Toward a representation theory of the group scheme represented by the dual Steenrod algebra

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Struggle over how to understand the theory of unstable modules over the Steenrod algebra from a viewpoint of the representation theory

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References

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§1. Motivation
Let $p$ be a prime number and let us denote by $A^*$ the Steenrod algebra over the prime field $F_p$. For a space $X$, we denote by $H^*(X)$ the cohomology group with coefficients in $F_p$. Then, $H^*(X)$ has a structure of left $A^*$-module.

Let us denote by $A_*$ the dual Steenrod algebra. J. Milnor defined a map $\lambda : H^*(X) \to H^*(X) \hat{\otimes} A_*$ from the $A^*$-module structure of $H^*(X)$. We call this map the Milnor coaction. He showed that $\lambda$ is coassociative and counital, that is, $\lambda$ is a representation of the affine group scheme represented by $A_*$ on $H^*(X)$. This motivates us to introduce various methods in representation theory to study the category of modules over the Steenrod algebra.
§2. The Milnor coaction
Let $A^*$ be a graded Hopf algebra over a field $K$ and $M^*$ a graded left $A^*$-module with structure map $\alpha : A^* \otimes M^* \to M^*$. We assume that $A^*$ and $M^*$ are finite type, namely, $A^n$ and $M^n$ are finite dimensional for all $n \in \mathbb{Z}$.

Put $\text{Hom}(A^n, K) = A_{-n}$ and consider the dual Hopf algebra $A_* = \bigoplus_{n \in \mathbb{Z}} A_n$ of $A^*$. We review how Milnor defined the coaction

$$\lambda : M^* \to M^* \hat{\otimes} A_* = \bigoplus_{j \in \mathbb{Z}} \prod_{i \in \mathbb{Z}} M^{i+j} \otimes A_{-i}$$

in this section.
In Milnor’s paper “The Steenrod algebra and its dual”, the original definition of the Milnor coaction is as follows.

4. The homomorphism $\lambda^*$

Let $H_*, H^*$ denote the homology and cohomology, with coefficients $\mathbb{Z}_p$, of a finite complex. The action of $\mathcal{S}^*$ on $H^*$ gives rise to an action of $\mathcal{S}^*$ on $H_*$ which is defined by the rule:

$$\langle \mu\theta, \alpha \rangle = \langle \mu, \theta\alpha \rangle$$

for all $\mu \in H_*$, $\theta \in \mathcal{S}^*$, $\alpha \in H^*$. This action can be considered as a homomorphism

$$\lambda_* : H_* \otimes \mathcal{S}^* \to H_* .$$

The dual homomorphism

$$\lambda^* : H^* \to H^* \otimes \mathcal{S}_*$$

will be the subject of this section.

He also remarked the following in the next paragraph.
Alternatively, the restricted homomorphism $H_{n+i} \otimes \mathcal{S}_i \to H_n$ has a
dual which we will denote by

$$\lambda^i : H^n \to H^{n+i} \otimes \mathcal{S}_i .$$

In this terminology we have

$$\lambda^* = \lambda^0 + \lambda^1 + \lambda^2 + \cdots$$

carrying $H^n$ into $\sum_i H^{n+i} \otimes \mathcal{S}_i$. The condition that $H^*$ be the cohomolo-
gy of a finite complex is essential here, since otherwise $\lambda^*$ would be an
infinite sum.

Namely, the target of the Milnor coaction should be the
completed tensor product in general, hence we have to give
suitable topologies on $H^*$ and $A_*$. Since the above definition of the Milnor coaction is very
brief, next we try to describe the Milnor coaction in detail.
For vector spaces $V$ and $W$, we consider the following maps.

\[ D_{v,w}: \text{Hom}(V, W) \to \text{Hom}(\text{Hom}(W, K), \text{Hom}(V, K)) \]
\[ \phi_{v,w}: \text{Hom}(V, K) \otimes \text{Hom}(W, K) \to \text{Hom}(V \otimes W, K) \]
\[ \chi_v: V \to \text{Hom}(\text{Hom}(V, K), K) \]
\[ T_{v,w}: V \otimes W \to W \otimes V \]

$D_{v,w}$ assigns a linear map to its dual map, $\phi_{v,w}$ is defined by $\phi_{v,w}(f \otimes g) = f \otimes g$, $\chi_{v,w}$ is defined by $\chi_{v,w}(x)(f) = f(x)$ and $T_{v,w}$ is the switching map.

We note that $D_{v,w}$, $\phi_{v,w}$ and $\chi_v$ are isomorphisms if $V$ and $W$ are finite dimensional.
Let $\tilde{\alpha}_{i,j} : A^i \to \text{Hom}(M^j, M^{i+j})$ be the adjoint of a component $\alpha_{i,j} : A^i \otimes M^j \to M^{i+j}$ of $\alpha : A^* \otimes M^* \to M^*$. We put

$$D_{j,i+j} = (-1)^{i(i+j)} D_{M^j,M^{i+j}} : \text{Hom}(M^j, M^{i+j}) \to \text{Hom}(\text{Hom}(M^{i+j}, K), \text{Hom}(M^j, K)).$$

Let $\beta_{i,j} : A^i \otimes \text{Hom}(M^{i+j}, K) \to \text{Hom}(M^j, K)$ be the adjoint of the following composition.

$$A^i \xrightarrow{\tilde{\alpha}_{i,j}} \text{Hom}(M^j, M^{i+j}) \xrightarrow{D_{j,i+j}} \text{Hom}(\text{Hom}(M^{i+j}, K), \text{Hom}(M^j, K)).$$
We put $\chi_i = (-1)^i \chi^i_A : A^i \to \text{Hom}(\text{Hom}(A^i, K), K) = \text{Hom}(A_{-i}, K)$, $\phi_{i,j} = (-1)^{(i+j)} \phi_{M^{i+j}, A^{-i}} : \text{Hom}(M^{i+j}, K) \otimes \text{Hom}(A_{-i}, K) \to \text{Hom}(M^{i+j} \otimes A_{-i}, K)$ and $T_{i,j} = (-1)^{(i+j)} T_{\text{Hom}(M^{i+j}, K), A^i} : \text{Hom}(M^{i+j}, K) \otimes A^i \to A^i \otimes \text{Hom}(M^{i+j}, K)$.

