\sigma_{f_0},\tau(f_0)_\sigma)$ and it follows from (1.3.33) that the following diagram (ii) is commutative.

$$\begin{tikzcd}
f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\hat{\mathrm{pr}}_1,\tau f_1 \hat{\mathrm{pr}}_2]} \arrow[r, "\bar{\beta}^2_{M_{[\sigma_{f_0},\tau(f_0)_\sigma]}}"] \arrow[d, "f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{id_{D_0}\times_{C_0} f_1}"'] \arrow[dr, "(\omega(f_1\,;\sigma,\tau)\bar{\beta}^2)_{M_{[\sigma_{f_0},\tau(f_0)_\sigma]}}" description] & (M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[f_0\sigma',f_0\tau']} \arrow[d, "(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{f_1}"] \\
f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[r, "\beta^2_{M_{[\sigma_{f_0},\tau(f_0)_\sigma]}}"] & (M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]}
\end{tikzcd}$$

diagram (ii)

The following diagram is commutative by (1.3.9).

$$\begin{tikzcd}
(M_{[\sigma',\tau']})_{[\hat{\mathrm{pr}}_1,\tau f_1 \hat{\mathrm{pr}}_2]} \arrow[r, "(\beta^1_M)_{[\hat{\mathrm{pr}}_1,\tau f_1 \hat{\mathrm{pr}}_2]}"] \arrow[d, "(M_{[\sigma',\tau']})_{id_{D_0}\times_{C_0} f_1}"] & f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\hat{\mathrm{pr}}_1,\tau f_1 \hat{\mathrm{pr}}_2]} \arrow[d, "f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{id_{D_0}\times_{C_0} f_1}"] \\
(M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[r, "(\beta^1_M)_{[\sigma_{f_0},\tau(f_0)_\sigma]}"'] & f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma_{f_0},\tau(f_0)_\sigma]}
\end{tikzcd}$$

diagram (iii)

Define a natural transformation $\gamma : S_0 \to D_{\hat{\mathrm{pr}}_1,\tau f_1 \hat{\mathrm{pr}}_2}$ by $\gamma_0 = (\sigma', id_{D_1})$, $\gamma_1 = id_{D_0}$, $\gamma_2 = f_0$, then we have $\bar{\beta}^2\gamma = \omega(\sigma',\tau'; f_0, f_0)$. It follows from (1.3.33) that

$$\begin{tikzcd}
f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma',\tau']} \arrow[d, "\gamma_{f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})}"] \arrow[dr, "\omega(\sigma',\tau';f_0,f_0)_{M_{[\sigma_{f_0},\tau(f_0)_\sigma]}}"] & \\
f_0^*(f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\hat{\mathrm{pr}}_1,\tau f_1 \hat{\mathrm{pr}}_2]}) \arrow[r, "f_0^*(\bar{\beta}^2_{M_{[\sigma_{f_0},\tau(f_0)_\sigma]}})"] & f_0^*((M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[f_0\sigma',f_0\tau']})
\end{tikzcd}$$

diagram (iv)

is commutative. Moreover, (1.3.31) implies that the following diagram is commutative.

$$\begin{tikzcd}
(M_{[\sigma',\tau']})_{[\sigma',\tau']} \arrow[r, "(\beta^1_M)_{[\sigma',\tau']}"] \arrow[d, "\gamma_{M_{[\sigma',\tau']}}"] & f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma',\tau']} \arrow[d, "\gamma_{f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})}"] \\
f_0^*((M_{[\sigma',\tau']})_{[\hat{\mathrm{pr}}_1,\tau f_1 \hat{\mathrm{pr}}_2]}) \arrow[r, "f_0^*((\beta^1_M)_{[\hat{\mathrm{pr}}_1,\tau f_1 \hat{\mathrm{pr}}_2]})"] & f_0^*(f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\hat{\mathrm{pr}}_1,\tau f_1 \hat{\mathrm{pr}}_2]})
\end{tikzcd}$$

diagram (v)

The following diagram is commutative by the definition of $\hat{\xi}_{\boldsymbol{f}}$ and (1.3.9), (1.3.21).

$$\begin{tikzcd}
f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma',\tau']} \arrow[r, "f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma',\tau']}"] \arrow[d, "\omega(\sigma',\tau',f_0,f_0)_{M_{[\sigma_{f_0},\tau(f_0)_\sigma]}}"] & f_0^*((M,\xi)_{\boldsymbol{f}})_{[\sigma',\tau']} \arrow[d, "\omega(\sigma',\tau',f_0,f_0)_{(M,\xi)_{\boldsymbol{f}}}"] \\
f_0^*((M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[f_0\sigma',f_0\tau']}) \arrow[r, "f_0^*((P^{\boldsymbol{f}}_{(M,\xi)})_{[f_0\sigma',f_0\tau']})"] \arrow[d, "f_0^*((M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{f_1})"] & f_0^*(((M,\xi)_{\boldsymbol{f}})_{[f_0\sigma',f_0\tau']}) \arrow[d, "f_0^*(((M,\xi)_{\boldsymbol{f}})_{f_1})"] \\
f_0^*((M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]}) \arrow[r, "f_0^*((P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]})"] \arrow[d, "f_0^*(\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma,\tau}(M))^{-1}"] & f_0^*(((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]}) \arrow[dd, "f_0^*(\hat{\xi}_{\boldsymbol{f}})"] \\
f_0^*(M_{[\sigma_{f_0}\hat{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\hat{\mathrm{pr}}_{23}]}) \arrow[d, "f_0^*(M_{id_{D_0}\times_{C_0}\mu})"] & \\
f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]}) \arrow[r, "f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})"] & f_0^*(((M,\xi)_{\boldsymbol{f}}))
\end{tikzcd}$$

diagram (vi)

Consider natural transformations $\omega(\varepsilon';\sigma',\tau'):S_2\to S_0$ and $\omega(id_{D_0}\times_{C_0}f_1;\sigma_{f_0},\tau(f_0)_\sigma):D_{\hat{\mathrm{pr}}_1,\tau f_1\hat{\mathrm{pr}}_2}\to T_2$. Then, we have the following equalities.

$$\alpha^2=\beta^1\omega(\varepsilon';\sigma',\tau')\qquad \omega(id_{D_0}\times_{C_0}f_1;\sigma_{f_0},\tau(f_0)_\sigma)\gamma=\beta^1=\omega((\sigma'\tilde{\mathrm{pr}}_1,\mu(f_1\times_{C_0}id_{C_1});\sigma_{f_0},\tau(f_0)_\sigma)\alpha^0$$

It follows from (1.3.33) that the following diagrams are commutative.

$$\begin{tikzcd}
M=M_{[id_{D_0},id_{D_0}]} \arrow[r, "M_{\varepsilon'}"] \arrow[dr, "\alpha^2_M"'] & M_{[\sigma',\tau']} \arrow[d, "\beta^1_M"] \\
 & f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})
\end{tikzcd}$$

diagram (vii)

$$\begin{tikzcd}
(M_{[\sigma',\tau']})_{[\sigma',\tau']} \arrow[r, "\gamma_{M_{[\sigma',\tau']}}"] \arrow[d, "\alpha^0_{M_{[\sigma',\tau']}}"] \arrow[dr, "\beta^1_{M_{[\sigma',\tau']}}"] & f_0^*((M_{[\sigma',\tau']})_{[\hat{\mathrm{pr}}_1,\tau f_1\hat{\mathrm{pr}}_2]}) \arrow[d, "f_0^*((M_{[\sigma',\tau']})_{id_{D_0}\times_{C_0}f_1})"] \\
f_0^*((M_{[\sigma',\tau']})_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]}) \arrow[r, "f_0^*((M_{[\sigma',\tau']})_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))})"'] & f_0^*((M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]})
\end{tikzcd}$$

diagram $(viii)$

We also have the following commutative diagrams by (1.3.31) and (1.3.9).

$$\begin{tikzcd}
M_{[\sigma',\tau']} \arrow[r, "(M_{\varepsilon'})_{[\sigma',\tau']}"] \arrow[d, "\alpha^0_M"] & (M_{[\sigma',\tau']})_{[\sigma',\tau']} \arrow[d, "\alpha^0_{M_{[\sigma',\tau']}}"] \\
f_0^*(M_{[\sigma'\tilde{\mathrm{pr}}_1,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]}) \arrow[r, "f_0^*((M_{\varepsilon'})_{[\sigma'\tilde{\mathrm{pr}}_1,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]})"] & f_0^*((M_{[\sigma',\tau']})_{[\sigma'\tilde{\mathrm{pr}}_1,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]})
\end{tikzcd}$$

diagram (ix)

$$\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]} \arrow[r, "(M_{\varepsilon'})_{[\sigma'\tilde{\mathrm{pr}}_1,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]}"] \arrow[d, "M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))}"] & (M_{[\sigma',\tau']})_{[\sigma'\tilde{\mathrm{pr}}_1,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]} \arrow[d, "(M_{[\sigma',\tau']})_{(\sigma'\tilde{\mathrm{pr}}_1,\mu(f_1\times_{C_0}id_{C_1}))}"] \\
M_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[r, "(M_{\varepsilon'})_{[\sigma_{f_0},\tau(f_0)_\sigma]}"] & (M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]}
\end{tikzcd}$$

diagram (x)

We put $\tilde{\xi}_{\boldsymbol{f}}=P_{\sigma',\tau'}(f_0^*((M,\xi)_{\boldsymbol{f}}))_{f_0^*((M,\xi)_{\boldsymbol{f}})}((\xi^l_{\boldsymbol{f}})_{\boldsymbol{f}})$. Then, $\tilde{\xi}_{\boldsymbol{f}}$ is the following composition by (3.3.5).

$$f_0^*((M,\xi)_{\boldsymbol{f}})_{[\sigma',\tau']}\xrightarrow{\omega(\sigma',\tau';f_0,f_0)_{(M,\xi)_{\boldsymbol{f}}}}f_0^*(((M,\xi)_{\boldsymbol{f}})_{[f_0\sigma',f_0\tau']})\xrightarrow{f_0^*(((M,\xi)_{\boldsymbol{f}})_{f_1})}f_0^*(((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]})\xrightarrow{f_0^*(\tilde{\xi}_{\boldsymbol{f}})}f_0^*((M,\xi)_{\boldsymbol{f}})$$

We note that $(id_{D_0}\times_{C_0}\mu)(\sigma'\tilde{\mathrm{pr}}_1,f_1\tilde{\mathrm{pr}}_1,\tilde{\mathrm{pr}}_2)=(\sigma'\tilde{\mathrm{pr}}_1,\mu(f_1\times_{C_0}id_{C_1}))$ holds and recall that $P^{\boldsymbol{f}}_{(M,\xi)}$ is a coequalizer of $M_{(\sigma'\tilde{\mathrm{pr}}_1,\mu(f_1\times_{C_0}id_{C_1}))}$ and $\hat{\xi}_{[\sigma_{f_0},\tau(f_0)_\sigma]}\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M)$. We also have $f_0^*(M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))})\alpha^0_M=\beta^1_M$ by (1.3.33). Therefore by the commutativity of diagrams $(i)\sim(ix)$ and (3.5.7), we have

$$\begin{aligned}
\tilde{\xi}_{\boldsymbol{f}}(f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})\alpha^2_M)_{[\sigma',\tau']}&=f_0^*(\tilde{\xi}_{\boldsymbol{f}})f_0^*(((M,\xi)_{\boldsymbol{f}})_{f_1})\omega(\sigma',\tau';f_0,f_0)_{(M,\xi)_{\boldsymbol{f}}}f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma',\tau']}(\beta^1_M)_{[\sigma',\tau']}(M_{\varepsilon'})_{[\sigma',\tau']}\\
&=f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})f_0^*(M_{id_{D_0}\times_{C_0}\mu})f_0^*(M_{(\sigma'\tilde{\mathrm{pr}}_1,f_1\tilde{\mathrm{pr}}_1,\tilde{\mathrm{pr}}_2)})f_0^*(\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M)^{-1})\\
&\quad f_0^*((M_{[\sigma',\tau']})_{id_{D_0}\times_{C_0}f_1})\gamma_{M_{[\sigma',\tau']}}(M_{\varepsilon'})_{[\sigma',\tau']}\\
&=f_0^*(P^{\boldsymbol{f}}_{(M,\xi)}M_{(\sigma'\tilde{\mathrm{pr}}_1,\mu(f_1\times_{C_0}id_{C_1}))}\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M)^{-1})\\
&\quad f_0^*((M_{[\sigma',\tau']})_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))})\alpha^0_{M_{[\sigma',\tau']}}(M_{\varepsilon'})_{[\sigma',\tau']}\\
&=f_0^*(P^{\boldsymbol{f}}_{(M,\xi)}\hat{\xi}_{[\sigma_{f_0},\tau(f_0)_\sigma]})f_0^*((M_{[\sigma',\tau']})_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))}(M_{\varepsilon'})_{(\sigma'\tilde{\mathrm{pr}}_1,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1}))})\alpha^0_M\\
&=f_0^*(P^{\boldsymbol{f}}_{(M,\xi)}\hat{\xi}_{[\sigma_{f_0},\tau(f_0)_\sigma]})f_0^*((M_{\varepsilon'})_{[\sigma_{f_0},\tau(f_0)_\sigma]}M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))})\alpha^0_M\\
&=f_0^*(P^{\boldsymbol{f}}_{(M,\xi)}(\hat{\xi}M_{\varepsilon'})_{[\sigma_{f_0},\tau(f_0)_\sigma]})f_0^*(M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))})\alpha^0_M\\
&=f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})\beta^1_M=f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})\alpha^2_M\hat{\xi}.
\end{aligned}$$

This shows that $f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})\alpha^2_M : M \to f_0^*((M,\xi)_{\boldsymbol{f}})$ defines a morphism $(M,\xi) \to (f_0^*((M,\xi)_{\boldsymbol{f}}), (\xi^l_{\boldsymbol{f}})_{\boldsymbol{f}})$ of representations of $\boldsymbol{D}$. $\square$

We put $(\eta_{\boldsymbol{f}})_{(M,\xi)} = f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})\alpha^2_M : M \to f_0^*((M,\xi)_{\boldsymbol{f}})$.

Remark 3.5.9 *If $\varphi : (M,\xi) \to (N,\zeta)$ is a morphism of representations of $\boldsymbol{D}$, the following diagram is commutative by (1.3.31) and the definition of $\varphi_{\boldsymbol{f}}$.*

$$\begin{tikzcd}
M \arrow[r, "\alpha^2_M"] \arrow[d, "\varphi"] \arrow[rr, bend left, "(\eta_{\boldsymbol{f}})_{(M,\xi)}"] & f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]}) \arrow[r, "f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})"] \arrow[d, "f_0^*(\varphi_{[\sigma_{f_0},\tau(f_0)_\sigma]})"] & f_0^*((M,\xi)_{\boldsymbol{f}}) \arrow[d, "f_0^*(\varphi_{\boldsymbol{f}})"] \\
N \arrow[r, "\alpha^2_N"] \arrow[rr, bend right, "(\eta_{\boldsymbol{f}})_{(N,\zeta)}"'] & f_0^*(N_{[\sigma_{f_0},\tau(f_0)_\sigma]}) \arrow[r, "f_0^*(P^{\boldsymbol{f}}_{(N,\zeta)})"] & f_0^*((N,\zeta)_{\boldsymbol{f}})
\end{tikzcd}$$

Define a functor $R : \mathcal{P} \to \mathcal{E}$ and a natural transformation $\kappa : U \to R$ by $R(0) = C_1 \times_{C_0} C_1$, $R(1) = C_1$, $R(2) = C_1$, $R(i) = C_0$ $(i = 3,4,5)$, $R(\tau_{01}) = \mathrm{pr}_1$, $R(\tau_{02}) = \mathrm{pr}_2$, $R(\tau_{13}) = R(\tau_{24}) = \sigma$, $R(\tau_{14}) = R(\tau_{25}) = \tau$ and $\kappa_0 = \tilde{\mathrm{pr}}_{23}$, $\kappa_1 = (f_0)_\sigma$, $\kappa_2 = id_{C_1}$, $\kappa_3 = f_0$, $\kappa_4 = \kappa_5 = id_{C_0}$. We also define functors $R_i : \mathcal{Q} \to \mathcal{E}$ and natural transformations $\kappa^i : U_i \to R_i$ for $i = 0,1,2$ by

$$\begin{array}{lllll}
R_0(0) = R(0) & R_0(1) = R(3) & R_0(2) = R(5) & R_0(\tau_{01}) = R(\tau_{13}\tau_{01}) & R_0(\tau_{02}) = R(\tau_{25}\tau_{02}) \\
R_1(0) = R(1) & R_1(1) = R(3) & R_1(2) = R(4) & R_1(\tau_{01}) = R(\tau_{13}) & R_1(\tau_{02}) = R(\tau_{14}) \\
R_2(0) = R(2) & R_2(1) = R(4) & R_2(2) = R(5) & R_2(\tau_{01}) = R(\tau_{24}) & R_2(\tau_{02}) = R(\tau_{25})
\end{array}$$
$$\kappa^0_0 = \kappa_0 \quad \kappa^0_1 = \kappa_3 \quad \kappa^0_2 = \kappa_5 \quad \kappa^1_0 = \kappa_1 \quad \kappa^1_1 = \kappa_3 \quad \kappa^1_2 = \kappa_4 \quad \kappa^2_0 = \kappa_2 \quad \kappa^2_1 = \kappa_4 \quad \kappa^2_2 = \kappa_5.$$

Proposition 3.5.10 *For an object N of $\mathcal{F}_{C_0}$, $\beta^2_N : f_0^*(N)_{[\sigma_{f_0},\tau(f_0)_\sigma]} \to N_{[\sigma,\tau]}$ defines a morphism of representations $(f_0^*(N)_{[\sigma_{f_0},\tau(f_0)_\sigma]}, \mu^l_{\boldsymbol{f}}(f_0^*(N))) \to (N_{[\sigma,\tau]}, \mu^l_N)$ under the assumption of (3.5.1) for $M = f_0^*(N)$ and the assumption of (3.3.10) for $M = N$.*

Proof. Since κ^2 is the identity natural transformation and $\kappa^1 = \beta^2$, we have a commutative diagram below by applying (1.3.34) to $\kappa : U \to R$.

$$\begin{tikzcd}
f_0^*(N)_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "\kappa^0_N"] \arrow[d, "\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma,\tau}(f_0^*(N))"] & N_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[d, "\theta_{\sigma,\tau,\sigma,\tau}(N)"] \\
(f_0^*(N)_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \arrow[r, "(\beta^2_N)_{[\sigma,\tau]}"] & (N_{[\sigma,\tau]})_{[\sigma,\tau]}
\end{tikzcd}$$

We consider functors $\omega(\mu;\sigma,\tau) : R_0 \to U_2$ and $\omega(id_{D_0} \times_{C_0} \mu; \sigma_{f_0}, \tau(f_0)_\sigma) : U_0 \to T_2$. Then we have $\omega(\mu;\sigma,\tau)\kappa^0 = \beta^2\omega(id_{D_0} \times_{C_0} \mu; \sigma_{f_0}, \tau(f_0)_\sigma)$. Hence it follows from (1.3.33) that the following diagram is commutative.

$$\begin{tikzcd}
f_0^*(N)_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "\kappa^0_N"] \arrow[d, "f_0^*(N)_{id_{D_0}\times_{C_0}\mu}"] \arrow[dr, "(\omega(\mu;\sigma,\tau)\kappa^0)_N = (\beta^2\omega(id_{D_0}\times_{C_0}\mu;\sigma_{f_0},\tau(f_0)_\sigma))_N" description] & N_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[d, "N_\mu"] \\
f_0^*(N)_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[r, "\beta^2_N"] & N_{[\sigma,\tau]}
\end{tikzcd}$$

Since $\hat{\mu}_{\boldsymbol{f}}(f_0^*(N)) = f_0^*(N)_{id_{D_0}\times_{C_0}\mu}\theta_{\sigma_{f_0},\tau(f_0)_\sigma,\sigma,\tau}(f_0^*(N))^{-1}$ and $\hat{\mu}_N = N_\mu\theta_{\sigma,\tau,\sigma,\tau}(N)^{-1}$, the commutativity of the above diagrams implies that the following diagram is commutative.

$$\begin{tikzcd}
(f_0^*(N)_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \arrow[r, "\hat{\mu}_{\boldsymbol{f}}(f_0^*(N))"] \arrow[d, "(\beta^2_N)_{[\sigma,\tau]}"] & f_0^*(N)_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[d, "\beta^2_N"] \\
(N_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\hat{\mu}_N"] & N_{[\sigma,\tau]}
\end{tikzcd}$$

Hence the assertion follows from (3.3.6). $\square$

Lemma 3.5.11 *Let (M,ξ) and (N,ζ) be representations of $\boldsymbol{D}$ and $\boldsymbol{C}$, respectively. We put $\hat{\xi} = P_{\sigma'\tau'}(M)_M(\xi)$ and $\hat{\zeta} = P_{\sigma,\tau}(N)_N(\zeta)$. For a morphism $\varphi : (M,\xi) \to \boldsymbol{f}^{\bullet}(N,\zeta)$ of representations of $\boldsymbol{D}$, the following diagram is commutative if $\theta_{\sigma,\tau,\sigma,\tau}(N) : N_{[\sigma \mathrm{pr}_1, \tau \mathrm{pr}_2]} \to (N_{[\sigma,\tau]})_{[\sigma,\tau]}$ is an isomorphism.*

$$\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1,\, \tau(f_0)_\sigma(\tau'\times_{C_0} id_{C_1})]} \arrow[rr, "M_{(\sigma'\tilde{\mathrm{pr}}_1,\, \mu(f_1\times_{C_0} id_{C_1}))}"] \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M)"] & & M_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[r, "\varphi_{[\sigma_{f_0},\, \tau(f_0)_\sigma]}"] & f_0^*(N)_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[d, "\beta_N^2"] \\
(M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[d, "\hat{\xi}_{[\sigma_{f_0},\tau(f_0)_\sigma]}"] & & & N_{[\sigma,\tau]} \arrow[d, "\hat{\zeta}"] \\
M_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[r, "\varphi_{[\sigma_{f_0},\, \tau(f_0)_\sigma]}"] & f_0^*(N)_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[r, "\beta_N^2"] & N_{[\sigma,\tau]} \arrow[r, "\hat{\zeta}"] & N
\end{tikzcd}$$

Proof. Since $P_{\sigma',\tau'}(f_0^*(N))_{f_0^*(N)}(\zeta_{\boldsymbol{f}})$ is a composition

$$f_0^*(N)_{[\sigma',\tau']} \xrightarrow{\omega(\sigma',\tau';f_0,f_0)_N} f_0^*(N_{[f_0\sigma',f_0\tau']}) \xrightarrow{f_0^*(N_{f_1})} f_0^*(N_{[\sigma,\tau]}) \xrightarrow{f_0^*(\hat{\zeta})} f_0^*(N)$$

by (3.3.5), the following diagram is commutative by (3.3.6).

$$\begin{tikzcd}
M_{[\sigma',\tau']} \arrow[rrr, "\hat{\xi}"] \arrow[d, "\varphi_{[\sigma',\tau']}"] & & & M \arrow[d, "\varphi"] \\
f_0^*(N)_{[\sigma',\tau']} \arrow[r, "\omega(\sigma',\tau';f_0,f_0)_N"] & f_0^*(N_{[f_0\sigma',f_0\tau']}) \arrow[r, "f_0^*(N_{f_1})"] & f_0^*(N_{[\sigma,\tau]}) \arrow[r, "f_0^*(\hat{\zeta})"] & f_0^*(N)
\end{tikzcd}$$

It follows from (1.3.31) that the following diagram is commutative.

$$\begin{tikzcd}
f_0^*(N_{[\sigma,\tau]})_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[r, "\beta^2_{N_{[\sigma,\tau]}}"] \arrow[d, "f_0^*(\hat{\zeta})_{[\sigma_{f_0},\, \tau(f_0)_\sigma]}"] & (N_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[d, "\hat{\zeta}_{[\sigma,\tau]}"] \\
f_0^*(N)_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[r, "\beta_N^2"] & N_{[\sigma,\tau]}
\end{tikzcd}$$

Hence the following diagram (i) is commutative by (1.3.4), (1.3.9) and (1.3.21).

$$\begin{tikzcd}
M_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[r, "\varphi_{[\sigma_{f_0},\, \tau(f_0)_\sigma]}"] & f_0^*(N)_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \\
M_{[\sigma'\tilde{\mathrm{pr}}_1,\, \tau(f_0)_\sigma(\tau'\times_{C_0} id_{C_1})]} \arrow[u, "M_{(\sigma'\tilde{\mathrm{pr}}_1,\, \mu(f_1\times_{C_0} id_{C_1}))}"] \arrow[r, "\varphi_{[\sigma'\tilde{\mathrm{pr}}_1,\, \tau(f_0)_\sigma(\tau'\times_{C_0} id_{C_1})]}"] \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M)"] & f_0^*(N)_{[\sigma'\tilde{\mathrm{pr}}_1,\, \tau(f_0)_\sigma(\tau'\times_{C_0} id_{C_1})]} \arrow[u, "f_0^*(N)_{(\sigma'\tilde{\mathrm{pr}}_1,\, \mu(f_1\times_{C_0} id_{C_1}))}"'] \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(f_0^*(N))"] \\
(M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[r, "(\varphi_{[\sigma',\tau']})_{[\sigma_{f_0},\, \tau(f_0)_\sigma]}"] \arrow[d, "\hat{\xi}_{[\sigma_{f_0},\tau(f_0)_\sigma]}"] & (f_0^*(N)_{[\sigma',\tau']})_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[d, "(\omega(\sigma',\tau';f_0,f_0)_N)_{[\sigma_{f_0},\, \tau(f_0)_\sigma]}"] \\
M_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[d, "\varphi_{[\sigma_{f_0},\, \tau(f_0)_\sigma]}"] & f_0^*(N_{[f_0\sigma',f_0\tau']})_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[d, "f_0^*(N_{f_1})_{[\sigma_{f_0},\, \tau(f_0)_\sigma]}"] \\
f_0^*(N)_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[ddr, "\beta_N^2"] & f_0^*(N_{[\sigma,\tau]})_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[l, "f_0^*(\hat{\zeta})_{[\sigma_{f_0},\, \tau(f_0)_\sigma]}"'] \arrow[d, "\beta^2_{N_{[\sigma,\tau]}}"] \\
& (N_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[d, "\hat{\zeta}_{[\sigma,\tau]}"] \\
& N_{[\sigma,\tau]}
\end{tikzcd}$$

diagram (i)

Define a functor $V : \mathcal{P} \to \mathcal{E}$ and a natural transformation $\lambda : T \to V$ by $V(0) = D_1 \times_{C_0} C_1$, $V(1) = D_1$, $V(2) = C_1$, $V(i) = C_0$ $(i = 3, 4, 5)$, $V(\tau_{01}) = \tilde{\mathrm{pr}}_1$, $V(\tau_{02}) = \tilde{\mathrm{pr}}_2$, $V(\tau_{13}) = f_0\sigma'$, $V(\tau_{14}) = f_0\tau'$, $V(\tau_{24}) = \sigma$, $V(\tau_{25}) = \tau$ and $\lambda_0 = id_{D_1 \times_{C_0} C_1}$, $\lambda_1 = id_{D_1}$, $\lambda_2 = (f_0)_\sigma$, $\lambda_3 = \lambda_4 = f_0$, $\lambda_5 = id_{C_0}$. We also define functors $V_i : \mathcal{Q} \to \mathcal{E}$ and natural transformations $\lambda^i : V_i \to T_i$ for $i = 0, 1, 2$ by

$$\begin{array}{lllll} V_0(0) = V(0) & V_0(1) = V(3) & V_0(2) = V(5) & V_0(\tau_{01}) = V(\tau_{13}\tau_{01}) & V_0(\tau_{02}) = V(\tau_{25}\tau_{02}) \\ V_1(0) = V(1) & V_1(1) = V(3) & V_1(2) = V(4) & V_1(\tau_{01}) = V(\tau_{13}) & V_1(\tau_{02}) = V(\tau_{14}) \\ V_2(0) = V(2) & V_2(1) = V(4) & V_2(2) = V(5) & V_2(\tau_{01}) = V(\tau_{24}) & V_2(\tau_{02}) = V(\tau_{25}) \end{array}$$

$$\lambda^0_0 = \lambda_0 \quad \lambda^0_1 = \lambda_3 \quad \lambda^0_2 = \lambda_5 \quad \lambda^1_0 = \lambda_1 \quad \lambda^1_1 = \lambda_3 \quad \lambda^1_2 = \lambda_4 \quad \lambda^2_0 = \lambda_2 \quad \lambda^2_1 = \lambda_4 \quad \lambda^2_2 = \lambda_5.$$

Then, $V_2 = U_2$, $\lambda^1 = \omega(\sigma', \tau' ; f_0, f_0)$ and $\lambda^2 = \beta^2$ and it follows from (1.3.34) that the following diagram is commutative.

$$\begin{tikzcd}
f_0^*(N)_{[\sigma'\tilde{\mathrm{pr}}_1, \tau(f_0)_\sigma(\tau' \times_{C_0} id_{C_1})]} \arrow[rr, "\lambda^0_N"] \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(f_0^*(N))"] & & N_{[f_0\sigma'\tilde{\mathrm{pr}}_1, \tau\tilde{\mathrm{pr}}_2]} \arrow[d, "\theta_{f_0\sigma',f_0\tau',\sigma,\tau}(N)"'] \\
(f_0^*(N)_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[r, "(\omega(\sigma',\tau' ; f_0, f_0)_N)_{[\sigma_{f_0},\tau(f_0)_\sigma]}"] & f_0^*(N_{[f_0\sigma',f_0\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[r, "\beta^2_{N_{[f_0\sigma',f_0\tau']}}"] & (N_{[f_0\sigma',f_0\tau']})_{[\sigma,\tau]}
\end{tikzcd}$$

Consider natural transformations $\omega(\mu(f_1 \times_{C_0} id_{C_1}) ; \sigma, \tau) : V_0 \to U_2$ and $\omega((\sigma'\tilde{\mathrm{pr}}_1,\ \mu(f_1 \times_{C_0} id_{C_1})) ; \sigma_{f_0}, \tau(f_0)_\sigma) : T_0 \to T_2$. Then, $\omega(\mu(f_1 \times_{C_0} id_{C_1}) ; \sigma, \tau)\lambda^0 = \beta^2\omega((\sigma'\tilde{\mathrm{pr}}_1,\ \mu(f_1 \times_{C_0} id_{C_1})) ; \sigma_{f_0}, \tau(f_0)_\sigma)$ holds and the following diagram is commutative by (1.3.33).

$$\begin{tikzcd}
f_0^*(N)_{[\sigma'\tilde{\mathrm{pr}}_1,\, \tau(f_0)_\sigma(\tau' \times_{C_0} id_{C_1})]} \arrow[r, "\lambda^0_N"] \arrow[d, "f_0^*(N)_{(\sigma'\tilde{\mathrm{pr}}_1,\, \mu(f_1 \times_{C_0} id_{C_1}))}"] \arrow[dr, "(\omega(\mu(f_1 \times_{C_0} id_{C_1}) ; \sigma, \tau)\lambda^0)_N" description] & N_{[f_0\sigma'\tilde{\mathrm{pr}}_1, \tau\tilde{\mathrm{pr}}_2]} \arrow[d, "N_{\mu(f_1 \times_{C_0} id_{C_1})}"] \\
f_0^*(N)_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[r, "\beta^2_N"] & N_{[\sigma,\tau]}
\end{tikzcd}$$

Moreover, the following diagrams are commutative by (3.3.2) and (1.3.31), respectively.

$$\begin{tikzcd}
N_{[\sigma\mathrm{pr}_1, \tau\mathrm{pr}_2]} \arrow[r, "N_\mu"] \arrow[d, "\theta_{\sigma,\tau,\sigma,\tau}(N)"] & N_{[\sigma,\tau]} \arrow[r, "\hat{\varsigma}"] & N \\
(N_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\hat{\varsigma}_{[\sigma,\tau]}"] & N_{[\sigma,\tau]} \arrow[ur, "\hat{\varsigma}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
f_0^*(N_{[f_0\sigma',f_0\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[r, "\beta^2_{N_{[f_0\sigma',f_0\tau']}}"] \arrow[d, "f_0^*(N_{f_1})_{[\sigma_{f_0},\tau(f_0)_\sigma]}"] & (N_{[f_0\sigma',f_0\tau']})_{[\sigma,\tau]}) \arrow[d, "(N_{f_1})_{[\sigma,\tau]}"] \\
f_0^*(N_{[\sigma,\tau]})_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[r, "\beta^2_{N_{[\sigma,\tau]}}"] & (N_{[\sigma,\tau]})_{[\sigma,\tau]})
\end{tikzcd}$$

Therefore the following diagram (ii) is commutative

$$\begin{tikzcd}
f_0^*(N)_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[rrr, "\beta^2_N"] & & & N_{[\sigma,\tau]} \arrow[dddddd, "\hat{\varsigma}"] \\
f_0^*(N)_{[\sigma'\tilde{\mathrm{pr}}_1,\, \tau(f_0)_\sigma(\tau' \times_{C_0} id_{C_1})]} \arrow[u, "f_0^*(N)_{(\sigma'\tilde{\mathrm{pr}}_1,\, \mu(f_1 \times_{C_0} id_{C_1}))}"] \arrow[r, "\lambda^0_N"] \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(f_0^*(N))"] & N_{[f_0\sigma'\tilde{\mathrm{pr}}_1, \tau\tilde{\mathrm{pr}}_2]} \arrow[dd, "\theta_{f_0\sigma',f_0\tau',\sigma,\tau}(N)"'] \arrow[ddddr, "N_{f_1 \times_{C_0} id_{C_1}}"] & & \\
(f_0^*(N)_{[\sigma',\tau']})_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[d, "(\omega(\sigma',\tau',f_0,f_0)_N)_{[\sigma_{f_0},\, \tau(f_0)_\sigma]}"] & & & \\
f_0^*(N_{[f_0\sigma',f_0\tau']})_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[r, "\beta^2_{N_{[f_0\sigma',f_0\tau']}}"'] \arrow[d, "f_0^*(N_{f_1})_{[\sigma_{f_0},\, \tau(f_0)_\sigma]}"] & (N_{[f_0\sigma',f_0\tau']})_{[\sigma,\tau]} \arrow[ddl, "(N_{f_1})_{[\sigma,\tau]}"] & & \\
f_0^*(N_{[\sigma,\tau]})_{[\sigma_{f_0},\, \tau(f_0)_\sigma]} \arrow[d, "\beta^2_{N_{[\sigma,\tau]}}"] & & & \\
(N_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[rr, "\theta_{\sigma,\tau,\sigma,\tau}(N)^{-1}"] \arrow[d, "\hat{\varsigma}_{[\sigma,\tau]}"] & & N_{[\sigma\mathrm{pr}_1, \tau\mathrm{pr}_2]} \arrow[uuuuur, "N_\mu"] & \\
N_{[\sigma,\tau]} \arrow[rrr, "\hat{\varsigma}"] & & & N
\end{tikzcd}$$

diagram (ii)

By glueing the right edge of diagram (i) and the left edge of diagram (ii), the assertion follows. $\square$

Recall that $P^{\boldsymbol{f}}_{(M,\xi)} : M_{[\sigma_{f_0},\tau(f_0)_\sigma]} \to (M,\xi)_{\boldsymbol{f}}$ is a coequalizer of the following morphisms.

$$M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0} id_{C_1})]} \xrightarrow{\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(M)} (M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]} \xrightarrow{\xi_{[\sigma_{f_0},\tau(f_0)_\sigma]}} M_{[\sigma_{f_0},\tau(f_0)_\sigma]}$$

$$M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0} id_{C_1})]} \xrightarrow{M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1}))}} M_{[\sigma_{f_0},\tau(f_0)_\sigma]}$$

Hence there exists unique morphism ${}^t\varphi : (M,\xi)_{\boldsymbol{f}} \to N$ that satisfies ${}^t\varphi P^{\boldsymbol{f}}_{(M,\xi)} = \hat{\zeta}\beta^2_N\varphi_{[\sigma_{f_0},\tau(f_0)_\sigma]}$.

Proposition 3.5.12 *Under the assumptions of (3.5.4) for M and the assumptions of (iv) and the first one of (v) of (3.5.4) for $f_0^*(N)$, ${}^t\varphi : (M,\xi)_{\boldsymbol{f}} \to N$ gives a morphism $((M,\xi)_{\boldsymbol{f}}, \xi^l_{\boldsymbol{f}}) \to (N,\zeta)$ of representations of $\boldsymbol{C}$.*

Proof. It follows from (3.3.10), (3.5.10) and (3.3.11) that $\hat{\zeta}\beta^2_N\varphi_{[\sigma_{f_0},\tau(f_0)_\sigma]} : M_{[\sigma_{f_0},\tau(f_0)_\sigma]} \to N$ gives a morphism $(M_{[\sigma_{f_0},\tau(f_0)_\sigma]}, \mu^l_{\boldsymbol{f}}(M)) \to (N,\zeta)$ of representations of $\boldsymbol{C}$. Hence the outer rectangle of the following diagram is commutative by (3.3.6).

$$\begin{tikzcd}
(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \arrow[r, "(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]}"] \arrow[d, "\hat{\mu}_{\boldsymbol{f}}(M)"] & ((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]} \arrow[r, "{}^t\varphi_{[\sigma,\tau]}"] \arrow[d, "\hat{\xi}_{\boldsymbol{f}}"] & N_{[\sigma,\tau]} \arrow[d, "\hat{\zeta}"] \\
M_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[r, "P^{\boldsymbol{f}}_{(M,\xi)}"] & (M,\xi)_{\boldsymbol{f}} \arrow[r, "{}^t\varphi"] & N
\end{tikzcd}$$

Since $(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]} : (M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \to ((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]}$ is an epimorphism and the left rectangle of the above diagram is commutative by the definition of $\hat{\xi}_{\boldsymbol{f}}$, the right rectangle of the above diagram is also commutative. Thus the assertion follows from (3.3.6). $\square$

For a morphism $f : X \to Y$ of $\mathcal{E}$, we define a natural transformation $\omega(f) : D_{id_X,id_X} \to D_{id_Y,id_Y}$ by $\omega(f)_0 = \omega(f)_1 = \omega(f)_2 = f$. Since $\iota_{id_Y,id_Y}(M) \in \mathcal{F}_Y(id_Y^*(M), id_Y^*(M_{[id_Y,id_Y]})) = \mathcal{F}_Y(M,M)$ is the identity morphism of $M \in \mathcal{F}_Y$, the following assertion is straightforward from the definition of $\omega(f)_M$.

Proposition 3.5.13 *For an object M of $\mathcal{F}_Y$, $\omega(f)_M : f^*(M) = f^*(M)_{[id_X,id_X]} \to f^*(M_{[id_Y,id_Y]}) = f^*(M)$ is the identity morphism of $f^*(M)$.*

Proposition 3.5.14 *For a morphism $\varphi : (M,\xi) \to \boldsymbol{f}^\bullet(N,\zeta)$ of representations of $\boldsymbol{D}$, the following composition coincides with φ.*

$$M \xrightarrow{(\eta_{\boldsymbol{f}})_{(M,\xi)}} f_0^*((M,\xi)_{\boldsymbol{f}}) \xrightarrow{f_0^*({}^t\varphi)} f_0^*(N)$$

Proof. We note that compositions $S_2 \xrightarrow{\alpha^2} T_2 \xrightarrow{\beta^2} U_2$ and $S_2 = D_{id_{D_0},id_{D_0}} \xrightarrow{\omega(f_0)} D_{id_{C_0},id_{C_0}} \xrightarrow{\omega(\varepsilon;\sigma,\tau)} U_2$ coincide. Hence the following diagram is commutative by (reffcwp21) and (1.3.33).

$$\begin{tikzcd}
M \arrow[rr, "\alpha^2_M"] \arrow[d, "\varphi"] & & f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]}) \arrow[r, "f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})"] \arrow[d, "f_0^*(\varphi_{[\sigma_{f_0},\tau(f_0)_\sigma]})"] & f_0^*((M,\xi)_{\boldsymbol{f}}) \arrow[dd, "f_0^*({}^t\varphi)"] \\
f_0^*(N) \arrow[rr, "\alpha^2_{f_0^*(N)}"] \arrow[d, "\omega(f_0)_N"] \arrow[drr, "(\beta^2\alpha^2)_N=(\omega(\varepsilon;\sigma,\tau)\omega(f_0))_N"] & & f_0^*(f_0^*(N)_{[\sigma_{f_0},\tau(f_0)_\sigma]}) \arrow[d, "f_0^*(\beta^2_N)"] & \\
f_0^*(N) \arrow[rr, "f_0^*(N_\varepsilon)"] & & f_0^*(N_{[\sigma,\tau]}) \arrow[r, "f_0^*(\hat{\zeta})"] & f_0^*(N)
\end{tikzcd}$$

Since $\omega(f_0)_N$ is the identity morphism of $f^*(N)$ by (3.5.13) and $\hat{\zeta}N_\varepsilon$ is the identity morphism of N by (3.3.2), the assertion follows. $\square$

Lemma 3.5.15 *For an object M of $\mathcal{F}_{D_0}$, a composition*

$$M_{[\sigma_{f_0},\tau(f_0)_\sigma]} \xrightarrow{(\alpha^2_M)_{[\sigma_{f_0},\tau(f_0)_\sigma]}} f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma_{f_0},\tau(f_0)_\sigma]} \xrightarrow{\beta^2_{M_{[\sigma_{f_0},\tau(f_0)_\sigma]}}} (M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \xrightarrow{\hat{\mu}_{\boldsymbol{f}}(M)} M_{[\sigma_{f_0},\tau(f_0)_\sigma]}$$

coincides with the identity morphism of $M_{[\sigma_{f_0},\tau(f_0)_\sigma]}$.