Since $A^*$ and $M^*$ are finite type, $\chi_i$ and $\phi_{i,j}$ are isomorphisms. Let $\gamma_{i,j} : \text{Hom}(M^{i+j} \otimes A_{-i}, K) \to \text{Hom}(M^j, K)$ be the following composition.

$$
\text{Hom}(M^{i+j} \otimes A_{-i}, K) \xrightarrow{\phi_{i,j}^{-1}} \text{Hom}(M^{i+j}, K) \otimes \text{Hom}(A_{-i}, K) \xrightarrow{(1 \otimes \chi_i)^{-1}} \\
\text{Hom}(M^{i+j}, K) \otimes A^i \xrightarrow{T_{i,j}} A^i \otimes \text{Hom}(M^{i+j}, K) \xrightarrow{\beta_{i,j}} \text{Hom}(M^j, K)
$$
Then, $\gamma_{i,j}$ is regarded as an element of
\[ \text{Hom}(\text{Hom}(M^{i+j} \otimes A_{-i}, K), \text{Hom}(M^{j}, K)). \]
Since
\[ D_{M^{i+j} \otimes A_{-i}} : \text{Hom}(M^{j}, M^{i+j} \otimes A_{-i}) \to \text{Hom}(\text{Hom}(M^{i+j} \otimes A_{-i}, K), \text{Hom}(M^{j}, K)) \]
is an isomorphism, there exists a unique element
\[ \lambda_{i,j} : M^{j} \to M^{i+j} \otimes A_{-i} \]
of $\text{Hom}(M^{j}, M^{i+j} \otimes A_{-i})$ that maps to $\gamma_{i,j}$ by $D_{M^{i+j} \otimes A_{-i}}$.
Let $\lambda_{j} : M^{j} \to \prod_{i \in \mathbb{Z}} M^{i+j} \otimes A_{-i}$ be the map whose $i$-th component is $\lambda_{i,j}$. Finally, the Milnor coaction
\[ \lambda : M^{*} \to M^{*} \hat{\otimes} A_{*} = \bigoplus_{j \in \mathbb{Z}} \prod_{i \in \mathbb{Z}} M^{i+j} \otimes A_{-i} \]
is the map whose component of degree $j$ is $\lambda_{j}$. 
Milnor showed that the left $A^*$-module structure
\[ \alpha : A^* \otimes M^* \rightarrow M^* \]
is recovered from the Milnor coaction as follows.

**Theorem 2.1**

For $x \in A^i$ and $m \in M^j$, $\alpha(x \otimes m) = (-1)^{ij} \sum_{k \in \mathbb{Z}} a_k(x)m_k$ if

\[ \lambda_j(m) = \sum_{k \in \mathbb{Z}} m_k \otimes a_k \in \prod_{l \in \mathbb{Z}} M^{j+l} \otimes A_{-l}. \]
§3. Topological graded rings and modules

Definition 3.1

(1) We say that a graded ring $K^*$ is commutative if $xy = (-1)^{mn}yx$ for any $m, n \in \mathbb{Z}$ and $x \in K^m, y \in K^n$.

(2) Let $K^*$ be a graded ring and $M^*$ a graded $K^*$-module. A submodule of $M^*$ is said to be homogeneous if it is generated by elements of $\bigcup_{n \in \mathbb{Z}} M^n$. Similarly, an ideal of $K^*$ is said to be homogeneous if it is generated by elements of $\bigcup_{n \in \mathbb{Z}} K^n$.

From now on, “an ideal” of a graded ring always means a homogeneous ideal and “a submodule” of a graded module means a homogeneous submodule unless otherwise stated.
Definition 3.2

(1) For a topological graded ring $A^*$, we denote by $I_{A^*}$ the set of open homogeneous two-sided ideals of $A^*$. If $I_{A^*}$ is a fundamental system of neighborhoods of 0, $A^*$ is said to be linearly topologized.

(2) Let $A^*$ and $K^*$ be linearly topologized graded rings and $\eta : K^* \rightarrow A^*$ a continuous homomorphism preserving degrees. If $\eta(x)y = (-1)^{mn} y \eta(x)$ holds for any $m, n \in \mathbb{Z}$ and $x \in K^m, y \in A^n$, $(A^*, \eta)$ (or $A^*$ for short) is called a topological $K^*$-algebra.
(3) Let \((A^*, \eta)\) and \((B^*, \iota)\) be topological \(K^*\)-algebras. If a continuous homomorphism \(f: A^* \to B^*\) preserving degrees satisfies \(f \eta = \iota\), we call \(f\) a homomorphism of topological \(K^*\)-algebras.

(4) For a commutative linearly topologized graded ring \(K^*\), we denote by \(\text{TopAlg}_{K^*}\) the category of commutative topological \(K^*\)-algebras and homomorphisms of topological \(K^*\)-algebras.
Definition 3.3

(1) Let $L^*$, $M^*$ and $N^*$ be graded abelian groups. A map $\beta : L^* \times M^* \to N^*$ is said to be biadditive if $\beta$ satisfies the following conditions (i) and (ii).

(i) $\beta (L^l \times M^m) \subset N^{l+m}$ for any $l, m \in \mathbb{Z}$.

(ii) $\beta (x+y, z) = \beta (x, z) + \beta (y, z)$, $\beta (x, z+w) = \beta (x, z) + \beta (x, w)$ for any $x, y \in L^*$ and $z, w \in M^*$.

(2) Suppose that $K^*$ is a commutative graded ring and $L^*$, $M^*$, $N^*$ are graded left $K^*$-modules. If $\beta : L^* \times M^* \to N^*$ is biadditive and satisfies the following condition (iii), we say that $\beta$ is bilinear.

(iii) $\beta (rx, z) = r \beta (x, z)$, $\beta (x, rz) = (-1)^{ln} r \beta (x, z)$ if $r \in K^n$, $x \in L^l$ and $z \in M^*$ for $l, n \in \mathbb{Z}$. 
Definition 3.4
(1) For a topological graded $K^*$-module $M^*$, let us denote by $V_{M^*}$ the set of homogeneous open submodules of $M^*$. If $V_{M^*}$ is a fundamental system of neighborhoods of 0, we say that $M^*$ is **linearly topologized**.
(2) Let $K^*$ be a linearly topologized graded ring and $M^*$ a topological graded left (resp. right) $K^*$-module. If 
\{IM^* | I \in I_{K^*}\} (resp. \{M^*I | I \in I_{K^*}\}) is a fundamental system of neighborhoods of 0, we say that the topology of $M^*$ is **induced by $K^*$**.

Remark 3.5
If $A^*$ is a topological $K^*$-algebra and we regard $A^*$ as a left (right) $K^*$-module, then the topology of $A^*$ is coarser than the topology induced by $K^*$. 
**Definition 3.6**

Let $L^*$, $M^*$ and $N^*$ be linearly topologized graded abelian groups. We say that a biadditive map $\beta : L^* \times M^* \rightarrow N^*$ is **strongly continuous** if, for any open subgroup $U^*$ of $N^*$, there exist an open subgroup $V^*$ of $L^*$ and an open subgroup $W^*$ of $M^*$ such that $\beta (V^* \times M^*)$ and $\beta (L^* \times W^*)$ are contained in $U^*$. 
Proposition 3.7
Let $K^*$ be a linearly topologized graded ring and $M^*, N^*$ linearly topologized graded left $K^*$-modules.

(1) The topology of $M^*$ is coarser than the topology induced by $K^*$ if and only if the structure map $\alpha : K^* \times M^* \to M^*$ is strongly continuous.

(2) Let $f : M^* \to N^*$ be a homomorphism of left $K^*$-modules. If the topology of $M^*$ is finer than the topology induced by $K^*$ and the topology of $N^*$ is coarser than the topology induced by $K^*$, then $f$ is continuous.
For a linearly topologized graded ring $K^*$, we denote by $\text{TopMod}_{K^*}$ the category of linearly topologized graded left $K^*$-modules and continuous homomorphisms preserving degrees. We denote by $\text{TopMod}_{iK^*}$ the full subcategory of $\text{TopMod}_{K^*}$ consisting of linearly topologized graded left $K^*$-modules whose topology are coarser than the topology induced by $K^*$.

For objects $M^*$ and $N^*$ of $\text{TopMod}_{K^*}$, we denote by $\text{Hom}_{K^*}(M^*, N^*)$

the set of all morphisms in $\text{TopMod}_{K^*}$ from $M^*$ to $N^*$.

Proposition 3.8
(1) $\text{TopMod}_{K^*}$ is complete and cocomplete.
(2) $\text{TopMod}_{iK^*}$ is complete and finitely cocomplete.
§4. Suspensions

Let $\tau_{K^*}: K^* \to K^*$ be a homomorphism of graded rings given by
$\tau_{K^*}(r) = (-1)^r$ if $r \in K^n$. Then, it is clear that $\tau_{K^*}$ is continuous
and $\tau_{K^*} \tau_{K^*} = \text{id}_{K^*}$. We call $\tau_{K^*}$ the conjugation of $K^*$.

For $m \in \mathbb{Z}$ and an object $M^*$ of $\text{TopMod}_{K^*}$, define an object
$\Sigma^m M^*$ of $\text{TopMod}_{K^*}$ as follows.
Definition 4.1

Put \((\sum^m M^*)_i = \{[m]\} \times M^i_m\) for \(i \in \mathbb{Z}\) and give \((\sum^m M^*)_i\) the structure of an abelian group such that the projection \((\sum^m M^*)_i = \{[m]\} \times M^i_m\) onto the second component is an isomorphism of abelian groups.

If \(\alpha : K^* \times M^* \rightarrow M^*\) is the \(K^*\)-module structure of \(M^*\), we define the \(K^*\)-module structure \(\alpha^m : K^* \times \sum^m M^* \rightarrow \sum^m M^*\) of \(\sum^m M^*\) by \(\alpha^m(r, ([m], x)) = ([m], \alpha(\tau_K^m(r), x))\) for \(r \in K^*\) and \(x \in M^*\), where \(\tau_K^m : K^* \rightarrow K^*\) is the \(m\) times composition of \(\tau_K^*\).

If \(U^*\) is an submodule of \(M^*\), we can regard \(\sum^m U^*\) as a submodule of \(\sum^m M^*\). We give a linear topology on \(\sum^m M^*\) such that the set of open submodules of \(\sum^m M^*\) is given by

\[ V_{\sum^m M^*} = \{ \sum^m U^* | U^* \in V_{M^*} \} . \]
If \( f: M^* \rightarrow N^* \) is a morphism in \( \text{TopMod}_{K^*} \), we denote by \( \Sigma^m f: \Sigma^m M^* \rightarrow \Sigma^m N^* \) the map which maps \(([m], x) \in (\Sigma^m M^*)^i\) to \(([m], f(x)) \in (\Sigma^m N^*)^i\). It is easy to verify that \( \Sigma^m f \) is a morphism in \( \text{TopMod}_{K^*} \). Thus we have a functor

\[
\Sigma^m: \text{TopMod}_{K^*} \rightarrow \text{TopMod}_{K^*}.
\]

We call \( \Sigma^m M^* \) and \( \Sigma^m f \) the m-fold suspension of \( M^* \) and \( f \), respectively.
§5. Completion of topological modules

Definition 5.1
We say that an object $M^*$ of $\text{TopMod}_{k^*}$ is complete if $M^n$ is complete for each $n \in \mathbb{Z}$.

Let $M^*$ an object of $\text{TopMod}_{k^*}$. Regarding $V_{M^*}$ as a category whose morphisms are inclusion maps, consider a functor $D_{M^*}: V_{M^*} \to \text{TopMod}_{k^*}$ given by $D_{M^*}(U^*) = M^*/U^*$. We denote by $\hat{M}^*$ the limit $\lim D_{M^*}$ of $D_{M^*}$, namely, there is a limiting cone $(\hat{M}^* \xrightarrow{\pi_{U^*}} M^*/U^*)_{U^* \in V_{M^*}}$. Since the quotient maps $p_{U^*}: M^* \to M^*/U^*$ for $U^* \in V_{M^*}$ define a cone of $D_{M^*}$, there is a unique map $\eta_{M^*}: M^* \to \hat{M}^*$ satisfying $\pi_{U^*} \eta_{M^*} = p_{U^*}$ for any $U^* \in V_{M^*}$. 
Proposition 5.2
The image of $\eta_{M^*}:M^* \to \hat{M}^*$ is dense and $\eta_{M^*}$ is an open map onto its image.

Proposition 5.3
(1) $M^*$ is Hausdorff if and only if $\eta_{M^*}:M^* \to \hat{M}^*$ is injective.
(2) $\hat{M}^*$ is complete Hausdorff.
(3) $M^*$ is complete Hausdorff if and only if $\eta_{M^*}:M^* \to \hat{M}^*$ is an isomorphism.
Definition 5.4

The limit $\hat{M}^*$ of $D_{M^*}: V_{M^*} \rightarrow \text{TopMod}_{K^*}$ is called the completion of $M^*$.