Proof. Define a functor $W:\mathcal{P}\to\mathcal{E}$ and a natural transformation $\nu: W\to U$ by $W(0)=W(2)=D_0\times_{C_0}C_1$, $W(i)=D_0$ $(i=1,3,4)$, $W(5)=C_0$, $W(\tau_{01})=\sigma_{f_0}$, $W(\tau_{02})=id_{D_0\times_{C_0}C_1}$, $W(\tau_{13})=W(\tau_{14})=id_{D_0}$, $W(\tau_{24})=\sigma_{f_0}$, $W(\tau_{25})=\tau(f_0)_\sigma$ and $\nu_0=(\sigma_{f_0},\varepsilon\sigma(f_0)_\sigma,(f_0)_\sigma)$, $\nu_1=(id_{D_0},\varepsilon f_0)$, $\nu_2=(f_0)_\sigma$, $\nu_3=id_{D_0}$, $\nu_4=f_0$, $\nu_5=id_{C_0}$. We also define functors $W_i:\mathcal{Q}\to\mathcal{E}$ and natural transformations $\nu^i: W_i\to T_i$ for $i=0,1,2$ by

$$\begin{array}{lllll}
W_0(0)=W(0) & W_0(1)=W(3) & W_0(2)=W(5) & W_0(\tau_{01})=W(\tau_{13}\tau_{01}) & W_0(\tau_{02})=W(\tau_{25}\tau_{02})\\
W_1(0)=W(1) & W_1(1)=W(3) & W_1(2)=W(4) & W_1(\tau_{01})=W(\tau_{13}) & W_1(\tau_{02})=W(\tau_{14})\\
W_2(0)=W(2) & W_2(1)=W(4) & W_2(2)=W(5) & W_2(\tau_{01})=W(\tau_{24}) & W_2(\tau_{02})=W(\tau_{25})
\end{array}$$

$$\nu^0_0=\nu_0\quad \nu^0_1=\nu_3\quad \nu^0_2=\nu_5\quad \nu^1_0=\nu_1\quad \nu^1_1=\nu_3\quad \nu^1_2=\nu_4\quad \nu^2_0=\nu_2\quad \nu^2_1=\nu_4\quad \nu^2_2=\nu_5.$$

Then, we have $W_1=S_2$, $W_2=T_2$, $\nu^1=\alpha^2$, $\nu^2=\beta^2$ and $\nu^0=\omega((\sigma_{f_0},\varepsilon\sigma(f_0)_\sigma,(f_0)_\sigma);\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23})$. It follows from (1.3.34) and the definition of $\hat{\mu}_{\boldsymbol{f}}(M)$ that the following diagram is commutative.

$$\begin{tikzcd}
M_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[rrr, "M_{(\sigma_{f_0},\varepsilon\sigma(f_0)_\sigma,(f_0)_\sigma)}"] \arrow[d, "\theta_{id_{D_0},id_{D_0},\sigma_{f_0},(f_0)_\sigma}(M)=id_{M_{[\sigma_{f_0},\tau(f_0)_\sigma]}}"'] & & & M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[d, "\theta_{\sigma_{f_0},(f_0)_\sigma,\sigma,\tau}(M)"'] \arrow[dr, "M_{id_{D_0}\times_{C_0}\mu}"] & \\
M_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[r, "(\alpha^2_M)_{[\sigma_{f_0},\tau(f_0)_\sigma]}"] & (f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]}))_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[rr, "\beta^2_{M_{[\sigma_{f_0},\tau(f_0)_\sigma]}}"] & & (M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \arrow[r, "\hat{\mu}_{\boldsymbol{f}}(M)"] & M_{[\sigma_{f_0},\tau(f_0)_\sigma]}
\end{tikzcd}$$

Since a composition $D_0\times_{C_0}C_1\xrightarrow{(\sigma_{f_0},\,\varepsilon\sigma(f_0)_\sigma,\,(f_0)_\sigma)}D_0\times_{C_0}C_1\times_{C_0}C_1\xrightarrow{id_{D_0}\times_{C_0}\mu}D_0\times_{C_0}C_1$ is the identity morphism of $D_0\times_{C_0}C_1$, the assertion follows from the commutativity of the above diagram and (1.3.7). □

Under the assumptions of (3.5.4) for M and the assumptions of (iv) and the first one of (v) of (3.5.4) for $f_0^*(N)$, we define a map

$$\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}:\mathrm{Rep}(\boldsymbol{C};\mathcal{F})(((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}}),(N,\zeta))\to\mathrm{Rep}(\boldsymbol{D};\mathcal{F})((M,\xi),\boldsymbol{f}^\bullet(N,\zeta))$$

by $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}(\psi)=f_0^*(\psi)(\eta_{\boldsymbol{f}})_{(M,\xi)}$.

Proposition 3.5.16 $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}$ *is bijective.*

Proof. We show that a map $\Phi:\mathrm{Rep}(\boldsymbol{D};\mathcal{F})((M,\xi),\boldsymbol{f}^\bullet(N,\zeta))\to\mathrm{Rep}(\boldsymbol{C};\mathcal{F})(((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}}),(N,\zeta))$ defined by $\Phi(\varphi)={}^t\varphi$ is the inverse of $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}$. $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}\Phi$ is the identity map of $\mathrm{Rep}(\boldsymbol{D};\mathcal{F})((M,\xi),\boldsymbol{f}^\bullet(N,\zeta))$ by (3.5.14). For $\psi\in\mathrm{Rep}(\boldsymbol{C};\mathcal{F})(((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}}),(N,\zeta))$, we put $\varphi=\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}(\psi)$. The following diagram is commutative by (1.3.4), (1.3.31), (3.3.6) and the definition of $\hat{\xi}_{\boldsymbol{f}}$.

$$\begin{tikzcd}
f_0^*(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[rr, "f_0^*(\psi P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma_{f_0},\tau(f_0)_\sigma]}"] \arrow[d, "\beta^2_{M_{[\sigma_{f_0},\tau(f_0)_\sigma]}}"] & & f_0^*(N)_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[d, "\beta^2_N"] \\
(M_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \arrow[rr, "(\psi P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]}"] \arrow[d, "\hat{\mu}_{\boldsymbol{f}}(M)"] \arrow[dr, "(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]}"'] & & N_{[\sigma,\tau]} \arrow[d, "\hat{\zeta}"] \\
M_{[\sigma_{f_0},\tau(f_0)_\sigma]} \arrow[dr, "P^{\boldsymbol{f}}_{(M,\xi)}"'] & ((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]} \arrow[ur, "\psi_{[\sigma,\tau]}"'] \arrow[d, "\hat{\xi}_{\boldsymbol{f}}"] & N \\
& (M,\xi)_{\boldsymbol{f}} \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Hence we have the following equalities by the commutativity of the above diagram and (3.5.15).

$$\begin{aligned}
\hat{\zeta}\beta^2_N\varphi_{[\sigma_{f_0},\tau(f_0)_\sigma]}&=\hat{\zeta}\beta^2_N f_0^*(\psi)_{[\sigma_{f_0},\tau(f_0)_\sigma]}((\eta_{\boldsymbol{f}})_{(M,\xi)})_{[\sigma_{f_0},\tau(f_0)_\sigma]}\\
&=\hat{\zeta}\beta^2_N f_0^*(\psi)_{[\sigma_{f_0},\tau(f_0)_\sigma]}f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma_{f_0},\tau(f_0)_\sigma]}(\alpha^2_M)_{[\sigma_{f_0},\tau(f_0)_\sigma]}\\
&=\hat{\zeta}\beta^2_N f_0^*(\psi P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma_{f_0},\tau(f_0)_\sigma]}(\alpha^2_M)_{[\sigma_{f_0},\tau(f_0)_\sigma]}\\
&=\psi P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M)\beta^2_{M_{[\sigma_{f_0},\tau(f_0)_\sigma]}}(\alpha^2_M)_{[\sigma_{f_0},\tau(f_0)_\sigma]}=\psi P^{\boldsymbol{f}}_{(M,\xi)}
\end{aligned}$$

Since we also have $\hat{\zeta}\beta^2_N\varphi_{[\sigma_{f_0},\tau(f_0)_\sigma]} = {}^t\varphi P^{\boldsymbol{f}}_{(M,\xi)}$ by the definition of ${}^t\varphi$, it follows that $\Phi(\varphi) = {}^t\varphi = \psi$ which implies that $\Phi\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}$ is the identity map of $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})(((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}}),(N,\zeta))$. $\square$

Definition 3.5.17 *For a representation (M,ξ) of $\boldsymbol{D}$, we call $((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}})$ the left induced representation of (M,ξ) by $\boldsymbol{f}:\boldsymbol{D}\to\boldsymbol{C}$.*

The following fact is straightforward from (3.5.9).

Proposition 3.5.18 *The following diagrams are commutative for a morphism $\varphi:(L,\chi)\to(M,\xi)$ of $\mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})$ and a morphism $\psi:(N,\zeta)\to(P,\rho)$ of $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$.*

$$\begin{tikzcd}
\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})(((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}}),(N,\zeta)) \arrow[r, "\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}"] \arrow[d, "\varphi^*_{\boldsymbol{f}}"] & \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})((M,\xi),\boldsymbol{f}^\bullet(N,\zeta)) \arrow[d, "\varphi^*"] \\
\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})(((L,\chi)_{\boldsymbol{f}},\chi^l_{\boldsymbol{f}}),(N,\zeta)) \arrow[r, "\mathrm{ad}^{(L,\chi)}_{(N,\zeta)}"] & \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})((L,\chi),\boldsymbol{f}^\bullet(N,\zeta))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})(((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}}),(N,\zeta)) \arrow[r, "\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}"] \arrow[d, "\psi_*"] & \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})((M,\xi),\boldsymbol{f}^\bullet(N,\zeta)) \arrow[d, "\boldsymbol{f}^\bullet(\psi)_*"] \\
\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})(((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}}),(P,\rho)) \arrow[r, "\mathrm{ad}^{(M,\xi)}_{(P,\rho)}"] & \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})((M,\xi),\boldsymbol{f}^\bullet(P,\rho))
\end{tikzcd}$$

3.6 Construction of right induced representations

Let $p:\mathcal{F}\to\mathcal{E}$ be a normalized cloven fibered category. For morphisms $f:X\to Y$, $g:X\to Z$ of $\mathcal{E}$ and an object N of $\mathcal{F}_Z$, we assume that (f,g) is a right fibered representable pair with respect to N if necessary.

Let $\boldsymbol{C}=(C_0,C_1;\sigma,\tau,\varepsilon,\mu)$ and $\boldsymbol{D}=(D_0,D_1;\sigma',\tau',\varepsilon',\mu')$ be internal categories in $\mathcal{E}$. For an internal functor $\boldsymbol{f}=(f_0,f_1):\boldsymbol{D}\to\boldsymbol{C}$ in $\mathcal{E}$, we consider the following diagram whose rectangles are all cartesian.

$$\begin{tikzcd}
C_1\times_{C_0}C_1\times_{C_0}C_1\times_{C_0}D_0 \arrow[r, "(\mathrm{pr}_2,\mathrm{pr}_3)\times_{C_0}id_{D_0}"] \arrow[d, "(f_0)_{\tau\mathrm{pr}_2(\mathrm{pr}_2,\mathrm{pr}_3)}"] & C_1\times_{C_0}C_1\times_{C_0}D_0 \arrow[r, "\mathrm{pr}_2\times_{C_0}id_{D_0}"] \arrow[d, "(f_0)_{\tau\mathrm{pr}_2}"] & C_1\times_{C_0}D_0 \arrow[r, "\tau_{f_0}"] \arrow[d, "(f_0)_\tau"] & D_0 \arrow[d, "f_0"] \\
C_1\times_{C_0}C_1\times_{C_0}C_1 \arrow[r, "(\mathrm{pr}_2,\mathrm{pr}_3)"] & C_1\times_{C_0}C_1 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & C_1 \arrow[r, "\tau"] \arrow[d, "\sigma"] & C_0 \\
& C_1 \arrow[r, "\tau"] & C_0 &
\end{tikzcd}$$

Diagram 3.6.1

We set $\tilde{\mathrm{pr}}_{234}=(\mathrm{pr}_2,\mathrm{pr}_3)\times_{C_0}id_{D_0}$, $\tilde{\mathrm{pr}}_{23}=\mathrm{pr}_2\times_{C_0}id_{D_0}$, $\tilde{\mathrm{pr}}_{123}=(f_0)_{\tau\mathrm{pr}_2(\mathrm{pr}_2,\mathrm{pr}_3)}$, $\tilde{\mathrm{pr}}_{12}=(f_0)_{\tau\mathrm{pr}_2}$ and $\mathrm{pr}_{23}=(\mathrm{pr}_2,\mathrm{pr}_3)$ for simplicity. Since $\mu\times_{C_0}id_{D_0}=(\mu\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23})$, we have $\sigma(f_0)_\tau(\mu\times_{C_0}id_{D_0})=\sigma\mu\tilde{\mathrm{pr}}_{12}=\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}$ and $\tau_{f_0}(\mu\times_{C_0}id_{D_0})=\tau_{f_0}\tilde{\mathrm{pr}}_{23}$. Let N be an object of $\mathcal{F}_{D_0}$. If $\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(N):(N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]}\to N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\,\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}$ is an isomorphism, we define a morphism $\check{\mu}_{\boldsymbol{f}}(N):N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}\to(N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]}$ to be the following composition.

$$N^{[\sigma(f_0)_\tau,\tau_{f_0}]}\xrightarrow{N^{\mu\times_{C_0}id_{D_0}}}N^{[\sigma(f_0)_\tau(\mu\times_{C_0}id_{D_0}),\tau_{f_0}(\mu\times_{C_0}id_{D_0})]}=N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}\xrightarrow{\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(N)^{-1}}(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]}$$

We consider the following commutative diagram.

$$\begin{tikzcd}[column sep=tiny]
& & & & & & C_1\times_{C_0}C_1\times_{C_0}C_1\times_{C_0}D_0 \arrow[dll, "\tilde{\mathrm{pr}}_{123}"'] \arrow[drr, "\tilde{\mathrm{pr}}_{234}"] & & & & & \\
& & & & C_1\times_{C_0}C_1\times_{C_0}C_1 \arrow[dll, "\mathrm{pr}_{12}"'] \arrow[drr, "\mathrm{pr}_{23}"] & & & & C_1\times_{C_0}C_1\times_{C_0}D_0 \arrow[dll, "\tilde{\mathrm{pr}}_{12}"'] \arrow[drr, "\tilde{\mathrm{pr}}_{23}"] & & & \\
& & C_1\times_{C_0}C_1 \arrow[dl, "\mathrm{pr}_1"'] \arrow[drr, "\mathrm{pr}_2"] & & & & C_1\times_{C_0}C_1 \arrow[dll, "\mathrm{pr}_1"'] \arrow[drr, "\mathrm{pr}_2"] & & & & C_1\times_{C_0}D_0 \arrow[dll, "(f_0)_\tau"'] \arrow[dr, "\tau_{f_0}"] & \\
& C_1 \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & & & C_1 \arrow[dl, "\sigma"'] \arrow[drr, "\tau"] & & & & C_1 \arrow[dll, "\sigma"'] & & & D_0 \\
C_0 & & C_0 & & & & C_0 & & & & &
\end{tikzcd}$$

Diagram 3.6.2

Proposition 3.6.1 *Assume that that* $\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(N) : (N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]} \to N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}$ *is an isomorphism and that* $\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\sigma(f_0)_\tau,\,\tau_{f_0}}(N) : (N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \to N^{[\sigma\mathrm{pr}_1\mathrm{pr}_{12}\tilde{\mathrm{pr}}_{123},\tau_{f_0}\tilde{\mathrm{pr}}_{23}\tilde{\mathrm{pr}}_{234}]}$ *is a monomorphism. We put* $\mu^r_{\boldsymbol{f}}(N) = E_{\sigma,\tau}(N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{-1}_{N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}}(\check{\mu}_{\boldsymbol{f}}(N)) : \sigma^*(N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}) \to \tau^*(N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})$*. Then,* $(N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}, \mu^r_{\boldsymbol{f}}(N))$ *is a representation of* $\boldsymbol{C}$*.*

Proof. It follows from (1.4.21) that the following diagram is commutative.

$$\begin{tikzcd}
N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]} \arrow[r, "N^{\mu\times_{C_0} id_{D_0}}"] \arrow[dr, "id_{N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}}"'] & N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\,\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[r, "\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(N)^{-1}"] \arrow[d, "N^{\varepsilon\times_{C_0} id_{C_1}\times_{C_0} D_0}"] & (N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[d, "(N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{\varepsilon}"] \\
& N^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "\theta_{\sigma\varepsilon,\tau\varepsilon,\sigma(f_0)_\tau,\tau_{f_0}}(N)^{-1}"] & (N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma\varepsilon,\tau\varepsilon]}
\end{tikzcd}$$

Hence a composition $N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]} \xrightarrow{\check{\mu}_{\boldsymbol{f}}(N)} (N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]} \xrightarrow{(N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{\varepsilon}} (N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma\varepsilon,\tau\varepsilon]} = N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}$ coincides with the identity morphism of $N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}$.
Note that we have the following equalities.

$$\begin{aligned}
\sigma\mathrm{pr}_1\mathrm{pr}_{12}\tilde{\mathrm{pr}}_{123} &= \sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}(\mu\times_{C_0} id_{C_0}\times_{C_0} id_{D_0}) = \sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}(id_{C_0}\times_{C_0}\mu\times_{C_0} id_{D_0})\\
\tau_{f_0}\tilde{\mathrm{pr}}_{23}\tilde{\mathrm{pr}}_{234} &= \tau_{f_0}\tilde{\mathrm{pr}}_{23}(\mu\times_{C_0} id_{C_0}\times_{C_0} id_{D_0}) = \tau_{f_0}\tilde{\mathrm{pr}}_{23}(id_{C_0}\times_{C_0}\mu\times_{C_0} id_{D_0})\\
\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12} &= \sigma(f_0)_\tau(\mu\times_{C_0} id_{D_0})\\
\tau_{f_0}\tilde{\mathrm{pr}}_{23} &= \tau_{f_0}(\mu\times_{C_0} id_{D_0})
\end{aligned}$$

It follows from (2) of (1.4.7), (1.4.21) and (1.4.25) that the following diagram commutes.

$$\begin{tikzcd}
N^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "N^{\mu\times_{C_0} id_{D_0}}"] \arrow[d, "N^{\mu\times_{C_0} id_{D_0}}"] & N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[d, "N^{\mu\times_{C_0} id_{C_0}\times_{C_0} id_{D_0}}"] & (N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[l, "\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(N)"'] \arrow[d, "(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{\mu}"'] \\
N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[r, "N^{id_{C_0}\times_{C_0}\mu\times_{C_0} id_{D_0}}"] & N^{[\sigma\mathrm{pr}_1\mathrm{pr}_{12}\tilde{\mathrm{pr}}_{123},\tau_{f_0}\tilde{\mathrm{pr}}_{23}\tilde{\mathrm{pr}}_{234}]} & (N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[l, "\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\sigma(f_0)_\tau,\tau_{f_0}}(N)"'] \\
(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[u, "\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(N)"] \arrow[r, "(N^{\mu\times_{C_0} id_{D_0}})^{[\sigma,\tau]}"] & (N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]})^{[\sigma,\tau]} \arrow[u, "\theta^{\sigma,\tau,\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}}(N)"] & ((N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[l, "\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(N)^{[\sigma,\tau]}"'] \arrow[u, "\theta^{\sigma,\tau,\sigma,\tau}(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})"']
\end{tikzcd}$$

Thus the following diagram commutes.

$$\begin{tikzcd}
N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]} \arrow[r, "\check{\mu}_{\boldsymbol{f}}(N)"] \arrow[dr, "\check{\mu}_{\boldsymbol{f}}(N)"'] & (N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "\check{\mu}_{\boldsymbol{f}}(N)^{[\sigma,\tau]}"] & ((N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[d, "\theta^{\sigma,\tau,\sigma,\tau}(N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})"] \\
& (N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "(N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{\mu}"] & (N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}
\end{tikzcd}$$

and $\check{\mu}_{\boldsymbol{f}}(N)$ satisfies the conditions of (3.4.2). □

Proposition 3.6.2 *Let* $\varphi : M \to N$ *be a morphisms in* $\mathcal{F}_{D_0}$*. Assume that that the following upper morphism is an isomorphism and that the lower morphism is a monomorphism for* $L = M, N$*.*

$$\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(L) : (L^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]} \longrightarrow L^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}$$
$$\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\sigma(f_0)_\tau,\,\tau_{f_0}}(L) : (L^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \longrightarrow L^{[\sigma\mathrm{pr}_1\mathrm{pr}_{12}\tilde{\mathrm{pr}}_{123},\tau_{f_0}\tilde{\mathrm{pr}}_{23}\tilde{\mathrm{pr}}_{234}]}$$

Then, $\varphi^{[\sigma(f_0)_\tau,\,\tau_{f_0}]} : (M^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}, \mu^r_{\boldsymbol{f}}(M)) \to (N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}, \mu^r_{\boldsymbol{f}}(N))$ *is a morphism of representations of* $\boldsymbol{C}$*.*

Proof. The following diagram is commutative by (1.4.9) and (1.4.21).

$$\begin{tikzcd}
M^{[\sigma(f_0)_\tau,\,\tau_{f_0}]} \arrow[r, "M^{\mu\times_{C_0} id_{D_0}}"] \arrow[d, "\varphi^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}"] & M^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[r, "\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(M)^{-1}"] \arrow[d, "\varphi^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}"] & (M^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[d, "(\varphi^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]}"] \\
N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]} \arrow[r, "N^{\mu\times_{C_0} id_{D_0}}"] & N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[r, "\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(N)^{-1}"] & (N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]}
\end{tikzcd}$$

Hence the assertion follows from (3.4.6). □

We consider the following cartesian square.

$$\begin{tikzcd}
C_1 \times_{C_0} D_1 \arrow[r, "\tilde{\mathrm{pr}}_2"] \arrow[d, "\tilde{\mathrm{pr}}_1"] & D_1 \arrow[d, "f_0\sigma'"] \\
C_1 \arrow[r, "\tau"] & C_0
\end{tikzcd}$$

There exists unique morphisms $id_{C_1} \times_{C_0} \sigma' : C_1 \times_{C_0} D_1 \to C_1 \times_{C_0} D_0$ and $id_{C_1} \times_{C_0} f_1 : C_1 \times_{C_0} D_1 \to C_1 \times_{C_0} C_1$ that satisfy $\tau_{f_0}(id_{C_1} \times_{C_0} \sigma') = \sigma'\tilde{\mathrm{pr}}_2$, $(f_0)_\tau(id_{C_1} \times_{C_0} \sigma') = \tilde{\mathrm{pr}}_1$ and $\mathrm{pr}_1(id_{C_1} \times_{C_0} f_1) = \tilde{\mathrm{pr}}_1$, $\mathrm{pr}_2(id_{C_1} \times_{C_0} f_1) = f_1\tilde{\mathrm{pr}}_2$.

$$\begin{tikzcd}
C_1 \times_{C_0} D_1 \arrow[rrd, bend left, "\tilde{\mathrm{pr}}_1"] \arrow[rd, dashed, "id_{C_1} \times_{C_0} \sigma'"] \arrow[dd, "\tilde{\mathrm{pr}}_2"] & & \\
& C_1 \times_{C_0} D_0 \arrow[r, "(f_0)_\tau"] \arrow[d, "\tau_{f_0}"] & C_1 \arrow[d, "\tau"] \\
D_1 \arrow[r, "\sigma'"] & D_0 \arrow[r, "f_0"] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \times_{C_0} D_1 \arrow[rrd, bend left, "\tilde{\mathrm{pr}}_1"] \arrow[rd, dashed, "id_{C_1} \times_{C_0} f_1"] \arrow[dd, "\tilde{\mathrm{pr}}_2"] & & \\
& C_1 \times_{C_0} C_1 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & C_1 \arrow[d, "\tau"] \\
D_1 \arrow[r, "f_1"] & C_1 \arrow[r, "\sigma"] & C_0
\end{tikzcd}$$

We note that the following diagrams are cartesian.

$$\begin{tikzcd}
C_1 \times_{C_0} D_1 \arrow[r, "id_{C_1} \times_{C_0} \sigma'"] \arrow[d, "\tilde{\mathrm{pr}}_2"] & C_1 \times_{C_0} D_0 \arrow[d, "\tau_{f_0}"] \\
D_1 \arrow[r, "\sigma'"] & D_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \times_{C_0} D_1 \arrow[r, "id_{C_1} \times_{C_0} f_1"] \arrow[d, "\tilde{\mathrm{pr}}_2"] & C_1 \times_{C_0} C_1 \arrow[d, "\mathrm{pr}_2"] \\
D_1 \arrow[r, "f_1"] & C_1
\end{tikzcd}$$

Since $\tau\mu(id_{C_1} \times_{C_0} f_1) = \tau\mathrm{pr}_2(id_{C_1} \times_{C_0} f_1) = \tau f_1\tilde{\mathrm{pr}}_2 = f_0\tau'\tilde{\mathrm{pr}}_2$, there exists unique morphism

$$(\mu(id_{C_1} \times_{C_0} f_1), \tau'\tilde{\mathrm{pr}}_2) : C_1 \times_{C_0} D_1 \to C_1 \times_{C_0} D_0$$

that satisfies $\tau_{f_0}(\mu(id_{C_1} \times_{C_0} f_1), \tau'\tilde{\mathrm{pr}}_2) = \tau'\tilde{\mathrm{pr}}_2$ and $(f_0)_\tau(\mu(id_{C_1} \times_{C_0} f_1), \tau'\tilde{\mathrm{pr}}_2) = \mu(id_{C_1} \times_{C_0} f_1)$. Hence we have

$$\sigma(f_0)_\tau(\mu(id_{C_1} \times_{C_0} f_1), \tau'\tilde{\mathrm{pr}}_2) = \sigma\mu(id_{C_1} \times_{C_0} f_1) = \sigma\mathrm{pr}_1(id_{C_1} \times_{C_0} f_1) = \sigma\tilde{\mathrm{pr}}_1 = \sigma(f_0)_\tau(id_{C_1} \times_{C_0} \sigma').$$

We also consider the following cartesian square.

$$\begin{tikzcd}
C_1 \times_{C_0} C_1 \times_{C_0} D_1 \arrow[r, "\check{\mathrm{pr}}_1"] \arrow[d, "\bar{\mathrm{pr}}_{23}"] & C_1 \arrow[d, "\tau"] \\
D_1 \times_{C_0} C_1 \arrow[r, "\sigma\bar{\mathrm{pr}}_1"] & C_0
\end{tikzcd}$$

Assumption 3.6.3 *For a representation (N, ζ) of $\boldsymbol{D}$, we put $\check{\zeta} = E_{\sigma',\tau'}(N)_N : N \to N^{[\sigma',\tau']}$. We assume the following.*

(i) An equalizer of the following morphisms in $\mathcal{F}_{C_0}$ exists.

$$N^{[\sigma(f_0)_\tau, \tau_{f_0}]} \xrightarrow{\check{\zeta}^{[\sigma(f_0)_\tau, \tau_{f_0}]}} (N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau, \tau_{f_0}]} \xrightarrow{\theta^{\sigma(f_0)_\tau, \tau_{f_0}, \sigma', \tau'}(N)} N^{[\sigma(f_0)_\tau(id_{C_1} \times_{C_0} \sigma'), \tau'\tilde{\mathrm{pr}}_2]}$$

$$N^{[\sigma(f_0)_\tau, \tau_{f_0}]} \xrightarrow{N^{(\mu(id_{C_1} \times_{C_0} f_1), \tau'\tilde{\mathrm{pr}}_2)}} N^{[\sigma(f_0)_\tau(\mu(id_{C_1} \times_{C_0} f_1), \tau'\tilde{\mathrm{pr}}_2), \tau_{f_0}(\mu(id_{C_1} \times_{C_0} f_1), \tau'\tilde{\mathrm{pr}}_2)]} = N^{[\sigma(f_0)_\tau(id_{C_1} \times_{C_0} \sigma'), \tau'\tilde{\mathrm{pr}}_2]}$$

(ii) Let us denote by $E^{\boldsymbol{f}}_{(N,\zeta)} : (N, \zeta)^{\boldsymbol{f}} \to N^{[\sigma(f_0)_\tau, \tau_{f_0}]}$ an equalizer of the above morphisms. Then

$$\big(E^{\boldsymbol{f}}_{(N,\zeta)}\big)^{[\sigma,\tau]} : ((N, \zeta)^{\boldsymbol{f}})^{[\sigma,\tau]} \to (N^{[\sigma(f_0)_\tau, \tau_{f_0}]})^{[\sigma,\tau]}$$

is an equalizer of the following morphisms.

$$(N^{[\sigma(f_0)_\tau, \tau_{f_0}]})^{[\sigma,\tau]} \xrightarrow{(\check{\zeta}^{[\sigma(f_0)_\tau, \tau_{f_0}]})^{[\sigma,\tau]}} ((N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau, \tau_{f_0}]})^{[\sigma,\tau]} \xrightarrow{\theta^{\sigma(f_0)_\tau, \tau_{f_0}, \sigma', \tau'}(N)^{[\sigma,\tau]}} (N^{[\sigma(f_0)_\tau(id_{C_1} \times_{C_0} \sigma'), \tau'\tilde{\mathrm{pr}}_2]})^{[\sigma,\tau]}$$

$$(N^{[\sigma(f_0)_\tau, \tau_{f_0}]})^{[\sigma,\tau]} \xrightarrow{(N^{(\mu(id_{C_1} \times_{C_0} f_1), \tau'\tilde{\mathrm{pr}}_2)})^{[\sigma,\tau]}} (N^{[\sigma(f_0)_\tau(id_{C_1} \times_{C_0} \sigma'), \tau'\tilde{\mathrm{pr}}_2]})^{[\sigma,\tau]}$$

(iii) The following map is injective.

$$(\tau\mu)^*\big(E^{\boldsymbol{f}}_{(N,\zeta)}\big)_* : \mathcal{F}_{C_1 \times_{C_0} C_1}((\sigma\mu)^*((N, \zeta)^{\boldsymbol{f}}), (\tau\mu)^*((N, \zeta)^{\boldsymbol{f}})) \to \mathcal{F}_{C_1 \times_{C_0} C_1}((\sigma\mu)^*((N, \zeta)^{\boldsymbol{f}}), (\tau\mu)^*(N^{[\sigma(f_0)_\tau, \tau_{f_0}]}))$$

(iv) $\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(N) : (N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]} \to N^{[\sigma \mathrm{pr}_1 \tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}$ *is an isomorphism.*
(v) The following morphisms are monomorphisms.

$$\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\sigma(f_0)_\tau,\,\tau_{f_0}}(N) : (N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \to N^{[\sigma\mathrm{pr}_1\mathrm{pr}_{12}\tilde{\mathrm{pr}}_{123},\tau_{f_0}\tilde{\mathrm{pr}}_{23}\tilde{\mathrm{pr}}_{234}]}$$
$$\theta^{\sigma,\tau,\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2}(N) : (N^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]})^{[\sigma,\tau]} \longrightarrow N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}(id_{C_1}\times_{C_0}id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2\tilde{\mathrm{pr}}_{23}]}$$

The following diagram commutes.

$$\begin{tikzcd}
C_1\times_{C_0}C_1 \arrow[d, "\mu"] & C_1\times_{C_0}C_1\times_{C_0}D_0 \arrow[l, "\tilde{\mathrm{pr}}_{12}"'] \arrow[r, "\tilde{\mathrm{pr}}_{23}"] \arrow[d, "\mu\times_{C_0}id_{D_0}"] & C_1\times_{C_0}D_0 \arrow[d, "\tau_{f_0}"] \\
C_1 & C_1\times_{C_0}D_0 \arrow[l, "(f_0)_\tau"'] \arrow[r, "\tau_{f_0}"] & D_0
\end{tikzcd}$$

Hence we have $\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12} = \sigma\mu\tilde{\mathrm{pr}}_{12} = \sigma(f_0)_\tau(\mu\times_{C_0} id_{D_0})$ and $\tau_{f_0}\tilde{\mathrm{pr}}_{23} = \tau_{f_0}(\mu\times_{C_0} id_{D_0})$.
Consider the following diagram whose rhombuses are all cartesian.

$$\begin{tikzcd}
 & & & & C_1\times_{C_0}C_1\times_{C_0}D_1 \arrow[dl, "id_{C_1}\times_{C_0}id_{C_1}\times_{C_0}\sigma'"'] \arrow[drr, "\tilde{\mathrm{pr}}_{23}"] & & & & \\
 & & & C_1\times_{C_0}C_1\times_{C_0}D_0 \arrow[dll, "\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}"'] \arrow[dr, "\tilde{\mathrm{pr}}_{23}"] & & & C_1\times_{C_0}D_1 \arrow[dll, "id_{C_1}\times_{C_0}\sigma'"'] \arrow[dr, "\tilde{\mathrm{pr}}_2"] & & \\
 & C_1 \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & & & C_1\times_{C_0}D_0 \arrow[dll, "\sigma(f_0)_\tau"'] \arrow[dr, "\tau_{f_0}"] & & & D_1 \arrow[dll, "\sigma'"'] \arrow[dr, "\tau'"] & \\
C_0 & & C_0 & & & D_0 & & & D_0
\end{tikzcd}$$

It follows from (1.4.25) that

$$\begin{tikzcd}
((N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(N^{[\sigma',\tau']})"] \arrow[d, "\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(N)^{[\sigma,\tau]}"] & (N^{[\sigma',\tau']})^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[d, "\theta^{\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23},\sigma',\tau'}(N)"] \\
(N^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2}(N)"] & N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}(id_{C_1}\times_{C_0}id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2\tilde{\mathrm{pr}}_{23}]}
\end{tikzcd}$$

is commutative. The following diagrams are commutative by (1.4.21), (1.4.19), (1.4.9), respectively.

$$\begin{tikzcd}
(N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "(N^{[\sigma',\tau']})^{\mu\times_{C_0}id_{D_0}}"] \arrow[d, "\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(N)"] & (N^{[\sigma',\tau']})^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[d, "\theta^{\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23},\sigma',\tau'}(N)"] \\
N^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]} \arrow[r, "N^{\mu\times_{C_0}id_{D_1}}"] & N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}(id_{C_1}\times_{C_0}id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2\tilde{\mathrm{pr}}_{23}]}
\end{tikzcd}$$

$$\begin{tikzcd}
(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(N)"] \arrow[d, "(\zeta^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]}"] & N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[d, "\zeta^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}"] \\
((N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(N^{[\sigma',\tau']})"] & (N^{[\sigma',\tau']})^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}
\end{tikzcd}$$

$$\begin{tikzcd}
N^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "N^{\mu\times_{C_0}id_{D_0}}"] \arrow[d, "\zeta^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] & N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[d, "\zeta^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}"] \\
(N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "(N^{[\sigma',\tau']})^{\mu\times_{C_0}id_{D_0}}"] & (N^{[\sigma',\tau']})^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}
\end{tikzcd}$$

The associativity of μ implies that a diagram

$$\begin{tikzcd}
C_1\times_{C_0}C_1\times_{C_0}D_1 \arrow[r, "\mu\times_{C_0}id_{D_1}"] \arrow[d, "id_{C_1}\times_{C_0}(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)"] & C_1\times_{C_0}D_1 \arrow[d, "(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)"] \\
C_1\times_{C_0}C_1\times_{C_0}D_0 \arrow[r, "\mu\times_{C_0}id_{D_0}"] & C_1\times_{C_0}D_0
\end{tikzcd}$$

is commutative. Hence the following diagram is commutative by (1.4.7).

$$\begin{tikzcd}
N^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "N^{\mu\times_{C_0} id_{D_0}}"] \arrow[d, "N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}"] & N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[d, "N^{id_{C_1}\times_{C_0}(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}"] \\
N^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]} \arrow[r, "N^{\mu\times_{C_0} id_{D_1}}"] & N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}(id_{C_1}\times_{C_0} id_{C_1}\times_{C_0}\sigma'),\,\tau'\tilde{\mathrm{pr}}_2\bar{\mathrm{pr}}_{23}]}
\end{tikzcd}$$

Moreover, it follows from (1.4.21) that the following diagram commutes.

$$\begin{tikzcd}
(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(N)"] \arrow[d, "(N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)})^{[\sigma,\tau]}"] & N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[d, "N^{id_{C_1}\times_{C_0}(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}"] \\
(N^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2}(N)"] & N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}(id_{C_1}\times_{C_0} id_{C_1}\times_{C_0}\sigma'),\,\tau'\tilde{\mathrm{pr}}_2\bar{\mathrm{pr}}_{23}]}
\end{tikzcd}$$

Since $E^{\boldsymbol{f}}_{(N,\zeta)}$ is an equalizer of $\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(N)\check{\zeta}^{[\sigma(f_0)_\tau,\tau_{f_0}]}$ and $N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}$, we have

$$\begin{aligned}
&\theta^{\sigma,\tau,\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2}(N)(\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(N)\check{\zeta}^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]}\check{\mu}_{\boldsymbol{f}}(N)E^{\boldsymbol{f}}_{(N,\zeta)} \\
&\quad= \theta^{\sigma,\tau,\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2}(N)\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(N)^{[\sigma,\tau]}(\check{\zeta}^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]}\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(N)^{-1}N^{\mu\times_{C_0} id_{D_0}}E^{\boldsymbol{f}}_{(N,\zeta)} \\
&\quad= \theta^{\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23},\sigma',\tau'}(N)\check{\zeta}^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}N^{\mu\times_{C_0} id_{D_0}}E^{\boldsymbol{f}}_{(N,\zeta)} \\
&\quad= \theta^{\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23},\sigma',\tau'}(N)(N^{[\sigma',\tau']})^{\mu\times_{C_0} id_{D_0}}\check{\zeta}^{[\sigma(f_0)_\tau,\tau_{f_0}]}E^{\boldsymbol{f}}_{(N,\zeta)} \\
&\quad= N^{\mu\times_{C_0} id_{D_1}}\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(N)\check{\zeta}^{[\sigma(f_0)_\tau,\tau_{f_0}]}E^{\boldsymbol{f}}_{(N,\zeta)} = N^{\mu\times_{C_0} id_{D_1}}N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}E^{\boldsymbol{f}}_{(N,\zeta)} \\
&\quad= N^{id_{C_1}\times_{C_0}(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}N^{\mu\times_{C_0} id_{D_0}}E^{\boldsymbol{f}}_{(N,\zeta)} \\
&\quad= N^{id_{C_1}\times_{C_0}(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(N)\check{\mu}_{\boldsymbol{f}}(N)E^{\boldsymbol{f}}_{(N,\zeta)} \\
&\quad= \theta^{\sigma,\tau,\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2}(N)(N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)})^{[\sigma,\tau]}\check{\mu}_{\boldsymbol{f}}(N)E^{\boldsymbol{f}}_{(N,\zeta)}.
\end{aligned}$$

Therefore, it follows from the assumption (v) of (3.6.3) that we have

$$(\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(N)\check{\zeta}^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]}\check{\mu}_{\boldsymbol{f}}(N)E^{\boldsymbol{f}}_{(N,\zeta)} = (N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)})^{[\sigma,\tau]}\check{\mu}_{\boldsymbol{f}}(N)E^{\boldsymbol{f}}_{(N,\zeta)}.$$

Hence (ii) of (3.6.3) implies that there exists unique morphism $\check{\zeta}_{\boldsymbol{f}} : (N,\zeta)^{\boldsymbol{f}} \to ((N,\zeta)^{\boldsymbol{f}})^{[\sigma,\tau]}$ that satisfies $(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}\check{\zeta}_{\boldsymbol{f}} = \check{\mu}_{\boldsymbol{f}}(N)E^{\boldsymbol{f}}_{(N,\zeta)}$. We put $\zeta^r_{\boldsymbol{f}} = E_{\sigma,\tau}((N,\zeta)^{\boldsymbol{f}})^{-1}_{(N,\zeta)^{\boldsymbol{f}}}(\check{\zeta}_{\boldsymbol{f}}) : \sigma^*((N,\zeta)^{\boldsymbol{f}}) \to \tau^*((N,\zeta)^{\boldsymbol{f}})$.