Let $f: M^* \rightarrow N^*$ be a morphism in $\text{TopMod}_{K^*}$. For each $U^* \in V_{N^*}$, we have a map $f_{U^*}: M^*/f^{-1}(U^*) \rightarrow N^*/U^*$ induced by $f$. Then, $(\hat{M}^* \xrightarrow{f_{U^*} \pi_f^{-1}(U^*)} N^*/U^*)_{U^* \in V_{N^*}}$ is a cone of $D_{N^*}: V_{N^*} \rightarrow \text{TopMod}_{K^*}$.

There exists a unique morphism $\hat{f}: \hat{M}^* \rightarrow \hat{N}^*$ which makes the following diagram commute for any $U^* \in V_{N^*}$.

\[
\begin{array}{ccc}
M^* & \xrightarrow{\eta_{M^*}} & \hat{M}^* \\
\downarrow f & & \downarrow \hat{f} \\
N^* & \xrightarrow{\eta_{N^*}} & \hat{N}^*
\end{array}
\]
Proposition 5.5
Let \( f: M^* \to N^* \) be a morphism in \( \text{TopMod}_{K^*} \) such that \( N^* \) is complete Hausdorff. Then, there exists a unique morphism \( g: \hat{M}^* \to N^* \) such that \( g \eta_{M^*} = f \).

Let us denote by \( \text{TopMod}_{cK^*} \) (resp. \( \text{TopMod}_i^{cK^*} \)) the full subcategory of \( \text{TopMod}_{K^*} \) (resp. \( \text{TopMod}_i^{K^*} \)) consisting of objects which are complete Hausdorff.

Proposition 5.6
A functor \( C: \text{TopMod}_{K^*} \to \text{TopMod}_{cK^*} \) (resp. \( C: \text{TopMod}_i^{K^*} \to \text{TopMod}_{cK^*} \)) defined by \( C(M^*) = \hat{M}^* \) and \( C(f) = \hat{f} \) is a left adjoint of the inclusion functor \( \text{TopMod}_{cK^*} \to \text{TopMod}_{K^*} \) (resp. \( \text{TopMod}_i^{cK^*} \to \text{TopMod}_i^{K^*} \)).
§6. Topologies on graded modules

Definition 6.1

(1) A linearly topologized graded ring $K^*$ is said to be finite if $K^*$ is discrete and artinian.

(2) For a linearly topologized graded ring $K^*$, we say that an ideal $I$ of $K^*$ is cofinite if $K^*/I$ is artinian. We say that $K^*$ has the cofinite topology if the set of all cofinite ideals of $K^*$ is a fundamental system of the neighborhood of $0$.

(3) If the topology of $K^*$ is coarser (resp. finer) than the cofinite topology, we say that $K^*$ is subcofinite (resp. supercofinite). Hence $K^*$ is subcofinite (resp. supercofinite) if and only if every open ideal is cofinite (resp. every cofinite ideal is open).
Definition 6.2

(1) An object $M^*$ of $\text{TopMod}_{k^*}$ is said to be finite if $M^*$ is discrete and of finite length.

(2) For an object $M^*$ of $\text{TopMod}_{k^*}$, we say that a submodule $N^*$ of $M^*$ is cofinite if $M^*/N^*$ is finite. We say that $M^*$ has the cofinite topology if the set of all cofinite submodules of $M^*$ is a fundamental system of the neighborhood of $0$.

(3) If the topology of $M^*$ is coarser (resp. finer) than the cofinite topology, we say that $M^*$ is subcofinite (resp. supercofinite). Hence $M^*$ is subcofinite (resp. supercofinite) if and only if every open submodule is cofinite (resp. every cofinite submodule is open).
(4) If \( M^* \) is complete Hausdorff and subcofinite, we say that \( M^* \) is profinite.

(5) For a non-negative integer \( n \), let us denote by \( M^*[n] \) the submodule of \( M^* \) generated by \( \bigcup_{|i| \geq n} M^i \). We say that an object \( M^* \) of \( \text{TopMod}_{k^*} \) has a skeletal topology if \( \{M^*[n]|n=0,1,2,...\} \) is a fundamental system of the neighborhood of 0.

(6) If the topology of \( M^* \) is coarser (resp. finer) than the skeletal topology, we say that \( M^* \) is subskeletal (resp. superskeletal). Hence \( M^* \) is subskeletal (resp. superskeletal) if and only if every open submodule contains \( M^*[n] \) for some \( n \) (resp. every submodule containing \( M^*[n] \) for some \( n \) is open).
Proposition 6.3

(1) If $M^*$ is a subcofinite $K^*$-module, then each submodule and quotient module of $M^*$ are subcofinite.

(2) If $(M_i^*)_{i \in I}$ is a family of subcofinite $K^*$-modules, then $\prod_{i \in I} M_i^*$ is also subcofinite.

(3) $M^*$ is isomorphic to a submodule of product of finite $K^*$-modules if and only if $M^*$ is subcofinite and Hausdorff.

(4) If $M^*$ is subcofinite, the completion $\hat{M}^*$ is also subcofinite.

Proposition 6.4

Let $M^*$ and $N^*$ be objects of $\text{TopMod}_{K^*}$. If “$M^*$ is supercofinite and $N^*$ is subcofinite” or “$M^*$ is superskeletal and $N^*$ is subskeletal”, then every linear map from $M^*$ to $N^*$ preserving degrees is continuous.
§7. Tensor products of topological modules

For objects $M^*$, $N^*$ of $\text{TopMod}_{K^*}$, we give a topology on $M^* \otimes_{K^*} N^*$ so that

$$\{\text{Ker}(p_{U^*} \otimes q_{V^*}: M^* \otimes_{K^*} N^* \to \frac{M^*}{U^*} \otimes_{K^*} \frac{N^*}{V^*}) | U^* \in V_{M^*}, V^* \in V_{N^*}\}$$

is a fundamental system of the neighborhood of 0.

Here, we denote by $p_{U^*}: M^* \to \frac{M^*}{U^*}$ and $q_{V^*}: N^* \to \frac{N^*}{V^*}$ the quotient maps for submodules $U^*$ of $M^*$ and $V^*$ of $N^*$.

We denote by $\hat{M^* \otimes_{K^*} N^*}$ the completion of $M^* \otimes_{K^*} N^*$ and call this the completed tensor product of $M^*$ and $N^*.$
Proposition 7.1

A map $\beta_{M^*,N^*}: M^* \times N^* \to M^* \otimes_{K^*} N^*$ defined by $\beta_{M^*,N^*}(x,y) = x \otimes y$ is a strongly continuous bilinear map and, for a strongly continuous bilinear map $B: M^* \times N^* \to L^*$, there exists a unique morphism $\tilde{B}: M^* \otimes_{K^*} N^* \to L^*$ in $\text{TopMod}_{K^*}$ satisfying $\tilde{B} \beta_{M^*,N^*} = B$.