Proposition 3.6.4 *$((N,\zeta)^{\boldsymbol{f}},\zeta^r_{\boldsymbol{f}})$ is a representation of $\boldsymbol{C}$ and $E^{\boldsymbol{f}}_{(N,\zeta)} : ((N,\zeta)^{\boldsymbol{f}},\zeta^r_{\boldsymbol{f}}) \to (N^{[\sigma(f_0)_\tau,\tau_{f_0}]},\mu^r_{\boldsymbol{f}}(N))$ is a morphism of representations of $\boldsymbol{C}$.*

Proof. It follows from (3.4.5) that $(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}\check{\zeta}_{\boldsymbol{f}} = \check{\mu}_{\boldsymbol{f}}(N)E^{\boldsymbol{f}}_{(N,\zeta)}$ implies the commutativity of the following diagram.

$$\begin{tikzcd}
\sigma^*((N,\zeta)^{\boldsymbol{f}}) \arrow[r, "\zeta^r_{\boldsymbol{f}}"] \arrow[d, "\sigma^*\left(E^{\boldsymbol{f}}_{(N,\zeta)}\right)"] & \tau^*((N,\zeta)^{\boldsymbol{f}}) \arrow[d, "\tau^*\left(E^{\boldsymbol{f}}_{(N,\zeta)}\right)"] \\
\sigma^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]}) \arrow[r, "\mu^r_{\boldsymbol{f}}(N)"] & \tau^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})
\end{tikzcd}$$

Hence the assertion follows from (iii) of (3.6.3) and (1) of (3.1.5). □

We assume (3.6.3) also for a representation (M,ξ) of $\boldsymbol{D}$. Let $\varphi : (M,\xi) \to (N,\zeta)$ be a morphism of representations of $\boldsymbol{D}$. The following diagrams are commutative by (1.4.21), (1.4.4) and (1.4.9).

$$\begin{tikzcd}
M^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "\check{\xi}^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] \arrow[d, "\varphi^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] & (M^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(M)"] \arrow[d, "(\varphi^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] & M^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]} \arrow[d, "\varphi^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}"] \\
N^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "\check{\zeta}^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] & (N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(N)"] & N^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}
\end{tikzcd}$$

$$\begin{array}{ccc} M^{[\sigma(f_0)_\tau,\,\tau_{f_0}]} & \xrightarrow{M^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}} & M^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]} \\ \downarrow{\scriptstyle\varphi^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}} & & \downarrow{\scriptstyle\varphi^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}} \\ N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]} & \xrightarrow{N^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}} & N^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]} \end{array}$$

Hence there exists unique morphism $\varphi^{\boldsymbol{f}} : (M,\xi)^{\boldsymbol{f}} \to (N,\zeta)^{\boldsymbol{f}}$ that satisfies $E^{\boldsymbol{f}}_{(N,\zeta)}\varphi^{\boldsymbol{f}} = \varphi^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}E^{\boldsymbol{f}}_{(M,\xi)}$.

Proposition 3.6.5 *$\varphi^{\boldsymbol{f}} : ((M,\xi)^{\boldsymbol{f}},\xi^r_{\boldsymbol{f}}) \to ((N,\zeta)^{\boldsymbol{f}},\zeta^r_{\boldsymbol{f}})$ is a morphism of representations of $\boldsymbol{C}$.*

Proof. It follows from (3.6.2) that the inner rectangle of the following diagram is commutative.

$$\begin{tikzcd}
(M,\xi)^{\boldsymbol{f}} \arrow[rrr, "\check{\xi}_{\boldsymbol{f}}"] \arrow[dr, "E^{\boldsymbol{f}}_{(M,\xi)}"] \arrow[ddd, "\varphi^{\boldsymbol{f}}"'] & & & ((M,\xi)^{\boldsymbol{f}})^{[\sigma,\tau]} \arrow[dl, "(E^{\boldsymbol{f}}_{(M,\xi)})^{[\sigma,\tau]}"'] \arrow[ddd, "(\varphi^{\boldsymbol{f}})^{[\sigma,\tau]}"] \\
& M^{[\sigma(f_0)_\tau,\,\tau_{f_0}]} \arrow[r, "\check{\mu}_{\boldsymbol{f}}(M)"] \arrow[d, "\varphi^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}"] & (M^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[d, "(\varphi^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]}"] & \\
& N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]} \arrow[r, "\check{\mu}_{\boldsymbol{f}}(N)"] & (N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]} & \\
(N,\zeta)^{\boldsymbol{f}} \arrow[rrr, "\check{\zeta}_{\boldsymbol{f}}"] \arrow[ur, "E^{\boldsymbol{f}}_{(N,\zeta)}"] & & & ((N,\zeta)^{\boldsymbol{f}})^{[\sigma,\tau]} \arrow[ul, "(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}"']
\end{tikzcd}$$

Then, by the definitions of $\check{\xi}_{\boldsymbol{f}}$, $\check{\zeta}_{\boldsymbol{f}}$ and $\varphi^{\boldsymbol{f}}$, we have

$$\begin{aligned}(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}\check{\zeta}_{\boldsymbol{f}}\varphi^{\boldsymbol{f}} &= \check{\mu}_{\boldsymbol{f}}(N)E^{\boldsymbol{f}}_{(N,\zeta)}\varphi^{\boldsymbol{f}} = \check{\mu}_{\boldsymbol{f}}(N)\varphi^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}E^{\boldsymbol{f}}_{(M,\xi)} = (\varphi^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]}\check{\mu}_{\boldsymbol{f}}(M)E^{\boldsymbol{f}}_{(M,\xi)} \\ &= (\varphi^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]}(E^{\boldsymbol{f}}_{(M,\xi)})^{[\sigma,\tau]}\check{\xi}_{\boldsymbol{f}} = (E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}\check{\xi}_{\boldsymbol{f}}(\varphi^{\boldsymbol{f}})^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}.\end{aligned}$$

Since $(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}$ is an epimorphism by (ii) of (3.6.3), the above equality implies $\check{\zeta}_{\boldsymbol{f}}\varphi^{\boldsymbol{f}} = (\varphi^{\boldsymbol{f}})^{[\sigma,\tau]}\check{\xi}_{\boldsymbol{f}}$. Therefore $\varphi^{\boldsymbol{f}}$ is a morphism of representations of $\boldsymbol{D}$ by (3.4.5). □

Define functors $S,T,U : \mathcal{P} \to \mathcal{E}$ and natural transformations $\alpha : S \to T$, $\beta : T \to U$ as follows.

$S(0)=D_1$	$S(1)=D_0$	$S(2)=D_1$	$S(3)=D_0$	$S(4)=D_0$	$S(5)=D_0$
$S(\tau_{01})=\sigma'$	$S(\tau_{02})=id_{D_1}$	$S(\tau_{13})=id_{D_0}$	$S(\tau_{14})=id_{D_0}$	$S(\tau_{24})=\sigma'$	$S(\tau_{25})=\tau'$
$T(0)=C_1\times_{C_0}D_1$	$T(1)=C_1\times_{C_0}D_0$	$T(2)=D_1$	$T(3)=C_0$	$T(4)=D_0$	$T(5)=D_0$
$T(\tau_{01})=id_{C_1}\times_{C_0}\sigma'$	$T(\tau_{02})=\tilde{\mathrm{pr}}_2$	$T(\tau_{13})=\sigma(f_0)_\tau$	$T(\tau_{14})=\tau_{f_0}$	$T(\tau_{24})=\sigma'$	$T(\tau_{25})=\tau'$
$U(0)=C_1\times_{C_0}C_1\times_{C_0}D_0$	$U(1)=C_1$	$U(2)=C_1\times_{C_0}D_0$	$U(3)=C_0$	$U(4)=C_0$	$U(5)=D_0$
$U(\tau_{01})=\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}$	$U(\tau_{02})=\tilde{\mathrm{pr}}_{23}$	$U(\tau_{13})=\sigma$	$U(\tau_{14})=\tau$	$U(\tau_{24})=\sigma(f_0)_\tau$	$U(\tau_{25})=\tau_{f_0}$
$\alpha_0=(f_1\varepsilon'\sigma', id_{D_1})$	$\alpha_1=(f_1\varepsilon', id_{D_0})$	$\alpha_2=id_{D_1}$	$\alpha_3=f_0$	$\alpha_4=id_{D_0}$	$\alpha_5=id_{D_0}$
$\beta_0=(\tilde{\mathrm{pr}}_1, f_1\tilde{\mathrm{pr}}_2, \tau'\tilde{\mathrm{pr}}_2)$	$\beta_1=(f_0)_\tau$	$\beta_2=(f_1,\tau')$	$\beta_3=id_{C_0}$	$\beta_4=f_0$	$\beta_5=id_{D_0}$

Hence if we define functors $S_i,T_i,U_i : \mathcal{Q} \to \mathcal{E}$ for $i=0,1,2$ by

$S_0(0)=S(0)$	$S_0(1)=S(3)$	$S_0(2)=S(5)$	$S_0(\tau_{01})=S(\tau_{13}\tau_{01})$	$S_0(\tau_{02})=S(\tau_{25}\tau_{02})$
$T_0(0)=T(0)$	$T_0(1)=T(3)$	$T_0(2)=T(5)$	$T_0(\tau_{01})=T(\tau_{13}\tau_{01})$	$T_0(\tau_{02})=T(\tau_{25}\tau_{02})$
$U_0(0)=U(0)$	$U_0(1)=U(3)$	$U_0(2)=U(5)$	$U_0(\tau_{01})=U(\tau_{13}\tau_{01})$	$U_0(\tau_{02})=U(\tau_{25}\tau_{02})$
$S_1(0)=S(1)$	$S_1(1)=S(3)$	$S_1(2)=S(4)$	$S_1(\tau_{01})=S(\tau_{13})$	$S_1(\tau_{02})=S(\tau_{14})$
$T_1(0)=T(1)$	$T_1(1)=T(3)$	$T_1(2)=T(4)$	$T_1(\tau_{01})=T(\tau_{13})$	$T_1(\tau_{02})=T(\tau_{14})$
$U_1(0)=U(1)$	$U_1(1)=U(3)$	$U_1(2)=U(4)$	$U_1(\tau_{01})=U(\tau_{13})$	$U_1(\tau_{02})=U(\tau_{14})$
$S_2(0)=S(2)$	$S_2(1)=S(4)$	$S_2(2)=S(5)$	$S_2(\tau_{01})=S(\tau_{24})$	$S_2(\tau_{02})=S(\tau_{25})$
$T_2(0)=T(2)$	$T_2(1)=T(4)$	$T_2(2)=T(5)$	$T_2(\tau_{01})=T(\tau_{24})$	$T_2(\tau_{02})=T(\tau_{25})$
$U_2(0)=U(2)$	$U_2(1)=U(4)$	$U_2(2)=U(5)$	$U_2(\tau_{01})=U(\tau_{24})$	$U_2(\tau_{02})=U(\tau_{25})$

and natural transformations $\alpha^i : S_i \to T_i$, $\beta^i : T_i \to U_i$ for $i=0,1,2$ by

$$\begin{array}{ccccccccc} \alpha^0_0=\alpha_0 & \alpha^0_1=\alpha_3 & \alpha^0_2=\alpha_5 & \alpha^1_0=\alpha_1 & \alpha^1_1=\alpha_3 & \alpha^1_2=\alpha_4 & \alpha^2_0=\alpha_2 & \alpha^2_1=\alpha_4 & \alpha^2_2=\alpha_5, \\ \beta^0_0=\beta_0 & \beta^0_1=\beta_3 & \beta^0_2=\beta_5 & \beta^1_0=\beta_1 & \beta^1_1=\beta_3 & \beta^1_2=\beta_4 & \beta^2_0=\beta_2 & \beta^2_1=\beta_4 & \beta^2_2=\beta_5, \end{array}$$

then we have $S_0 = S_2 = T_2$, $U_2 = T_1$.

We note that $\omega(k\,;f,g)^N = N^k : N^{[f,g]} \to N^{[fk,gk]}$ for morphisms $f : X \to Y$, $g : X \to Z$ and $k : W \to X$ of $\mathcal{E}$ and $N \in \operatorname{Ob}\mathcal{F}_Z$ by (1.4.29).

Lemma 3.6.6 *For a representation (N,ζ) of $\boldsymbol{D}$, the following diagram is commutative.*

$$\begin{tikzcd}
f_0^*((N,\zeta)^{\boldsymbol{f}}) \arrow[r, "f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})"] \arrow[d, "f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})"] & f_0^*(N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}) \arrow[r, "\alpha^{1N}"] & N^{[id_{D_0},\,id_{D_0}]} \arrow[d, equal] \\
f_0^*(N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}) \arrow[r, "\beta^{2N}"] & N^{[\sigma',\tau']} & N \arrow[l, "\check{\zeta}"']
\end{tikzcd}$$

Proof. The following diagram is commutative by the definition of $E^{\boldsymbol{f}}_{(N,\zeta)}$.

$$\begin{tikzcd}
(N,\zeta)^{\boldsymbol{f}} \arrow[rr, "E^{\boldsymbol{f}}_{(N,\zeta)}"] \arrow[d, "E^{\boldsymbol{f}}_{(N,\zeta)}"] & & N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]} \arrow[d, "N^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}"] \\
N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]} \arrow[r, "\check{\zeta}^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] & (N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(N)"] & N^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}
\end{tikzcd}$$

It follows from (1.4.33) that the following diagram is commutative.

$$\begin{tikzcd}
f_0^*((N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]}) \arrow[r, "\alpha^{1N^{[\sigma',\tau']}}"] \arrow[d, "f_0^*(\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(N))"] & (N^{[\sigma',\tau']})^{[id_{D_0},id_{D_0}]} \arrow[r, "(\alpha^{2N})^{[id_{D_0},id_{D_0}]}"] & (N^{[\sigma',\tau']})^{[id_{D_0},id_{D_0}]} \arrow[d, "\theta^{\sigma',\tau',id_{D_0},id_{D_0}}(N)"] \\
f_0^*(N^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}) \arrow[rr, "\alpha^{0N}"] & & N^{[\sigma' id_{D_1},id_{D_0}\tau']}
\end{tikzcd}$$

We note that $\theta^{\sigma',\tau',id_{D_0},id_{D_0}}(N)$ and $(\alpha^{2N})^{[id_{D_0},id_{D_0}]}$ are the identity morphism of $N^{[\sigma',\tau']}$ by (1.4.26) and the definition of α^{2N}. Therefore the following diagram commutes by the commutativity of the above diagrams and (1.4.31).

$$\begin{tikzcd}
f_0^*((N,\zeta)^{\boldsymbol{f}}) \arrow[rr, "f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})"] \arrow[d, "f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})"] & & f_0^*(N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}) \arrow[d, "f_0^*(N^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)})"] \\
f_0^*(N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}) \arrow[r, "f_0^*(\check{\zeta}^{[\sigma(f_0)_\tau,\tau_{f_0}]})"] \arrow[d, "\alpha^{1N}"] & f_0^*((N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]}) \arrow[r, "f_0^*(\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(N))"] \arrow[d, "\alpha^{1N^{[\sigma',\tau']}}"] & f_0^*(N^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}) \arrow[d, "\alpha^{0N}"] \\
N \arrow[r, "\check{\zeta}"] & N^{[\sigma',\tau']} \arrow[r, "\theta^{\sigma',\tau',id_{D_0},id_{D_0}}(N)=id_{N^{[\sigma',\tau']}}"] & N^{[\sigma',\tau']}
\end{tikzcd}$$

We put $\bar{\beta} = \omega((\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)\,;\sigma(f_0)_\tau,\tau_{f_0}) : T_0 \to T_1$. Then, $\beta^2 = \bar{\beta}\alpha^0$ holds. It follows from (1.4.32) that the following diagram is commutative.

$$\begin{tikzcd}
f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]}) \arrow[r, "f_0^*(\bar{\beta}^N)"] \arrow[d, "c_{id_{C_0},f_0}(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})=id_{N^{[\sigma(f_0)_\tau,\tau_{f_0}]}}"] & f_0^*(N^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}) \arrow[r, "\alpha^{0N}"] & N^{[\sigma',\tau']} \arrow[d, "c_{id_{D_0},id_{D_0}}(N)^{[\sigma',\tau']}=id_{N^{[\sigma',\tau']}}"] \\
f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]}) \arrow[rr, "\beta^{2N}=(\bar{\beta}\alpha^0)^N"] & & N^{[\sigma',\tau']}
\end{tikzcd}$$

Since $\bar{\beta}^N = \omega((\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)\,;\sigma(f_0)_\tau,\tau_{f_0})_N = N^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}$ by (1.4.29), we have

$$\check{\zeta}\alpha^{1N}f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}) = \alpha^{0N}f_0^*(N^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)})f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}) = \alpha^{0N}f_0^*(\bar{\beta}^N)f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}) = \beta^{2N}f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}).$$

$\square$

Proposition 3.6.7 *A composition*

$$f_0^*((N,\zeta)^{\boldsymbol{f}}) \xrightarrow{f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})} f_0^*(N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}) \xrightarrow{\alpha^{1N}} N^{[id_{D_0},\,id_{D_0}]} = N$$

defines a morphism $(f_0^*((N,\zeta)^{\boldsymbol{f}}), (\zeta^r_{\boldsymbol{f}})_{\boldsymbol{f}}) \to (N,\zeta)$ of representations of $\boldsymbol{D}$.

Proof. By applying (1.4.33) to $\beta:\mathcal{P}\to\mathcal{E}$, we see that the following diagram (i) is commutative.

$$\begin{tikzcd}
(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "\beta^{1N^{[\sigma(f_0)_\tau,\tau_{f_0}]}}"] \arrow[d, "\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(N)"'] & f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "(\beta^{2N})^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] & (N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(N)"] \\
N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[rr, "\beta^{0N}=N^{(\tilde{\mathrm{pr}}_1,f_1\tilde{\mathrm{pr}}_2,\tau'\tilde{\mathrm{pr}}_2)}"] & & N^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}
\end{tikzcd}$$

diagram (i)

Let $D_1 \xleftarrow{\hat{\mathrm{pr}}_1} D_1\times_{C_0} D_0 \xrightarrow{\hat{\mathrm{pr}}_2} D_0$ be a limit of a diagram $D_1 \xrightarrow{\tau f_1} C_0 \xleftarrow{f_0} D_0$. Define a natural transformation $\bar\beta^1 : D_{\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2}\to D_{\sigma f_1,\tau f_1}$ by $\bar\beta^1_0=\hat{\mathrm{pr}}_1$, $\bar\beta^1_1=id_{C_0}$, $\bar\beta^1_2=f_0$. We also consider natural transformations $\omega(f_1\times_{C_0} id_{D_0}\,;\sigma(f_0)_\tau,\tau_{f_0}) : D_{\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2}\to D_{\sigma(f_0)_\tau,\tau_{f_0}}=T_1$ and $\omega(f_1\,;\sigma,\tau):D_{\sigma f_1,\tau f_1}\to D_{\sigma,\tau}=U_1$. Then, we have $\omega(f_1\,;\sigma,\tau)\bar\beta^1=\beta^1\omega(f_1\times_{C_0} id_{D_0}\,;\sigma(f_0)_\tau,\tau_{f_0})$ and it follows from (1.4.32) that the following diagram (ii) is commutative.

$$\begin{tikzcd}
(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "\beta^{1N^{[\sigma(f_0)_\tau,\tau_{f_0}]}}"] \arrow[d, "(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{f_1}"] \arrow[dr, "(\omega(f_1\,;\sigma,\tau)\bar\beta^1)^{N^{[\sigma(f_0)_\tau,\tau_{f_0}]}}" description] & f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{f_1\times_{C_0} id_{D_0}}"] \\
(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma f_1,\tau f_1]} \arrow[r, "\bar\beta^{1N^{[\sigma(f_0)_\tau,\tau_{f_0}]}}"] & f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2]}
\end{tikzcd}$$

diagram (ii)

The following diagram is commutative by (1.4.9).

$$\begin{tikzcd}
f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "(\beta^{2N})^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] \arrow[d, "f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{f_1\times_{C_0} id_{D_0}}"] & (N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "(N^{[\sigma',\tau']})^{f_1\times_{C_0} id_{D_0}}"] \\
f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2]} \arrow[r, "(\beta^{2N})^{[\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2]}"] & (N^{[\sigma',\tau']})^{[\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2]}
\end{tikzcd}$$

diagram (iii)

Define a natural transformation $\gamma : S_0\to D_{\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2}$ by $\gamma_0=(id_{D_1},\tau')$, $\gamma_1=f_0$, $\gamma_2=id_{D_0}$, then we have $\bar\beta^1\gamma=\omega(\sigma',\tau';f_0,f_0)$. It follows from (1.4.32) that

$$\begin{tikzcd}
f_0^*((N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma f_1,\tau f_1]}) \arrow[r, "f_0^*(\bar\beta^{1N^{[\sigma(f_0)_\tau,\tau_{f_0}]}})"] \arrow[dr, "\omega(\sigma',\tau';f_0,f_0)^{N^{[\sigma(f_0)_\tau,\tau_{f_0}]}}"'] & f_0^*(f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2]}) \arrow[d, "\gamma^{f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})}"] \\
& f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma',\tau']}
\end{tikzcd}$$

diagram (iv)

is commutative. Moreover, (1.4.31) implies that the following diagram is commutative.

$$\begin{tikzcd}
f_0^*(f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2]}) \arrow[r, "f_0^*((\beta^{2N})^{[\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2]})"] \arrow[d, "\gamma^{f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})}"] & f_0^*((N^{[\sigma',\tau']})^{[\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2]}) \arrow[d, "\gamma^{N^{[\sigma',\tau']}}"] \\
f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma',\tau']} \arrow[r, "(\beta^{2N})^{[\sigma',\tau']}"] & (N^{[\sigma',\tau']})^{[\sigma',\tau']}
\end{tikzcd}$$

diagram (v)

The following diagram is commutative by the definition of $\check\zeta_{\boldsymbol{f}}$ and (1.4.9), (1.4.21).

$$\begin{tikzcd}
f_0^*(((N,\zeta)^{\boldsymbol{f}})) \arrow[r, "f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})"] \arrow[dd, "f_0^*(\check{\zeta}_{\boldsymbol{f}})"] & f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]}) \arrow[d, "f_0^*(N^{\mu\times_{C_0} id_{D_0}})"] \\
 & f_0^*(N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}) \arrow[d, "f_0^*(\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(N))^{-1}"] \\
f_0^*(((N,\zeta)^{\boldsymbol{f}})^{[\sigma,\tau]}) \arrow[r, "f_0^*((E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]})"] \arrow[d, "f_0^*(((N,\zeta)_{\boldsymbol{f}})^{f_1})"] & f_0^*((N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]}) \arrow[d, "f_0^*((N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{f_1})"] \\
f_0^*(((N,\zeta)^{\boldsymbol{f}})^{[f_0\sigma',f_0\tau']}) \arrow[r, "f_0^*((E^{\boldsymbol{f}}_{(N,\zeta)})^{[f_0\sigma',f_0\tau']})"] \arrow[d, "\omega(\sigma',\tau',f_0,f_0)^{(N,\zeta)^{\boldsymbol{f}}}"] & f_0^*((N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[f_0\sigma',f_0\tau']}) \arrow[d, "\omega(\sigma',\tau',f_0,f_0)_{N^{[\sigma(f_0)_\tau,\tau_{f_0}]}}"] \\
f_0^*((N,\zeta)^{\boldsymbol{f}})^{[\sigma',\tau']} \arrow[r, "f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma',\tau']}"] & f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma',\tau']}
\end{tikzcd}$$

diagram (vi)

Consider natural transformations $\omega(\varepsilon'\,;\sigma',\tau') : S_1 \to S_2$ and $\omega(f_1\times_{C_0} id_{D_0}\,;\sigma(f_0)_\tau,\tau_{f_0}) : D_{\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2} \to T_2$. Then, we have the following equalities

$$\alpha^1 = \beta^2\omega(\varepsilon'\,;\sigma',\tau') \qquad \omega(f_1\times_{C_0} id_{D_0}\,;\sigma(f_0)_\tau,\tau_{f_0})\gamma = \beta^2 = \omega((\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)\,;\,\sigma(f_0)_\tau,\tau_{f_0})\alpha^0$$

It follows from (1.4.32) that the following diagrams are commutative.

$$\begin{tikzcd}
f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]}) \arrow[r, "\beta^{2N}"] \arrow[dr, "\alpha^{1N}"'] & N^{[\sigma',\tau']} \arrow[d, "N^{\varepsilon'}"] \\
 & N^{[id_{D_0},id_{D_0}]} = N
\end{tikzcd}$$

diagram (vii)

$$\begin{tikzcd}
f_0^*((N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]}) \arrow[r, "f_0^*((N^{[\sigma',\tau']})^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)})"] \arrow[d, "f_0^*((N^{[\sigma',\tau']})^{f_1\times_{C_0} id_{D_0}})"] \arrow[dr, "\beta^{2N^{[\sigma',\tau']}}"] & f_0^*((N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}) \arrow[d, "\alpha^{0N^{[\sigma',\tau']}}"] \\
f_0^*((N^{[\sigma',\tau']})^{[\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2]}) \arrow[r, "\gamma^{N^{[\sigma',\tau']}}"] & (N^{[\sigma',\tau']})^{[\sigma',\tau']}
\end{tikzcd}$$

diagram $(viii)$

We also have the following commutative diagrams by (1.4.31) and (1.4.9).

$$\begin{tikzcd}
f_0^*((N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}) \arrow[r, "f_0^*((N^{\varepsilon'})^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]})"] \arrow[d, "\alpha^{0N^{[\sigma',\tau']}}"] & f_0^*(N^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}) \arrow[d, "\alpha^{0N}"] \\
(N^{[\sigma',\tau']})^{[\sigma',\tau']} \arrow[r, "(N^{\varepsilon'})^{[\sigma',\tau']}"] & N^{[\sigma',\tau']}
\end{tikzcd}$$

diagram (ix)

$$\begin{tikzcd}
(N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "(N^{\varepsilon'})^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] \arrow[d, "(N^{[\sigma',\tau']})^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}"] & N^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}"] \\
(N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2)]} \arrow[r, "(N^{\varepsilon'})^{(\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2))}"] & N^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2)]}
\end{tikzcd}$$

diagram (x)

We put $\tilde{\zeta}_{\boldsymbol{f}} = E_{\sigma',\tau'}(f_0^*((N,\zeta)^{\boldsymbol{f}}))_{f_0^*((N,\zeta)^{\boldsymbol{f}})}((\zeta^r_{\boldsymbol{f}})_{\boldsymbol{f}})$. Then, $\tilde{\zeta}_{\boldsymbol{f}}$ is the following composition by (3.4.4).

$$f_0^*((N,\zeta)^{\boldsymbol{f}}) \xrightarrow{f_0^*(\check{\zeta}_{\boldsymbol{f}})} f_0^*(((N,\zeta)^{\boldsymbol{f}})^{[\sigma,\tau]}) \xrightarrow{f_0^*(((N,\zeta)^{\boldsymbol{f}})^{f_1})} f_0^*(((N,\zeta)^{\boldsymbol{f}})^{[f_0\sigma',f_0\tau']}) \xrightarrow{\omega(\sigma',\tau';f_0,f_0)^{(N,\zeta)^{\boldsymbol{f}}}} f_0^*((N,\zeta)^{\boldsymbol{f}})^{[\sigma',\tau']}$$

We note that $(\mu\times_{C_0} id_{D_0})(\tilde{\mathrm{pr}}_1, f_1\tilde{\mathrm{pr}}_2, \tau'\tilde{\mathrm{pr}}_2) = (\mu(id_{C_1}\times_{C_0} f_1), \tau'\tilde{\mathrm{pr}}_2)$ holds and recall that $E^{\boldsymbol{f}}_{(N,\zeta)}$ is an equalizer of $N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}$ and $\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(N)\check{\zeta}^{[\sigma(f_0)_\tau,\tau_{f_0}]}$. We also have $\alpha^{0N} f_0^*(N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}) = \beta^{2N}$ by (1.4.32). Therefore by the commutativity of diagrams $(i) \sim (ix)$ and (3.6.6), we have

$$\begin{aligned}
(\alpha^{1N} f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}))^{[\sigma',\tau']}\tilde{\zeta}_{\boldsymbol{f}} &= (N^{\varepsilon'})^{[\sigma',\tau']}(\beta^{2N})^{[\sigma',\tau']} f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma',\tau']}\omega(\sigma',\tau'; f_0, f_0)^{(N,\zeta)^{\boldsymbol{f}}} f_0^*(((N,\zeta)^{\boldsymbol{f}})^{f_1}) f_0^*(\check{\zeta}^{\boldsymbol{f}}) \\
&= (N^{\varepsilon'})^{[\sigma',\tau']}\gamma_{N^{[\sigma',\tau']}} f_0^*((N^{[\sigma',\tau']})^{f_1\times_{C_0} id_{D_0}}) f_0^*(\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(N)^{-1}) \\
&\quad f_0^*(N^{(\tilde{\mathrm{pr}}_1, f_1\tilde{\mathrm{pr}}_2,\tau'\tilde{\mathrm{pr}}_2)}) f_0^*(N^{\mu\times_{C_0} id_{D_0}}) f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}) \\
&= (N^{\varepsilon'})^{[\sigma',\tau']}\alpha^{0N^{[\sigma',\tau']}} f_0^*((N^{[\sigma',\tau']})^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}) \\
&\quad f_0^*(\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(N)^{-1} N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)} E^{\boldsymbol{f}}_{(N,\zeta)}) \\
&= \alpha^{0N} f_0^*((N^{\varepsilon'})^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}(N^{[\sigma',\tau']})^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}\check{\zeta}^{[\sigma(f_0)_\tau,\tau_{f_0}]} E^{\boldsymbol{f}}_{(N,\zeta)}) \\
&= \alpha^{0N} f_0^*(N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}(N^{\varepsilon'})^{[\sigma(f_0)_\tau,\tau_{f_0}]}\check{\zeta}^{[\sigma(f_0)_\tau,\tau_{f_0}]} E^{\boldsymbol{f}}_{(N,\zeta)}) \\
&= \alpha^{0N} f_0^*(N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}) f_0^*((N^{\varepsilon'}\check{\zeta})^{[\sigma(f_0)_\tau,\tau_{f_0}]} E^{\boldsymbol{f}}_{(N,\zeta)}) \\
&= \beta^{2N} f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}) = \check{\zeta}\alpha^{1N} f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}).
\end{aligned}$$

This shows that $\alpha^{1N} f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}) : f_0^*((N,\zeta)^{\boldsymbol{f}}) \to N$ defines a morphism $(f_0^*((N,\zeta)^{\boldsymbol{f}}), (\zeta^r_{\boldsymbol{f}})_{\boldsymbol{f}}) \to (N,\zeta)$ of representations of $\boldsymbol{D}$. □

We put $\varepsilon^{\boldsymbol{f}}_{(N,\zeta)} = \alpha^{1N} f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}) : f_0^*((N,\zeta)^{\boldsymbol{f}}) \to N$.

Remark 3.6.8 *If $\varphi : (M,\xi) \to (N,\zeta)$ is a morphism of representations of $\boldsymbol{D}$, the following diagram is commutative by (1.4.31) and the definition of $\varphi^{\boldsymbol{f}}$.*

$$\begin{tikzcd}
f_0^*((M,\xi)^{\boldsymbol{f}}) \arrow[r, "f_0^*(E^{\boldsymbol{f}}_{(M,\xi)})"] \arrow[d, "\varphi"] \arrow[rr, bend left, "\varepsilon^{\boldsymbol{f}}_{(M,\xi)}"] & f_0^*(M^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}) \arrow[r, "\alpha^{1M}"] \arrow[d, "f_0^*(\varphi^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})"] & M \arrow[d, "f_0^*(\varphi^{\boldsymbol{f}})"] \\
f_0^*((N,\zeta)^{\boldsymbol{f}}) \arrow[r, "f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})"] \arrow[rr, bend right, "\varepsilon^{\boldsymbol{f}}_{(N,\zeta)}"'] & f_0^*(N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}) \arrow[r, "\alpha^{1N}"] & N
\end{tikzcd}$$

Define a functor $R : \mathcal{P} \to \mathcal{E}$ and a natural transformation $\kappa : U \to R$ by $R(0) = C_1 \times_{C_0} C_1$, $R(1) = C_1$, $R(2) = C_1$, $R(i) = C_0$ $(i = 3,4,5)$, $R(\tau_{01}) = \mathrm{pr}_1$, $R(\tau_{02}) = \mathrm{pr}_2$, $R(\tau_{13}) = R(\tau_{24}) = \sigma$, $R(\tau_{14}) = R(\tau_{25}) = \tau$ and $\kappa_0 = \tilde{\mathrm{pr}}_{12}$, $\kappa_1 = id_{C_1}$, $\kappa_2 = (f_0)_\tau$, $\kappa_3 = \kappa_4 = id_{C_0}$, $\kappa_5 = f_0$. We also define functors $R_i : \mathcal{Q} \to \mathcal{E}$ and natural transformations $\kappa^i : U_i \to R_i$ for $i = 0,1,2$ by

$$\begin{array}{lllll}
R_0(0) = R(0) & R_0(1) = R(3) & R_0(2) = R(5) & R_0(\tau_{01}) = R(\tau_{13}\tau_{01}) & R_0(\tau_{02}) = R(\tau_{25}\tau_{02}) \\
R_1(0) = R(1) & R_1(1) = R(3) & R_1(2) = R(4) & R_1(\tau_{01}) = R(\tau_{13}) & R_1(\tau_{02}) = R(\tau_{14}) \\
R_2(0) = R(2) & R_2(1) = R(4) & R_2(2) = R(5) & R_2(\tau_{01}) = R(\tau_{24}) & R_2(\tau_{02}) = R(\tau_{25})
\end{array}$$
$$\kappa^0_0 = \kappa_0 \quad \kappa^0_1 = \kappa_3 \quad \kappa^0_2 = \kappa_5 \quad \kappa^1_0 = \kappa_1 \quad \kappa^1_1 = \kappa_3 \quad \kappa^1_2 = \kappa_4 \quad \kappa^2_0 = \kappa_2 \quad \kappa^2_1 = \kappa_4 \quad \kappa^2_2 = \kappa_5.$$

Proposition 3.6.9 *For an object M of $\mathcal{F}_{C_0}$, $\beta^{1M} : M^{[\sigma,\tau]} \to f_0^*(M)^{[\sigma(f_0)_\tau,\tau_{f_0}]}$ defines a morphism of representations $(M^{[\sigma,\tau]}, \mu^r_M) \to (f_0^*(M)^{[\sigma(f_0)_\tau,\tau_{f_0}]}, \mu^r_{\boldsymbol{f}}(f_0^*(M)))$ under the assumption of (3.6.1) for $N = f_0^*(M)$ and the assumption of (3.4.9).*

Proof. Since κ^1 is the identity natural transformation and $\kappa^2 = \beta^1$, we have a commutative diagram below by applying (1.4.33) to $\kappa : U \to R$.

$$\begin{tikzcd}
(M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[r, "(\beta^{1M})^{[\sigma,\tau]}"] \arrow[d, "\theta^{\sigma,\tau,\sigma,\tau}(M)"] & (f_0^*(M)^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[d, "\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(f_0^*(M))"] \\
M^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[r, "\kappa^{0M}"] & f_0^*(M)^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}
\end{tikzcd}$$

We consider functors $\omega(\mu\,;\sigma,\tau) : R_0 \to U_1$ and $\omega(\mu \times_{C_0} id_{D_0}\,;\sigma(f_0)_\tau,\tau_{f_0}) : U_0 \to T_1$. Then we have $\omega(\mu\,;\sigma,\tau)\kappa^0 = \beta^1\omega(\mu \times_{C_0} id_{D_0}\,;\sigma(f_0)_\tau,\tau_{f_0})$. Hence it follows from (1.4.32) that the following diagram is commutative.

$$\begin{tikzcd}
M^{[\sigma,\tau]} \arrow[rr, "\beta^{1M}"] \arrow[d, "M^{\mu}"] \arrow[drr, "(\omega(\mu\,;\sigma,\tau)\kappa^0)^M = (\beta^1\omega(\mu\times_{C_0} id_{D_0}\,;\sigma(f_0)_\tau,\tau_{f_0}))^M"] & & f_0^*(M)^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "f_0^*(M)^{\mu\times_{C_0} id_{D_0}}"] \\
M^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[rr, "\kappa^{0M}"] & & f_0^*(M)^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}
\end{tikzcd}$$

Since $\check{\mu}_{\boldsymbol{f}}(f_0^*(M)) = \theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(f_0^*(M))^{-1}f_0^*(M)^{\mu\times_{C_0} id_{D_0}}$ and $\check{\mu}_M = \theta^{\sigma,\tau,\sigma,\tau}(M)^{-1}M^{\mu}$, the commutativity of the above diagrams implies that the following diagram is commutative.

$$\begin{tikzcd}
M^{[\sigma,\tau]} \arrow[r, "\beta^{1M}"] \arrow[d, "\check{\mu}_M"] & f_0^*(M)^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "\check{\mu}_{\boldsymbol{f}}(f_0^*(M))"] \\
(M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[r, "(\beta^{1M})^{[\sigma,\tau]}"] & (f_0^*(M)^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]}
\end{tikzcd}$$

Hence the assertion follows from (3.4.5). □

Lemma 3.6.10 *Let (M,ξ) and (N,ζ) be representations of $\boldsymbol{C}$ and $\boldsymbol{D}$, respectively. We put $\check{\xi} = E_{\sigma,\tau}(M)_M(\xi)$ and $\check{\zeta} = E_{\sigma',\tau'}(N)_N(\zeta)$. For a morphism $\varphi : \boldsymbol{f}^{\bullet}(M,\xi) \to (N,\zeta)$ of representations of $\boldsymbol{D}$, the following diagram is commutative if $\theta^{\sigma,\tau,\sigma,\tau}(M) : (M^{[\sigma,\tau]})^{[\sigma,\tau]} \to M^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$ is an isomorphism.*

$$\begin{tikzcd}
M \arrow[r, "\check{\xi}"] \arrow[d, "\check{\xi}"] & M^{[\sigma,\tau]} \arrow[r, "\beta^{1M}"] & f_0^*(M)^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "\varphi^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] & N^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "\check{\zeta}^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] \\
M^{[\sigma,\tau]} \arrow[d, "\beta^{1M}"] & & & (N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\,\tau_{f_0}]} \arrow[d, "\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(N)"] \\
f_0^*(M)^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[rr, "\varphi^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] & & N^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}"] & N^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}
\end{tikzcd}$$

Proof. Since $E_{\sigma',\tau'}(f_0^*(M))^{f_0^*(M)}(\xi_{\boldsymbol{f}})$ is a composition

$$f_0^*(M) \xrightarrow{f_0^*(\check{\xi})} f_0^*(M^{[\sigma,\tau]}) \xrightarrow{f_0^*(M^{f_1})} f_0^*(M^{[f_0\sigma',f_0\tau']}) \xrightarrow{\omega(\sigma',\tau';f_0,f_0)^M} f_0^*(M)^{[\sigma',\tau']}$$

by (3.4.4), the following diagram is commutative by (3.4.5).

$$\begin{tikzcd}
f_0^*(M) \arrow[r, "f_0^*(\check{\xi})"] \arrow[d, "\varphi"] & f_0^*(M^{[\sigma,\tau]}) \arrow[r, "f_0^*(M^{f_1})"] & f_0^*(M^{[f_0\sigma',f_0\tau']}) \arrow[r, "\omega(\sigma',\tau'\,;f_0,f_0)^M"] & f_0^*(M)^{[\sigma',\tau']} \arrow[d, "\varphi^{[\sigma',\tau']}"] \\
N \arrow[rrr, "\check{\zeta}"] & & & N^{[\sigma',\tau']}
\end{tikzcd}$$

It follows from (1.4.31) that the following diagram is commutative.

$$\begin{tikzcd}
M^{[\sigma,\tau]} \arrow[r, "\beta^{1M}"] \arrow[d, "\check{\xi}^{[\sigma,\tau]}"] & f_0^*(M)^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "f_0^*(\check{\xi})^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] \\
(M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[r, "\beta^{1M^{[\sigma,\tau]}}"] & f_0^*(M^{[\sigma,\tau]})^{[\sigma(f_0)_\tau,\tau_{f_0}]}
\end{tikzcd}$$