Proposition 7.2

(1) If $M^*$ or $N^*$ has a topology coarser than the topology induced by $K^*$, the topology on $M^* \otimes_{K^*} N^*$ is coarser than the topology induced by $K^*$.

(2) $M^*$ has a topology coarser than the topology induced by $K^*$ if and only if there is an isomorphism $K^* \otimes_{K^*} M^* \to M^*$. 
For objects $M^*$ and $N^*$ of $\text{TopMod}_{K^*}$, define a morphism

$$
\tau_{M^*, N^*}^{m,n} : \Sigma^m M^* \otimes_{K^*} \Sigma^n N^* \to \Sigma^{m+n} (M^* \otimes_{K^*} N^*)
$$

as follows. Define $\tilde{\tau}_{M^*, N^*}^{m,n} : \Sigma^m M^* \times \Sigma^n N^* \to \Sigma^{m+n} (M^* \otimes_{K^*} N^*)$ by

$$
\tilde{\tau}_{M^*, N^*}^{m,n} \left( ([m], x), ([n], y) \right) = ([m+n], (-1)^{n(i-m)} \beta_{M^*, N^*}(x, y))
$$

for $(x, y) \in M^{i-m} \times N^{j-n}$. Then, it is easy to verify that $\tilde{\tau}_{M^*, N^*}^{m,n}$ is bilinear and strongly continuous.

Let $\tau_{M^*, N^*}^{m,n}$ be the unique morphism satisfying

$$
\tau_{M^*, N^*}^{m,n} \beta_{\Sigma^m M^*, \Sigma^n N^*} = \tilde{\tau}_{M^*, N^*}^{m,n}
$$

Clearly, $\tau_{M^*, N^*}^{m,n}$ is a natural isomorphism.
For an object $M^*$ of $\text{TopMod}_{K^*}$, define a morphism

$$s_M^*: \Sigma^m M^* \to (\Sigma^m K^*) \otimes_{K^*} M^*$$

by $s_M^m([m], x) = \beta_{\Sigma^m K^*, M^*}(([m], 1), x)$ for $x \in M^{i-m}$. Then, $s_M^m$ is a homomorphism of $K^*$-modules.

We note that $s_M^m$ is a natural isomorphism if and only if the topology on $M^*$ is coarser than the topology induced by $K^*$. 
**§8. Spaces of homomorphisms**

For \( r \in K^l \) and a morphism \( f: \Sigma^m M^* \to N^* \), define a morphism \( rf: \Sigma^{l+m} M^* \to N^* \) in \( \text{TopMod}_{K^*} \) by \( (rf)([l+m], x) = rf([m], x) \) for \( x \in M^* \).

**Definition 8.1**

For objects \( M^* \) and \( N^* \) of \( \text{TopMod}_{K^*} \), we define an object \( \text{Hom}^*(M^*, N^*) \) of \( \text{TopMod}_{K^*} \) as follows. Put

\[
\text{Hom}^n(M^*, N^*) = (\text{Hom}^*(M^*, N^*))^n = \text{Hom}_{K^*}(\Sigma^n M^*, N^*).
\]

The maps \( K^l \times \text{Hom}^n(M^*, N^*) \to \text{Hom}^{l+n}(M^*, N^*) \) for \( l, n \in \mathbb{Z} \) given by \( (r, f) \to rf \) define a left \( K^* \)-module structure of \( \text{Hom}^*(M^*, N^*) \).
For morphisms $f: M^* \rightarrow N^*$, $g: N^* \rightarrow L^*$ in $\text{TopMod}_{k^*}$, define maps $f^*: \text{Hom}^*(N^*, L^*) \rightarrow \text{Hom}^*(M^*, L^*)$ and $g_*: \text{Hom}^*(M^*, N^*) \rightarrow \text{Hom}^*(M^*, L^*)$ by $f^*(\varphi) = \varphi \Sigma n^* f$ and $g_*(\psi) = g \psi$ for $\varphi \in \text{Hom}^n(N^*, L^*)$ and $\psi \in \text{Hom}^m(M^*, N^*)$. It is easy to verify that $f^*$ and $g_*$ are maps of $K^*$-modules.

For a submodule $S^*$ of $M^*$ and a submodule $U^*$ of $N^*$, we put $O(S^*, U^*) = \text{Ker}(i_{S^*}^* p_{U^*}: \text{Hom}^*(M^*, N^*) \rightarrow \text{Hom}^*(S^*, N^*/U^*))$.

Here we denote by $i_{S^*}: S^* \rightarrow M^*$ the inclusion map and by $p_{U^*}: N^* \rightarrow N^*/U^*$ the quotient map.
Let $F_{M^*}$ be the set of finitely generated submodules of $M^*$. Define a topology on $\text{Hom}^*(M^*, N^*)$ such that

$$\{O(S^*, U^*) \mid S^* \in F_{M^*}, U^* \in V_{N^*}\}$$

is a fundamental system of neighborhoods of 0. We denote by $M^{**}$ the dual module $\text{Hom}^*(M^*, K^*)$ of $M^*$.

**Proposition 8.2**

Define a map $\delta_{N^*} : N^* \to \text{Hom}^*(K^*, N^*)$ by $(\delta_{N^*}(x))(\langle n \rangle, s) = (-1)^n s x$ for $x \in N^n$ and $s \in K^*$. Then, $\delta_{N^*}$ is an isomorphism whose inverse is the evaluation map $E_1 : \text{Hom}^*(K^*, N^*) \to N^*$ defined by $E_1(f) = f([k], 1)$ for $f \in \text{Hom}^k(K^*, N^*)$. 

§9. Adjointness

Let $M^*$, $N^*$, $L^*$ be objects of $\text{TopMod}_{K^*}$ and $f: M^* \otimes_{K^*} N^* \rightarrow L^*$ a morphism in $\text{TopMod}_{K^*}$. For $x \in M^k$, define a map $f_x: \Sigma^k N^* \rightarrow L^*$ by $f_x(y) = f(x \otimes y)$ for $y \in (\Sigma^k N^*)^n = N^{n-k}$. Then, $f_x$ is an element of $\text{Hom}_K^k(N^*, L^*) = \text{Hom}_{K^*}(\Sigma^k N^*, L^*)$.

Thus we have a map $(f^a)^k: M^k \rightarrow \text{Hom}_{K^*}(\Sigma^k N^*, L^*)$ given by $(f^a)^k(x) = f_x$ and a family of linear maps $((f^a)^k)_{k \in \mathbb{Z}}$ defines a morphism $f^a: M^* \rightarrow \text{Hom}^*(N^*, L^*)$ in $\text{TopMod}_{K^*}$. Define a map

$$\Phi = \Phi_{M^*, N^*, L^*}: \text{Hom}_{K^*}(M^* \otimes_{K^*} N^*, L^*) \rightarrow \text{Hom}_{K^*}(M^*, \text{Hom}^*(N^*, L^*))$$

by $\Phi(f) = f^a$. 
Proposition 9.1

$\Phi_{M^*,N^*,L^*}$ is injective and if one of the following conditions is satisfied, $\Phi_{M^*,N^*,L^*}$ is an isomorphism.

(i) $M^* \times_K N^*$ is supercofinite and $L^*$ is subcofinite.
(ii) $M^* \times_K N^*$ is superskeletal and $L^*$ is subskeletal.
(iii) The topology on $M^* \times_K N^*$ is finer than the topology induced by $K^*$ and the topology on $L^*$ is coarser than the topology induced by $K^*$. 
§10. Completion of spaces of homomorphisms

Proposition 10.1
If $N^*$ is Hausdorff, so is $\text{Hom}^*(M^*, N^*)$.

Proposition 10.2
If $N^*$ is complete Hausdorff, $\eta_{M^*}^*: \text{Hom}^*(\hat{M}^*, N^*) \to \text{Hom}^*(M^*, N^*)$ is an isomorphism in $\text{TopMod}_{K^*}$.

Proposition 10.3
Suppose that $N^*$ is complete Hausdorff. If there exists a finitely generated open submodule of $M^*$, $\text{Hom}^*(M^*, N^*)$ is complete Hausdorff.
Proposition 10.4

If one of the following conditions (i) or (ii) is satisfied, there exists a unique monomorphism

$$\lambda_{M^*, N^*} : \text{Hom}^*(M^*, N^*)^\wedge \to \text{Hom}^*(M^*, \hat{N}^*)$$

that makes a diagram

$$\begin{array}{ccc}
\text{Hom}^*(M^*, N^*) & \xrightarrow{\eta_{\text{Hom}^*(M^*, N^*)}} & \text{Hom}^*(M^*, N^*)^\wedge \\
\downarrow{\eta_{N^*}} & & \downarrow{\lambda_{M^*, N^*}} \\
\text{Hom}^*(M^*, \hat{N}^*) & & \\
\end{array}$$

commute.

(i) $M^*$ is supercofinite and $N^*$ is subcofinite.

(ii) $M^*$ has a finitely generated open submodule.
Definition 10.5

(1) We say that a pair \((M^*, N^*)\) of objects of \(\text{TopMod}_{K^*}\) is **nice** if there exists a cofinal subset \(C\) of \(F_{M^*} \times V_{N^*}^{op}\) such that \(i^*_S p_{U^*}: \text{Hom}^*(M^*, N^*) \to \text{Hom}^*(S^*, N^*/U^*)\) is surjective for each \((S^*, U^*) \in C\).

(2) We say that a pair \((M^*, N^*)\) of objects of \(\text{TopMod}_{K^*}\) is **very nice** if there exists a cofinal subset \(C\) of \(F_{M^*} \times V_{N^*}^{op}\) such that \(i^*_S p_{U^*}: \text{Hom}^*(M^*, N^*) \to \text{Hom}^*(S^*, N^*/U^*)\) is surjective and \(S^*\) is projective for each \((S^*, U^*) \in C\).
Remark 10.6

(1) A pair \((M^*, N^*)\) is nice if one of the following conditions is satisfied.

(i) \(N^*\) is injective and there exists a cofinal subset \(S\) of \(F_M^*\) such that every element of \(S\) is projective.

(ii) \(M^*\) is projective and there exists a cofinal subset \(M\) of \(V_N^{\text{op}}\) such that \(N^*/U^*\) is injective for every \(U^* \in M\).

(iii) There exists a cofinal subset \(S\) of \(F_M^*\) such that every element of \(S\) is a direct summand of \(M^*\) and there exists a cofinal subset \(M\) of \(V_N^{\text{op}}\) such that every element of \(M\) is a direct summand of \(N^*\).

(2) \((M^*, N^*)\) is a very nice pair if the above (i) is satisfied.

(3) The above (iii) is satisfied for \(S = F_M^*\) and \(M = V_N^{\text{op}}\) if \(K^*\) is a field and \(M^*\) is supercofinite.
Theorem 10.7
Suppose that \((M^*, N^*)\) is nice. If one of the following conditions (i) or (ii) is satisfied, then the morphism

\[ \lambda_{M^*, N^*} : \text{Hom}^*(M^*, N^*)^\wedge \rightarrow \text{Hom}^*(M^*, \hat{N}^*) \]

given in (10.4) is an isomorphism.

(i) \(M^*\) is supercofinite and \(N^*\) is subcofinite.

(ii) \(M^*\) has a finitely generated open submodule.
§11. Tensor products and spaces of homomorphisms

Let $M_s, N_s$ $(s=1, 2)$ be objects of $\text{TopMod}_{K^*}$. We define a map

$$\phi: \text{Hom}^*(M_1^*, N_1^*) \otimes_{K^*} \text{Hom}^*(M_2^*, N_2^*) \rightarrow \text{Hom}^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*)$$

of graded $K^*$-modules by

$$\phi^{m+n}(f \otimes g) = (f \otimes g)(\tau_{M_1^*, M_2^*}^{m,n})^{-1}$$

for $f \in \text{Hom}^m(M_1^*, N_1^*)$ and $g \in \text{Hom}^n(M_2^*, N_2^*)$. In other words,

$$\phi^{m+n}(f \otimes g)([m+n], x \otimes y) = (-1)^{n(i-m)}f([m], x) \otimes g([n], y)$$

if $x \in M_1^{i-m}$, $y \in M_2^{j-n}$. Then, $\phi$ is a morphism in $\text{TopMod}_{K^*}$.