Hence the following diagram (i) is commutative by (1.4.4), (1.4.9) and (1.4.21).

$$\begin{tikzcd}
M^{[\sigma,\tau]} \arrow[d, "\check{\xi}^{[\sigma,\tau]}"] \arrow[ddr, "\beta^{1M}"] & \\
(M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[d, "\beta^{1M^{[\sigma,\tau]}}"] & \\
f_0^*(M^{[\sigma,\tau]})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "f_0^*(M^{f_1})^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] & f_0^*(M)^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[l, "f_0^*(\check{\xi})^{[\sigma(f_0)_\tau,\tau_{f_0}]}"'] \arrow[d, "\varphi^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] \\
f_0^*(M^{[f_0\sigma',f_0\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "(\omega(\sigma',\tau';f_0,f_0)^M)^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] & N^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "\check{\zeta}^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] \\
(f_0^*(M)^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "(\varphi^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] \arrow[d, "\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(f_0^*(M))"] & (N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(N)"] \\
f_0^*(M)^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]} \arrow[r, "\varphi^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}"] & N^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]} \\
f_0^*(M)^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[u, "f_0^*(M)^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}"] \arrow[r, "\varphi^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] & N^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[u, "N^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}"']
\end{tikzcd}$$

diagram (i)

Define a functor $V:\mathcal{P}\to\mathcal{E}$ and a natural transformation $\lambda:T\to V$ by $V(0)=C_1\times_{C_0}D_1$, $V(1)=C_1$, $V(2)=D_1$, $V(i)=C_0$ $(i=3,4,5)$, $V(\tau_{01})=\tilde{\mathrm{pr}}_1$, $V(\tau_{02})=\tilde{\mathrm{pr}}_2$, $V(\tau_{13})=\sigma$, $V(\tau_{14})=\tau$, $V(\tau_{24})=f_0\sigma'$, $V(\tau_{25})=f_0\tau'$ and $\lambda_0=id_{C_1\times_{C_0}D_1}$, $\lambda_1=(f_0)_\tau$, $\lambda_2=id_{D_1}$, $\lambda_3=id_{C_0}$, $\lambda_4=\lambda_5=f_0$. We also define functors $V_i:\mathcal{Q}\to\mathcal{E}$ and natural transformations $\lambda^i:V_i\to T_i$ for $i=0,1,2$ by

$$\begin{array}{lllll}
V_0(0)=V(0) & V_0(1)=V(3) & V_0(2)=V(5) & V_0(\tau_{01})=V(\tau_{13}\tau_{01}) & V_0(\tau_{02})=V(\tau_{25}\tau_{02})\\
V_1(0)=V(1) & V_1(1)=V(3) & V_1(2)=V(4) & V_1(\tau_{01})=V(\tau_{13}) & V_1(\tau_{02})=V(\tau_{14})\\
V_2(0)=V(2) & V_2(1)=V(4) & V_2(2)=V(5) & V_2(\tau_{01})=V(\tau_{24}) & V_2(\tau_{02})=V(\tau_{25})
\end{array}$$

$$\lambda^0_0=\lambda_0\quad \lambda^0_1=\lambda_3\quad \lambda^0_2=\lambda_5\quad \lambda^1_0=\lambda_1\quad \lambda^1_1=\lambda_3\quad \lambda^1_2=\lambda_4\quad \lambda^2_0=\lambda_2\quad \lambda^2_1=\lambda_4\quad \lambda^2_2=\lambda_5.$$

Then, $V_1=U_1$, $\lambda^2=\omega(\sigma',\tau';f_0,f_0)$ and $\lambda^1=\beta^1$ and it follows from (1.4.33) that the following diagram is commutative.

$$\begin{tikzcd}
(M^{[f_0\sigma',f_0\tau']})^{[\sigma,\tau]} \arrow[r, "\beta^{1M^{[f_0\sigma',f_0\tau']}}"] \arrow[d, "\theta^{\sigma,\tau,f_0\sigma',f_0\tau'}(M)"] & f_0^*(M^{[f_0\sigma',f_0\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "(\omega(\sigma',\tau';f_0,f_0)^M)^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] & (f_0^*(M)^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(f_0^*(M))"'] \\
M^{[\sigma\tilde{\mathrm{pr}}_1,f_0\tau'\mathrm{pr}_2]} \arrow[rr, "\lambda^{0M}"] & & f_0^*(M)^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}
\end{tikzcd}$$

Consider natural transformations $\omega(\mu(id_{C_1}\times_{C_0}f_1);\sigma,\tau):V_0\to U_1$ and $\omega((\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2);\sigma(f_0)_\tau,\tau_{f_0}):T_0\to T_1$. Then, $\omega(\mu(id_{C_1}\times_{C_0}f_1);\sigma,\tau)\lambda^0=\beta^1\omega((\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2);\sigma(f_0)_\tau,\tau_{f_0})$ holds and the following diagram is commutative by (1.4.32).

$$\begin{tikzcd}
M^{[\sigma,\tau]} \arrow[r, "\beta^{1M}"] \arrow[d, "M^{\mu(id_{C_1}\times_{C_0}f_1)}"] \arrow[dr, "(\omega(\mu(id_{C_1}\times_{C_0}f_1);\sigma,\tau)\lambda^0)^M"] & f_0^*(M)^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "f_0^*(M)^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}"] \\
M^{[\sigma'\tilde{\mathrm{pr}}_1,f_0\tau'\tilde{\mathrm{pr}}_2]}) \arrow[r, "\lambda^{0M}"] & f_0^*(M)^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}
\end{tikzcd}$$

Moreover, the following diagrams are commutative by (3.4.2) and (1.4.31), respectively.

$$\begin{tikzcd}
M \arrow[r, "\check{\xi}"] \arrow[dr, "\check{\xi}"'] & M^{[\sigma,\tau]} \arrow[r, "\check{\xi}^{[\sigma,\tau]}"] & (M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[d, "\theta^{\sigma,\tau,\sigma,\tau}(M)"] \\
& M^{[\sigma,\tau]} \arrow[r, "M^\mu"] & M^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}
\end{tikzcd}
\qquad
\begin{tikzcd}
(M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[r, "\beta^{1M^{[\sigma,\tau]}}"] \arrow[d, "(M^{f_1})^{[\sigma,\tau]}"] & f_0^*(M^{[\sigma,\tau]})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "f_0^*(M^{f_1})^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] \\
(M^{[f_0\sigma',f_0\tau']})^{[\sigma,\tau]} \arrow[r, "\beta^{1M^{[f_0\sigma',f_0\tau']}}"] & f_0^*(M^{[f_0\sigma',f_0\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]}
\end{tikzcd}$$

Therefore the following diagram (ii) is commutative

$$
\begin{tikzcd}
M \arrow[rrr, "\check{\xi}"] \arrow[dddddd, "\check{\xi}"'] & & & M^{[\sigma,\tau]} \arrow[d, "\check{\xi}^{[\sigma,\tau]}"] \\
& M^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[rr, "\theta^{\sigma,\tau,\sigma,\tau}(M)^{-1}"] \arrow[ddddr, "M^{id_{C_1}\times_{C_0}f_1}"] & & (M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[d, "\beta^{1M^{[\sigma,\tau]}}"] \arrow[ddl, "(M^{f_1})^{[\sigma,\tau]}"'] \\
& & & f_0^*(M^{[\sigma,\tau]})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "f_0^*(M^{f_1})^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] \\
& & (M^{[f_0\sigma',f_0\tau']})^{[\sigma,\tau]} \arrow[r, "\beta^{1M^{[f_0\sigma',f_0\tau']}}"] \arrow[dd, "\theta^{\sigma,\tau,f_0\sigma',f_0\tau'}(M)"] & f_0^*(M^{[f_0\sigma',f_0\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "(\omega(\sigma',\tau';f_0,f_0)^M)^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] \\
& & & (f_0^*(M)^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(f_0^*(M))"] \\
& & M^{[\sigma\tilde{\mathrm{pr}}_1,f_0\tau'\mathrm{pr}_2]} \arrow[r, "\lambda^{0M}"] & f_0^*(M)^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]} \\
M^{[\sigma,\tau]} \arrow[rrr, "\beta^{1M}"] \arrow[uuuuur, "M^\mu"] & & & f_0^*(M)^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[u, "f_0^*(M)^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}"']
\end{tikzcd}
$$

diagram (ii)

By glueing the left edge of diagram (i) and the right edge of diagram (ii), the assertion follows. $\square$

Recall that $E^{\boldsymbol{f}}_{(N,\zeta)} : (N,\zeta)^{\boldsymbol{f}} \to N^{[\sigma(f_0)_\tau,\tau_{f_0}]}$ is an equalizer of the following morphisms.

$$N^{[\sigma(f_0)_\tau,\tau_{f_0}]} \xrightarrow{\check{\zeta}^{[\sigma(f_0)_\tau,\tau_{f_0}]}} (N^{[\sigma',\tau']})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \xrightarrow{\theta^{\sigma(f_0)_\tau,\tau_{f_0},\sigma',\tau'}(N)} N^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}$$

$$N^{[\sigma(f_0)_\tau,\tau_{f_0}]} \xrightarrow{N^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}} N^{[\sigma(f_0)_\tau(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}$$

Hence there exists unique morphism ${}^t\varphi : M \to (N,\zeta)^{\boldsymbol{f}}$ that satisfies $E^{\boldsymbol{f}}_{(N,\zeta)}\,{}^t\varphi = \varphi^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}\beta^{1M}\check{\xi}$.

Proposition 3.6.11 *Under the assumptions of (3.6.3) for N and the assumptions of (iii) and the first one of (iv) of (3.6.3) for $f_0^*(M)$, ${}^t\varphi : M \to (N,\zeta)^{\boldsymbol{f}}$ gives a morphism $(M,\xi) \to ((N,\zeta)^{\boldsymbol{f}}, \zeta^r_{\boldsymbol{f}})$ of representations of $\boldsymbol{C}$.*

Proof. It follows from (3.4.9), (3.6.9) and (3.4.10) that $\varphi^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}\beta^{1N}\check{\xi} : M \to N^{[\sigma(f_0)_\tau,\tau_{f_0}]}$ gives a morphism $(M,\xi) \to (N^{[\sigma(f_0)_\tau,\tau_{f_0}]}, \mu^r_{\boldsymbol{f}}(N))$ of representations of $\boldsymbol{C}$. Hence the outer rectangle of the following diagram is commutative by (3.4.5).

$$
\begin{tikzcd}
M \arrow[r, "{}^t\varphi"] \arrow[d, "\check{\zeta}"] & (N,\xi)_{\boldsymbol{f}} \arrow[r, "E^{\boldsymbol{f}}_{(N,\zeta)}"] \arrow[d, "\check{\xi}_{\boldsymbol{f}}"] & N^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "\check{\mu}_{\boldsymbol{f}}(N)"] \\
M^{[\sigma,\tau]} \arrow[r, "{}^t\varphi^{[\sigma,\tau]}"] & ((N,\xi)_{\boldsymbol{f}})^{[\sigma,\tau]} \arrow[r, "(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}"] & (N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]}
\end{tikzcd}
$$

Since $(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]} : (M^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]} \to ((M,\xi)_{\boldsymbol{f}})^{[\sigma,\tau]}$ is a monomorphism and the right rectangle of the above diagram is commutative by the definition of $\check{\xi}_{\boldsymbol{f}}$, the left rectangle of the above diagram is also commutative. Thus the assertion follows from (3.4.5). $\square$

Proposition 3.6.12 *For a morphism $\varphi : \boldsymbol{f}^\bullet(M,\xi) \to (N,\zeta)$ of representations of $\boldsymbol{D}$, the following composition coincides with φ.*

$$f_0^*(M) \xrightarrow{f_0^*({}^t\varphi)} f_0^*((N,\zeta)^{\boldsymbol{f}}) \xrightarrow{\varepsilon^{\boldsymbol{f}}_{(M,\xi)}} N$$

Proof. We note that compositions $S_1 \xrightarrow{\alpha^1} T_1 \xrightarrow{\beta^1} U_1$ and $S_1 = D_{id_{D_0},id_{D_0}} \xrightarrow{\omega(f_0)} D_{id_{C_0},id_{C_0}} \xrightarrow{\omega(\varepsilon;\sigma,\tau)} U_1$ coincide. Hence the following diagram is commutative by (1.4.31) and (1.4.32).

$$
\begin{tikzcd}
f_0^*(M) \arrow[r, "f_0^*(\check{\xi})"] \arrow[dd, "f_0^*({}^t\varphi)"] & f_0^*(M^{[\sigma,\tau]}) \arrow[rr, "f_0^*(M^{\varepsilon})"] \arrow[d, "f_0^*(\beta^{1M})"] \arrow[drr, "(\beta^1\alpha^1)^M = (\omega(\varepsilon\,;\sigma,\tau)\omega(f_0))^M"] & & f_0^*(M) \arrow[d, "\omega(f_0)^M"] \\
& f_0^*(f_0^*(M)^{[\sigma(f_0)_\tau,\tau_{f_0}]}) \arrow[rr, "\alpha^{1 f_0^*(M)}"] \arrow[d, "f_0^*(\varphi^{[\sigma(f_0)_\tau,\tau_{f_0}]})"] & & f_0^*(M) \arrow[d, "\varphi"] \\
f_0^*((N,\zeta)^{\boldsymbol{f}}) \arrow[r, "f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})"] & f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]}) \arrow[rr, "\alpha^{1N}"] & & N
\end{tikzcd}
$$

Since $\omega(f_0)^N$ is the identity morphism of $f^*(N)$ by (3.5.13) and $M^{\varepsilon}\check{\zeta}$ is the identity morphism of N by (3.4.2), the assertion follows. $\square$

Lemma 3.6.13 *For an object N of $\mathcal{F}_{D_0}$, a composition*

$$
N^{[\sigma(f_0)_\tau,\tau_{f_0}]} \xrightarrow{\check{\mu}_{\boldsymbol{f}}(N)} (N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]} \xrightarrow{\beta^{1N^{[\sigma(f_0)_\tau,\tau_{f_0}]}}} f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \xrightarrow{(\alpha^{1N})^{[\sigma(f_0)_\tau,\tau_{f_0}]}} N^{[\sigma(f_0)_\tau,\tau_{f_0}]}
$$

coincides with the identity morphism of $N^{[\sigma(f_0)_\tau,\tau_{f_0}]}$.

Proof. Define a functor $W : \mathcal{P} \to \mathcal{E}$ and a natural transformation $\nu : W \to U$ by $W(0) = W(1) = C_1 \times_{C_0} D_0$, $W(i) = D_0$ $(i = 2,4,5)$, $W(3) = C_0$, $W(\tau_{01}) = id_{C_1 \times_{C_0} D_0}$, $W(\tau_{02}) = \tau_{f_0}$, $W(\tau_{13}) = \sigma(f_0)_\tau$, $W(\tau_{14}) = \tau_{f_0}$, $W(\tau_{24}) = W(\tau_{25}) = id_{D_0}$ and $\nu_0 = ((f_0)_\tau, \varepsilon\tau(f_0)_\tau, \tau_{f_0})$, $\nu_1 = (f_0)_\tau$, $\nu_2 = (\varepsilon f_0, id_{D_0})$, $\nu_3 = id_{C_0}$, $\nu_4 = f_0$, $\nu_5 = id_{D_0}$. We also define functors $W_i : \mathcal{Q} \to \mathcal{E}$ and natural transformations $\nu^i : W_i \to T_i$ for $i = 0,1,2$ by

$$
\begin{array}{lllll}
W_0(0) = W(0) & W_0(1) = W(3) & W_0(2) = W(5) & W_0(\tau_{01}) = W(\tau_{13}\tau_{01}) & W_0(\tau_{02}) = W(\tau_{25}\tau_{02}) \\
W_1(0) = W(1) & W_1(1) = W(3) & W_1(2) = W(4) & W_1(\tau_{01}) = W(\tau_{13}) & W_1(\tau_{02}) = W(\tau_{14}) \\
W_2(0) = W(2) & W_2(1) = W(4) & W_2(2) = W(5) & W_2(\tau_{01}) = W(\tau_{24}) & W_2(\tau_{02}) = W(\tau_{25})
\end{array}
$$

$$
\nu^0_0 = \nu_0 \quad \nu^0_1 = \nu_3 \quad \nu^0_2 = \nu_5 \quad \nu^1_0 = \nu_1 \quad \nu^1_1 = \nu_3 \quad \nu^1_2 = \nu_4 \quad \nu^2_0 = \nu_2 \quad \nu^2_1 = \nu_4 \quad \nu^2_2 = \nu_5.
$$

Then, we have $W_1 = T_1$, $W_2 = S_1$, $\nu^1 = \beta^1$, $\nu^2 = \alpha^1$ and $\nu^0 = \omega(((f_0)_\tau, \varepsilon\tau(f_0)_\tau, \tau_{f_0})\,; \sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}, \tau_{f_0}\tilde{\mathrm{pr}}_{23})$. It follows from (1.4.33) and the definition of $\check{\mu}_{\boldsymbol{f}}(N)$ that the following diagram is commutative.

$$
\begin{tikzcd}
N^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "\check{\mu}_{\boldsymbol{f}}(N)"] \arrow[dr, "N^{\mu} \times_{C_0} id_{D_0}"] & (N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "\beta^{1N^{[\sigma(f_0)_\tau,\tau_{f_0}]}}"] \arrow[d, "\theta^{\sigma,\tau,\sigma(f_0)_\tau,\tau_{f_0}}(N)"] & f_0^*(N^{[\sigma(f_0)_\tau,\tau_{f_0}]})^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[r, "(\alpha^{1N})^{[\sigma(f_0)_\tau,\tau_{f_0}]}"] & N^{[\sigma(f_0)_\tau,\tau_{f_0}]} \arrow[d, "\theta^{\sigma(f_0)_\tau,\tau_{f_0},id_{D_0},id_{D_0}}(id^*_{D_0}(N)) = id_{N^{[\sigma(f_0)_\tau,\tau_{f_0}]}}"] \\
& N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[rr, "N^{((f_0)_\tau,\varepsilon\tau(f_0)_\tau,\tau_{f_0})}"] & & N^{[\sigma(f_0)_\tau,\tau_{f_0}]}
\end{tikzcd}
$$

Since a composition $C_1 \times_{C_0} D_0 \xrightarrow{((f_0)_\tau,\varepsilon\tau(f_0)_\tau,\tau_{f_0})} C_1 \times_{C_0} C_1 \times_{C_0} D_0 \xrightarrow{\mu \times_{C_0} id_{D_0}} C_1 \times_{C_0} D_0$ is the identity morphism of $C_1 \times_{C_0} D_0$, the assertion follows from the commutativity of the above diagram and (1.4.7). $\square$

Under the assumptions of (3.6.3) for N and the assumptions of (iii) and the first one of (iv) of (3.6.3) for $f_0^*(M)$, we define a map

$$
\mathrm{ad}^{(M,\xi)}_{(N,\zeta)} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\xi),((N,\zeta)^{\boldsymbol{f}},\zeta^r_{\boldsymbol{f}})) \to \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})(\boldsymbol{f}^{\bullet}(M,\xi),(N,\zeta))
$$

by $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}(\psi) = \varepsilon^{\boldsymbol{f}}_{(M,\xi)} f_0^*(\psi)$.

Proposition 3.6.14 $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}$ *is bijective.*

Proof. We show that a map $\Phi : \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})(\boldsymbol{f}^{\bullet}(M,\xi),(N,\zeta)) \to \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\xi),((N,\zeta)^{\boldsymbol{f}},\zeta^r_{\boldsymbol{f}}))$ defined by $\Phi(\varphi) = {}^t\varphi$ is the inverse of $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}$. $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}\Phi$ is the identity map of $\mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})(\boldsymbol{f}^{\bullet}(M,\xi),(N,\zeta))$ by (3.6.12). For $\psi \in \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\xi),((N,\zeta)^{\boldsymbol{f}},\zeta^r_{\boldsymbol{f}}))$, we put $\varphi = \mathrm{ad}^{(M,\xi)}_{(N,\zeta)}(\psi)$. The following diagram is commutative by (1.4.4), (1.4.31), (3.4.5) and the definition of $\check{\zeta}_{\boldsymbol{f}}$.

$$
\begin{tikzcd}
 & & (N,\zeta)_{\boldsymbol{f}} \arrow[d, "\check{\zeta}_{\boldsymbol{f}}"] \arrow[drr, "E^{\boldsymbol{f}}_{(N,\zeta)}"] & & \\
M \arrow[urr, "\psi"] \arrow[d, "\check{\xi}"] & & ((N,\zeta)_{\boldsymbol{f}})^{[\sigma,\tau]} \arrow[drr, "(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}"] & & N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]} \arrow[d, "\check{\mu}_{\boldsymbol{f}}(N)"] \\
M^{[\sigma,\tau]} \arrow[urr, "\psi^{[\sigma,\tau]}"] \arrow[rrrr, "(E^{\boldsymbol{f}}_{(N,\zeta)}\psi)^{[\sigma,\tau]}"] \arrow[d, "\beta^{1M}"] & & & & (N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[d, "\beta^{1N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}}"] \\
f_0^*(M)^{[\sigma(f_0)_\tau,\,\tau_{f_0}]} \arrow[rrrr, "f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}\psi)^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}"] & & & & f_0^*(N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]})^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}
\end{tikzcd}
$$

Hence we have the following equalities by the commutativity of the above diagram and (3.6.13).

$$
\begin{aligned}
\varphi^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}\beta^{1M}\check{\xi} &= (\varepsilon^{\boldsymbol{f}}_{(M,\xi)})^{[\sigma(f_0)_\tau,\,\tau_{f_0}]} f_0^*(\psi)^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}\beta^{1M}\check{\xi} \\
&= (\alpha^{1N})^{[\sigma(f_0)_\tau,\,\tau_{f_0}]} f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma(f_0)_\tau,\,\tau_{f_0}]} f_0^*(\psi)^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}\beta^{1M}\check{\xi} \\
&= (\alpha^{1N})^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}\beta^{1N^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}}\check{\mu}_{\boldsymbol{f}}(N)E^{\boldsymbol{f}}_{(N,\zeta)}\psi = E^{\boldsymbol{f}}_{(N,\zeta)}\psi
\end{aligned}
$$

Since we also have $\varphi^{[\sigma(f_0)_\tau,\,\tau_{f_0}]}\beta^{1M}\check{\xi} = E^{\boldsymbol{f}}_{(M,\xi)}\,{}^t\varphi$ by the definition of ${}^t\varphi$, it follows that $\Phi(\varphi) = {}^t\varphi = \psi$ which implies that $\Phi\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}$ is the identity map of $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\xi),((N,\zeta)^{\boldsymbol{f}},\zeta^r_{\boldsymbol{f}}))$. □

Definition 3.6.15 *For a representation (N,ζ) of $\boldsymbol{D}$, we call $((N,\zeta)^{\boldsymbol{f}},\xi^r_{\boldsymbol{f}})$ the left induced representation of (N,ζ) by $\boldsymbol{f}:\boldsymbol{D}\to\boldsymbol{C}$.*

The following fact is straightforward from (3.6.8).

Proposition 3.6.16 *The following diagrams are commutative for a morphism $\varphi:(L,\chi)\to(M,\xi)$ of $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ and a morphism $\psi:(N,\zeta)\to(P,\rho)$ of $\mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})$.*

$$
\begin{tikzcd}
\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\xi),((N,\zeta)^{\boldsymbol{f}},\zeta^r_{\boldsymbol{f}})) \arrow[r, "\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}"] \arrow[d, "\varphi^*"] & \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})(\boldsymbol{f}^\bullet(M,\xi),(N,\zeta)) \arrow[d, "\boldsymbol{f}^\bullet(\varphi)^*"] \\
\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((L,\chi),((N,\zeta)^{\boldsymbol{f}},\zeta^r_{\boldsymbol{f}})) \arrow[r, "\mathrm{ad}^{(L,\chi)}_{(N,\zeta)}"] & \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})(\boldsymbol{f}^\bullet(L,\chi),(N,\zeta))
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\xi),((N,\zeta)^{\boldsymbol{f}},\zeta^r_{\boldsymbol{f}})) \arrow[r, "\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}"] \arrow[d, "\psi^{\boldsymbol{f}}_*"] & \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})(\boldsymbol{f}^\bullet(M,\xi),(N,\zeta)) \arrow[d, "\psi_*"] \\
\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\xi),((P,\rho)^{\boldsymbol{f}},\rho^r_{\boldsymbol{f}})) \arrow[r, "\mathrm{ad}^{(M,\xi)}_{(P,\rho)}"] & \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})(\boldsymbol{f}^\bullet(M,\xi),(P,\rho))
\end{tikzcd}
$$

4 Representations in fibered category of modules

4.1 Hopf algebroids and comodules

We call an internal category in $\mathcal{A}lg_{K_*}^{op}$ a Hopf algebroid. Namely, a Hopf algebroid $\boldsymbol{\Gamma}$ consists of two objects A_*, Γ_* of $\mathcal{A}lg_{K_*}$ and four morphisms $\sigma, \tau : A_* \to \Gamma_*$, $\varepsilon : \Gamma_* \to A_*$, $\mu : \Gamma_* \to \Gamma_* \otimes_{A_*} \Gamma_*$ of $\mathcal{A}lg_{K_*}$ which satisfy $\varepsilon\sigma = \varepsilon\tau = id_{A_*}$ and make the following diagrams commute. We regard Γ_* as a left A_*-module by σ and a right A_*-module by τ.

$$\begin{tikzcd}
A_* \arrow[r, "\sigma"] \arrow[d, "\sigma"] & \Gamma_* \arrow[d, "\mu"] & A_* \arrow[l, "\tau"'] \arrow[d, "\tau"] \\
\Gamma \arrow[r, "i_1"] & \Gamma_* \otimes_{A_*} \Gamma_* & \Gamma_* \arrow[l, "i_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma_* \arrow[r, "\mu"] \arrow[d, "\mu"] & \Gamma_* \otimes_{A_*} \Gamma_* \arrow[d, "id_{\Gamma_*} \otimes_{A_*} \mu"] \\
\Gamma_* \otimes_{A_*} \Gamma_* \arrow[r, "\mu \otimes_{A_*} id_{\Gamma_*}"] & \Gamma_* \otimes_{A_*} \Gamma_* \otimes_{A_*} \Gamma_*
\end{tikzcd}$$

$$\begin{tikzcd}
& \Gamma_* \arrow[dl, "j_1"'] \arrow[d, "\mu"] \arrow[dr, "j_2"] & \\
\Gamma_* \otimes_{A_*} A_* & \Gamma_* \otimes_{A_*} \Gamma_* \arrow[l, "id_{\Gamma_*} \otimes_{A_*} \varepsilon"'] \arrow[r, "\varepsilon \otimes_{A_*} id_{\Gamma_*}"] & A_* \otimes_{A_*} \Gamma_*
\end{tikzcd}$$

Here, $i_1, i_2 : \Gamma_* \to \Gamma_* \otimes_{A_*} \Gamma_*$ and $j_1 : A_* \to A_* \otimes_{A_*} \Gamma_*$, $j_2 : A_* \to \Gamma_* \otimes_{A_*} A_*$ are maps defined by $i_1(x) = x \otimes 1$, $i_2(x) = 1 \otimes x$ and $j_1(a) = a \otimes 1$, $j_2(a) = 1 \otimes a$.

We assume that a subcategory $\mathcal{C}$ of $\mathcal{A}lg_{K_*}$ has finite colimits. We also assume that a subcategory $\mathcal{M}$ of $\mathcal{M}od_{K_*}$ is additive, satisfies (2.1.1) and that every morphism in $\mathcal{M}$ has a kernel in $\mathcal{M}$.

Let $\boldsymbol{\Gamma} = (A_*, \Gamma_*, \sigma, \tau, \varepsilon, \mu)$ be a Hopf algebroid in $\mathcal{C}$ and $\boldsymbol{M} = (A_*, M_*, \alpha)$ an object of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{A_*}$. For a morphism $\boldsymbol{\xi} : \sigma^*(\boldsymbol{M}) \to \tau^*(\boldsymbol{M})$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{A_*}$, we put $\hat{\boldsymbol{\xi}} = P_{\sigma,\tau}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) \in \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{A_*}(\boldsymbol{M}_{[\sigma,\tau]}, \boldsymbol{M})$. For a morphism $f : A_* \to B_*$ of $\mathcal{A}lg_{K_*}$, we denote by ${}_fB_*$ a left A_*-module defined by ${}_fB_* = B_*$ as a K_*-module, with left A_*-module structure map $A_* \otimes_{K_*} {}_fB_* \to {}_fB_*$ given by $a \otimes b \to f(a)b$. Then, if we put $\boldsymbol{\xi} = (id_{\Gamma_*}, \xi)$, ξ is a right Γ_*-module homomorphism from $M_* \otimes_{A_*\tau} \Gamma_*$ to $M_* \otimes_{A_*\sigma} \Gamma_*$. Since $\boldsymbol{M}_{[\sigma,\tau]} = (A_*, M_* \otimes_{A_*} \Gamma_*, \alpha_\sigma(id_{M_* \otimes_{A_*} \Gamma_*} \otimes_{K_*} \tau))$ and $\hat{\boldsymbol{\xi}} = (id_{A_*}, \hat{\xi})$ for a homomorphism $\hat{\xi} = \xi i_\tau(\boldsymbol{M}) : M_* \to M_* \otimes_{A_*\sigma} \Gamma_*$ of right A_*-modules by (3) of (2.1.8), the following result follows from (3.3.2) and (2.1.8).

Proposition 4.1.1 *$\boldsymbol{\xi}$ defines a representation of $\boldsymbol{\Gamma}$ on $\boldsymbol{M}$ if and only if a composition*

$$M_* \xrightarrow{\hat{\xi}} M_* \otimes_{A_*} \Gamma_* \xrightarrow{id_{M_*} \otimes_{A_*} \varepsilon} M_* \otimes_{A_*} A_* \xrightarrow{\bar{\alpha}} M_*$$

is the identity morphism of M_* and the following diagram commute.

$$\begin{tikzcd}
M_* \arrow[r, "\hat{\xi}"] \arrow[dr, "\hat{\xi}"'] & M_* \otimes_{A_*} \Gamma_* \arrow[r, "\hat{\xi} \otimes_{A_*} id_{\Gamma_*}"] & (M_* \otimes_{A_*} \Gamma_*) \otimes_{A_*} \Gamma_* \arrow[d, "\tilde{\theta}_{\sigma,\tau,\sigma,\tau}(\boldsymbol{M})"] \\
& M_* \otimes_{A_*} \Gamma_* \arrow[r, "id_{M_*} \otimes_{A_*} \mu"] & M_* \otimes_{A_*} (\Gamma_* \otimes_{A_*} \Gamma_*)
\end{tikzcd}$$

Here, $\bar{\alpha} : M_* \otimes_{A_*} A_* \to M_*$ is the isomorphism induced by α and $\Gamma_* \otimes_{A_*} \Gamma_*$ is regarded as a left A_*-module by $i_1\sigma$, a right A_*-module by $i_2\tau$.

The following result follows from (3.3.6) and (2.1.8).

Proposition 4.1.2 *Let $(\boldsymbol{M}, \boldsymbol{\xi})$ and $(\boldsymbol{N}, \boldsymbol{\zeta})$ be representations of $\boldsymbol{\Gamma}$ and $\boldsymbol{\varphi} : \boldsymbol{M} \to \boldsymbol{N}$ a morphism in $\mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{A_*}$. Suppose that $\boldsymbol{M} = (A_*, M_*, \alpha)$, $\boldsymbol{N} = (A_*, N_*, \beta)$ and $\boldsymbol{\varphi} = (id_{A_*}, \varphi)$ for objects M_*, N_* and a morphism $\varphi : N_* \to M_*$ of $\mathcal{M}$. We put $P_{\sigma,\tau}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{A_*}, \hat{\xi}) \in \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{A_*}(\boldsymbol{M}_{[\sigma,\tau]}, \boldsymbol{M})$ and $P_{\sigma,\tau}(\boldsymbol{N})_{\boldsymbol{N}}(\boldsymbol{\zeta}) = (id_{A_*}, \hat{\zeta}) \in \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{A_*}(\boldsymbol{N}_{[\sigma,\tau]}, \boldsymbol{N})$. Then, $\boldsymbol{\varphi}$ gives a morphism $(\boldsymbol{M}, \boldsymbol{\xi}) \to (\boldsymbol{N}, \boldsymbol{\zeta})$ of representations if and only if the following diagram in $\mathcal{M}$ is commutative.*

$$\begin{tikzcd}
N_* \arrow[r, "\hat{\zeta}"] \arrow[d, "\varphi"] & N_* \otimes_{A_*} \Gamma_* \arrow[d, "\varphi \otimes_{A_*} id_{\Gamma_*}"] \\
M_* \arrow[r, "\hat{\xi}"] & M_* \otimes_{A_*} \Gamma_*
\end{tikzcd}$$

If a morphism $\hat{\xi} : M_* \to M_* \otimes_{A_*} \Gamma_*$ of right A_*-modules satisfies the conditions of (4.1.1), a pair $(M_*, \hat{\xi})$ is usually called a right Γ_*-comodule. It follows from the above fact that, the category of representations of $\boldsymbol{\Gamma}$ is isomorphic to the opposite category of the category of right Γ_*-comodules.

Proposition 4.1.3 *Suppose that K_* is an object of $\mathcal{C}$ and let $\boldsymbol{M} = (K_*, M_*, \alpha)$ be an object of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K_*}$.*
(1) The trivial representation $(\eta^*_{A_*}(\boldsymbol{M}), \boldsymbol{\phi_M})$ associated with $\boldsymbol{M}$ is described as follows. Define a map

$$\phi_{\boldsymbol{M}} : (M_* \otimes_{K_*} A_*) \otimes_{A_*\,\tau}\Gamma \to (M_* \otimes_{K_*} A_*) \otimes_{A_*\,\sigma}\Gamma$$

by $\phi_{\boldsymbol{M}}((x \otimes a) \otimes b) = (x \otimes 1) \otimes \tau(a)b$, then the morphism $\boldsymbol{\phi_M} : \sigma^*\eta^*_{A_*}(\boldsymbol{M}) \to \tau^*\eta^*_{A_*}(\boldsymbol{M})$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{\Gamma_*}$ is $(id_{A_*}, \phi_{\boldsymbol{M}})$.
(2) Define a map $\hat{\phi}_{\boldsymbol{M}} : M_* \otimes_{K_*} A_* \to (M_* \otimes_{K_*} A_*) \otimes_{A_*\,\sigma}\Gamma$ by $\hat{\phi}_{\boldsymbol{M}}(x \otimes a) = (x \otimes 1) \otimes \tau(a)$. If we put $\hat{\boldsymbol{\phi}}_{\boldsymbol{M}} = P_{\sigma,\tau}(\eta^*_{A_*}(\boldsymbol{M}))_{\eta^*_{A_*}(\boldsymbol{M})}(\boldsymbol{\phi_M}) : \eta^*_{A_*}(\boldsymbol{M})_{[\sigma,\tau]} \to \eta^*_{A_*}(\boldsymbol{M})$, then we have $\hat{\boldsymbol{\phi}}_{\boldsymbol{M}} = (id_{A_*}, \hat{\phi}_{\boldsymbol{M}})$.

Proof. (1) Since $\boldsymbol{\phi_M} = \boldsymbol{c}_{\eta_{A_*},\tau}(\boldsymbol{M})^{-1}\boldsymbol{c}_{\eta_{A_*},\sigma}(\boldsymbol{M})$, the assertion follows from (2.1.7).
(2) This is a direct consequence of (3) of (2.1.8). □

Definition 4.1.4 *Suppose that K_* is an object of $\mathcal{C}$ and that $\Sigma^n K_*$ an object of $\mathcal{M}$. We denote by $\Sigma^n\boldsymbol{K}$ an object $(K_*, \Sigma^n K_*, \Sigma^n \mu_{K_*})$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K_*}$ and consider the trivial representation $(\eta^*_{A_*}(\Sigma^n\boldsymbol{K}), \boldsymbol{\phi}_{\Sigma^n\boldsymbol{K}})$ associated with $\Sigma^n\boldsymbol{K}$. For a representation $(\boldsymbol{M}, \boldsymbol{\xi})$ of $\boldsymbol{\Gamma}$, we call a morphism $(\boldsymbol{M}, \boldsymbol{\xi}) \to (\eta^*_{A_*}(\Sigma^n\boldsymbol{K}), \boldsymbol{\phi}_{\Sigma^n\boldsymbol{K}})$ of representations an n-dimensional primitive element of $(\boldsymbol{M}, \boldsymbol{\xi})$.*

Proposition 4.1.5 *Let $(\boldsymbol{M}, \boldsymbol{\xi})$ be a representation of $\boldsymbol{\Gamma}$ and put $\boldsymbol{M} = (A_*.M_*, \alpha)$. For a morphism $\varphi : \Sigma^n K_* \to M_*$ of $\mathcal{M}$, $(id_*, \varphi) : (\boldsymbol{M}, \boldsymbol{\xi}) \to (\eta^*_{A_*}(\Sigma^n\boldsymbol{K}), \boldsymbol{\phi}_{\Sigma^n\boldsymbol{K}})$ is a primitive element of $(\boldsymbol{M}, \boldsymbol{\xi})$ if and only if $\hat{\xi}(\varphi([n], 1)) = \varphi([n], 1) \otimes 1$. Hence if we define a subset $P_n(\boldsymbol{M}, \boldsymbol{\xi})$ of M_n by $P_n(\boldsymbol{M}, \boldsymbol{\xi}) = \{x \in M_n \,|\, \hat{\xi}(x) = x \otimes 1\}$, a correspondence $(id_*, \varphi) \mapsto \varphi([n], 1)$ gives a bijection from the set of n-dimensional primitive elements of $(\boldsymbol{M}, \boldsymbol{\xi})$ to $P_n(\boldsymbol{M}, \boldsymbol{\xi})$.*

Proof. We identify $\eta^*_{A_*}(\Sigma^n\boldsymbol{K})$ with $(A_*, \Sigma^n A_*, \Sigma^n \mu_{A_*})$. It follws from (4.1.3) that the Γ_*-comodule structure $\hat{\phi}_{\Sigma^n\boldsymbol{K}} : \Sigma^n A_* \to \Sigma^n A_* \otimes_{A_*} \Gamma_*$ is a homomorphism in right A_*-modules which is given by $\hat{\phi}_{\boldsymbol{K}}([n], a) = ([n], 1) \otimes \tau(a)$. Hence a morphism $(id_{A_*}, \varphi) : \boldsymbol{M} \to \eta^*_{A_*}(\Sigma^n\boldsymbol{K})$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{A_*}$ gives a morphism $(\boldsymbol{M}, \boldsymbol{\xi}) \to (\eta^*_{A_*}(\Sigma^n\boldsymbol{K}), \boldsymbol{\phi}_{\Sigma^n\boldsymbol{K}})$ of representations of $\boldsymbol{\Gamma}$ if and only if $\varphi : A_* \to M_*$ is a homomorphism in right A_*-modules and $\hat{\xi}(\varphi([n], 1)) = \varphi([n], 1) \otimes 1$ □

We also call an element of $\bigcup\limits_{n \in \boldsymbol{Z}} P_n(\boldsymbol{M}, \boldsymbol{\xi})$ a primitive element of $(\boldsymbol{M}, \boldsymbol{\xi})$.

Proposition 4.1.6 *Let $\boldsymbol{f} = (f_0, f_1) : \boldsymbol{\Gamma} \to \boldsymbol{\Delta}$ be a morphism in Hopf algebroids. We put $\boldsymbol{\Gamma} = (A_*, \Gamma_*, \sigma, \tau, \varepsilon, \mu)$ and $\boldsymbol{\Delta} = (B_*, \Delta_*, \sigma', \tau', \varepsilon', \mu')$. For an object $\boldsymbol{M} = (A_*, M_*, \alpha)$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{A_*}$ and a representation of $\boldsymbol{\Gamma}$ $(\boldsymbol{M}, \boldsymbol{\xi})$ on $\boldsymbol{M}$, we put $P_{\sigma,\tau}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{A_*}, \hat{\xi})$ and $P_{\sigma',\tau'}(f_0^*(\boldsymbol{M}))_{f_0^*(\boldsymbol{M})}(\boldsymbol{\xi_f}) = (id_{B_*}, \hat{\xi}_{\boldsymbol{f}})$. Then, $\hat{\xi}_{\boldsymbol{f}}$ is the following composition.*

$$M_* \otimes_{A_*} B_* \xrightarrow{\hat{\xi} \otimes_{A_*} id_{B_*}} (M_* \otimes_{A_*} \Gamma_*) \otimes_{A_*} B_* \xrightarrow{(id_{M_*} \otimes_{A_*} f_1) \otimes_{A_*} id_{B_*}} (M_* \otimes_{A_*} \Delta_*) \otimes_{A_*} B_*$$
$$\xrightarrow{\tilde{\omega}(\sigma', \tau'; f_0, f_0)_{\boldsymbol{M}}} (M_* \otimes_{A_*} B_*) \otimes_{B_*} \Delta_*$$

Here, $\tilde{\omega}(\sigma', \tau'; f_0, f_0)_{\boldsymbol{M}}$ is a map given by $\tilde{\omega}(\sigma', \tau'; f_0, f_0)_{\boldsymbol{M}}((x \otimes r) \otimes s) = (x \otimes 1) \otimes r\tau'(s)$.