We denote by

$$\hat{\phi}: \text{Hom}^*(M_1^*, N_1^*) \hat{\otimes}_{K^*} \text{Hom}^*(M_2^*, N_2^*) \rightarrow \text{Hom}^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*)^\wedge$$

the completion of $\phi$. 
Let us define a map \( \iota_1: M^* \to M^* \otimes_{K^*} K^* \) by \( \iota_1(x) = x \otimes 1 \). Then, \( \iota_1 \) is a morphism in \( \text{TopMod}_{K^*} \) and it is an isomorphism if the topology on \( M^* \) is coarser than the topology induced by \( K^* \). Suppose that the topology on \( N^* \) is coarser than the topology induced by \( K^* \). Then, the \( K^* \)-module structure map of \( N^* \) induces an isomorphism \( \tilde{\alpha}: K^* \otimes_{K^*} N^* \to N^* \) by (7.2). Let

\[
\varphi_{M^*}^{N^*}: \text{Hom}^*(M^*, K^*) \otimes_{K^*} N^* \to \text{Hom}^*(M^*, N^*)
\]

be the following composition of morphisms.

\[
\begin{array}{c}
\text{Hom}^*(M^*, K^*) \otimes_{K^*} N^* \xrightarrow{1 \otimes \delta_{N^*}} \text{Hom}^*(M^*, K^*) \otimes_{K^*} \text{Hom}^*(K^*, N^*) \xrightarrow{\phi} \\
\text{Hom}^*(M^* \otimes_{K^*} K^*, K^* \otimes_{K^*} N^*) \xrightarrow{\iota_1^*} \text{Hom}^*(M^*, K^* \otimes_{K^*} N^*) \xrightarrow{\tilde{\alpha}^*} \text{Hom}^*(M^*, N^*)
\end{array}
\]
Theorem 11.1
If both \((M^*_1, N^*_1)\) and \((M^*_2, N^*_2)\) are very nice pairs, then
\[
\hat{\phi}: \text{Hom}^*(M^*_1, N^*_1) \otimes_K^* \text{Hom}^*(M^*_2, N^*_2) \to \text{Hom}^*(M^*_1 \otimes_K^* M^*_2, N^*_1 \otimes_K^* N^*_2)^\wedge
\]
is an isomorphism.

Suppose that the topology on \(N^*\) is coarser than the topology induced by \(K^*\). Let
\[
\hat{\phi}^M_{N^*}: \text{Hom}^*(M^*, K^*) \hat{\otimes}_K^* N^* \to \text{Hom}^*(M^*, N^*)^\wedge
\]
be the completion of \(\varphi^M_{N^*}: \text{Hom}^*(M^*, K^*) \otimes_K^* N^* \to \text{Hom}^*(M^*, N^*).\)

Corollary 11.2
Let \((M^*, K^*)\) be a very nice pair.

(1) \(\hat{\phi}: \text{Hom}^*(M^*, K^*) \hat{\otimes}_K^* \text{Hom}^*(M^*, K^*) \to \text{Hom}^*(M^* \otimes_K^* M^*, K^*)^\wedge\) is an isomorphism.

(2) If \(M^*\) and \(N^*\) are objects of \(\text{TopMod}_{K^*}^i\), then
\[
\hat{\phi}^M_{N^*}: \text{Hom}^*(M^*, K^*) \hat{\otimes}_K^* N^* \to \text{Hom}^*(M^*, N^*)^\wedge
\]
is an isomorphism.
Suppose that “$M_1^* \otimes_{k^*} M_2^*$ is supercofinite and both $N_1^*$ and $N_2^*$ are subcofinite” or “$M_1^* \otimes_{k^*} M_2^*$ has a finitely generated open submodule”. By (10.7), there exists the following morphism.

$\lambda_{M_1^* \otimes_{k^*} M_2^*, N_1^* \otimes_{k^*} N_2^*} : \text{Hom}^*(M_1^* \otimes_{k^*} M_2^*, N_1^* \otimes_{k^*} N_2^*)^\wedge \rightarrow \text{Hom}^*(M_1^* \otimes_{k^*} M_2^*, N_1^* \otimes_{k^*} N_2^*)$

Composing $\lambda_{M_1^* \otimes_{k^*} M_2^*, N_1^* \otimes_{k^*} N_2^*}$ and an isomorphism 

$(\eta_{M_1^* \otimes_{k^*} M_2^*}^*)^{-1} : \text{Hom}^*(M_1^* \otimes_{k^*} M_2^*, N_1^* \otimes_{k^*} N_2^*) \rightarrow \text{Hom}^*(M_1^* \otimes_{k^*} M_2^*, N_1^* \otimes_{k^*} N_2^*)$

with $\hat{\phi}$, we have the following morphism.

$\tilde{\phi} : \text{Hom}^*(M_1^*, N_1^*) \otimes_{k^*} \text{Hom}^*(M_2^*, N_2^*) \rightarrow \text{Hom}^*(M_1^* \otimes_{k^*} M_2^*, N_1^* \otimes_{k^*} N_2^*)$
Combining (11.1) and (10.7), we have the following result.

**Corollary 11.3**

Suppose that both \((M_1^*, N_1^*)\) and \((M_2^*, N_2^*)\) are very nice pairs. If one of the following conditions (i) or (ii) is satisfied, then the morphism

\[ \tilde{\phi} : \text{Hom}^*(M_1^*, N_1^*) \hat{\otimes}_k \text{Hom}^*(M_2^*, N_2^*) \to \text{Hom}^*(M_1^* \hat{\otimes}_k M_2^*, N_1^* \hat{\otimes}_k N_2^*) \]

is an isomorphism.

(i) \(M_1^* \hat{\otimes}_k M_2^*\) is supercofinite and both \(N_1^*\) and \(N_2^*\) are subcofinite.

(ii) \(M_1^* \hat{\otimes}_k M_2^*\) has a finitely generated open submodule.
Suppose that “M* is supercofinite and N* is subcofinite” or “M* has a finitely generated open submodule”. By (10.7), there exists a morphism \( \lambda_{\text{M}, N} : \text{Hom}^*(\text{M}, N^*)^! \to \text{Hom}^*(\text{M}, \hat{N})^* \). We define a morphism

\[
\tilde{\phi}_{\text{N}}^\text{M} : \text{Hom}^*(\text{M}, K^*) \widehat{\otimes} K N^* \to \text{Hom}^*(\text{M}, \hat{N})^*
\]

by \( \tilde{\phi}_{\text{N}^*}^\text{M} = \lambda_{\text{M}, N} \tilde{\phi}_{\text{N}}^\text{M} \). The next result follows from (2) of (11.2) and (10.7).