Proof. It follows from (3.3.5) and (5) of (2.1.8) that we have the following equalities in $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{B_*}$.

$$\begin{aligned}
P_{\sigma',\tau'}(f_0^*(\boldsymbol{M}))_{f_0^*(\boldsymbol{M})}(\boldsymbol{\xi_f}) &= \omega(\sigma', \tau'; f_0, f_0)_{\boldsymbol{M}}\, f_0^*(\boldsymbol{M}_{f_1}\hat{\boldsymbol{\xi}}) \\
&= (id_{B_*}, \tilde{\omega}(\sigma', \tau'; f_0, f_0)_{\boldsymbol{M}}) f_0^*((id_{A_*}, id_{M_*} \otimes_{A_*} f_1)(id_{A_*}, \hat{\xi})) \\
&= (id_{B_*}, \tilde{\omega}(\sigma', \tau'; f_0, f_0)_{\boldsymbol{M}}) f_0^*(id_{A_*}, (id_{M_*} \otimes_{A_*} f_1)\hat{\xi}) \\
&= (id_{B_*}, \tilde{\omega}(\sigma', \tau'; f_0, f_0)_{\boldsymbol{M}}((id_{M_*} \otimes_{A_*} f_1)\hat{\xi} \otimes_{A_*} id_{B_*}))
\end{aligned}$$

Hence the assertion follows from (2.1.12). □

For a Hopf algebroid $\boldsymbol{\Gamma}$, we call an internal diagram on $\boldsymbol{\Gamma}$ in $\mathcal{A}lg^{op}_{K_*}$ a $\boldsymbol{\Gamma}$-comodule algebra. Namely, if $\boldsymbol{\Gamma} = (A_*, \Gamma_*, \sigma, \tau, \varepsilon, \mu)$, a $\boldsymbol{\Gamma}$-commdule algebra consists of a pair $(\pi : A_* \to B_*, \gamma : B_* \to B_* \otimes_{A_*} \Gamma_*)$ of morphisms in $\mathcal{A}lg_{K_*}$ which make the following diagrams commute.

$$\begin{array}{ccc} \begin{tikzcd} A_* \arrow[r, "\tau"] \arrow[d, "\pi"] & \Gamma_* \arrow[d, "j_2"] \\ B_* \arrow[r, "\gamma"] & B_* \otimes_{A_*} \Gamma_* \end{tikzcd} & \begin{tikzcd} B_* \arrow[r, "\gamma"] \arrow[d, "\gamma"] & B_* \otimes_{A_*} \Gamma_* \arrow[d, "id_{B_*} \otimes_{A_*} \mu"] \\ B_* \otimes_{A_*} \Gamma_* \arrow[r, "\gamma \otimes_{A_*} id_{\Gamma_*}"] & B_* \otimes_{A_*} \Gamma_* \otimes_{A_*} \Gamma_* \end{tikzcd} & \begin{tikzcd} B_* \arrow[r, "\gamma"] \arrow[dr, "\tilde{j}_1"] & B_* \otimes_{A_*} \Gamma_* \arrow[d, "id_{B_*} \otimes_{A_*} \varepsilon"] \\ & B_* \otimes_{A_*} A_* \end{tikzcd} \end{array}$$

Here, $\tilde{j}_1 : B_* \to B_* \otimes_{A_*} A_*$ and $j_2 : \Gamma_* \to B_* \otimes_{A_*} \Gamma_*$ are maps defined by $\tilde{j}_1(b) = b \otimes 1$, $j_2(x) = 1 \otimes x$. We define a functor $D_\gamma : \mathcal{P} \to \mathcal{A}lg_{K_*}^{op}$ by $D_\gamma(0) = B_* \otimes_{A_*} \Gamma_*$, $D_\gamma(1) = \Gamma_*$, $D_\gamma(2) = B_*$, $D_\gamma(3) = D_\gamma(4) = D_\gamma(5) = A_*$, $D_\gamma(\tau_{01}) = j_2$, $D_\gamma(\tau_{02}) = \gamma$, $D_\gamma(\tau_{13}) = \sigma$, $D_\gamma(\tau_{14}) = \tau$, $D_\gamma(\tau_{24}) = D_\gamma(\tau_{25}) = \pi$. We also define a map $j_1 : B_* \to B_* \otimes_{A_*} \Gamma_*$ by $j_1(b) = b \otimes 1$. For a representation $(\boldsymbol{M}, \boldsymbol{\xi})$ of $\boldsymbol{C}$, we put $\hat{\boldsymbol{\xi}} = P_{\sigma,\tau}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi})$. We define a morphism $\hat{\boldsymbol{\xi}}_\gamma : \boldsymbol{M}_{[\pi,\pi]} \to (\boldsymbol{M}_{[\pi,\pi]})_{[\sigma,\tau]}$ of $\mathcal{M}od(\mathcal{A}lg_{K_*}, \mathcal{M}od_{K_*})_{B_*}$ to be the following composition.

$$\boldsymbol{M}_{[\pi,\pi]} \xrightarrow{\hat{\boldsymbol{\xi}}_{[\pi,\pi]}} (\boldsymbol{M}_{[\sigma,\tau]})_{[\pi,\pi]} \xrightarrow{\theta_{D_\gamma}(\boldsymbol{M})} \boldsymbol{M}_{[j_2\sigma,\gamma\pi]} = \boldsymbol{M}_{[j_1\pi,\, j_2\tau]} \xrightarrow{\theta_{\pi,\pi,\sigma,\tau}(\boldsymbol{M})^{-1}} (\boldsymbol{M}_{[\pi,\pi]})_{[\sigma,\tau]}$$

Proposition 4.1.7 *If $\boldsymbol{M} = (A_*, M_*, \alpha)$ and $\hat{\boldsymbol{\xi}} = (id_{A*}, \hat{\xi})$ for a map $\hat{\xi} : M_* \to M_* \otimes_{A_*} \Gamma_*$, we define a map $\hat{\xi}_\gamma : M_* \otimes_{A_*} B_* \to (M_* \otimes_{A_*} B_*) \otimes_{A_*} \Gamma_*$ to be a composition of $\hat{\xi} \otimes_{A_*} id_{B_*} : M_* \otimes_{A_*} B_* \to (M_* \otimes_{A_*} \Gamma_*) \otimes_{A_*} B_*$ and a map $(M_* \otimes_{A_*} \Gamma_*) \otimes_{A_*} B_* \to (M_* \otimes_{A_*} B_*) \otimes_{A_*} \Gamma_*$ given by $x \otimes g \otimes b \mapsto x \otimes (1 \otimes g)\gamma(b)$ Then, we have $\hat{\boldsymbol{\xi}}_\gamma = (id_{A*}, \hat{\xi}_\gamma)$.*

Proof. It follows from the definition of $\hat{\boldsymbol{\xi}}_\gamma$ that $\hat{\xi}_\gamma$ is the following composition.

$$M_* \otimes_{A_*} B_* \xrightarrow{\hat{\xi} \otimes_{A_*} id_{B_*}} (M_* \otimes_{A_*} \Gamma_*) \otimes_{A_*} B_* \xrightarrow{\tilde{\theta}_{D_\gamma}(\boldsymbol{M})} M_* \otimes_{A_*} (B_* \otimes_{A_*} \Gamma_*) \xrightarrow{\tilde{\theta}_{\pi,\pi,\sigma,\tau}(\boldsymbol{M})^{-1}} (M_* \otimes_{A_*} B_*) \otimes_{A_*} \Gamma_*$$

Hence the assertion follows from (2.1.10). □

We define a morphism $\hat{\boldsymbol{\mu}}_{\boldsymbol{M}} : \boldsymbol{M}_{[\sigma,\tau]} \to (\boldsymbol{M}_{[\sigma,\tau]})_{[\sigma,\tau]}$ to be the following composition.

$$\boldsymbol{M}_{[\sigma,\tau]} \xrightarrow{\boldsymbol{M}_\mu} \boldsymbol{M}_{[\mu\sigma,\mu\tau]} = \boldsymbol{M}_{[i_1\sigma, i_2\tau]} \xrightarrow{\theta_{\sigma,\tau,\sigma,\tau}(\boldsymbol{M})^{-1}} (\boldsymbol{M}_{[\sigma,\tau]})_{[\sigma,\tau]}$$

Proposition 4.1.8 *If $\boldsymbol{M} = (A_*, M_*, \alpha)$, we define a map $\hat{\mu}_{\boldsymbol{M}} : M_* \otimes_{A_*} \Gamma_* \to (M_* \otimes_{A_*} \Gamma_*) \otimes_{A_*} \Gamma_*$ to be the following composition.*

$$M_* \otimes_{A_*} \Gamma_* \xrightarrow{id_{M_*} \otimes_{A_*} \mu} M_* \otimes_{A_*} (\Gamma_* \otimes_{A_*} \Gamma_*) \xrightarrow{\tilde{\theta}_{\sigma,\tau,\sigma,\tau}(\boldsymbol{M})^{-1}} (M_* \otimes_{A_*} \Gamma_*) \otimes_{A_*} \Gamma_*$$

Then, we have $\hat{\boldsymbol{\mu}}_{\boldsymbol{M}} = (id_{A_*}, \hat{\mu}_{\boldsymbol{M}})$.

Proof. The assertion is a direct consequence of (2.1.8) and (2.1.12). □

(3.3.14) implies the following result.

Proposition 4.1.9 *Let $(\boldsymbol{M}, \boldsymbol{\xi})$ and $(\boldsymbol{M}, \boldsymbol{\zeta})$ be representations of $\boldsymbol{\Gamma}$ on $\boldsymbol{M} = (A_*, M_*, \alpha) \in \operatorname{Ob} \mathcal{M}od(\mathcal{C}, \mathcal{M})$. We put $P_{\sigma,\tau}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{A_*}, \hat{\xi})$ and $P_{\sigma,\tau}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\zeta}) = (id_{A_*}, \hat{\zeta})$. Assume that $\sigma : A_* \to \Gamma_*$ is flat.*

(1) Let $\kappa_{\boldsymbol{\xi},\boldsymbol{\zeta}} : M_{(\boldsymbol{\xi}:\boldsymbol{\zeta})*} \to M_*$ be the kernel of $\hat{\xi} - \hat{\zeta} : M_* \to M_* \otimes_{A_*} \Gamma_*$. There exists unique homomorphism $\hat{\lambda} : M_{(\boldsymbol{\xi}:\boldsymbol{\zeta})*} \to M_{(\boldsymbol{\xi}:\boldsymbol{\zeta})*} \otimes_{A_*} \Gamma_*$ of right A_*-modules that makes the following diagram commute. Here we put $\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})} = (A_*, M_{(\boldsymbol{\xi}:\boldsymbol{\zeta})*}, \bar{\alpha})$ where $\bar{\alpha} : M_{(\boldsymbol{\xi}:\boldsymbol{\zeta})*} \otimes_{K_*} A_* \to M_{(\boldsymbol{\xi}:\boldsymbol{\zeta})*}$ is the map induced by $\alpha : M_* \otimes_{K_*} A_* \to M_*$.

$$\begin{tikzcd} \boldsymbol{M} \arrow[d, "(id_{A_*}, \hat{\xi})"] & \boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})} \arrow[l, "\kappa_{\boldsymbol{\xi},\boldsymbol{\zeta}}"'] \arrow[r, "\kappa_{\boldsymbol{\xi},\boldsymbol{\zeta}}"] \arrow[d, "(id_{A_*}, \hat{\lambda})"] & \boldsymbol{M} \arrow[d, "(id_{A_*}, \hat{\zeta})"] \\ \boldsymbol{M}_{[\sigma,\tau]} & (\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})})_{[\sigma,\tau]} \arrow[l, "(\kappa_{\boldsymbol{\xi},\boldsymbol{\zeta}})_{[\sigma,\tau]}"'] \arrow[r, "(\kappa_{\boldsymbol{\xi},\boldsymbol{\zeta}})_{[\sigma,\tau]}"] & \boldsymbol{M}_{[\sigma,\tau]} \end{tikzcd}$$

(2) Put $\hat{\boldsymbol{\lambda}} = (id_{A_*}, \hat{\lambda}) : \boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})} \to (\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})})_{[\sigma,\tau]}$ and $\boldsymbol{\lambda} = P_{\sigma,\tau}(\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})})^{-1}_{\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}}(\hat{\boldsymbol{\lambda}}) : \sigma^*(\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}) \to \tau^*(\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})})$. Then, $(\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}, \boldsymbol{\lambda})$ is a representation of $\boldsymbol{\Gamma}$ and a morphism $\boldsymbol{\kappa}_{\boldsymbol{\xi},\boldsymbol{\zeta}} = (id_{A_*}, \kappa_{\boldsymbol{\xi},\boldsymbol{\zeta}}) : \boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})} \to \boldsymbol{M}$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})$ defines morphisms in representations $(\boldsymbol{M}, \boldsymbol{\xi}) \to (\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}, \boldsymbol{\lambda})$ and $(\boldsymbol{M}, \boldsymbol{\zeta}) \to (\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}, \boldsymbol{\lambda})$.

(3) Let $(\boldsymbol{N}, \boldsymbol{\nu})$ be a representation of $\boldsymbol{\Gamma}$. Suppose that a morphism $\boldsymbol{\varphi} : \boldsymbol{M} \to \boldsymbol{N}$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{A_*}$ gives morphisms $(\boldsymbol{M}, \boldsymbol{\xi}) \to (\boldsymbol{N}, \boldsymbol{\nu})$ and $(\boldsymbol{M}, \boldsymbol{\zeta}) \to (\boldsymbol{N}, \boldsymbol{\nu})$ of representations of $\boldsymbol{\Gamma}$. Then, there exists unique morphism $\tilde{\boldsymbol{\varphi}} : (\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}, \lambda) \to (\boldsymbol{N}, \boldsymbol{\nu})$ of representations of $\boldsymbol{\Gamma}$ that satisfies $\tilde{\boldsymbol{\varphi}}\pi_{\boldsymbol{\xi},\boldsymbol{\zeta}} = \boldsymbol{\varphi}$.

4.2 Left induced representation of Hopf algebroids

Let $\mathbf{\Gamma} = (A_*, \Gamma_*, \sigma, \tau, \varepsilon, \mu)$ and $\mathbf{\Delta} = (B_*, \Delta_*, \sigma', \tau', \varepsilon', \mu')$ be Hopf algebroids. We regard $\mathbf{\Gamma}$ as a left A_*-module by σ and a right A_*-module by τ. Similarly, we regard $\mathbf{\Delta}$ as a left A_*-module by σ' and a right A_*-module by τ'. Let $\boldsymbol{f} = (f_0, f_1) : \mathbf{\Gamma} \to \mathbf{\Delta}$ be a morphism in Hopf algebroids. Regard B_* as an A_*-algebra by f_0 and define maps $f_{0\sigma} : \Gamma_* \to B_* \otimes_{A_*} \Gamma_*$ and $\sigma_{f_0} : B_* \to B_* \otimes_{A_*} \Gamma_*$ by $f_{0\sigma}(x) = 1 \otimes x$ and $\sigma_{f_0}(b) = b \otimes 1$, respectively. Let us consider the following diagram in $\mathcal{C}$ whose rectangles are all cocartesian.

$$\begin{tikzcd}
 & A_* \arrow[r, "\sigma"] \arrow[d, "\tau"] & \Gamma_* \arrow[d, "i_2"] & \\
A_* \arrow[r, "\sigma"] \arrow[d, "f_0"] & \Gamma_* \arrow[r, "i_1"] \arrow[d, "f_{0\sigma}"] & \Gamma_* \otimes_{A_*} \Gamma_* \arrow[r, "id_{\Gamma_*} \otimes_{A_*} i_1"] \arrow[d, "f_{0\sigma} \otimes_{A_*} id_{\Gamma_*}"] & \Gamma_* \otimes_{A_*} \Gamma_* \otimes_{A_*} \Gamma_* \arrow[d, "f_{0\sigma} \otimes_{A_*} id_{\Gamma_* \otimes_{A_*} \Gamma_*}"] \\
B_* \arrow[r, "\sigma_{f_0}"] & B_* \otimes_{A_*} \Gamma_* \arrow[r, "id_{B_*} \otimes_{A_*} i_1"] & B_* \otimes_{A_*} \Gamma_* \otimes_{A_*} \Gamma_* \arrow[r, "id_{B_* \otimes_{A_*} \Gamma_*} \otimes_{A_*} i_1"] & B_* \otimes_{A_*} \Gamma_* \otimes_{A_*} \Gamma_* \otimes_{A_*} \Gamma_*
\end{tikzcd}$$

Let $\boldsymbol{M} = (B_*, M_*, \alpha)$ be an object of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{B_*}$. We regard M_* as a right A_*-module by $\alpha(id_{M_*} \otimes_{K^*} f_0)$ and we denote by χ the following composition, where $\otimes_{f_0}$ is the quotient map induced by $f_0 : A_* \to B_*$.

$$M_* \otimes_{A_*} \Gamma_* \xrightarrow{id_{M_*} \otimes_{A_*} f_{0\sigma}} M_* \otimes_{A_*} (B_* \otimes_{A_*} \Gamma_*) \xrightarrow{\otimes_{f_0}} M_* \otimes_{B_*} (B_* \otimes_{A_*} \Gamma_*)$$

Then, χ is an isomorphism whose inverse is the following composition, where $\bar{\alpha} : M_* \otimes_{B_*} B_* \to M_*$ is the isomorphism induced by α.

$$M_* \otimes_{B_*} (B_* \otimes_{A_*} \Gamma_*) \xrightarrow{\tilde{\theta}_{id_{B_*}, f_0, \sigma, \tau}(\boldsymbol{M})^{-1}} (M_* \otimes_{B_*} B_*) \otimes_{A_*} \Gamma_* \xrightarrow{\bar{\alpha} \otimes_{A_*} id_{\Gamma_*}} M_* \otimes_{A_*} \Gamma_*$$

We also define a map $\alpha_{\boldsymbol{f}} : (M_* \otimes_{A_*} \Gamma_*) \otimes_{K_*} (B_* \otimes_{A_*} \Gamma_*) \to M_* \otimes_{A_*} \Gamma_*$ to be the following composition.

$$(M_* \otimes_{A_*} \Gamma_*) \otimes_{K_*} (B_* \otimes_{A_*} \Gamma_*) \xrightarrow{\chi \otimes_{K_*} id_{B_* \otimes_{A_*} \Gamma_*}} (M_* \otimes_{B_*} (B_* \otimes_{A_*} \Gamma_*)) \otimes_{K_*} (B_* \otimes_{A_*} \Gamma_*) \xrightarrow{\alpha_{\sigma_{f_0}}} M_* \otimes_{B_*} (B_* \otimes_{A_*} \Gamma_*)$$
$$\xrightarrow{\chi^{-1}} M_* \otimes_{A_*} \Gamma_*$$

Then, the following diagram is commutative.

$$\begin{tikzcd}
(M_* \otimes_{A_*} \Gamma_*) \otimes_{K_*} A_* \arrow[r, "\chi \otimes_{K_*} id_{A_*}"] \arrow[d, "id_{M_* \otimes_{A_*} \Gamma_*} \otimes_{K_*} f_{0\sigma} \tau"] & (M_* \otimes_{B_*} (B_* \otimes_{A_*} \Gamma_*)) \otimes_{K_*} A_* \arrow[d, "id_{M_* \otimes_{B_*} (B_* \otimes_{A_*} \Gamma_*)} \otimes_{K_*} f_{0\sigma} \tau"] \\
(M_* \otimes_{A_*} \Gamma_*) \otimes_{K_*} (B_* \otimes_{A_*} \Gamma_*) \arrow[r, "\chi \otimes_{K_*} id_{B_* \otimes_{A_*} \Gamma_*}"] \arrow[d, "\alpha_{\boldsymbol{f}}"] & (M_* \otimes_{B_*} (B_* \otimes_{A_*} \Gamma_*)) \otimes_{K_*} (B_* \otimes_{A_*} \Gamma_*) \arrow[d, "\alpha_{\sigma_{f_0}}"] \\
M_* \otimes_{A_*} \Gamma_* \arrow[r, "\chi"] & M_* \otimes_{B_*} (B_* \otimes_{A_*} \Gamma_*)
\end{tikzcd}$$

Thus we have shown the following.

Proposition 4.2.1 *$(id_{A_*}, \chi) : (A_*, M_* \otimes_{A_*} \Gamma_*, \alpha_{\boldsymbol{f}}(id_{M_* \otimes_{A_*} \Gamma_*} \otimes_{K_*} f_{0\sigma}\tau)) \to \boldsymbol{M}_{[\sigma_{f_0}, f_{0\sigma}\tau]}$ is an isomorphism.*

It follows from (4.2.1) that $(id_{A_*}, \chi)_{[\sigma,\tau]} : (A_*, M_* \otimes_{A_*} \Gamma_*, \alpha_{\boldsymbol{f}}(id_{M_* \otimes_{A_*} \Gamma_*} \otimes_{K_*} f_{0\sigma}\tau))_{[\sigma,\tau]} \to (\boldsymbol{M}_{[\sigma_{f_0}, f_{0\sigma}\tau]})_{[\sigma,\tau]}$ is also an isomorphism. Hence we identify $\boldsymbol{M}_{[\sigma_{f_0}, f_{0\sigma}\tau]}$ with $(A_*, M_* \otimes_{A_*} \Gamma_*, \alpha_{\boldsymbol{f}}(id_{M_* \otimes_{A_*} \Gamma_*} \otimes_{K_*} f_{0\sigma}\tau))$ by the isomorphism (id_{A_*}, χ) and we also identify $(\boldsymbol{M}_{[\sigma_{f_0}, f_{0\sigma}\tau]})_{[\sigma,\tau]}$ with

$$(A_*, M_* \otimes_{A_*} \Gamma_*, \alpha_{\boldsymbol{f}}(id_{M_* \otimes_{A_*} \Gamma_*} \otimes_{K_*} f_{0\sigma}\tau))_{[\sigma,\tau]} = (A_*, (M_* \otimes_{A_*} \Gamma_*) \otimes_{A_*} \Gamma_*, (\tilde{\alpha}_{\boldsymbol{f}})_\sigma(id_{(M_* \otimes_{A_*} \Gamma_*) \otimes_{A_*} \Gamma_*} \otimes_{K_*} \tau)).$$

Here we put $\tilde{\alpha}_{\boldsymbol{f}} = \alpha_{\boldsymbol{f}}(id_{M_* \otimes_{A_*} \Gamma_*} \otimes_{K_*} f_{0\sigma}\tau)$.

We note that the following diagram is commutative.

$$\begin{tikzcd}
B_* \arrow[r, "\sigma_{f_0}"] \arrow[d, "\sigma_{f_0}"] & B_* \otimes_{A_*} \Gamma_* \arrow[d, "id_{B_*} \otimes_{A_*} \mu"] & \Gamma_* \arrow[l, "f_{\sigma 0}"'] \arrow[d, "i_2"] \\
B_* \otimes_{A_*} \Gamma_* \arrow[r, "id_{B_*} \otimes_{A_*} i_1"] & B_* \otimes_{A_*} \Gamma_* \otimes_{A_*} \Gamma_* & \Gamma_* \otimes_{A_*} \Gamma_* \arrow[l, "f_{\sigma 0} \otimes_{A_*} id_{\Gamma_*}"']
\end{tikzcd}$$

Hence we can define a morphism $\hat{\boldsymbol{\mu}}_{\boldsymbol{f}}(\boldsymbol{M}) : \boldsymbol{M}_{[\sigma_{f_0},\, f_{0\sigma}\tau]} \to (\boldsymbol{M}_{[\sigma_{f_0},\, f_{0\sigma}\tau]})_{[\sigma,\tau]}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{A_*}$ to be the following composition.

$$\boldsymbol{M}_{[\sigma_{f_0},\, f_{0\sigma}\tau]} \xrightarrow{\boldsymbol{M}_{id_{B_*}\otimes_{A_*}\mu}} \boldsymbol{M}_{[(id_{B_*}\otimes_{A_*}\mu)\sigma_{f_0},(id_{B_*}\otimes_{A_*}\mu)f_{0\sigma}\tau]} = \boldsymbol{M}_{[(id_{B_*}\otimes_{A_*}i_1)\sigma_{f_0},(f_{0\sigma}\otimes_{A_*}id_{\Gamma_*})i_2\tau]} \xrightarrow{\theta_{\sigma_{f_0},\, f_{0\sigma}\tau,\sigma,\tau}(\boldsymbol{M})^{-1}} (\boldsymbol{M}_{[\sigma_{f_0},\, f_{0\sigma}\tau]})_{[\sigma,\tau]}$$

We consider the following commutative diagram below.

$$\begin{tikzcd}[column sep=small]
&&& B_*\otimes_{A_*}\Gamma_*\otimes_{A_*}\Gamma_*\otimes_{A_*}\Gamma_* &&&& \\
&& B_*\otimes_{A_*}\Gamma_*\otimes_{A_*}\Gamma_* \arrow[ur, "id_{B_*}\otimes_{A_*}\Gamma_*\otimes_{A_*}i_1"] && \Gamma_*\otimes_{A_*}\Gamma_*\otimes_{A_*}\Gamma_* \arrow[ul, "f_{0\sigma}\otimes_{A_*}id_{\Gamma_*\otimes_{A_*}\Gamma_*}"'] &&& \\
& B_*\otimes_{A_*}\Gamma_* \arrow[ur, "id_{B_*}\otimes_{A_*}i_1"] && \Gamma_*\otimes_{A_*}\Gamma_* \arrow[ul, "f_{0\sigma}\otimes_{A_*}id_{\Gamma_*}"'] \arrow[ur, "id_{\Gamma_*}\otimes_{A_*}i_1"] && \Gamma_*\otimes_{A_*}\Gamma_* \arrow[ul, "i_2\otimes_{A_*}id_{\Gamma_*}"'] && \\
B_* \arrow[ur, "\sigma_{f_0}"] && \Gamma_* \arrow[ul, "f_{0\sigma}"'] \arrow[ur, "i_1"] && \Gamma_* \arrow[ul, "i_2"'] \arrow[ur, "i_1"] && \Gamma_* \arrow[ul, "i_2"'] & \\
&&& A_* \arrow[ul, "\tau"'] \arrow[ur, "\sigma"] && A_* \arrow[ul, "\tau"'] \arrow[ur, "\sigma"] && A_* \arrow[ul, "\tau"']
\end{tikzcd}$$

The following result is a direct consequence of (2.1.8) and (2.1.10).

Proposition 4.2.2 *We define a map* $\hat{\mu}_{\boldsymbol{f}}(\boldsymbol{M}) : M_*\otimes_{A_*}\Gamma_* \to (M_*\otimes_{A_*}\Gamma_*)\otimes_{A_*}\Gamma_*$ *to be the following composition.*

$$M_*\otimes_{A_*}\Gamma_* \xrightarrow{id_{M_*}\otimes_{A_*}\mu} M_*\otimes_{A_*}(\Gamma_*\otimes_{A_*}\Gamma_*) \xrightarrow{\tilde{\theta}_{\sigma_{f_0},\, f_{0\sigma}\tau,\sigma,\tau}(\boldsymbol{M})^{-1}} (M_*\otimes_{A_*}\Gamma_*)\otimes_{A_*}\Gamma_*$$

Then, $\hat{\boldsymbol{\mu}}_{\boldsymbol{f}}(\boldsymbol{M}) = (id_{A_*}, \hat{\mu}_{\boldsymbol{f}}(\boldsymbol{M})) : (A_*, M_*\otimes_{A_*}\Gamma_*, \alpha_{\boldsymbol{f}}) \to (A_*, (M_*\otimes_{A_*}\Gamma_*)\otimes_{A_*}\Gamma_*, (\alpha_{\boldsymbol{f}})_\sigma(id_{(M_*\otimes_{A_*}\Gamma_*)\otimes_{A_*}\Gamma_*}\otimes_{K_*}\tau))$.

We also have the following result from (3.5.1) and (3.5.2), but it is easy to verify it directly.

Proposition 4.2.3 *We put*

$$\boldsymbol{\mu}^l_{\boldsymbol{f}}(\boldsymbol{M}) = P_{\sigma,\tau}(\boldsymbol{M}_{[\sigma_{f_0},\tau f_{0\sigma}]})^{-1}_{\boldsymbol{M}_{[\sigma_{f_0},\tau f_{0\sigma}]}}(\hat{\boldsymbol{\mu}}_{\boldsymbol{f}}(\boldsymbol{M})) : \sigma^*(\boldsymbol{M}_{[\sigma_{f_0},\tau f_{0\sigma}]}) \to \tau^*(\boldsymbol{M}_{[\sigma_{f_0},\tau f_{0\sigma}]}).$$

Then, $(\boldsymbol{M}_{[\sigma_{f_0},\tau f_{0\sigma}]}, \boldsymbol{\mu}^l_{\boldsymbol{f}}(\boldsymbol{M}))$ *is a representation of* $\boldsymbol{\Gamma}$*. In other words,* $\hat{\mu}_{\boldsymbol{f}}(\boldsymbol{M}) : M_*\otimes_{A_*}\Gamma_* \to (M_*\otimes_{A_*}\Gamma_*)\otimes_{A_*}\Gamma_*$ *is a right* Γ_**-comodule structure on* M_**. If* $\boldsymbol{\varphi} = (id_{B_*}, \varphi) : \boldsymbol{N} \to \boldsymbol{M}$ *is a morphism in* $\mathcal{M}od(\mathcal{C},\mathcal{M})_{B_*}$*, then* $\boldsymbol{\varphi}_{[\sigma_{f_0},\tau f_{0\sigma}]} = (id_{A_*}, \varphi\otimes_{A_*} id_{\Gamma_*}) : \boldsymbol{N}_{[\sigma_{f_0},\tau f_{0\sigma}]} \to \boldsymbol{M}_{[\sigma_{f_0},\tau f_{0\sigma}]}$ *is a morphism in representations of* $\boldsymbol{\Gamma}$*, namely* $\varphi\otimes_{A_*} id_{\Gamma_*} : N_*\otimes_{A_*}\Gamma_* \to M_*\otimes_{A_*}\Gamma_*$ *is a morphism in right* Γ_**-comodules if* $\boldsymbol{N} = (B_*, N_*, \beta)$*.*

Let us regard Δ_* as a right A_*-module by $\tau' f_0 : A_* \to \Delta$ and define maps $j_1 : \Delta_* \to \Delta_*\otimes_{A_*}\Gamma_*$ and $j_2 : \Gamma_* \to \Delta_*\otimes_{A_*}\Gamma_*$ by $j_1(x) = x\otimes 1$ and $j_2(y) = 1\otimes y$, respectively. Then, $\tau'\otimes_{A_*} id_{\Gamma_*} : B_*\otimes_{A_*}\Gamma_* \to \Delta_*\otimes_{A_*}\Gamma_*$ is unique morphism that satisfies $(\tau'\otimes_{A_*} id_{\Gamma_*})\sigma_{f_0} = j_1\tau'$ and $(\tau'\otimes_{A_*} id_{\Gamma_*})f_{0\sigma} = j_2$. We note that the left and right diagrams below are cocartesian.

$$\begin{tikzcd}
A_* \arrow[r, "\sigma"] \arrow[d, "\tau' f_0"'] & \Gamma_* \arrow[d, "j_2"] \\
\Delta_* \arrow[r, "j_1"] & \Delta_*\otimes_{A_*}\Gamma_*
\end{tikzcd}
\qquad
\begin{tikzcd}
A_* \arrow[r, "f_0"] \arrow[d, "\sigma"] & B_* \arrow[r, "\tau'"] \arrow[d, "\sigma_{f_0}"] & \Delta_* \arrow[dd, "j_1"] \\
\Gamma_* \arrow[r, "f_{0\sigma}"] \arrow[drr, "j_2"'] & B_*\otimes_{A_*}\Gamma_* \arrow[dr, dotted, "\tau'\otimes_{A_*}id_{\Gamma_*}"] & \\
& & \Delta_*\otimes_{A_*}\Gamma_*
\end{tikzcd}
\qquad
\begin{tikzcd}
B_* \arrow[r, "\tau'"] \arrow[d, "\sigma_{f_0}"'] & \Delta_* \arrow[d, "j_1"] \\
B_*\otimes_{A_*}\Gamma_* \arrow[r, "\tau'\otimes_{A_*}id_{\Gamma_*}"] & \Delta_*\otimes_{A_*}\Gamma_*
\end{tikzcd}$$

Since (f_0, f_1) is an internal functor, we also note that $f_1\otimes_{A_*} id_{\Gamma_*} : \Gamma_*\otimes_{A_*}\Gamma_* \to \Delta_*\otimes_{A_*}\Gamma_*$ is unique morphism that makes the following diagrams commute.

$$\begin{tikzcd}
\Gamma_* \arrow[r, "i_1"] \arrow[d, "f_1"'] & \Gamma_*\otimes_{A_*}\Gamma_* \arrow[d, "f_1\otimes_{A_*}id_{\Gamma_*}"'] & \Gamma_* \arrow[l, "i_2"'] \arrow[dl, "j_2"] \\
\Delta_* \arrow[r, "j_1"] & \Delta_*\otimes_{A_*}\Gamma_* &
\end{tikzcd}$$

We remark that $f_1 \otimes_{A_*} id_{\Gamma_*}$ is a homomorphism in left A_*-modules if we regard $\Delta_* \otimes_{A_*} \Gamma_*$ as a left A_*-module by $a \otimes (x \otimes y) \mapsto \sigma'(f_0(a))x \otimes y$, By the commutativity of the above diagram, we have

$$(f_1 \otimes_{A_*} id_{\Gamma_*})\mu\sigma = (f_1 \otimes_{A_*} id_{\Gamma_*})i_1\sigma = j_1 f_1 \sigma = j_1 \sigma' f_0$$

which implies that there exists unique morphism $(j_1\sigma', (f_1 \otimes_{A_*} id_{\Gamma_*})\mu) : B_* \otimes_{A_*} \Gamma_* \to \Delta_* \otimes_{A_*} \Gamma_*$ that makes the following diagram commute.

$$\begin{tikzcd}
A_* \arrow[r, "f_0"] \arrow[d, "\sigma"] & B_* \arrow[rr, "\sigma'"] \arrow[d, "\sigma_{f_0}"] & & \Delta_* \arrow[dd, "j_1"] \\
\Gamma_* \arrow[r, "f_{0\sigma}"] \arrow[d, "\mu"] & B_* \otimes_{A_*} \Gamma_* \arrow[rrd, dotted, "{(j_1\sigma', (f_1 \otimes_{A_*} id_{\Gamma_*})\mu)}"] & & \\
\Gamma_* \otimes_{A_*} \Gamma_* \arrow[rrr, "f_1 \otimes_{A_*} id_{\Gamma_*}"] & & & \Delta_* \otimes_{A_*} \Gamma_*
\end{tikzcd}$$

Hence we have

$$(j_1\sigma', (f_1 \otimes_{A_*} id_{\Gamma_*})\mu) f_{0\sigma}\tau = (f_1 \otimes_{A_*} id_{\Gamma_*})\mu\tau = (f_1 \otimes_{A_*} id_{\Gamma_*}) i_2 \tau = j_2 \tau = (\tau' \otimes_{A_*} id_{\Gamma_*}) f_{0\sigma}\tau.$$

For a representation $(\boldsymbol{M}, \boldsymbol{\xi})$ of $\boldsymbol{\Delta}$ on $\boldsymbol{M} = (B_*, M_*, \alpha)$, we put $P_{\sigma',\tau'}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = \hat{\boldsymbol{\xi}} : \boldsymbol{M} \to \boldsymbol{M}_{[\sigma',\tau']}$ and $\hat{\boldsymbol{\xi}} = (id_{B_*}, \hat{\xi}) : (B_*, M_*, \alpha) \to (B_*, M_* \otimes_{B_*} \Delta_*, \alpha_{\sigma'}(id_{M_* \otimes_{B_*} \Delta_*} \otimes_{K_*} \tau'))$. As we identify $\boldsymbol{M}_{[\sigma_{f_0}, f_{0\sigma}\tau]}$ with $(A_*, M_* \otimes_{A_*} \Gamma_*, \alpha_{\boldsymbol{f}})$, we identify $(\boldsymbol{M}_{[\sigma',\tau']})_{[\sigma_{f_0}, f_{0\sigma}\tau]}$ with $(A_*, (M_* \otimes_{B_*} \Delta_*) \otimes_{A_*} \Gamma_*, \alpha'_{\boldsymbol{f}})$. Here the right A_*-module structure of $M_* \otimes_{B_*} \Delta_*$ is given by $(x \otimes y) \otimes a \mapsto x \otimes y\tau' f_0(a)$ and we put $\alpha' = \alpha_{\sigma'}(id_{M_* \otimes_{B_*} \Delta_*} \otimes_{K_*} \tau')$. Then, it follows from (2.1.8) that $\hat{\boldsymbol{\xi}}_{[\sigma_{f_0}, f_{0\sigma}\tau]} : \boldsymbol{M}_{[\sigma_{f_0}, f_{0\sigma}\tau]} \to (\boldsymbol{M}_{[\sigma',\tau']})_{[\sigma_{f_0}, f_{0\sigma}\tau]}$ is identified with

$$(id_{A_*}, \hat{\xi} \otimes_{A_*} id_{\Gamma_*}) : (A_*, M_* \otimes_{A_*} \Gamma_*, \alpha_{\boldsymbol{f}}) \to (A_*, (M_* \otimes_{B_*} \Delta_*) \otimes_{A_*} \Gamma_*, \alpha'_{\boldsymbol{f}}).$$

It also follows from (2.1.10) that if we put

$$\theta_{\sigma',\tau',\sigma_{f_0}, f_{0\sigma}\tau}(\boldsymbol{M}) = (id_{A_*}, \tilde{\theta}_{\sigma',\tau',\sigma_{f_0}, f_{0\sigma}\tau}(\boldsymbol{M})) : (\boldsymbol{M}_{[\sigma',\tau']})_{[\sigma_{f_0}, f_{0\sigma}\tau]} \to \boldsymbol{M}_{[j_1\sigma', (\tau' \otimes_{A_*} id_{\Gamma_*}) f_{0\sigma}\tau]},$$

$\tilde{\theta}_{\sigma',\tau',\sigma_{f_0}, f_{0\sigma}\tau}(\boldsymbol{M})$ is identified with a map $(M_* \otimes_{B_*} \Delta_*) \otimes_{A_*} \Gamma_* \to M_* \otimes_{B_*} (\Delta_* \otimes_{A_*} \Gamma_*)$ which maps $(x \otimes y) \otimes z$ to $x \otimes (y \otimes z)$.

Let $\otimes_{f_0} : M_* \otimes_{A_*} (\Delta_* \otimes_{A_*} \Gamma_*) \to M_* \otimes_{B_*} (\Delta_* \otimes_{A_*} \Gamma_*)$ be the quotient map induced by f_0. Then, the following diagram is commutative.

$$\begin{tikzcd}
M_* \otimes_{A_*} \Gamma_* \arrow[r, "{id_{M_*} \otimes_{A_*} ((f_1 \otimes_{A_*} id_{\Gamma_*})\mu)}"] \arrow[d, "\chi"] & M_* \otimes_{A_*} (\Delta_* \otimes_{A_*} \Gamma_*) \arrow[d, "\otimes_{f_0}"] \\
M_* \otimes_{B_*} (B_* \otimes_{A_*} \Gamma_*) \arrow[r, "{id_{M_*} \otimes_{B_*} (j_1\sigma', (f_1 \otimes_{A_*} id_{\Gamma_*})\mu)}"] & M_* \otimes_{B_*} (\Delta_* \otimes_{A_*} \Gamma_*)
\end{tikzcd}$$

Hence if we put $\boldsymbol{M}_{(j_1\sigma', (f_1 \otimes_{A_*} id_{\Gamma_*})\mu)} = (id_{A_*}, \Phi) : \boldsymbol{M}_{[\sigma_{f_0}, f_{0\sigma}\tau]} \to \boldsymbol{M}_{[j_1\sigma', (\tau' \otimes_{A_*} id_{\Gamma_*}) f_{0\sigma}\tau]}$, Φ is identified with the following composition.

$$M_* \otimes_{A_*} \Gamma_* \xrightarrow{id_{M_*} \otimes_{A_*} ((f_1 \otimes_{A_*} id_{\Gamma_*})\mu)} M_* \otimes_{A_*} (\Delta_* \otimes_{A_*} \Gamma_*) \xrightarrow{\otimes_{f_0}} M_* \otimes_{B_*} (\Delta_* \otimes_{A_*} \Gamma_*)$$

From now, we assume that $\sigma : A_* \to \Gamma_*$ is flat. Then, the assumptions of (3.5.4) are all satisfied for a representation $(\boldsymbol{M}, \boldsymbol{\xi})$ of $\boldsymbol{\Gamma}$. Let us denote by $\kappa^{\boldsymbol{f}}_{(\boldsymbol{M},\xi)} : K(\boldsymbol{M}, \boldsymbol{\xi}; \boldsymbol{f})_* \to M_* \otimes_{A_*} \Gamma_*$ the kernel of

$$\tilde{\theta}_{\sigma',\tau',\sigma_{f_0}, f_{0\sigma}\tau}(\boldsymbol{M})(\hat{\xi} \otimes_{A_*} id_{\Gamma_*}) - \Phi : M_* \otimes_{A_*} \Gamma_* \to M_* \otimes_{B_*} (\Delta_* \otimes_{A_*} \Gamma_*).$$

Let $\alpha_{\boldsymbol{\xi},\boldsymbol{f}} : K(\boldsymbol{M}, \boldsymbol{\xi}; \boldsymbol{f})_* \otimes_{K_*} A_* \to K(\boldsymbol{M}, \boldsymbol{\xi}; \boldsymbol{f})_*$ be the right A_*-module structure of $K(\boldsymbol{M}, \boldsymbol{\xi}; \boldsymbol{f})_*$ defined from the right A_*-module structure of $M_* \otimes_{A_*} \Gamma_*$. We put $(\boldsymbol{M}, \xi)_{\boldsymbol{f}} = (A_*, K(\boldsymbol{M}, \boldsymbol{\xi}; \boldsymbol{f})_*, \alpha_{\boldsymbol{\xi},\boldsymbol{f}})$ and define a morphism $P^{\boldsymbol{f}}_{(\boldsymbol{M},\boldsymbol{\xi})} : (\boldsymbol{M}, \boldsymbol{\xi})_{\boldsymbol{f}} \to \boldsymbol{M}_{[\sigma_{f_0}, \tau f_{0\sigma}]}$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{A_*}$ to be $(id_{A_*}, \kappa^{\boldsymbol{f}}_{(\boldsymbol{M},\xi)})$. Then, $P^{\boldsymbol{f}}_{(\boldsymbol{M},\boldsymbol{\xi})}$ is an equalizer of the following morphisms.

$$\theta_{\sigma',\tau',\sigma_{f_0}, f_{0\sigma}\tau}(\boldsymbol{M})\hat{\boldsymbol{\xi}}_{[\sigma_{f_0}, f_{0\sigma}\tau]},\ \boldsymbol{M}_{(j_1\sigma', (f_1 \otimes_{A_*} id_{\Gamma_*})\mu)} : \boldsymbol{M}_{[\sigma_{f_0}, f_{0\sigma}\tau]} \to \boldsymbol{M}_{[j_1\sigma', (\tau' \otimes_{A_*} id_{\Gamma_*}) f_{0\sigma}\tau]}$$

Hence $(P^{\boldsymbol{f}}_{(\boldsymbol{M},\boldsymbol{\xi})})_{[\sigma,\tau]} : ((\boldsymbol{M},\boldsymbol{\xi})_{\boldsymbol{f}})_{[\sigma,\tau]} \to (\boldsymbol{M}_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]}$ is an equalizer of the following morphisms.

$$(\theta_{\sigma',\tau',\sigma_{f_0},f_{0\sigma}\tau}(\boldsymbol{M})\hat{\boldsymbol{\xi}}_{[\sigma_{f_0},f_{0\sigma}\tau]})_{[\sigma,\tau]},\ (\boldsymbol{M}_{(j_1\sigma',(f_1\otimes_{A_*}id_{\Gamma_*})\mu)})_{[\sigma,\tau]} : (\boldsymbol{M}_{[\sigma_{f_0},\,f_{0\sigma}\tau]})_{[\sigma,\tau]} \to (\boldsymbol{M}_{[j_1\sigma',\,(\tau'\otimes_{A_*}id_{\Gamma_*})f_{0\sigma}\tau]})_{[\sigma,\tau]}$$

It follows from the argument after (3.5.4) that the following diagrams are commutative.

$$\begin{tikzcd}
 & \boldsymbol{M}_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[r, "\hat{\boldsymbol{\mu}}_{\boldsymbol{f}}(\boldsymbol{M})"] & (\boldsymbol{M}_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]} \arrow[dr, "(\theta_{\sigma',\tau',\sigma_{f_0},f_{0\sigma}\tau}(\boldsymbol{M})\hat{\boldsymbol{\xi}}_{[\sigma_{f_0},f_{0\sigma}\tau]})_{[\sigma,\tau]}"] & \\
(\boldsymbol{M},\boldsymbol{\xi})_{\boldsymbol{f}} \arrow[ur, "P^{\boldsymbol{f}}_{(\boldsymbol{M},\boldsymbol{\xi})}"] \arrow[dr, "P^{\boldsymbol{f}}_{(\boldsymbol{M},\boldsymbol{\xi})}"'] & & & (\boldsymbol{M}_{[j_1\sigma',\,(\tau'\otimes_{A_*}id_{\Gamma_*})f_{0\sigma}\tau]})_{[\sigma,\tau]} \\
 & \boldsymbol{M}_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[r, "\hat{\boldsymbol{\mu}}_{\boldsymbol{f}}(\boldsymbol{M})"] & (\boldsymbol{M}_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]} \arrow[ur, "(\boldsymbol{M}_{(j_1\sigma',(f_1\otimes_{A_*}id_{\Gamma_*})\mu)})_{[\sigma,\tau]}"'] &
\end{tikzcd}$$

Thus there exists unique morphism $\hat{\boldsymbol{\xi}}^l_{\boldsymbol{f}} : (\boldsymbol{M},\boldsymbol{\xi})_{\boldsymbol{f}} \to ((\boldsymbol{M},\boldsymbol{\xi})_{\boldsymbol{f}})_{[\sigma,\tau]}$ that satisfies $(P^{\boldsymbol{f}}_{(\boldsymbol{M},\boldsymbol{\xi})})_{[\sigma,\tau]}\hat{\boldsymbol{\xi}}^l_{\boldsymbol{f}} = \hat{\mu}_{\boldsymbol{f}}(\boldsymbol{M})P^{\boldsymbol{f}}_{(\boldsymbol{M},\boldsymbol{\xi})}$. If put $\hat{\boldsymbol{\xi}}^l_{\boldsymbol{f}} = (id_{A_*}, \hat{\xi}^l_{\boldsymbol{f}})$, $\hat{\xi}^l_{\boldsymbol{f}} : K(\boldsymbol{M},\boldsymbol{\xi}\,;\boldsymbol{f})_* \to K(\boldsymbol{M},\boldsymbol{\xi}\,;\boldsymbol{f})_*\otimes_{A_*}\Gamma_*$ is a right Γ_*-comodule structure on $K(\boldsymbol{M},\boldsymbol{\xi}\,;\boldsymbol{f})_*$. We put $\boldsymbol{\xi}^l_{\boldsymbol{f}} = P_{\sigma,\tau}((\boldsymbol{M},\boldsymbol{\xi})_{\boldsymbol{f}})^{-1}_{(\boldsymbol{M},\boldsymbol{\xi})_{\boldsymbol{f}}}(\hat{\boldsymbol{\xi}}^l_{\boldsymbol{f}}) : \sigma^*((\boldsymbol{M},\boldsymbol{\xi})_{\boldsymbol{f}}) \to \tau^*((\boldsymbol{M},\boldsymbol{\xi})_{\boldsymbol{f}})$. Here we regard $\boldsymbol{\xi}^l_{\boldsymbol{f}}$ as a morphism in $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{A_*}$. The following results is a special case of (3.5.5).

Proposition 4.2.4 *$((\boldsymbol{M},\boldsymbol{\xi})_{\boldsymbol{f}}, \boldsymbol{\xi}^l_{\boldsymbol{f}})$ is a representation of $\boldsymbol{\Gamma}$ and $P^{\boldsymbol{f}}_{(\boldsymbol{M},\boldsymbol{\xi})} : (\boldsymbol{M}_{[\sigma_{f_0},\,\tau f_{0\sigma}]}, \mu^l_{\boldsymbol{f}}(\boldsymbol{M})) \to ((\boldsymbol{M},\boldsymbol{\xi})_{\boldsymbol{f}}, \boldsymbol{\xi}^l_{\boldsymbol{f}})$ is a morphism in representations of $\boldsymbol{\Gamma}$.*

Let $\boldsymbol{\varphi} : (\boldsymbol{M},\boldsymbol{\xi}) \to (\boldsymbol{N},\boldsymbol{\zeta})$ be a morphism in representations of $\boldsymbol{\Delta}$. By the argument after (3.5.5), there exists unique morphism $\boldsymbol{\varphi}_{\boldsymbol{f}} : (\boldsymbol{M},\boldsymbol{\xi})_{\boldsymbol{f}} \to (\boldsymbol{N},\boldsymbol{\zeta})_{\boldsymbol{f}}$ that satisfies $P^{\boldsymbol{f}}_{(\boldsymbol{N},\boldsymbol{\zeta})}\boldsymbol{\varphi}_{\boldsymbol{f}} = \boldsymbol{\varphi}_{[\sigma_{f_0},\,\tau f_{0\sigma}]}P^{\boldsymbol{f}}_{(\boldsymbol{M},\boldsymbol{\xi})}$.

The following results is a special case of (3.5.6).

Proposition 4.2.5 *$\boldsymbol{\varphi}_{\boldsymbol{f}} : ((\boldsymbol{M},\boldsymbol{\xi})_{\boldsymbol{f}}, \boldsymbol{\xi}^l_{\boldsymbol{f}}) \to ((\boldsymbol{N},\boldsymbol{\zeta})_{\boldsymbol{f}}, \zeta^l_{\boldsymbol{f}})$ is a morphism in representations of $\boldsymbol{\Gamma}$.*

Remark 4.2.6 *If $x \in M_n$ is a primitive element of $(\boldsymbol{M},\boldsymbol{\xi})$, $x\otimes 1 \in M_*\otimes_{A_*}\Gamma_*$ belongs to $K(\boldsymbol{M},\boldsymbol{\xi}\,;\boldsymbol{f})_*$ and it is a primitive element of $((\boldsymbol{M},\boldsymbol{\xi})_{\boldsymbol{f}}, \boldsymbol{\xi}^l_{\boldsymbol{f}})$.*

For a representation $(\boldsymbol{M},\boldsymbol{\xi})$ of $\boldsymbol{\Delta}$ and a morphism $\boldsymbol{f} = (f_0, f_1) : \Gamma \to \Delta$ of Hopf algebroids, we define a map $\tilde{\omega}_{\boldsymbol{M}} : (M_*\otimes_{A_*}\Gamma_*)\otimes_{A_*}B_* \to M_*$ by $\tilde{\omega}_{\boldsymbol{M}}((x\otimes y)\otimes b) = \alpha(x\otimes f_0(\varepsilon(y))b)$ if $\boldsymbol{M} = (B_*, M_*, \alpha)$. We note that $f_0^*(\boldsymbol{M}_{[\sigma_{f_0},\,\tau f_{0\sigma}]})$ is identified with $(B_*, (M_*\otimes_{A_*}\Gamma_*)\otimes_{A_*}B_*, (\tilde{\alpha}_{\boldsymbol{f}})_{f_0})$ by (4.2.1). Then, $(id_{B_*}, \tilde{\omega}_{\boldsymbol{M}}) : f_0^*(\boldsymbol{M}_{[\sigma_{f_0},\,\tau f_{0\sigma}]}) \to \boldsymbol{M}$ is a morphism in $\mathcal{M}od(\mathcal{C},\mathcal{M})_{B_*}$. We denote by $(\boldsymbol{\eta}_{\boldsymbol{f}})_{(\boldsymbol{M},\boldsymbol{\xi})}$ the following composition.

$$f_0^*((\boldsymbol{M},\boldsymbol{\xi})_{\boldsymbol{f}}) \xrightarrow{f_0^*(P^{\boldsymbol{f}}_{(\boldsymbol{M},\boldsymbol{\xi})})} f_0^*(\boldsymbol{M}_{[\sigma_{f_0},\,\tau f_{0\sigma}]}) \xrightarrow{(id_{B_*},\tilde{\omega}_{\boldsymbol{M}})} \boldsymbol{M}$$

It follows from (3.5.8) that $(\boldsymbol{\eta}_{\boldsymbol{f}})_{(\boldsymbol{M},\boldsymbol{\xi})}$ defines a morphism $(\boldsymbol{M},\boldsymbol{\xi}) \to (f_0^*((\boldsymbol{M},\boldsymbol{\xi})_{\boldsymbol{f}}), (\boldsymbol{\xi}^l_{\boldsymbol{f}})_{\boldsymbol{f}})$ of representations of $\boldsymbol{\Delta}$. By (3.5.9), $(\boldsymbol{\eta}_{\boldsymbol{f}})_{(\boldsymbol{M},\boldsymbol{\xi})}$ is natural in $(\boldsymbol{M},\boldsymbol{\xi})$. We denote by $\mathrm{Comod}(\Gamma_*)$ the category of right Γ_*-comodules and recall that the opposite category of $\mathrm{Comod}(\Gamma_*)$ is isomorphic to the category of representations of $\boldsymbol{\Gamma}$. We denote by $\mathrm{Rep}(\boldsymbol{\Gamma})$ the category of representations of $\boldsymbol{\Gamma}$ for short. For a representation $(\boldsymbol{M},\boldsymbol{\xi})$ of $\boldsymbol{\Delta}$ and a representation $(\boldsymbol{N},\boldsymbol{\zeta})$ of $\boldsymbol{\Gamma}$, we put $\boldsymbol{M} = (B_*, M_*, \alpha)$ and $\boldsymbol{N} = (A_*, N_*, \beta)$ and define a map

$$\mathrm{ad}^{(\boldsymbol{M},\boldsymbol{\xi})}_{(\boldsymbol{N},\boldsymbol{\zeta})} : \mathrm{Rep}(\boldsymbol{\Gamma})(((\boldsymbol{M},\boldsymbol{\xi})_{\boldsymbol{f}}, \boldsymbol{\xi}^l_{\boldsymbol{f}}), (\boldsymbol{N},\boldsymbol{\zeta})) \to \mathrm{Rep}(\boldsymbol{\Delta})((\boldsymbol{M},\boldsymbol{\xi}), \boldsymbol{f}^{\bullet}(\boldsymbol{N},\boldsymbol{\zeta}))$$

by giving a map

$$\mathrm{Comod}(\Gamma_*)((N_*,\hat{\zeta}), (K(\boldsymbol{M},\boldsymbol{\xi};\,\boldsymbol{f})_*, \hat{\xi}^l_{\boldsymbol{f}})) \to \mathrm{Comod}(\Delta_*)((N_*\otimes_{A_*}B_*, \hat{\zeta}_{\boldsymbol{f}}), (M_*, \hat{\xi}))$$

which maps $\psi \in \mathrm{Comod}(\Gamma_*)((N_*,\hat{\zeta}), (K(\boldsymbol{M},\boldsymbol{\xi};\,\boldsymbol{f})_*, \hat{\xi}^l_{\boldsymbol{f}}))$ to the following composition.

$$N_*\otimes_{A_*}B_* \xrightarrow{\psi\otimes_{A_*}id_{B_*}} K(\boldsymbol{M},\boldsymbol{\xi};\,\boldsymbol{f})_*\otimes_{A_*}B_* \xrightarrow{\kappa^{\boldsymbol{f}}_{(\boldsymbol{M},\xi)}\otimes_{A_*}id_{B_*}} (M_*\otimes_{A_*}\Gamma_*)\otimes_{A_*}B_* \xrightarrow{\tilde{\omega}_{\boldsymbol{M}}} M_*$$

Finally, we have the following result by (3.5.16).

Theorem 4.2.7 *$\mathrm{ad}^{(\boldsymbol{M},\boldsymbol{\xi})}_{(\boldsymbol{N},\boldsymbol{\zeta})} : \mathrm{Rep}(\boldsymbol{\Gamma})(((\boldsymbol{M},\boldsymbol{\xi})_{\boldsymbol{f}}, \boldsymbol{\xi}^l_{\boldsymbol{f}}), (\boldsymbol{N},\boldsymbol{\zeta})) \to \mathrm{Rep}(\boldsymbol{\Delta})((\boldsymbol{M},\boldsymbol{\xi}), \boldsymbol{f}^{\bullet}(\boldsymbol{N},\boldsymbol{\zeta}))$ is a bijection. Hence a correspondence $(\boldsymbol{M},\boldsymbol{\xi}) \mapsto ((\boldsymbol{M},\boldsymbol{\xi})_{\boldsymbol{f}}, \boldsymbol{\xi}^l_{\boldsymbol{f}})$ gives a left adjoint of the restriction functor $\boldsymbol{f}^{\bullet} : \mathrm{Rep}(\boldsymbol{\Gamma}) \to \mathrm{Rep}(\boldsymbol{\Delta})$.*

4.3 Sample calculation

Let BP be the Brown-Peterson spectrum ([3], [15], [18]) at a prime p and

$$\mathbf{\Gamma}_{BP} = (BP_*, BP_*BP, \sigma_{BP}, \tau_{BP}, \varepsilon_{BP}, \mu_{BP}, \iota_{BP})$$

the Hopf algebroid associated with BP [1]. We recall the structure of $\mathbf{\Gamma}_{BP}$ below (See [2],[14],[18]). The ordinary homology $H_*(BP)$ of BP is a polynomial algebra $\boldsymbol{Z}_{(p)}[m_1, m_2, \ldots, m_i, \ldots]$ for canonical generators m_i of degree $2(p^i - 1)$. $BP_* = \pi(BP)$.

The Hurewicz homomorphism $BP_* = \pi_*(BP) \to H_*(BP)$ is injective and if we identify $\pi_*(BP)$ with the image of the Hurewicz homomorphism, $\pi_*(BP)$ is a polynomial subring $\boldsymbol{Z}_{(p)}[v_1, v_2, \ldots, v_i, \ldots]$ of $H_*(BP)$, where v_i are Hazewinkel's generators which are determined inductively by the following equality in $H_*(BP)$.

$$v_n = pm_n - \sum_{i=1}^{n-1} v_{n-i}^{p^i} m_i$$

BP_*BP is a polynomial algebra $BP_*[t_1, t_2, \ldots, t_i, \ldots]$ with $\deg t_i = 2(p^i - 1)$. $\sigma_{BP} : BP_* \to BP_*BP$ and $\varepsilon_{BP} : BP_*BP \to BP_*$ are given by $\sigma_{BP}(v_i) = v_i$ and $\varepsilon_{BP}(v_i) = v_i$, $\varepsilon_{BP}(t_i) = 0$ for $i \geqq 1$. $\tau_{BP} : BP_* \to BP_*BP$, $\mu_{BP} : BP_*BP \to BP_*BP \otimes_{BP_*} BP_*BP$ and $\iota_{BP} : BP_*BP \to BP_*BP$ are given by the following equalities.

$$\tau_{BP}(m_n) = \sum_{i+j=n} m_i\, t_j^{p^i}, \quad \sum_{i+j=n} m_i\, \mu_{BP}(t_j)^{p^i} = \sum_{i+j+k=n} m_i\, t_j^{p^i} \otimes t_k^{p^{i+j}}, \quad \sum_{i+j+k=n} m_i\, t_j^{p^i}\, \iota_{BP}(t_k)^{p^{i+j}} = m_n$$

Here we set $m_0 = t_0 = 1$ and embed BP_* into $H_*(BP)$, hence BP_*BP is regarded as a subalgebra of $H_*(BP_*)[t_1, t_2, \ldots, t_i, \ldots]$.

Let Seq be the set of all infinite sequences $(j_1, j_2, \ldots, j_n, \ldots)$ of non-negative integers such that $j_n = 0$ for all but finite number of n's. Seq is regarded as an abelian monoid with unit $\mathbf{0} = (0, 0, \ldots)$ by componentwise addition. For $J = (j_1, j_2, \ldots, j_n, \ldots) \in \mathrm{Seq}$, we put

$$|J| = \sum_{n \geqq 0} j_n, \qquad \|J\| = \sum_{k \geqq 1} j_k(p^k - 1), \qquad t(J) = t_1^{j_1} t_2^{j_2} \cdots t_k^{j_k} \cdots \in BP_*BP.$$

Let I_∞ be the kernel of $t_0 : BP_* \to \boldsymbol{F}_p$. Then, $I_\infty = (p, v_1, v_2, \ldots, v_k, \ldots)$ and I_∞ is an invariant prime ideal. It follows from the formula for μ_{BP} that we have

$$\mu_{BP}(t_n) \equiv \sum_{k=0}^{n} t_k \otimes t_{n-k}^{p^k} \quad \text{modulo} \quad I_\infty BP_*BP.$$

Hence, the proof of theorem4b of [12] shows the following result.

Proposition 4.3.1 *Let X range over all infinite matrices*

$$\left\| \begin{array}{cccccc} * & x_{01} & x_{02} & \cdot & \cdot & \cdot \\ x_{10} & x_{11} & \cdot & \cdot & \cdot & \cdot \\ x_{20} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \right\|$$

of non-negative integers, almost all zero, with leading entry omitted. For each such matrix X, let us define $R(X) = (r_1, r_2, \ldots, r_n, \ldots)$, $S(X) = (s_1, s_2, \ldots, s_n, \ldots)$, $T(X) = (t_1, t_2, \ldots, t_n, \ldots)$ and $b(X)$ as follows.

$$r_i = \sum_{j \geqq 0} p^j x_{ij}, \qquad s_j = \sum_{i \geqq 0} x_{ij}, \qquad t_n = \sum_{i+j=n} x_{ij}, \qquad b(X) = \left(\prod_{n \geqq 1} t_n! \right) \left(\prod_{i,j \geqq 0} x_{ij}! \right)^{-1}$$

Then, the following congruence holds for $J \in \mathrm{Seq}$.

$$\mu_{BP}(t(J)) \equiv \sum_{T(X)=J} b(X) t(S(X)) \otimes t(R(X)) \quad \textit{modulo} \quad I_\infty BP_*BP.$$

Remark 4.3.2 *By the definition of $R(X)$, $S(X)$ and $T(X)$ above, $\|S(X)\| + \|R(X)\| = \|T(X)\|$ holds.*

We denote by H the Eilenberg-MacLane spectrum with coefficients in the prime field $\boldsymbol{F}_p$ and by $\mathcal{A}_{p*}$ the dual Steenrod algebra with coproduct $\mu_H : \mathcal{A}_{p*} \to \mathcal{A}_{p*} \otimes_{\boldsymbol{F}_p} \mathcal{A}_{p*}$. Let $\eta_H : \boldsymbol{F}_p \to \mathcal{A}_{p*}$ and $\varepsilon_H : \mathcal{A}_{p*} \to \boldsymbol{F}_p$ the unit and the counit of $\mathcal{A}_{p*}$. Then the Hopf algebroid $(\boldsymbol{F}_p, \mathcal{A}_{p*}, \eta_H, \eta_H, \varepsilon_H, \mu_H, \iota_H)$ associated with H is a Hopf algebra which we denote by $\boldsymbol{\Gamma}_H$. The structure of $\boldsymbol{\Gamma}_H$ is described by Milnor [12] as follows. We have

$$\mathcal{A}_{p*} = E(\tau_0, \tau_1, \ldots, \tau_i, \ldots) \otimes \boldsymbol{F}_p[\xi_1, \xi_2, \ldots, \xi_i, \ldots] \qquad (\deg \tau_i = 2p^i - 1,\ \deg \xi_i = 2(p^i - 1))$$

if p is an odd prime and

$$\mathcal{A}_{2*} = \boldsymbol{F}_2[\zeta_1, \zeta_2, \ldots, \zeta_i, \ldots] \qquad (\deg \zeta_i = 2^i - 1).$$

The counit ε_H is given by $\varepsilon_H(\tau_i) = 0$ $(i \geqq 0)$, $\varepsilon_H(\xi_i) = 0$ $(i \geqq 1)$ and $\varepsilon_H(\zeta_i) = 0$ $(i \geqq 1)$. $\mu_H : \mathcal{A}_{p*} \to \mathcal{A}_{p*} \otimes \mathcal{A}_{p*}$ is given by the following formulas.

$$\mu_H(\xi_n) = \sum_{k=0}^{n} \xi_{n-k}^{p^k} \otimes \xi_k, \qquad \mu_H(\tau_n) = \sum_{k=0}^{n} \xi_{n-k}^{p^k} \otimes \tau_k + \tau_n \otimes 1, \qquad \mu_H(\zeta_n) = \sum_{k=0}^{n} \zeta_{n-k}^{2^k} \otimes \zeta_k$$

$\iota_H : \mathcal{A}_{p*} \to \mathcal{A}_{p*}$ is determined by the following equalities. (See [12] for more explicit formula for $\iota_H(\xi_n)$.)

$$\sum_{i=0}^{n} \xi_{n-k}^{p^k} \iota_H(\xi_k) = 0, \qquad \iota_H(\tau_n) = -\sum_{k=0}^{n} \iota_H(\xi_{n-k})^{p^k} \tau_k, \qquad \sum_{k=0}^{n} \zeta_{n-k}^{2^k} \iota_H(\zeta_k) = 0$$

Here we set $\xi_0 = \zeta_0 = 1$.

Let $T : BP \to H$ be the Thom map. We denote by $T_0 : BP_* \to H_* = \boldsymbol{F}_p$ and $T_1 : BP_*BP \to H_*H = \mathcal{A}_{p*}$ the maps induced by T and $T \wedge T : BP \wedge BP \to H \wedge H$, respectively. Then, $\boldsymbol{T} = (T_0, T_1) : \boldsymbol{\Gamma}_{BP} \to \boldsymbol{\Gamma}_H$ is a morphism in Hopf algebroids.

$BP_*(\boldsymbol{C}P^\infty)$ is a free BP_*-module generated by $\beta_0^{BP}, \beta_1^{BP}, \ldots, \beta_i^{BP}, \ldots$ $(\deg \beta_i^{BP} = 2i)$ and $H_*(\boldsymbol{C}P^\infty)$ is a vector space over $\boldsymbol{F}_p$ spanned by $\beta_0^H, \beta_1^H, \ldots, \beta_i^H, \ldots$ $(\deg \beta_i^H = 2i)$ ([2]). $T_* : BP_*(\boldsymbol{C}P^\infty) \to H_*(\boldsymbol{C}P^\infty)$ maps β_i^{BP} to $\deg \beta_i^H$.

Let us denote by F_{BP} the formal group law associated with BP. We put $x +_F y = F_{BP}(x, y)$ and

$$t^F = 1 +_F t_1 +_F t_2 +_F \cdots +_F t_i +_F \cdots = {\sum_{i \geqq 0}}^{F} t_i, \qquad \beta^{BP} = \beta_0^{BP} + \beta_1^{BP} + \cdots + \beta_i^{BP} + \cdots = \sum_{i \geqq 0} \beta_i^{BP}.$$

For a spectrum X, $BP_*(X)$ has a left BP_*BP-comodule structure defined in [1]. The left BP_*BP-comodule structure on $BP_*(\boldsymbol{C}P^\infty)$ is given as follows.

Proposition 4.3.3 ([16]) *The left BP_*BP-comodule structure*

$$\psi'_{BP} : BP_*(\boldsymbol{C}P^\infty) \to BP_*BP \otimes_{BP_*} BP_*(\boldsymbol{C}P^\infty)$$

on $BP_*(\boldsymbol{C}P^\infty)$ is given by $\psi'_{BP}(\beta^{BP}) = \sum_{i \geqq 0} \iota_{BP}(t^F)^i \otimes \beta_i^{BP}$.

We denote by $\mathcal{A}_p^*$ the mod p Steenrod algebra and consider the Milnor basis [12] of $\mathcal{A}_p^*$ below. We put $E_k = (i_1, i_2, \ldots, i_n, \ldots) \in \mathrm{Seq}$ where $i_k = 1$ and $i_s = 0$ if $s \neq k$.

Lemma 4.3.4 *Let X be a topological space. For $R \in \mathrm{Seq}$ and $x \in H^2(X)$, the following equality holds.*

$$\wp(R)x = \begin{cases} x^{p^k} & R = E_k \\ 0 & |R| \geqq 2 \end{cases}$$

Proof. We first remark that the following equality is obtained by theorem4b of [12].

$$\wp^{p^k} \wp(E_k) = \wp(E_{k+1}) + \wp(p^k E_1 + E_k) \ \cdots (*)$$

Since the excess of $\wp(R)$ is $2|R|$ by [10], it follows $\wp(R)x = 0$ if $|R| \geqq 2$. In particular, we have $\wp(p^k E_1 + E_k)x = 0$, hence $\wp^{p^k} \wp(E_k)x = \wp(E_{k+1})x$ by $(*)$. Since $\wp(E_1) = \wp^1$, $\wp(E_k)x = x^{p^k}$ follows from the induction on k. $\square$

For $R = (r_1, r_2, \ldots, r_k, \ldots) \in \mathrm{Seq}$ and an integer n, we put

$$\binom{n}{R} = \begin{cases} \dfrac{n!}{(n - |R|)!\, r_1! r_2! \cdots r_k! \cdots} & |R| \leqq n \\ 0 & |R| > n \end{cases}.$$

The following result is a consequence of the above definition.

Proposition 4.3.5 *The following equality holds for $R \in \mathrm{Seq}$.*

$$\binom{n}{R} = \binom{n-1}{R} + \sum_{k \geqq 0} \binom{n-1}{R-E_k}$$

Lemma 4.3.6 *Let X be a topological space. For $R \in \mathrm{Seq}$ and $x \in H^2(X)$, we have $\wp(R)x^n = \binom{n}{R} x^{n+\|R\|}$.*

Proof. We show the assertion by the induction on n. The assertion holds for $n = 1$ by (4.3.4). It follow from (4.3.6), (4.3.5) and the inductive assumption that we have

$$\begin{aligned}
\wp(R)x^n &= \sum_{S+T=R} (\wp(S)x)(\wp(T)x^{n-1}) = x(\wp(R)x^{n-1}) + \sum_{k\geqq 0}(\wp(E_k)x)(\wp(R-E_k)x^{n-1}) \\
&= \binom{n-1}{R} x^{n+\|R\|} + \sum_{k\geqq 0} \binom{n-1}{R-E_k} x^{n+\|R-E_k\|+p^k-1} \\
&= \left(\binom{n-1}{R} + \sum_{k\geqq 0}\binom{n-1}{R-E_k}\right) x^{n+\|R\|} = \binom{n}{R} x^{n+\|R\|}
\end{aligned}$$

Thus the assertion follows. □

We put

$$\beta^H = \beta_0^H + \beta_1^H + \cdots + \beta_i^H + \cdots,$$
$$\xi^H = \begin{cases} 1 + \xi_1 + \xi_2 + \cdots + \xi_i + \cdots & (p \neq 2) \\ 1 + \zeta_1^2 + \zeta_2^2 + \cdots + \zeta_i^2 + \cdots & (p = 2) \end{cases}.$$

For $R = (r_1, r_2, \ldots, r_k, \ldots) \in \mathrm{Seq}$, we put $\xi(R) = \begin{cases} \xi_1^{r_1}\xi_2^{r_2}\cdots\xi_k^{r_k}\cdots & p \neq 2 \\ \zeta_1^{2r_1}\zeta_2^{2r_2}\cdots\zeta_k^{2r_k}\cdots & p = 2 \end{cases}$. Then, $\deg \xi(R) = 2\|R\|$.

Proposition 4.3.7 *The left $\mathcal{A}_{p*}$-comodule structure*

$$\psi'_H : H_*(\boldsymbol{C}P^\infty) \to \mathcal{A}_{p*} \otimes_{\boldsymbol{F}_p} H_*(\boldsymbol{C}P^\infty)$$

on $H_*(\boldsymbol{C}P^\infty)$ is given by $\psi'_H(\beta^H) = \sum_{n\geqq 0} (\xi^H)^n \otimes \beta_n^H$.

Proof. Since ψ'_H is the dual of the cohomology operation $\mathcal{A}_p^* \otimes_{\boldsymbol{F}_p} H^*(\boldsymbol{C}P^\infty) \to H^*(\boldsymbol{C}P^\infty)$, we have the following equalities for $R = (r_1, r_2, \ldots, r_k, \ldots)$ and non-negative integers m, n by (4.3.6).

$$\langle \wp(R) \otimes x^n, \psi'_H(\beta_m^H)\rangle = \langle \wp(R)x^n, \beta_m^H\rangle = \binom{n}{R}\langle x^{n+\|R\|}, \beta_m^H\rangle$$

Thus $\psi'_H(\beta_m^H) = \sum_{n+\|R\|=m} \binom{n}{R} \xi(R) \otimes \beta_n^H$ and the assertion follows from $(\xi^H)^n = \sum_{|R|\leqq n} \binom{n}{R}\xi(R)$. □

The following fact is a folklore.

Proposition 4.3.8 *$T_1 : BP_*BP \to H_*H = \mathcal{A}_{p*}$ maps t_i to $\iota_H(\xi_i)$ if p is an odd prime and to $\iota_H(\zeta_i)^2$ if $p = 2$.*

Proof. It follows from (4.3.3) and the commutativity of the following diagram that we have $T_1(\iota_{BP}(t^F)) = \xi^H$.

$$\begin{array}{ccc}
BP_*(\boldsymbol{C}P^\infty) & \xrightarrow{\psi'_{BP}} & BP_*BP \otimes_{BP_*} BP_*(\boldsymbol{C}P^\infty) \\
\big\downarrow{\scriptstyle T_*} & & \big\downarrow{\scriptstyle t_1 \otimes T_*} \\
H_*(\boldsymbol{C}P^\infty) & \xrightarrow{\psi'_H} & \mathcal{A}_{p*} \otimes_{\boldsymbol{F}_p} H_*(\boldsymbol{C}P^\infty)
\end{array}$$

Since $T_*F_{BP}(x,y)$ is the additive formal group law, $T_1(x+_F y) = T_1(x) + T_1(y)$ holds for $x, y \in BP_*BP$. Thus we have

$$\xi^H = T_1(\iota_{BP}(t^F)) = 1 + T_1(\iota_{BP}(t_1)) + T_1(\iota_{BP}(t_2)) + \cdots + T_1(\iota_{BP}(t_i)) + \cdots.$$

Therefore $T_1(\iota_{BP}(t_i)) = \xi_i$ if p is an odd prime and $T_1(\iota_{BP}(t_i)) = \zeta_i^2$ if $p=2$. Since $\boldsymbol{T} = (T_0, T_1) : \boldsymbol{\Gamma}_{BP} \to \boldsymbol{\Gamma}_H$ is a morphism in Hopf algebroids and $\iota_H\iota_H$ is the identity map of $\mathcal{A}_{p*}$, $\iota_H(\xi_i) = \iota_H(T_1(\iota_{BP}(t_i))) = T_1(t_i)$ holds if p is odd and $\iota_H(\zeta_i)^2 = \iota_H(T_1(\iota_{BP}(t_i))) = T_1(t_i)$ holds if $p=2$. $\square$

For a spectrum X consider the right BP_*BP-comodule structure on $BP_*(X)$ as in [14] below. Similarly, we consider the right $\mathcal{A}_{p*}$-comodule structure on $H_*(X)$. Then, (4.3.3) and (4.3.7) imply the following result.

Corollary 4.3.9 *The right BP_*BP-comodule structure*

$$\psi_{BP} : BP_*(\boldsymbol{C}P^\infty) \to BP_*(\boldsymbol{C}P^\infty) \otimes_{BP_*} BP_*BP$$

on $BP_*(\boldsymbol{C}P^\infty)$ is given by $\psi_{BP}(\beta^{BP}) = \sum_{i\geqq 0} \beta_i^{BP} \otimes (t^F)^i$. The right $\mathcal{A}_{p*}$-comodule structure

$$\psi_H : H_*(\boldsymbol{C}P^\infty) \to H_*(\boldsymbol{C}P^\infty) \otimes_{\boldsymbol{F}_p} \mathcal{A}_{p*}$$

on $H_*(\boldsymbol{C}P^\infty)$ is given by $\psi_H(\beta^H) = \sum_{i\geqq 0} \beta_i^H \otimes \iota_H(\xi^H)^i$, in other words, $\psi_H(\beta_l^H) = \sum_{j+\|I\|=l} \binom{j}{I} \beta_j^H \otimes \iota_H(\xi(I))$. In particular, $\psi_H(\beta_l^H) = \sum_{0\leqq i\leqq \frac{l}{p}} (-1)^i \binom{l-i(p-1)}{i} \beta^H_{l-i(p-1)} \otimes \xi_1^i$ if $l < p^2$.

For a positive integer n, $H_*(\boldsymbol{C}P^n)$ is a right $\mathcal{A}_{p*}$-subcomodule of $H_*(\boldsymbol{C}P^\infty)$ spanned by $\beta_0^H, \beta_1^H, \dots, \beta_n^H$. We denote by $\psi_H^n : H_*(\boldsymbol{C}P^n) \to H_*(\boldsymbol{C}P^n) \otimes_{\boldsymbol{F}_p} \mathcal{A}_{p*}$ (n is a positive integer or ∞) the comodule structure map. Let $\bar{\psi}_H^n : H_*(\boldsymbol{C}P^n) \otimes_{\boldsymbol{F}_p} \mathcal{A}_{p*} \to H_*(\boldsymbol{C}P^n) \otimes_{\boldsymbol{F}_p} \mathcal{A}_{p*}$ be the right $\mathcal{A}_{p*}$-module homomorphism induced by ψ_H^n. We put $\boldsymbol{H}(\boldsymbol{C}P^n) = (\boldsymbol{F}_p, H_*(\boldsymbol{C}P^n), \alpha)$ which is an object of $\mathcal{M}od(\mathcal{A}lg_{\boldsymbol{Z}_{(p)}}, \mathcal{M}od_{\boldsymbol{Z}_{(p)}})_{\boldsymbol{F}_p}$ (recall (2.1.2)) and put $\boldsymbol{\psi}_H^n = (id_{\mathcal{A}_{p*}}, \bar{\psi}_H^n) : \eta_H^*(\boldsymbol{H}(\boldsymbol{C}P^n)) \to \eta_H^*(\boldsymbol{H}(\boldsymbol{C}P^n))$ which is a morphism in $\mathcal{M}od(\mathcal{A}lg_{\boldsymbol{Z}_{(p)}}, \mathcal{M}od_{\boldsymbol{Z}_{(p)}})_{\mathcal{A}_{p*}}$. If we regard $\boldsymbol{\psi}_H^n$ as an morphism in the opposite category of $\mathcal{M}od(\mathcal{A}lg_{\boldsymbol{Z}_{(p)}}, \mathcal{M}od_{\boldsymbol{Z}_{(p)}})_{\mathcal{A}_{p*}}$, then $(\boldsymbol{H}(\boldsymbol{C}P^n), \boldsymbol{\psi}_H^n)$ is a representation of $\boldsymbol{\Gamma}_H$ on $\boldsymbol{H}(\boldsymbol{C}P^n)$.

We regard $H_*(\boldsymbol{C}P^n)$ as a right BP_*-module by $T_0 : BP_* \to \boldsymbol{F}_p$. Define a homomorphisms

$$\Theta_1, \Theta_2 : H_*(\boldsymbol{C}P^n) \otimes_{BP_*} BP_*BP \to H_*(\boldsymbol{C}P^n) \otimes_{\boldsymbol{F}_p} (\mathcal{A}_{p*} \otimes_{BP_*} BP_*BP)$$

of right BP_*BP-modules to be the following compositions, respectively.

$$\begin{aligned}
H_*(\boldsymbol{C}P^n) \otimes_{BP_*} BP_*BP &\xrightarrow{\psi_H^n \otimes_{BP_*} id_{BP_*BP}} (H_*(\boldsymbol{C}P^n) \otimes_{\boldsymbol{F}_p} \mathcal{A}_{p*}) \otimes_{BP_*} BP_*BP \\
&\xrightarrow{\cong} H_*(\boldsymbol{C}P^n) \otimes_{\boldsymbol{F}_p} (\mathcal{A}_{p*} \otimes_{BP_*} BP_*BP) \\
H_*(\boldsymbol{C}P^n) \otimes_{BP_*} BP_*BP &\xrightarrow{id_{H_*(\boldsymbol{C}P^n)} \otimes_{BP_*} \mu_{BP}} H_*(\boldsymbol{C}P^n) \otimes_{BP_*} (BP_*BP \otimes_{BP_*} BP_*BP) \\
&\xrightarrow{id_{H_*(\boldsymbol{C}P^n)} \otimes_{BP_*} (T_1 \otimes_{BP_*} id_{BP_*BP})} H_*(\boldsymbol{C}P^n) \otimes_{BP_*} (\mathcal{A}_{p*} \otimes_{BP_*} BP_*BP) \\
&\xrightarrow{\otimes_{T_0}} H_*(\boldsymbol{C}P^n) \otimes_{\boldsymbol{F}_p} (\mathcal{A}_{p*} \otimes_{BP_*} BP_*BP)
\end{aligned}$$

Here, $\otimes_{T_0}$ is the quotient map induced by $T_0 : BP_* \to \boldsymbol{F}_p$.

It follows from (4.3.8), (4.3.9) and (4.3.1) that Θ_1 and Θ_2 are described as follows.

$$\Theta_1(\beta_k^H \otimes t(J)) = \sum_{\|I\| \leqq k} \binom{k - \|I\|}{I} \beta^H_{k-\|I\|} \otimes \iota_H(\xi(I)) \otimes t(J)$$

$$\Theta_2(\beta_k^H \otimes t(J)) = \sum_{T(X)=J} b(X) \beta_k^H \otimes \iota_H(\xi(S(X))) \otimes t(R(X))$$

We note that $\{\beta_k^H \otimes t(J) \,|\, 0 \leqq k \leqq n,\ J \in \mathrm{Seq}\}$ is a bais of $H_*(\boldsymbol{C}P^n) \otimes_{BP_*} BP_*BP$ over $\boldsymbol{F}_p$. Hence each element w of $H_*(\boldsymbol{C}P^n) \otimes_{BP_*} BP_*BP$ of can be expressed as

$$w = \sum_{l-n \leqq \|J\| \leqq l} z_J \beta^H_{l-\|J\|} \otimes t(J) = \sum_{0 \leqq j \leqq n} \beta_l^H \otimes \left(\sum_{\|J\|=l-j} z_J t(J) \right)$$

for $z_J \in \boldsymbol{F}_p$ if $\deg w = 2l$. Then, we have the following equalities by (4.3.2).

$$\begin{aligned}
\Theta_1(w) &= \sum_{0 \leqq j \leqq n} \left(\sum_{\|I\| \leqq j} \binom{j - \|I\|}{I} \beta^H_{j-\|I\|} \otimes \iota_H(\xi(I)) \right) \otimes \left(\sum_{\|J\|=l-j} z_J t(J) \right) \\
&= \sum_{l-n \leqq \|J\| \leqq l-\|I\|} \binom{l - \|I\| - \|J\|}{I} z_J \beta^H_{l-\|I\|-\|J\|} \otimes \iota_H(\xi(I)) \otimes t(J) \\
\Theta_2(w) &= \sum_{l-n \leqq \|T(X)\| \leqq l} b(X) z_{T(X)} \beta^H_{l-\|T(X)\|} \otimes \iota_H(\xi(S(X))) \otimes t(R(X)) \\
&= \sum_{l-n \leqq \|I\|+\|J\| \leqq l} \left(\sum_{S(X)=I, R(X)=J} b(X) z_{T(X)} \right) \beta^H_{l-\|I\|-\|J\|} \otimes \iota_H(\xi(I)) \otimes t(J)
\end{aligned}$$

Let $\kappa^{\boldsymbol{T}}_{(\boldsymbol{H}(\boldsymbol{C}P^n), \boldsymbol{\psi}^n_H)} : K(\boldsymbol{H}(\boldsymbol{C}P^n), \boldsymbol{\psi}^n_H; \boldsymbol{T})_* \to H_*(\boldsymbol{C}P^n) \otimes_{BP_*} BP_*BP$ be the kernel of $\Theta_1 - \Theta_2$. The above equalities imply the following.

Proposition 4.3.10 *An element* $\sum_{l-n \leqq \|J\| \leqq n} z_J \beta^H_{l-\|J\|} \otimes t(J)$ *of* $H_*(\boldsymbol{C}P^n) \otimes_{BP_*} BP_*BP$ *of degree* $2l$ *belongs to* $K(\boldsymbol{H}(\boldsymbol{C}P^n), \boldsymbol{\psi}^n_H; \boldsymbol{T})_*$ *if and only if* z_J*'s satisfy the following equations.*

$$\sum_{S(X)=I, R(X)=J} b(X) z_{T(X)} = \begin{cases} \binom{n-\|I\|-\|J\|}{I} z_J & \text{if } l-n \leqq \|J\| \leqq l - \|I\| \\ 0 & \text{if } \|J\| < l-n \leqq \|I\| + \|J\| \leqq l \end{cases}$$

Since it is not easy to solve the above linear equations of z_J's, we partially solve this for the case $l < p^2$ and describe $K(\boldsymbol{H}(\boldsymbol{C}P^n), \boldsymbol{\psi}^n_H; \boldsymbol{T})_{2l}$ for $l < p^2$.

Let w be a homogeneous element of $H_*(\boldsymbol{C}P^n) \otimes_{BP_*} BP_*BP$ and put $\deg w = 2l$. If $l < p^2 - 1$, then there exist $z_k \in \boldsymbol{F}_p$ for $\max\left\{\frac{l-n}{p-1}, 0\right\} \leqq k \leqq \frac{l}{p-1}$ such that

$$w = \sum_{\max\left\{\frac{l-n}{p-1}, 0\right\} \leqq k \leqq \frac{l}{p-1}} z_k \beta^H_{l-k(p-1)} \otimes t_1^k.$$

If $l = p^2 - 1$, then there exist $z_k \in \boldsymbol{F}_p$ for $\max\left\{p+1-\frac{n}{p-1}, 0\right\} \leqq k \leqq p+2$ such that

$$w = \sum_{\max\left\{p+1-\frac{n}{p-1}, 0\right\} \leqq k \leqq p+1} z_k \beta^H_{(p-1)(p+1-k)} \otimes t_1^k + z_{p+2} \beta^H_0 \otimes t_2.$$

Hence we have the following equalities if $l < p^2 - 1$.

$$\begin{aligned}
\Theta_1(w) &= \sum_{\max\{\frac{l-n}{p-1}, 0\} \leqq k \leqq \frac{l-pi}{p-1}} (-1)^i \binom{l-(i+k)(p-1)}{i} z_k \beta^H_{l-(p-1)(i+k)} \otimes \xi_1^i \otimes t_1^k \\
\Theta_2(w) &= \sum_{\max\left\{\frac{l-n}{p-1}, 0\right\} \leqq i+k \leqq \frac{l}{p-1}} (-1)^i \binom{i+k}{i} z_{i+k} \beta^H_{l-(p-1)(i+k)} \otimes \xi_1^i \otimes t_1^k
\end{aligned}$$

We also have the following equalities if $l = p^2 - 1$.

$$\begin{aligned}
\Theta_1(w) &= \sum_{\max\left\{p+1-\frac{n}{p-1}, 0\right\} \leqq k \leqq p+1-i} (-1)^i \binom{(p-1)(p+1-i-k)}{i} z_k \beta^H_{(p-1)(p+1-i-k)} \otimes \xi_1^i \otimes t_1^k + z_{p+2} \beta^H_0 \otimes 1 \otimes t_2 \\
\Theta_2(w) &= \sum_{\max\left\{p+1-\frac{n}{p-1}, 0\right\} \leqq i+k \leqq p+1} (-1)^i \binom{i+k}{i} z_{i+k} \beta^H_{(p-1)(p+1-i-k)} \otimes \xi_1^i \otimes t_1^k \\
&\quad + z_{p+2} \beta^H_0 \otimes (1 \otimes t_2 - \xi_1 \otimes t_1^p + \xi_1^{p+1} \otimes 1 - \xi_2 \otimes 1)
\end{aligned}$$

We assume that $w \in K(\boldsymbol{H}(\boldsymbol{C}P^n), \boldsymbol{\psi}_H^n; \boldsymbol{T})_{2l}$ for $l \leqq p^2-1$ below. It follows from the above equalities that $z_{p+2}=0$ if $l=p^2-1$ and that we have the following equations of $z_0, z_1, \ldots, z_{[\frac{l}{p-1}]}$.

$$\binom{i+k}{i} z_{i+k} = \begin{cases} \binom{l-(p-1)(i+k)}{i} z_k & \max\left\{\frac{l-n}{p-1}, 0\right\} \leqq k \leqq \frac{l-pi}{p-1} \\ 0 & k < \max\left\{\frac{l-n}{p-1}, 0\right\} \text{ or } \frac{l-i}{p-1} < i+k \leqq \frac{l}{p-1} \end{cases} \tag{4.3.1}$$

Lemma 4.3.11 $\binom{j}{i}\binom{l-j(p-1)}{j} \equiv \binom{l-j(p-1)}{i}\binom{l-(j-i)(p-1)}{j-i}$ *holds modulo p if $j-i<p$.*

Proof. Put $m = l-j(p-1)-i$, then we have the following equalities.

$$\binom{j}{i}\binom{l-j(p-1)}{j} = \binom{j}{i}\binom{i+m}{j} = \binom{i+m}{i}\binom{m}{j-i}$$
$$\binom{l-j(p-1)}{i}\binom{l-(j-i)(p-1)}{j-i} = \binom{i+m}{i}\binom{m+ip}{j-i}$$

Since $\binom{m+ip}{j-i} = \dfrac{(m+ip)(m+ip-1)\cdots(m+ip-j+i+1)}{(j-i)!}$ and $\binom{m}{j-i} = \dfrac{m(m-1)\cdots(m-j+i+1)}{(j-i)!}$, we have $\binom{m+ip}{j-i} \equiv \binom{m}{j-i}$ modulo p if $j-i<p$. □

If $l = l_0 + l_1 p$ for $0 \leqq l_0, l_1 \leqq p-1$, we have the following equalities.

$$\left[\frac{l}{p}\right] = l_1, \qquad \left[\frac{l}{p-1}\right] = \left[\frac{l_0+l_1}{p-1} + l_1\right] = \begin{cases} l_1 & l_0+l_1 \leqq p-2 \\ l_1+1 & p-1 \leqq l_0+l_1 \leqq 2p-3 \\ l_1+2 & l_0=l_1=p-1 \end{cases}$$

Lemma 4.3.12 *The solution of* (4.3.1) *is given as follows if $n \geqq l$ and $l \leqq p^2-1$.*

(1) *The case $l_0+l_1 \leqq p-2$; $z_i = \binom{l-i(p-1)}{i} a$ for $a \in \boldsymbol{F}_p$, $i=0,1,\ldots,l_1$.*

(2) *The case $p-1 \leqq l_0+l_1 \leqq 2p-3$; $z_i = \binom{l-i(p-1)}{i} a$ for $a \in \boldsymbol{F}_p$, $i=0,1,\ldots,p-l_0-1$ and $z_i=0$ for $i=p-l_0, p-l_0+1, \ldots, l_1+1$.*

(3) *The case $l=p^2-1$; $z_0=a$, $z_p=b$ for $a,b \in \boldsymbol{F}_p$, $z_i=0$ for $i=1,2,\ldots,p-1,p+1,p+2$.*

Proof. Put $j=i+k$. Then, (4.3.1) is equivalent to the following equation $(*)$.

$$(*) \quad \begin{cases} z_i = \binom{l-i(p-1)}{i} z_0 & 1 \leqq i \leqq l_1 \\ \binom{j}{i} z_j = \binom{l-j(p-1)}{i} z_{j-i} & 1 \leqq i \leqq \frac{l_0+1}{p} + l_1 - 1,\ i+1 \leqq j \leqq \frac{l_0+l_1-i}{p-1} + l_1 \\ \binom{j}{i} z_j = 0 & 1 \leqq i \leqq j,\ \frac{l_0+l_1-i}{p-1} + l_1 < j \leqq \frac{l_0+l_1}{p-1} + l_1 \end{cases}$$

(1) Suppose $l_0+l_1 \leqq p-2$. Since $\frac{l_0+1}{p} < 1$, $\frac{l_0+l_1}{p-1} < 1$ and $\frac{l_0+l_1-i}{p-1} > 0$ if $i \leqq l_1$, $(*)$ is equivalent to

$$\begin{cases} z_i = \binom{l-i(p-1)}{i} z_0 & 1 \leqq i \leqq l_1 \\ \binom{j}{i} z_j = \binom{l-j(p-1)}{i} z_{j-i} & 1 \leqq i \leqq l_1-1,\ i+1 \leqq j \leqq l_1 \end{cases}.$$

Hence the assertion follows from (4.3.11).

(2) Suppose $p-1 \leqq l_0+l_1 \leqq 2p-3$. Then $1 \leqq \dfrac{l_0+l_1}{p-1} < 2$ and $\dfrac{l_0+1}{p}+l_1-1 \geqq l_0+l_1-p+1$ hold. In fact, $\dfrac{l_0+1}{p}+l_1-1-(l_0+l_1-p+1) = p-2-\dfrac{l_0(p-1)-1}{p} \geqq 0$. Hence $(*)$ is equivalent to the following equation.

$$\begin{cases} z_i = \dbinom{l-i(p-1)}{i} z_0 & 1 \leqq i \leqq l_1 \\ \dbinom{j}{i} z_j = \dbinom{l-j(p-1)}{i} z_{j-i} & 1 \leqq i \leqq l_0+l_1-p+1,\ i+1 \leqq j \leqq l_1 \\ z_{l_1+1-i} = 0 & 1 \leqq i \leqq l_0+l_1-p+1 \\ \dbinom{j}{i} z_j = \dbinom{l-j(p-1)}{i} z_{j-i} & l_0+l_1-p+1 < i \leqq \dfrac{l_0+1}{p}+l_1-1,\ i+1 \leqq j \leqq l_1 \\ z_{l_1+1} = 0 \end{cases}$$

This is also equivalent to

$$\begin{cases} z_i = \dbinom{l-i(p-1)}{i} z_0 & 1 \leqq i \leqq l_1 \\ \dbinom{j}{i} z_j = \dbinom{l-j(p-1)}{i} z_{j-i} & 1 \leqq i \leqq \dfrac{l_0+1}{p}+l_1-1,\ i+1 \leqq j \leqq l_1 \\ z_i = 0 & p-l_0 \leqq i \leqq l_1+1 \end{cases}.$$

By (4.3.11), the above equation to the following equation.

$$\begin{cases} z_i = \dbinom{l-i(p-1)}{i} z_0 & 1 \leqq i \leqq l_1 \\ z_i = 0 & p-l_0 \leqq i \leqq l_1+1 \end{cases}$$

If $p-l_0 \leqq i \leqq l_1$, then we have $0 \leqq l_0+i-p < i \leqq l_1 \leqq p-1$, $1 \leqq l_1-i+1 \leqq p-1$ and $l_0+i-p < i$. Hence $l-i(p-1) = l_0+i-p+(l_1-i+1)p$ implies $\dbinom{l-i(p-1)}{i} \equiv \dbinom{l_0+i-p}{i}\dbinom{l_1-i+1}{0} \equiv 0$ modulo p. Thus if we put $z_0 = a$, $z_i = \dbinom{l-i(p-1)}{i} a$ for $1 \leqq i \leqq p-l_0-1$ and $z_i = 0$ for $p-l_0 \leqq i \leqq l_1+1$.

(3) Suppose $l = p^2-1$, then $l_0 = l_1 = p-1$. If $1 \leqq i \leqq p-1$, it follows from $p^2-1-i(p-1) = i-1+(p-i)p$ that $\dbinom{p^2-1-i(p-1)}{i} \equiv \dbinom{i-1}{i}\dbinom{p-i}{0} \equiv 0$ modulo p. Hence $(*)$ is equivalent to $z_i = 0$ for $1 \leqq i \leqq p-1$ or $i = p+1$ and the assertion follows. □

The above result implies the following.

Proposition 4.3.13 *If $n \geqq l$ and $l < p^2-1$, $K(\boldsymbol{H}(\boldsymbol{C}P^n), \boldsymbol{\psi}_H^n; \boldsymbol{T})_{2l}$ is a 1-dimensional vector space over $\boldsymbol{F}_p$. A basis of $K(\boldsymbol{H}(\boldsymbol{C}P^n), \boldsymbol{\psi}_H^n; \boldsymbol{T})_{2l}$ is given by $\sum_{k=0}^{l_1} \dbinom{l-k(p-1)}{k} \beta^H_{l-k(p-1)} \otimes t_1^k$ if $l_0+l_1 \leqq p-2$ and by $\sum_{k=0}^{p-l_0-1} \dbinom{l-k(p-1)}{k} \beta^H_{l-k(p-1)} \otimes t_1^k$ if $p-1 \leqq l_0+l_1 \leqq 2p-3$. On the other hand, $K(\boldsymbol{H}(\boldsymbol{C}P^n), \boldsymbol{\psi}_H^n; \boldsymbol{T})_{2p^2-2}$ is a 2-dimensional vector space over $\boldsymbol{F}_p$ spanned by $\beta^H_{p^2-1} \otimes 1$, $\beta^H_{p-1} \otimes t_1^p$.*

5 Representations in fibered category of morphisms

In this section, we consider a category $\mathcal{E}$ with finite limits and the category $p : \mathcal{E}^{(2)} \to \mathcal{E}$ given in (2.4.3). It follows from (2.4.8) that $p : \mathcal{E}^{(2)} \to \mathcal{E}$ is a bifibered category.

5.1 Restrictions and trivial representations

Let $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathcal{E}$ and $\boldsymbol{E} = (E \xrightarrow{\pi} C_0)$ an object of $\mathcal{E}^{(2)}_{C_0}$. We consider the following cartesian squares.

$$\begin{tikzcd}
E \times^{\sigma}_{C_0} C_1 \arrow[r, "\sigma_\pi"] \arrow[d, "\pi_\sigma"] & E \arrow[d, "\pi"] \\
C_1 \arrow[r, "\sigma"] & C_0
\end{tikzcd}
\quad
\begin{tikzcd}
(E \times^{\sigma}_{C_0} C_1) \times^{\mu}_{C_1} (C_1 \times_{C_0} C_1) \arrow[r, "\mu_{\pi_\sigma}"] \arrow[d, "(\pi_\sigma)_\mu"] & E \times^{\sigma}_{C_0} C_1 \arrow[d, "\pi_\sigma"] \\
C_1 \times_{C_0} C_1 \arrow[r, "\mu"] & C_1
\end{tikzcd}
\quad
\begin{tikzcd}
E \times^{\sigma\mu}_{C_0} (C_1 \times_{C_0} C_1) \arrow[r, "(\sigma\mu)_\pi"] \arrow[d, "\pi_{\sigma\mu}"] & E \arrow[d, "\pi"] \\
C_1 \times_{C_0} C_1 \arrow[r, "\sigma\mu"] & C_0
\end{tikzcd}$$

$$\begin{tikzcd}
E \times^{\tau}_{C_0} C_1 \arrow[r, "\tau_\pi"] \arrow[d, "\pi_\tau"] & E \arrow[d, "\pi"] \\
C_1 \arrow[r, "\tau"] & C_0
\end{tikzcd}
\quad
\begin{tikzcd}
(E \times^{\tau}_{C_0} C_1) \times^{\mu}_{C_1} (C_1 \times_{C_0} C_1) \arrow[r, "\mu_{\pi_\tau}"] \arrow[d, "(\pi_\tau)_\mu"] & E \times^{\tau}_{C_0} C_1 \arrow[d, "\pi_\tau"] \\
C_1 \times_{C_0} C_1 \arrow[r, "\mu"] & C_1
\end{tikzcd}
\quad
\begin{tikzcd}
E \times^{\tau\mu}_{C_0} (C_1 \times_{C_0} C_1) \arrow[r, "(\tau\mu)_\pi"] \arrow[d, "\pi_{\tau\mu}"] & E \arrow[d, "\pi"] \\
C_1 \times_{C_0} C_1 \arrow[r, "\tau\mu"] & C_0
\end{tikzcd}$$

$$\begin{tikzcd}
(E \times^{\sigma}_{C_0} C_1) \times^{\mathrm{pr}_i}_{C_1} (C_1 \times_{C_0} C_1) \arrow[r, "(\mathrm{pr}_i)_{\pi_\sigma}"] \arrow[d, "(\pi_\sigma)_{\mathrm{pr}_i}"] & E \times^{\sigma}_{C_0} C_1 \arrow[d, "\pi_\sigma"] \\
C_1 \times_{C_0} C_1 \arrow[r, "\mathrm{pr}_i"] & C_1
\end{tikzcd}
\quad
\begin{tikzcd}
E \times^{\sigma\mathrm{pr}_i}_{C_0} (C_1 \times_{C_0} C_1) \arrow[r, "(\sigma\mathrm{pr}_i)_\pi"] \arrow[d, "\pi_{\sigma\mathrm{pr}_i}"] & E \arrow[d, "\pi"] \\
C_1 \times_{C_0} C_1 \arrow[r, "\sigma\mathrm{pr}_i"] & C_0
\end{tikzcd}$$

$$\begin{tikzcd}
(E \times^{\tau}_{C_0} C_1) \times^{\mathrm{pr}_i}_{C_1} (C_1 \times_{C_0} C_1) \arrow[r, "(\mathrm{pr}_i)_{\pi_\tau}"] \arrow[d, "(\pi_\tau)_{\mathrm{pr}_i}"] & E \times^{\tau}_{C_0} C_1 \arrow[d, "\pi_\tau"] \\
C_1 \times_{C_0} C_1 \arrow[r, "\mathrm{pr}_i"] & C_1
\end{tikzcd}
\quad
\begin{tikzcd}
E \times^{\tau\mathrm{pr}_i}_{C_0} (C_1 \times_{C_0} C_1) \arrow[r, "(\tau\mathrm{pr}_i)_\pi"] \arrow[d, "\pi_{\tau\mathrm{pr}_i}"] & E \arrow[d, "\pi"] \\
C_1 \times_{C_0} C_1 \arrow[r, "\tau\mathrm{pr}_i"] & C_0
\end{tikzcd}$$

We note that $\sigma\mu = \sigma\mathrm{pr}_1$, $\tau\mu = \tau\mathrm{pr}_2$ and $\tau\mathrm{pr}_1 = \sigma\mathrm{pr}_2$ hold. The following assertion follows from (2.4.6).

Proposition 5.1.1 *For a morphism $\boldsymbol{\xi} : \sigma^*(\boldsymbol{E}) \to \tau^*(\boldsymbol{E})$ in $\mathcal{E}^{(2)}_{C_1}$, we put $\boldsymbol{\xi} = \langle \xi : E \times^{\sigma}_{C_0} C_1 \to E \times^{\tau}_{C_0} C_1, id_{C_1} \rangle$. $\boldsymbol{\xi}$ satisfies condition (A) of (3.1.2) if and only if the following diagram is commutative.*

$$\begin{tikzcd}
E \times^{\sigma\mu}_{C_0} (C_1 \times_{C_0} C_1) \arrow[r, equal] \arrow[d, "(id_E \times_{C_0} \mu, \pi_{\sigma\mu})"] & E \times^{\sigma\mathrm{pr}_1}_{C_0} (C_1 \times_{C_0} C_1) \arrow[r, "(id_E \times_{C_0} \mathrm{pr}_1, \pi_{\sigma\mathrm{pr}_1})"] & (E \times^{\sigma}_{C_0} C_1) \times^{\mathrm{pr}_1}_{C_1} (C_1 \times_{C_0} C_1) \arrow[d, "\xi \times_{C_1} id_{C_1 \times_{C_0} C_1}"'] \\
(E \times^{\sigma}_{C_0} C_1) \times^{\mu}_{C_1} (C_1 \times_{C_0} C_1) \arrow[d, "\xi \times_{C_1} id_{C_1 \times_{C_0} C_1}"] & & (E \times^{\tau}_{C_0} C_1) \times^{\mathrm{pr}_1}_{C_1} (C_1 \times_{C_0} C_1) \arrow[d, "((\tau_\pi(\mathrm{pr}_1)_{\pi_\tau}, \mathrm{pr}_2(\pi_\tau)_{\mathrm{pr}_1}), (\pi_\tau)_{\mathrm{pr}_1})"'] \\
(E \times^{\tau}_{C_0} C_1) \times^{\mu}_{C_1} (C_1 \times_{C_0} C_1) \arrow[d, "\tau_\pi \times_{C_1} id_{C_1 \times_{C_0} C_1}"] & & (E \times^{\sigma}_{C_0} C_1) \times^{\mathrm{pr}_2}_{C_1} (C_1 \times_{C_0} C_1) \arrow[d, "\xi \times_{C_1} id_{C_1 \times_{C_0} C_1}"'] \\
E \times^{\tau\mu}_{C_0} (C_1 \times_{C_0} C_1) \arrow[r, equal] & E \times^{\tau\mathrm{pr}_2}_{C_0} (C_1 \times_{C_0} C_1) & (E \times^{\tau}_{C_0} C_1) \times^{\mathrm{pr}_2}_{C_1} (C_1 \times_{C_0} C_1) \arrow[l, "\tau_\pi \times_\tau id_{C_1 \times_{C_0} C_1}"']
\end{tikzcd}$$

Lemma 5.1.2 *The following diagrams are cartesian*

$$\begin{tikzcd}
E \arrow[r, "(id_E, \varepsilon\pi)"] \arrow[d, "\pi"] & E \times^{\sigma}_{C_0} C_1 \arrow[d, "\pi_\sigma"] \\
C_0 \arrow[r, "\varepsilon"] & C_1
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "(id_E, \varepsilon\pi)"] \arrow[d, "\pi"] & E \times^{\tau}_{C_0} C_1 \arrow[d, "\pi_\tau"] \\
C_0 \arrow[r, "\varepsilon"] & C_1
\end{tikzcd}$$

Proof. Since $\sigma\varepsilon = \tau\varepsilon = id_{C_0}$ and $\sigma_\pi(id_E, \varepsilon\pi) = \tau_\pi(id_E, \varepsilon\pi) = id_E$, the outer rectangles of the following diagrams are cartesian. Since the right rectangles of the following diagrams are also cartesian, so are the left rectangles.

$$\begin{tikzcd} E \arrow[r, "(id_E,\,\varepsilon\pi)"] \arrow[d, "\pi"] & E\times^{\sigma}_{C_0}C_1 \arrow[r, "\sigma_\pi"] \arrow[d, "\pi_\sigma"] & E \arrow[d, "\pi"] \\ C_0 \arrow[r, "\varepsilon"] & C_1 \arrow[r, "\sigma"] & C_0 \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "(id_E,\,\varepsilon\pi)"] \arrow[d, "\pi"] & E\times^{\tau}_{C_0}C_1 \arrow[r, "\tau_\pi"] \arrow[d, "\pi_\tau"] & E \arrow[d, "\pi"] \\ C_0 \arrow[r, "\varepsilon"] & C_1 \arrow[r, "\tau"] & C_0 \end{tikzcd}$$

□

Proposition 5.1.3 *For a morphism* $\boldsymbol{\xi} : \sigma^*(\boldsymbol{E}) \to \tau^*(\boldsymbol{E})$ *in* $\mathcal{E}^{(2)}_{C_1}$*, we put* $\boldsymbol{\xi} = \langle \xi : E \times^{\sigma}_{C_0} C_1 \to E \times^{\tau}_{C_0} C_1, id_{C_1}\rangle$*.* $\boldsymbol{\xi}$ *satisfies condition* (U) *of (3.1.2) if and only if the following diagram is commutative.*

$$\begin{tikzcd} & E \arrow[dl, "(id_E,\,\varepsilon\pi)"'] \arrow[dr, "(id_E,\,\varepsilon\pi)"] & \\ E\times^{\sigma}_{C_0}C_1 \arrow[rr, "\xi"] & & E\times^{\tau}_{C_0}C_1 \end{tikzcd}$$

Proof. We consider the following commutative diagram whose upper and lower trapezoids are cartesian.

$$\begin{tikzcd} (E\times^{\sigma}_{C_0}C_1)\times_{C_1}C_0 \arrow[rrr, "\varepsilon_{\pi_\sigma}"] \arrow[dr, "(\pi_\sigma)_\varepsilon"] \arrow[dd, "\xi\times_{C_1}id_{C_0}"'] & & & E\times^{\sigma}_{C_0}C_1 \arrow[dl, "\pi_\sigma"'] \arrow[dd, "\xi"] \\ & C_0 \arrow[r, "\varepsilon"] & C_1 & \\ (E\times^{\tau}_{C_0}C_1)\times_{C_1}C_0 \arrow[rrr, "\varepsilon_{\pi_\tau}"] \arrow[ur, "(\pi_\tau)_\varepsilon"] & & & E\times^{\tau}_{C_0}C_1 \arrow[ul, "\pi_\tau"] \end{tikzcd}$$

Then $\varepsilon^*(\boldsymbol{\xi}) : \varepsilon^*(\sigma^*(\boldsymbol{E})) \to \varepsilon^*(\tau^*(\boldsymbol{E})$ is given by $\langle \xi \times_{C_1} id_{C_0} : (E\times^{\sigma}_{C_0}C_1)\times_{C_1}C_0 \to (E\times^{\tau}_{C_0}C_1)\times_{C_1}C_0, id_{C_0}\rangle$. Since $((id_E, \varepsilon\pi), \pi) : E \to (E\times^{\sigma}_{C_0}C_1)\times_{C_1}C_0$ and $((id_E, \varepsilon\pi), \pi) : E \to (E\times^{\tau}_{C_0}C_1)\times_{C_1}C_0$ are isomorphisms by (5.1.2), there exists unique morphism $\langle \xi', id_{C_0}\rangle : \boldsymbol{E} \to \boldsymbol{E}$ that makes the following diagram commute.

$$\begin{tikzcd} E \arrow[rrrr, bend left, "(id_E,\,\varepsilon\pi)"] \arrow[dr, "((id_E,\,\varepsilon\pi),\,\pi)"] \arrow[dddd, "\xi'"'] & & & & \\ & (E\times^{\sigma}_{C_0}C_1)\times_{C_1}C_0 \arrow[rrr, "\varepsilon_{\pi_\sigma}"] \arrow[dr, "(\pi_\sigma)_\varepsilon"] \arrow[dd, "\xi\times_{C_1}id_{C_0}"'] & & & E\times^{\sigma}_{C_0}C_1 \arrow[dl, "\pi_\sigma"'] \arrow[dd, "\xi"] \\ & & C_0 \arrow[r, "\varepsilon"] & C_1 & \\ & (E\times^{\tau}_{C_0}C_1)\times_{C_1}C_0 \arrow[rrr, "\varepsilon_{\pi_\tau}"] \arrow[ur, "(\pi_\tau)_\varepsilon"] & & & E\times^{\tau}_{C_0}C_1 \arrow[ul, "\pi_\tau"] \\ E \arrow[ur, "((id_E,\,\varepsilon\pi),\,\pi)"'] \arrow[urrrr, bend right, "(id_E,\,\varepsilon\pi)"'] & & & & \end{tikzcd}$$

Since the outer rectangles of the both diagrams in the proof of (5.1.2), $(\sigma\varepsilon)^*(\boldsymbol{E}) = (\tau\varepsilon)^*(\boldsymbol{E}) = id^*_{C_0}(\boldsymbol{E})$ is identified with $\boldsymbol{E}$. Hence $\langle \xi', id_{C_0}\rangle : \boldsymbol{E} \to \boldsymbol{E}$ is identified with $\boldsymbol{\xi}_\varepsilon : (\sigma\varepsilon)^*(\boldsymbol{E}) \to (\tau\varepsilon)^*(\boldsymbol{E})$. It follows that condition (U) of (3.1.2) is equivalent to $\xi' = id_E$. □

Let $\boldsymbol{D} = (D_0, D_1; \sigma', \tau', \varepsilon', \mu')$ be an internal category in $\mathcal{E}$ and $\boldsymbol{f} = (f_0, f_1) : \boldsymbol{D} \to \boldsymbol{C}$ an internal functor. For an object $\boldsymbol{E} = (E \xrightarrow{\pi} C_0)$ of $\mathcal{E}^{(2)}_{C_0}$, we consider the following diagrams such that each rectangle is cartesian.

$$\begin{tikzcd} (E\times_{C_0}D_0)\times^{\sigma'}_{D_0}D_1 \arrow[r, "\sigma'_{\pi_{f_0}}"] \arrow[d, "(\pi_{f_0})_{\sigma'}"] & E\times_{C_0}D_0 \arrow[r, "(f_0)_\pi"] \arrow[d, "\pi_{f_0}"] & E \arrow[d, "\pi"] \\ D_1 \arrow[r, "\sigma'"] & D_0 \arrow[r, "f_0"] & C_0 \end{tikzcd} \qquad \begin{tikzcd} (E\times^{\tau}_{C_0}C_1)\times_{C_1}D_1 \arrow[r, "(f_1)_{\pi_\tau}"] \arrow[d, "(\pi_\tau)_{f_1}"] & E\times^{\tau}_{C_0}C_1 \arrow[r, "\tau_\pi"] \arrow[d, "\pi_\tau"] & E \arrow[d, "\pi"] \\ D_1 \arrow[r, "f_1"] & C_1 \arrow[r, "\tau"] & C_0 \end{tikzcd}$$

$$\begin{tikzcd} (E\times^{\sigma}_{C_0}C_1)\times_{C_1}D_1 \arrow[r, "(f_1)_{\pi_\sigma}"] \arrow[d, "(\pi_\sigma)_{f_1}"] & E\times^{\sigma}_{C_0}C_1 \arrow[r, "\sigma_\pi"] \arrow[d, "\pi_\sigma"] & E \arrow[d, "\pi"] \\ D_1 \arrow[r, "f_1"] & C_1 \arrow[r, "\sigma"] & C_0 \end{tikzcd} \qquad \begin{tikzcd} (E\times_{C_0}D_0)\times^{\tau'}_{D_0}D_1 \arrow[r, "\tau'_{\pi_{f_0}}"] \arrow[d, "(\pi_{f_0})_{\tau'}"] & E\times_{C_0}D_0 \arrow[r, "(f_0)_\pi"] \arrow[d, "\pi_{f_0}"] & E \arrow[d, "\pi"] \\ D_1 \arrow[r, "\tau'"] & D_0 \arrow[r, "f_0"] & C_0 \end{tikzcd}$$

$$\begin{tikzcd}
E \times_{C_0} D_1 \arrow[r, "(f_0\sigma')_\pi = (\sigma f_1)_\pi"] \arrow[d, "\pi_{f_0\sigma'} = \pi_{\sigma f_1}"] & E \arrow[d, "\pi"] \\
D_1 \arrow[r, "f_0\sigma' = \sigma f_1"] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
E \times_{C_0} D_1 \arrow[r, "(\tau f_1)_\pi = (f_0\tau')_\pi"] \arrow[d, "\pi_{\tau f_1} = \pi_{f_0\tau'}"] & E \arrow[d, "\pi"] \\
D_1 \arrow[r, "\tau f_1 = f_0\tau'"] & C_0
\end{tikzcd}$$

Proposition 5.1.4 *For a representation $(\boldsymbol{E},\boldsymbol{\xi})$ of $\boldsymbol{C}$ on $\boldsymbol{E}$, we define a morphism*

$$\xi_{\boldsymbol{f}} : (E \times_{C_0} D_0) \times^{\sigma'}_{D_0} D_1 \to (E \times_{C_0} D_0) \times^{\tau'}_{D_0} D_1$$

in $\mathcal{E}$ to be the following composition.

$$\begin{aligned}(E \times_{C_0} D_0) \times^{\sigma'}_{D_0} D_1 &\xrightarrow{(((f_0)_\pi \sigma'_{\pi_{f_0}},\, f_1(\pi_{f_0})_{\sigma'}),(\pi_{f_0})_{\sigma'})} (E \times^{\sigma}_{C_0} C_1) \times_{C_1} D_1 \xrightarrow{\xi \times_{C_1} id_{D_1}} (E \times^{\tau}_{C_0} C_1) \times_{C_1} D_1 \\ &\xrightarrow{((\tau_\pi(f_1)_{\pi_\tau},\tau'(\pi_\tau)_{f_1}),(\pi_\tau)_{f_1})} (E \times_{C_0} D_0) \times^{\tau'}_{D_0} D_1\end{aligned}$$

Then, the restriction $(f_0^*(\boldsymbol{E}), \boldsymbol{\xi_f})$ of $\boldsymbol{\xi}$ along $\boldsymbol{f}$ is given by $\boldsymbol{\xi_f} = \langle \xi_{\boldsymbol{f}}, id_{D_1}\rangle : \sigma'^*(f_0^*(\boldsymbol{E})) \to \tau'^*(f_0^*(\boldsymbol{E}))$. Moreover, the following diagram is commutative.

$$\begin{tikzcd}
 & D_1 \arrow[r, "\tau'"] & D_0 \\
(E \times_{C_0} D_0) \times^{\sigma'}_{D_0} D_1 \arrow[ur, "(\pi_{f_0})_{\sigma'}"] \arrow[r, "\xi_{\boldsymbol{f}}"] \arrow[d, "((f_0)_\pi \sigma'_{\pi_{f_0}},\, f_1(\pi_{f_0})_{\sigma'})"] & (E \times_{C_0} D_0) \times^{\tau'}_{D_0} D_1 \arrow[r, "\tau'_{\pi_{f_0}}"] & E \times_{C_0} D_0 \arrow[u, "\pi_{f_0}"] \arrow[d, "(f_0)_\pi"] \\
E \times^{\sigma}_{C_0} C_1 \arrow[r, "\xi"] & E \times^{\tau}_{C_0} C_1 \arrow[r, "\tau_\pi"] & E
\end{tikzcd}$$

Hence $\xi_{\boldsymbol{f}} = ((\tau_\pi \xi((f_0)_\pi \sigma'_{\pi_{f_0}}, f_1(\pi_{f_0})_{\sigma'}), \tau'(\pi_{f_0})_{\sigma'}), (\pi_{f_0})_{\sigma'})$ holds.

Proof. Recall that $\boldsymbol{\xi_f}$ is the following composition.

$$\begin{aligned}\sigma'^*(f_0^*(\boldsymbol{E})) &\xrightarrow{c_{f_0,\sigma'}(\boldsymbol{E})} (f_0\sigma')^*(\boldsymbol{E}) = (\sigma f_1)^*(\boldsymbol{E}) \xrightarrow{c_{\sigma,f_1}(\boldsymbol{E})^{-1}} f_1^*(\sigma^*(\boldsymbol{E})) \xrightarrow{f_1^*(\xi)} f_1^*(\tau^*(\boldsymbol{E})) \\ &\xrightarrow{c_{\tau,f_1}(\boldsymbol{E})} (\tau f_1)^*(\boldsymbol{E}) = (f_0\tau')^*(\boldsymbol{E}) \xrightarrow{c_{f_0,\tau'}(\boldsymbol{E})^{-1}} \tau'^*(f_0^*(\boldsymbol{E}))\end{aligned}$$

The first assertion follows from (2.4.6) and the proof of (2.4.3). There are the following commutative diagrams.

$$\begin{tikzcd}
(E \times^{\sigma}_{C_0} C_1) \times_{C_1} D_1 \arrow[rrr, "(f_1)_{\pi_\sigma}"] \arrow[dr, dashed, "\xi \times_{C_1} id_{D_1}"] \arrow[ddr, "(\pi_\sigma)_{f_1}"'] & & & E \times^{\sigma}_{C_0} C_1 \arrow[dl, "\xi"'] \arrow[ddl, "\pi_\sigma"] \\
 & (E \times^{\tau}_{C_0} C_1) \times_{C_1} D_1 \arrow[r, "(f_1)_{\pi_\tau}"] \arrow[d, "(\pi_\tau)_{f_1}"] & E \times^{\tau}_{C_0} C_1 \arrow[d, "\pi_\tau"] & \\
 & D_1 \arrow[r, "f_1"] & C_1 &
\end{tikzcd}$$

$$\begin{tikzcd}
(E \times_{C_0} D_0) \times^{\sigma'}_{D_0} D_1 \arrow[rr, "(((f_0)_\pi \sigma'_{\pi_{f_0}},\, f_1(\pi_{f_0})_{\sigma'}),\, (\pi_{f_0})_{\sigma'})"] \arrow[dr, "((f_0)_\pi \sigma'_{\pi_{f_0}},\, f_1(\pi_{f_0})_{\sigma'})"] \arrow[ddr, "\xi((f_0)_\pi \sigma'_{\pi_{f_0}},\, f_1(\pi_{f_0})_{\sigma'})"'] & & (E \times^{\sigma}_{C_0} C_1) \times_{C_1} D_1 \arrow[dl, "(f_1)_{\pi_\sigma}"'] \arrow[d, "\xi \times_{C_1} id_{D_1}"] \arrow[dr, "(\pi_\sigma)_{f_1}"] & \\
 & E \times^{\sigma}_{C_0} C_1 \arrow[d, "\xi"] & (E \times^{\tau}_{C_0} C_1) \times_{C_1} D_1 \arrow[r, "(\pi_\tau)_{f_1}"] \arrow[dl, "(f_1)_{\pi_\tau}"'] \arrow[d, "((\tau_\pi(f_1)_{\pi_\tau},\, \tau'(\pi_\tau)_{f_1}),\, (\pi_\tau)_{f_1})"] & D_1 \arrow[d, "id_{D_1}"] \\
 & E \times^{\tau}_{C_0} C_1 \arrow[d, "\tau_\pi"] & (E \times_{C_0} D_0) \times^{\tau'}_{D_0} D_1 \arrow[r, "(\pi_{f_0})_{\tau'}"] \arrow[d, "\tau'_{\pi_{f_0}}"] & D_1 \arrow[d, "\tau'"] \\
 & E & E \times_{C_0} D_0 \arrow[l, "(f_0)_\pi"'] \arrow[r, "\pi_{f_0}"] & D_0
\end{tikzcd}$$

The second assertion follows from the commutativity of the above diagram. $\square$

Since $\hat{\xi} = \tau_\pi \xi$ and $\hat{\xi}_{\boldsymbol{f}} = \tau'_{\pi_{f_0}} \xi_{\boldsymbol{f}}$, the following result is a direct consequence of (5.1.4).

Corollary 5.1.5 *Under the situation of (5.1.4), we put*

$$P_{\sigma,\tau}(\boldsymbol{E})_{\boldsymbol{E}}(\boldsymbol{\xi}) = \hat{\boldsymbol{\xi}} = \langle \hat{\xi} : E \times^{\sigma}_{C_0} C_1 \to E, id_{C_0} \rangle$$
$$P_{\sigma',\tau'}(f_0^*(\boldsymbol{E}))_{f_0^*(\boldsymbol{E})}(\boldsymbol{\xi_f}) = \hat{\boldsymbol{\xi}}_{\boldsymbol{f}} = \langle \hat{\xi}_{\boldsymbol{f}} : (E \times_{C_0} D_0) \times^{\sigma'}_{D_0} D_1 \to E \times_{C_0} D_0, id_{D_0} \rangle.$$

Then $\hat{\xi}_{\boldsymbol{f}} = (\hat{\xi}((f_0)_\pi \sigma'_{\pi_{f_0}}, f_1(\pi_{f_0})_{\sigma'}), \tau'(\pi_{f_0})_{\sigma'})$ *holds.*

For an object X of $\mathcal{E}$, consider the following cartesian squares.

$$\begin{tikzcd}
(X\times C_0)\times^{\sigma}_{C_0} C_1 \arrow[r, "\sigma_{\mathrm{pr}_{C_0}}"] \arrow[d, "(\mathrm{pr}_{C_0})_\sigma"] & X\times C_0 \arrow[d, "\mathrm{pr}_{C_0}"] \\
C_1 \arrow[r, "\sigma"] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
(X\times C_0)\times^{\tau}_{C_0} C_1 \arrow[r, "\tau_{\mathrm{pr}_{C_0}}"] \arrow[d, "(\mathrm{pr}_{C_0})_\tau"] & X\times C_0 \arrow[d, "\mathrm{pr}_{C_0}"] \\
C_1 \arrow[r, "\tau"] & C_0
\end{tikzcd}$$

The following result is a direct consequence of (2) of (2.4.5).

Proposition 5.1.6 *For an object X of $\mathcal{E}$, the trivial representation $(s_X(C_0), (s_X)_{\boldsymbol{C}})$ associated with X is given by $s_X(C_0) = (X \times C_0 \xrightarrow{\mathrm{pr}_{C_0}} C_0)$ and $(s_X)_{\boldsymbol{C}} = (s_X)_\tau (s_X)_\sigma^{-1} = \langle ((\mathrm{pr}_X \sigma_{\mathrm{pr}_{C_0}}, \tau(\mathrm{pr}_{C_0})_\sigma), (\mathrm{pr}_{C_0})_\sigma), id_{C_1} \rangle$.*

$$\begin{tikzcd}
(X\times C_0)\times^{\sigma}_{C_0} C_1 \arrow[rr, "\sigma_{\mathrm{pr}_{C_0}}"] \arrow[dr, "{(\mathrm{pr}_X\sigma_{\mathrm{pr}_{C_0}}, (\mathrm{pr}_{C_0})_\sigma)}"] \arrow[ddr, bend right, "(\mathrm{pr}_{C_0})_\sigma"'] & & X\times C_0 \arrow[d, "\mathrm{pr}_X"] \\
& X\times C_1 \arrow[r, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_{C_1}"] & X \arrow[d, "o_X"] \\
& C_1 \arrow[r, "o_{C_1}"] & 1_{\mathcal{E}}
\end{tikzcd}
\qquad
\begin{tikzcd}
X\times C_1 \arrow[dr, "{(id_X\times\tau, \mathrm{pr}_{C_1})}"] \arrow[drr, bend left, "id_X\times\tau"] \arrow[ddr, bend right, "\mathrm{pr}_{C_1}"'] & & \\
& (X\times C_0)\times^{\tau}_{C_0} C_1 \arrow[r, "\tau_{\mathrm{pr}_{C_0}}"] \arrow[d, "(\mathrm{pr}_{C_0})_\tau"] & X\times C_0 \arrow[d, "\mathrm{pr}_{C_0}"] \\
& C_1 \arrow[r, "\tau"] & C_0
\end{tikzcd}$$

5.2 Left induced representations in fibered category of morphisms

Let $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathcal{E}$. For an object $\boldsymbol{E} = (E \xrightarrow{\pi} C_0)$ of $\mathcal{E}^{(2)}_{C_0}$, we consider the following cartesian squares. Then, we have $\sigma^*(\boldsymbol{E}) = (E \times^{\sigma}_{C_0} C_1 \xrightarrow{\pi_\sigma} C_1)$ and $\tau^*(\boldsymbol{E}) = (E \times^{\tau}_{C_0} C_1 \xrightarrow{\pi_\tau} C_1)$.

$$\begin{tikzcd}
E\times^{\sigma}_{C_0} C_1 \arrow[r, "\sigma_\pi"] \arrow[d, "\pi_\sigma"] & E \arrow[d, "\pi"] \\
C_1 \arrow[r, "\sigma"] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
E\times^{\tau}_{C_0} C_1 \arrow[r, "\tau_\pi"] \arrow[d, "\pi_\tau"] & E \arrow[d, "\pi"] \\
C_1 \arrow[r, "\tau"] & C_0
\end{tikzcd}$$

For a morphism $\boldsymbol{\xi} : \sigma^*(\boldsymbol{E}) \to \tau^*(\boldsymbol{E})$ in $\mathcal{E}^{(2)}_{C_1}$, we put $\boldsymbol{\xi} = \langle \xi, id_{C_1} \rangle$, where $\xi : E \times^{\sigma}_{C_0} C_1 \to E \times^{\tau}_{C_0} C_1$ is a morphism in $\mathcal{E}$ which makes the following diagram commute.

$$\begin{tikzcd}
E\times^{\sigma}_{C_0} C_1 \arrow[rr, "\xi"] \arrow[dr, "\pi_\sigma"] & & E\times^{\tau}_{C_0} C_1 \arrow[dl, "\pi_\tau"'] \\
& C_1 &
\end{tikzcd}$$

Note that $\boldsymbol{E}_{[\sigma,\tau]} = \tau_*\sigma^*(\boldsymbol{E}) = (E \times^{\sigma}_{C_0} C_1 \xrightarrow{\tau\pi_\sigma} C_0)$ holds by (2.4.10). We denote by $\hat{\boldsymbol{\xi}} = \langle \hat{\xi}, id_{C_0} \rangle : \boldsymbol{E}_{[\sigma,\tau]} \to \boldsymbol{E}$ the image of $\boldsymbol{\xi}$ by the bijection $P_{\sigma,\tau}(\boldsymbol{E})_{\boldsymbol{E}} : \mathcal{E}^{(2)}_{C_1}(\sigma^*(\boldsymbol{E}), \tau^*(\boldsymbol{E})) \to \mathcal{E}^{(2)}_{C_0}(\boldsymbol{E}_{[\sigma,\tau]}, \boldsymbol{E})$. It follows from (2.4.10) that $\hat{\xi}$ is a composition $E \times^{\sigma}_{C_0} C_1 \xrightarrow{\xi} E \times^{\tau}_{C_0} C_1 \xrightarrow{\tau_\pi} E$.

We consider the following cartesian squares which give $(\boldsymbol{E}_{[\sigma,\tau]})_{[\sigma,\tau]} = ((E \times^{\sigma}_{C_0} C_1) \times^{\sigma}_{C_0} C_1 \xrightarrow{\tau(\tau\pi_\sigma)_\sigma} C_0)$ and $\boldsymbol{E}_{[\sigma\mathrm{pr}_1, \tau\mathrm{pr}_2]} = \boldsymbol{E}_{[\sigma\mu, \tau\mu]} = (E \times_{C_0} (C_1 \times_{C_0} C_1) \xrightarrow{\tau\mu\pi_{\sigma\mu}} C_0)$.

$$\begin{tikzcd}
(E\times^{\sigma}_{C_0} C_1)\times^{\sigma}_{C_0} C_1 \arrow[r, "\sigma_{\tau\pi_\sigma}"] \arrow[d, "(\tau\pi_\sigma)_\sigma"] & E\times^{\sigma}_{C_0} C_1 \arrow[d, "\tau\pi_\sigma"] \\
C_1 \arrow[r, "\sigma"] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
E\times^{\sigma\mu}_{C_0}(C_1\times_{C_0} C_1) \arrow[r, "(\sigma\mu)_\pi=(\sigma\mathrm{pr}_1)_\pi"] \arrow[d, "\pi_{\sigma\mu}=\pi_{\sigma\mathrm{pr}_1}"] & E \arrow[d, "\pi"] \\
C_1\times_{C_0} C_1 \arrow[r, "\sigma\mu=\sigma\mathrm{pr}_1"] & C_0
\end{tikzcd}$$

We have morphisms $\hat\xi \times_{C_0} id_{C_1} : (E \times^{\sigma}_{C_0} C_1) \times^{\sigma}_{C_0} C_1 \to E \times^{\sigma}_{C_0} C_1$ and $id_E \times_{C_0} \mu : E \times^{\sigma\mu}_{C_0} (C_1 \times_{C_0} C_1) \to E \times^{\sigma}_{C_0} C_1$.

$$\begin{tikzcd}
& (E \times^{\sigma}_{C_0} C_1) \times^{\sigma}_{C_0} C_1 \arrow[rr, "\sigma_{\tau\pi_\sigma}"] \arrow[dr, dashed, "\hat\xi \times_{C_0} id_{C_1}"] \arrow[ddr, "(\tau\pi_\sigma)_\sigma"'] & & E \times^{\sigma}_{C_0} C_1 \arrow[d, "\hat\xi"'] \arrow[dd, bend left, "\tau\pi_\sigma"] \\
& & E \times^{\sigma}_{C_0} C_1 \arrow[r, "\sigma_\pi"] \arrow[d, "\pi_\sigma"] & E \arrow[d, "\pi"'] \\
C_0 & & C_1 \arrow[ll, "\tau"'] \arrow[r, "\sigma"] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
E \times^{\sigma\mu}_{C_0} (C_1 \times_{C_0} C_1) \arrow[d, "\pi_{\sigma\mu}"'] \arrow[drr, dashed, "id_E \times_{C_0} \mu"] \arrow[drrr, bend left, "(\sigma\mu)_\pi"] & & & \\
C_1 \times_{C_0} C_1 \arrow[drr, "\mu"] & & E \times^{\sigma}_{C_0} C_1 \arrow[r, "\sigma_\pi"] \arrow[d, "\pi_\sigma"] & E \arrow[d, "\pi"] \\
C_0 & & C_1 \arrow[ll, "\tau"'] \arrow[r, "\sigma"] & C_0
\end{tikzcd}$$

It follows from (2.4.11) that $\hat{\boldsymbol{\xi}}_{[\sigma,\tau]} : (\boldsymbol{E}_{[\sigma,\tau]})_{[\sigma,\tau]} \to \boldsymbol{E}_{[\sigma,\tau]}$ and $\boldsymbol{E}_\mu : \boldsymbol{E}_{[\sigma \mathrm{pr}_1, \tau \mathrm{pr}_2]} = \boldsymbol{E}_{[\sigma\mu,\tau\mu]} \to \boldsymbol{E}_{[\sigma,\tau]}$ are given by $\langle \hat\xi \times_{C_0} id_{C_1}, id_{C_0}\rangle$ and $\langle id_E \times_{C_0} \mu, id_{C_0}\rangle$, respectively.

By (2.4.13), $\theta_{\sigma,\tau,\sigma,\tau}(\boldsymbol{E}) : \boldsymbol{E}_{[\sigma \mathrm{pr}_1, \tau \mathrm{pr}_2]} = \boldsymbol{E}_{[\sigma\mu,\tau\mu]} \to (\boldsymbol{E}_{[\sigma,\tau]})_{[\sigma,\tau]}$ is given by $\langle (id_E \times_{C_0} \mathrm{pr}_1, \mathrm{pr}_2\pi_{\sigma \mathrm{pr}_1}), id_{C_0}\rangle$.

$$\begin{tikzcd}
E \times^{\sigma\mu}_{C_0} (C_1 \times_{C_0} C_1) \arrow[d, "\pi_{\sigma \mathrm{pr}_1}"] \arrow[dr, dashed, "(id_E \times_{C_0} \mathrm{pr}_1, \mathrm{pr}_2\pi_{\sigma \mathrm{pr}_1})"] \arrow[drr, bend left, "id_E \times_{C_0} \mathrm{pr}_1"] & & \\
C_1 \times_{C_0} C_1 \arrow[dr, "\mathrm{pr}_2"] & (E \times^{\sigma}_{C_0} C_1) \times^{\sigma}_{C_0} C_1 \arrow[r, "\sigma_{\tau\pi_\sigma}"] \arrow[d, "\mathrm{pr}_2(\pi_\sigma \times_{C_0} id_{C_1}) = (\tau\pi_\sigma)_\sigma"] & E \times^{\sigma}_{C_0} C_1 \arrow[d, "\tau\pi_\sigma"] \\
& C_1 \arrow[r, "\sigma"] & C_0
\end{tikzcd}$$

Suppose that the following left diagram is cartesian. There exists unique morphism $id_E \times_{C_0} \varepsilon : E \times_{C_0} C_0 \to E \times^{\sigma}_{C_0} C_1$ that makes the following right diagram commute.

$$\begin{tikzcd}
E \times_{C_0} C_0 \arrow[r, "\mathrm{pr}_E"] \arrow[d, "\mathrm{pr}_{C_0}"] & E \arrow[d, "\pi"] \\
C_0 \arrow[r, "\sigma\varepsilon = id_{C_0}"] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
E \times_{C_0} C_0 \arrow[d, "\mathrm{pr}_{C_0}"] \arrow[dr, dashed, "id_E \times_{C_0} \varepsilon"] \arrow[drr, bend left, "\mathrm{pr}_E"] & & \\
C_0 \arrow[dr, "\varepsilon"] & E \times^{\sigma}_{C_0} C_1 \arrow[r, "\sigma_\pi"] \arrow[d, "\pi_\sigma"] & E \arrow[d, "\pi"] \\
& C_1 \arrow[r, "\sigma"] & C_0
\end{tikzcd}$$

The following is a direct consequence of (3.3.2).

Proposition 5.2.1 *For an object $\boldsymbol{E} = (E \xrightarrow{\pi} C_0)$ of $\mathcal{E}^{(2)}_{C_0}$ and a morphism $\boldsymbol{\xi} = \langle \xi, id_{C_1}\rangle : \sigma^*(\boldsymbol{E}) \to \tau^*(\boldsymbol{E})$ in $\mathcal{E}^{(2)}_{C_1}$, let $\hat{\boldsymbol{\xi}} = \langle \hat\xi, id_{C_0}\rangle : \boldsymbol{E}_{[\sigma,\tau]} = \tau_*\sigma^*(\boldsymbol{E}) \to \boldsymbol{E}$ be the image of $\boldsymbol{\xi}$ by the bijection*

$$P_{\sigma,\tau}(\boldsymbol{E})_{\boldsymbol{E}} : \mathcal{E}^{(2)}_{C_1}(\sigma^*(\boldsymbol{E}), \tau^*(\boldsymbol{E})) \to \mathcal{E}^{(2)}_{C_0}(\boldsymbol{E}_{[\sigma,\tau]}, \boldsymbol{E}).$$

Then $\boldsymbol{\xi}$ is a representation of $\boldsymbol{C}$ if and only if the following diagrams are commutative.

$$\begin{tikzcd}
E \times_{C_0} C_0 \arrow[r, "id_E \times_{C_0} \varepsilon"] \arrow[dr, "\mathrm{pr}_E"'] & E \times^{\sigma}_{C_0} C_1 \arrow[d, "\hat\xi"] \\
& E
\end{tikzcd}
\qquad
\begin{tikzcd}
E \times^{\sigma\mu}_{C_0} (C_1 \times_{C_0} C_1) \arrow[r, "id_E \times_{C_0} \mu"] \arrow[d, "(id_E \times_{C_0} \mathrm{pr}_1, \mathrm{pr}_2\pi_{\sigma \mathrm{pr}_1})"] & E \times^{\sigma}_{C_0} C_1 \arrow[r, "\hat\xi"] & E \\
(E \times^{\sigma}_{C_0} C_1) \times^{\sigma}_{C_0} C_1 \arrow[r, "\hat\xi \times_{C_0} id_{C_1}"] & E \times^{\sigma}_{C_0} C_1 \arrow[ur, "\hat\xi"'] &
\end{tikzcd}$$

We also have the following result by (3.3.6).

Proposition 5.2.2 *Let $\boldsymbol{E} = (E \xrightarrow{\pi} C_0)$ and $\boldsymbol{F} = (F \xrightarrow{\rho} C_0)$ be objects of $\mathcal{E}^{(2)}_{C_0}$ and $\boldsymbol{\varphi} = \langle \varphi, id_{C_0}\rangle : \boldsymbol{E} \to \boldsymbol{F}$ a morphism in $\mathcal{E}^{(2)}_{C_0}$. For representations $\boldsymbol{\xi} = \langle \xi, id_{C_1}\rangle : \sigma^*(\boldsymbol{E}) \to \tau^*(\boldsymbol{E})$ and $\boldsymbol{\zeta} = \langle \zeta, id_{C_1}\rangle : \sigma^*(\boldsymbol{F}) \to \tau^*(\boldsymbol{F})$ of $\boldsymbol{C}$ on $\boldsymbol{E}$ ane $\boldsymbol{F}$ respectively, we put $P_{\sigma,\tau}(\boldsymbol{E})_{\boldsymbol{E}}(\boldsymbol{\xi}) = \hat{\boldsymbol{\xi}} = \langle \hat\xi, id_{C_0}\rangle$ and $P_{\sigma,\tau}(\boldsymbol{F})_{\boldsymbol{F}}(\boldsymbol{\zeta}) = \hat{\boldsymbol{\zeta}} = \langle \hat\zeta, id_{C_0}\rangle$. Let $\varphi \times_{C_0} id_{C_1} : E \times^{\sigma}_{C_0} C_1 \to F \times^{\sigma}_{C_0} C_1$ be unique morphism which makes the following left diagram commute, where the outer trapezoid and the lower rectangle are cartesian. Then, $\boldsymbol{\varphi}$ is a morphism of representations if and only if the following right diagram is commutative.*

$$\begin{tikzcd}
E \times^{\sigma}_{C_0} C_1 \arrow[rrr, "\sigma_\pi"] \arrow[dr, dashed, "\varphi\times_{C_0} id_{C_0}"] \arrow[ddr, "\pi_\sigma"'] & & & E \arrow[dl, "\varphi"'] \arrow[ddl, "\pi"] \\
& F \times^{\sigma}_{C_0} C_1 \arrow[r, "\sigma_\rho"] \arrow[d, "\rho_\sigma"] & F \arrow[d, "\rho"'] & \\
& C_1 \arrow[r, "\sigma"] & C_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
E \times^{\sigma}_{C_0} C_1 \arrow[r, "\hat{\xi}"] \arrow[d, "\varphi\times_{C_0} id_{C_1}"] & E \arrow[d, "\varphi"] \\
F \times^{\sigma}_{C_0} C_1 \arrow[r, "\hat{\zeta}"] & F
\end{tikzcd}$$

For an object $\boldsymbol{E} = (E \xrightarrow{\pi} C_0)$ of $\mathcal{E}^{(2)}_{C_0}$, define a morphism $\hat{\mu}_{\boldsymbol{E}} : (E \times^{\sigma}_{C_0} C_1) \times^{\sigma}_{C_0} C_1 \to E \times^{\sigma}_{C_0} C_1$ to be a composition $(E \times^{\sigma}_{C_0} C_1) \times^{\sigma}_{C_0} C_1 \xrightarrow{(id_E \times_{C_0} \mathrm{pr}_1, \mathrm{pr}_2\pi_{\sigma\mathrm{pr}_1})^{-1}} E \times^{\sigma\mu}_{C_0} (C_1 \times_{C_0} C_1) \xrightarrow{id_E \times_{C_0} \mu} E \times^{\sigma}_{C_0} C_1$. Then, we have a morphism $\hat{\boldsymbol{\mu}}_{\boldsymbol{E}} = \langle \hat{\mu}_{\boldsymbol{E}}, id_{C_0}\rangle : (\boldsymbol{E}_{[\sigma,\tau]})_{[\sigma,\tau]} \to \boldsymbol{E}_{[\sigma,\tau]}$. We have the following result by (3.3.10) and (3.3.13).

Proposition 5.2.3 *Put $\boldsymbol{\mu}_{\boldsymbol{E}} = P_{\sigma,\tau}(\boldsymbol{E}_{[\sigma,\tau]})^{-1}_{\boldsymbol{E}_{[\sigma,\tau]}}(\hat{\boldsymbol{\mu}}_{\boldsymbol{E}}) : \sigma^*(\boldsymbol{E}_{[\sigma,\tau]}) \to \tau^*(\boldsymbol{E}_{[\sigma,\tau]})$. Then, $(\boldsymbol{E}_{[\sigma,\tau]}, \boldsymbol{\mu}_{\boldsymbol{E}})$ is a representation of $\boldsymbol{C}$. For a representation $(\boldsymbol{F}, \boldsymbol{\zeta})$ of $\boldsymbol{C}$, a map $\Phi : \mathrm{Rep}(\boldsymbol{C}; \mathcal{E}^{(2)})((\boldsymbol{E}_{[\sigma,\tau]}, \boldsymbol{\mu}_{\boldsymbol{E}}), (\boldsymbol{F}, \boldsymbol{\zeta})) \to \mathcal{E}^{(2)}_{C_0}(\boldsymbol{E}, \boldsymbol{F})$ defined by $\Phi(\langle \varphi, id_{C_0}\rangle) = \langle \varphi(id_E, \varepsilon), id_{C_0}\rangle$ is bijective.*

Let $\boldsymbol{D} = (D_0, D_1; \sigma', \tau', \varepsilon', \mu')$ be an internal category in $\mathcal{E}$ and $\boldsymbol{f} = (f_0, f_1) : \boldsymbol{D} \to \boldsymbol{C}$ an internal functor. We consider Diagram 3.5.1 and Diagram 3.5.2 of page 108 and 109, respectively. For an object $\boldsymbol{E} = (E \xrightarrow{\pi} D_0)$ of $\mathcal{E}^{(2)}_{D_0}$, suppose that the rectangles of the following diagrams are cartesian.

$$\begin{tikzcd}
(E \times_{D_0} (D_0 \times_{C_0} C_1)) \times_{C_0} C_1 \arrow[r, "\sigma_{\tau(f_0)_\sigma \pi_{\sigma_{f_0}}}"] \arrow[dd, "(\tau(f_0)_\sigma \pi_{\sigma_{f_0}})_\sigma"] & E \times_{D_0} (D_0 \times_{C_0} C_1) \arrow[r, "(\sigma_{f_0})_\pi"] \arrow[d, "\pi_{\sigma_{f_0}}"] & E \arrow[d, "\pi"] \\
& D_0 \times_{C_0} C_1 \arrow[r, "\sigma_{f_0}"] \arrow[d, "\tau(f_0)_\sigma"] & D_0 \\
C_1 \arrow[r, "\sigma"] & C_0 &
\end{tikzcd}$$

$$\begin{tikzcd}
E \times_{D_0} (D_0 \times_{C_0} C_1 \times_{C_0} C_1) \arrow[r, "(\tilde{\mathrm{pr}}_{12}\sigma_{f_0})_\pi"] \arrow[d, "\pi_{\tilde{\mathrm{pr}}_{12}\sigma_{f_0}}"] & E \arrow[d, "\pi"] \\
D_0 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "\tilde{\mathrm{pr}}_{12}\sigma_{f_0}"] & D_0
\end{tikzcd}$$

Then, we have the following.

$$\boldsymbol{E}_{[\sigma_{f_0}, \tau(f_0)_\sigma]} = (E \times_{D_0} (D_0 \times_{C_0} C_1) \xrightarrow{\tau(f_0)_\sigma \pi_{\sigma_{f_0}}} C_0)$$
$$(\boldsymbol{E}_{[\sigma_{f_0}, \tau(f_0)_\sigma]})_{[\sigma,\tau]} = ((E \times_{D_0} (D_0 \times_{C_0} C_1)) \times_{C_0} C_1 \xrightarrow{\tau(\tau(f_0)_\sigma \pi_{\sigma_{f_0}})_\sigma} C_0)$$
$$\boldsymbol{E}_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12}, \tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} = (E \times_{D_0} (D_0 \times_{C_0} C_1 \times_{C_0} C_1) \xrightarrow{\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}\pi_{\tilde{\mathrm{pr}}_{12}\sigma_{f_0}}} C_0)$$

It follows from (2.4.15) and (2.4.16) that $\theta_{\sigma_{f_0}, \tau(f_0)_\sigma, \sigma, \tau}(\boldsymbol{E}) : \boldsymbol{E}_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12}, \tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \to (\boldsymbol{E}_{[\sigma_{f_0}, \tau(f_0)_\sigma]})_{[\sigma,\tau]}$ is an isomorphism whose inverse $\theta_{\sigma_{f_0}, \tau(f_0)_\sigma, \sigma, \tau}(\boldsymbol{E})^{-1} : (\boldsymbol{E}_{[\sigma_{f_0}, \tau(f_0)_\sigma]})_{[\sigma,\tau]} \to \boldsymbol{E}_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12}, \tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]}$ is given by

$$\langle ((\sigma_{f_0})_\pi \sigma_{\tau(f_0)_\sigma \pi_{\sigma_{f_0}}}, \pi_{\sigma_{f_0}} \times_{C_0} id_{C_1}) : (E \times_{D_0} (D_0 \times_{C_0} C_1)) \times_{C_0} C_1 \to E \times_{D_0} (D_0 \times_{C_0} C_1 \times_{C_0} C_1), id_{C_0}\rangle.$$

$$\begin{tikzcd}
(E \times_{D_0} (D_0 \times_{C_0} C_1)) \times_{C_0} C_1 \arrow[rr, "\sigma_{\tau(f_0)_\sigma \pi_{\sigma_{f_0}}}"] \arrow[dr, "\pi_{\sigma_{f_0}} \times_{C_0} id_{C_1}"] \arrow[dd, "(\tau(f_0)_\sigma \pi_{\sigma_{f_0}})_\sigma"'] & & E \times_{D_0} (D_0 \times_{C_0} C_1) \arrow[d, "\pi_{\sigma_{f_0}}"] \\
& D_0 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "\tilde{\mathrm{pr}}_{12}"] \arrow[dl, "\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}"'] & D_0 \times_{C_0} C_1 \arrow[d, "\tau(f_0)_\sigma"] \\
C_1 \arrow[rr, "\sigma"] & & C_0
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
(E\times_{D_0}(D_0\times_{C_0}C_1))\times_{C_0}C_1 \arrow[rr, "\sigma_{\tau(f_0)_\sigma\pi\sigma_{f_0}}"] \arrow[dr, dashed, "((\sigma_{f_0})_\pi\sigma_{\tau(f_0)_\sigma\pi\sigma_{f_0}},\,\pi_{\sigma_{f_0}}\times_{C_0}id_{C_1})"] \arrow[ddr, "\pi_{\sigma_{f_0}}\times_{C_0}id_{C_1}"'] & & E\times_{D_0}(D_0\times_{C_0}C_1) \arrow[dr, "(\sigma_{f_0})_\pi"] \arrow[dd, "\pi_{\sigma_{f_0}}"] & \\
& E\times_{D_0}(D_0\times_{C_0}C_1\times_{C_0}C_1) \arrow[ur, "id_E\times_{D_0}\tilde{\mathrm{pr}}_{12}"'] \arrow[rr, "(\sigma_{f_0}\tilde{\mathrm{pr}}_{12})_\pi"] \arrow[d, "\pi_{\sigma_{f_0}\tilde{\mathrm{pr}}_{12}}"] & & E \arrow[d, "\pi"] \\
& D_0\times_{C_0}C_1\times_{C_0}C_1 \arrow[r, "\tilde{\mathrm{pr}}_{12}"] & D_0\times_{C_0}C_1 \arrow[r, "\sigma_{f_0}"] & D_0
\end{tikzcd}$$

Thus we see the following fact by (2.4.11).

Proposition 5.2.4 *Let $\hat{\boldsymbol{\mu}}_{\boldsymbol{f}}(\boldsymbol{E}) : (\boldsymbol{E}_{[\sigma_{f_0},\tau(f_0)_\sigma]})_{[\sigma,\tau]} \to \boldsymbol{E}_{[\sigma_{f_0},\tau(f_0)_\sigma]}$ be the morphism defined in subsection 3.5. We put $\hat{\boldsymbol{\mu}}_{\boldsymbol{f}}(\boldsymbol{E}) = \langle \hat{\mu}_{\boldsymbol{f}}(\boldsymbol{E}), id_{C_0}\rangle$. Then, $\hat{\mu}_{\boldsymbol{f}}(\boldsymbol{E}) : (E\times_{D_0}(D_0\times_{C_0}C_1))\times_{C_0}C_1 \to E\times_{D_0}(D_0\times_{C_0}C_1)$ is the following composition.*

$$(E\times_{D_0}(D_0\times_{C_0}C_1))\times_{C_0}C_1 \xrightarrow{((\sigma_{f_0})_\pi\sigma_{\tau(f_0)_\sigma\pi\sigma_{f_0}},\,\pi_{\sigma_{f_0}}\times_{C_0}id_{C_1})} E\times_{D_0}(D_0\times_{C_0}C_1\times_{C_0}C_1)$$
$$\xrightarrow{id_E\times_{D_0}(id_{D_0}\times_{C_0}\mu)} E\times_{D_0}(D_0\times_{C_0}C_1)$$

For an object $\boldsymbol{E} = (E \xrightarrow{\pi} D_0)$ of $\mathcal{E}^{(2)}_{D_0}$, we consider the following diagrams whose rectangles are all cartesian.

$$\begin{tikzcd}
E\times_{D_0}(D_1\times_{C_0}C_1) \arrow[r, "id_E\times_{D_0}\tilde{\mathrm{pr}}_1"] \arrow[d, "\pi_{\sigma'\tilde{\mathrm{pr}}_1}"] & E\times^{\sigma'}_{D_0}D_1 \arrow[r, "\sigma'_\pi"] \arrow[d, "\pi_{\sigma'}"] & E \arrow[d, "\pi"] \\
D_1\times_{C_0}C_1 \arrow[r, "\tilde{\mathrm{pr}}_1"] \arrow[d, "\tilde{\mathrm{pr}}_2"] & D_1 \arrow[r, "\sigma'"] \arrow[d, "f_0\tau'"] & D_0 \\
C_1 \arrow[r, "\sigma"] & C_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
E\times^{\sigma}_{C_0}C_1 \arrow[r, "\sigma_{f_0\pi}"] \arrow[d, "(\pi f_0)_\sigma"] & E \arrow[d, "f_0\pi"] \\
C_1 \arrow[r, "\sigma"] & C_0
\end{tikzcd}$$

$$\begin{tikzcd}
(E\times^{\sigma'}_{D_0}D_1)\times_{D_0}(D_0\times_{C_0}C_1) \arrow[r, "(\sigma_{f_0})_{\tau'\pi_{\sigma'}}"] \arrow[d, "\pi_{\sigma'}\times_{f_0}(f_0)_\sigma"] & E\times^{\sigma'}_{D_0}D_1 \arrow[r, "\sigma'_\pi"] \arrow[d, "\pi_{\sigma'}"] & E \arrow[d, "\pi"] \\
D_1\times_{C_0}C_1 \arrow[r, "\tilde{\mathrm{pr}}_1"] \arrow[d, "\tau'\times_{C_0}id_{C_1}"] & D_1 \arrow[r, "\sigma'"] \arrow[d, "\tau'"] & D_0 \\
D_0\times_{C_0}C_1 \arrow[r, "\sigma_{f_0}"] & D_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
E\times_{D_0}(D_0\times_{C_0}C_1) \arrow[r, "(\sigma_{f_0})_\pi"] \arrow[d, "\pi_{\sigma_{f_0}}"] & E \arrow[d, "\pi"] \\
D_0\times_{C_0}C_1 \arrow[r, "\sigma_{f_0}"] \arrow[d, "(f_0)_\sigma"] & D_0 \arrow[d, "f_0"] \\
C_1 \arrow[r, "\sigma"] & C_0
\end{tikzcd}$$

Thus $\boldsymbol{E}_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]}$, $(\boldsymbol{E}_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]}$, $\boldsymbol{E}_{[\sigma_{f_0},\,\tau(f_0)_\sigma]}$ and $\sigma^*(f_0)_*(\boldsymbol{E})$ are given as follows.

$$\boldsymbol{E}_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]} = (E\times_{D_0}(D_1\times_{C_0}C_1) \xrightarrow{\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})\pi_{\sigma'\tilde{\mathrm{pr}}_1}} C_0)$$
$$(\boldsymbol{E}_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]} = ((E\times^{\sigma'}_{D_0}D_1)\times_{D_0}(D_0\times_{C_0}C_1) \xrightarrow{\tau(f_0)_\sigma(\tau'\pi_{\sigma'}\times_{f_0}(f_0)_\sigma)} C_0)$$
$$\boldsymbol{E}_{[\sigma_{f_0},\,\tau(f_0)_\sigma]} = (E\times_{D_0}(D_0\times_{C_0}C_1) \xrightarrow{\tau(f_0)_\sigma\pi_{\sigma_{f_0}}} C_0)$$
$$\tau_*\sigma^*(f_0)_*(\boldsymbol{E}) = (E\times^{\sigma}_{C_0}C_1 \xrightarrow{\tau(\pi f_0)_\sigma} C_0)$$

There exists unique isomorphism $id_E\times_{f_0}(f_0)_\sigma : E\times_{D_0}(D_0\times_{C_0}C_1) \to E\times^{\sigma}_{C_0}C_1$ that makes the following diagram commute.

$$\begin{tikzcd}
& E \\
E\times_{D_0}(D_0\times_{C_0}C_1) \arrow[ur, "(\sigma_{f_0})_\pi"] \arrow[r, dashed, "id_E\times_{f_0}(f_0)_\sigma"] \arrow[dr, "(f_0)_\sigma\pi_{\sigma_{f_0}}"'] & E\times^{\sigma}_{C_0}C_1 \arrow[u, "\sigma_{f_0\pi}"'] \arrow[d, "(\pi f_0)_\sigma"] \\
& C_1
\end{tikzcd}$$

Thus we have an isomorphism $\langle id_E\times_{f_0}(f_0)_\sigma, id_{C_0}\rangle : \boldsymbol{E}_{[\sigma_{f_0},\,\tau(f_0)_\sigma]} \to \tau_*\sigma^*(f_0)_*(\boldsymbol{E})$ in $\mathcal{E}^{(2)}_{C_0}$.

Proposition 5.2.5 *For a representation* $(\boldsymbol{E},\boldsymbol{\xi})$ *of* $\boldsymbol{D}$*, we put* $P_{\sigma',\tau'}(\boldsymbol{E})_{\boldsymbol{E}}(\boldsymbol{\xi})=\hat{\boldsymbol{\xi}}=\langle\hat{\xi},id_{D_0}\rangle:\boldsymbol{E}_{[\sigma',\tau']}\to\boldsymbol{E}$.

(1) A composition $\boldsymbol{E}_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]}\xrightarrow{\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(\boldsymbol{E})}(\boldsymbol{E}_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]}\xrightarrow{\hat{\boldsymbol{\xi}}_{[\sigma_{f_0},\tau(f_0)_\sigma]}}\boldsymbol{E}_{[\sigma_{f_0},\,\tau(f_0)_\sigma]}$ *is given by* $\langle(\hat{\xi}(id_E\times_{D_0}\tilde{\mathrm{pr}}_1),(\tau'\times_{C_0}id_{C_1})\pi_{\sigma'\tilde{\mathrm{pr}}_1}):E\times_{D_0}(D_1\times_{C_0}C_1)\to E\times_{D_0}(D_0\times_{C_0}C_1),id_{C_0}\rangle$.

(2) A morphism

$$\boldsymbol{E}_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]}=\boldsymbol{E}_{[\sigma_{f_0}(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1})),\,\tau(f_0)_\sigma(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))]}\xrightarrow{\boldsymbol{E}_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))}}\boldsymbol{E}_{[\sigma_{f_0},\,\tau(f_0)_\sigma]}$$

in $\mathcal{E}^{(2)}_{C_0}$ *is given by* $\langle id_E\times_{D_0}(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1})):E\times_{D_0}(D_1\times_{C_0}C_1)\to E\times_{D_0}(D_0\times_{C_0}C_1),id_{C_0}\rangle$.

Proof. (1) It follows from (2.4.11) that $\hat{\boldsymbol{\xi}}_{[\sigma_{f_0},\tau(f_0)_\sigma]}:(\boldsymbol{E}_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]}\to\boldsymbol{E}_{[\sigma_{f_0},\,\tau(f_0)_\sigma]}$ is given by

$$\hat{\boldsymbol{\xi}}_{[\sigma_{f_0},\tau(f_0)_\sigma]}=\langle\hat{\xi}\times_{D_0}id_{D_0\times_{C_0}C_1}:(E\times^{\sigma'}_{D_0}D_1)\times_{D_0}(D_0\times_{C_0}C_1)\to E\times_{D_0}(D_0\times_{C_0}C_1),id_{C_0}\rangle.$$

We also see by (2.4.13) that $\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(\boldsymbol{E}):\boldsymbol{E}_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]}\to(\boldsymbol{E}_{[\sigma',\tau']})_{[\sigma_{f_0},\tau(f_0)_\sigma]}$ is given by

$$\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(\boldsymbol{E})=\langle(id_E\times_{D_0}\tilde{\mathrm{pr}}_1,(\tau'\times_{C_0}id_{C_1})\pi_{\sigma'\tilde{\mathrm{pr}}_1}):E\times_{D_0}(D_1\times_{C_0}C_1)\to(E\times^{\sigma'}_{D_0}D_1)\times_{D_0}(D_0\times_{C_0}C_1),id_{C_0}\rangle.$$

Thus the assertion follows.

(2) The assertion is a direct consequence of (2.4.11). □

Remark 5.2.6 *(1) A composition*

$$\langle id_E\times_{f_0}(f_0)_\sigma,id_{C_0}\rangle\hat{\boldsymbol{\xi}}_{[\sigma_{f_0},\tau(f_0)_\sigma]}\theta_{\sigma',\tau',\sigma_{f_0},\tau(f_0)_\sigma}(\boldsymbol{E}):\boldsymbol{E}_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]}\to\tau_*\sigma^*(f_0)_*(\boldsymbol{E})$$

is given by $\langle(\hat{\xi}(id_E\times_{D_0}\tilde{\mathrm{pr}}_1),\tilde{\mathrm{pr}}_2\pi_{\sigma'\tilde{\mathrm{pr}}_1}):E\times_{D_0}(D_1\times_{C_0}C_1)\to E\times^{\sigma}_{C_0}C_1,id_{C_0}\rangle$.

(2) A composition

$$\langle id_E\times_{f_0}(f_0)_\sigma,id_{C_0}\rangle\boldsymbol{E}_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))}:\boldsymbol{E}_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau(f_0)_\sigma(\tau'\times_{C_0}id_{C_1})]}\to\tau_*\sigma^*(f_0)_*(\boldsymbol{E})$$

is given by $\langle id_E\times_{D_0}\mu(f_1\times_{C_0}id_{C_1}):E\times_{D_0}(D_1\times_{C_0}C_1)\to E\times^{\sigma}_{C_0}C_1,id_{C_0}\rangle$.

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