Corollary 11.4
Suppose that (M*, K*) is a very nice pair and that both M* and N* are objects of \( \text{TopMod}^i_k \). If “M* is supercofinite and N* is subcofinite” or “M* has a finitely generated open submodule”, then \( \tilde{\phi}_{\text{N}}^\text{M} : \text{Hom}^*(\text{M}, K^*) \widehat{\otimes} K N^* \to \text{Hom}^*(\text{M}, \hat{N})^* \) is an isomorphism.
§12. Reformulation of the Milnor coaction

Definition 12.1

Let $C^*$ be an object of $\text{TopMod}_{K^*}$. Suppose that morphisms $\gamma : C^* \rightarrow C^* \hat{\otimes}_K C^*$ and $\varepsilon : C^* \rightarrow K^*$ in $\text{TopMod}_{K^*}$ are given. We call a triple $(C^*, \gamma, \varepsilon)$ a $K^*$-coalgebra if the following diagrams commute.

For the rest of this section, we assume that $K^*$ is complete Hausdorff.
We assume that $\text{Hom}^*(C^* \otimes K^* C^*, K^*)$ is complete.

For morphisms $\gamma : C^* \to C^* \hat{\otimes}_K C^*$ and $\varepsilon : C^* \to K^*$ in $\text{TopMod}_{K^*}$, we define morphisms $\tilde{\gamma} : C^{**} \hat{\otimes}_K C^{**} \to C^{**}$ and $\tilde{\varepsilon} : K^* \to C^{**}$ to be the following compositions of morphisms, respectively.

\[
\begin{align*}
\text{Hom}^*(C^*, K^*) \otimes_{K^*} \text{Hom}^*(C^*, K^*) & \xrightarrow{\eta} \text{Hom}^*(C^*, K^*) \hat{\otimes}_{K^*} \text{Hom}^*(C^*, K^*) \\
\text{Hom}^*(C^* \hat{\otimes}_{K^*} C^*, K^*) & \xrightarrow{(\eta^*)^{-1}} \text{Hom}^*(C^* \hat{\otimes}_{K^*} C^*, K^*) \xrightarrow{\gamma^*} \text{Hom}^*(C^*, K^*) \\
K^* & \xrightarrow{\delta_{K^*}} \text{Hom}^*(K^*, K^*) \xrightarrow{\varepsilon^*} \text{Hom}^*(C^*, K^*)
\end{align*}
\]

**Proposition 12.2**

$(C^{**}, \tilde{\gamma}, \tilde{\varepsilon})$ is a $K^*$-algebra if and only if $(C^*, \gamma, \varepsilon)$ is a $K^*$-coalgebra.
We assume that $\text{Hom}^*(A^* \otimes_{K^*} A^*, K^*)$ is complete and that $\hat{\phi}: \text{Hom}^*(A^*, K^*) \hat{\otimes}_{K^*} \text{Hom}^*(A^*, K^*) \to \text{Hom}^*(A^* \otimes_{K^*} A^*, K^*)$ is an isomorphism.

For morphisms $\mu: A^* \otimes_{K^*} A^* \to A^*$ and $\eta: K^* \to A^*$ in $\text{TopMod}_{K^*}$, we define morphisms $\tilde{\mu}: A^{**} \to A^{**} \hat{\otimes}_{K^*} A^{**}$ and $\tilde{\eta}: A^{**} \to K^*$ to be the following compositions of morphisms, respectively.

$$\begin{align*}
\text{Hom}^*(A^*, K^*) & \xrightarrow{\mu^*} \text{Hom}^*(A^* \otimes_{K^*} A^*, K^*) \xrightarrow{\hat{\phi}^{-1}} \text{Hom}^*(A^*, K^*) \hat{\otimes}_{K^*} \text{Hom}^*(A^*, K^*) \\
\text{Hom}^*(A^*, K^*) & \xrightarrow{\eta^*} \text{Hom}^*(K^*, K^*) \xrightarrow{\delta_{K^*}^{-1}} K^*
\end{align*}$$

**Proposition 12.3**

$(A^{**}, \tilde{\mu}, \tilde{\eta})$ is a $K^*$-coalgebra if and only if $(A^*, \mu, \eta)$ is a $K^*$-algebra.
Let $M^*$ and $N^*$ be objects of $\text{TopMod}_{K^*}^i$. We assume that $(M^*, K^*)$ is a very nice pair and that one of the following conditions is satisfied.

(i) $M^*$ is supercofinite and $N^*$ is subcofinite.
(ii) $M^*$ has a finitely generated open submodule.

Then, $\tilde{\varphi}^M_{N^*}_K: \text{Hom}^*(M^*, K^*) \hat{\otimes}_K N^* \to \text{Hom}^*(M^*, \hat{N}^*)$ is defined and it is an isomorphism by (11.4). Under these assumptions, let

$$\Lambda = \Lambda_{M^*, L^*, N^*}: \text{Hom}_K^*(M^* \otimes K^* L^*, N^*) \to \text{Hom}_K^*(L^*, N^* \hat{\otimes}_K M^*)$$

be the following composition.
\[
\begin{align*}
\text{Hom}_K^*(M^* \otimes K^* L^*, N^*) & \xrightarrow{\eta_{N^*}} \text{Hom}_K^*(M^* \otimes K^* L^*, \hat{N}^*) \\
& \xrightarrow{T^*_L, M^*} \text{Hom}_K^*(L^* \otimes K^* M^*, \hat{N}^*) \\
& \xrightarrow{\Phi_{L^*, M^*, \hat{N}^*}} \text{Hom}_K^*(L^*, \text{Hom}^*(M^*, \hat{N}^*)) \\
& \xrightarrow{(\phi_{N^*})^{-1}} \text{Hom}_K^*(L^*, M^{**} \hat{\otimes}_K N^*) \\
& \xrightarrow{\hat{T}_{M^{**}, N^*}} \text{Hom}_K^*(L^*, N^{**} \hat{\otimes}_K M^{**})
\end{align*}
\]

We remark that \( \Lambda \) is injective if \( N^* \) is Hausdorff and it is an isomorphism if \( N^* \) is profinite and \( L^* \hat{\otimes}_K M^* \) is supercofinite.

**Definition 12.4**

Let \((C^*, \gamma, \varepsilon)\) be a \( K^* \)-coalgebra and \( M^* \) an object of \( \text{TopMod}_{K^*} \). A right \( C^* \)-comodule in \( \text{TopMod}_{K^*} \) is a pair \((M^*, \lambda)\) of an that the following diagrams commute.

\[
\begin{align*}
M^* & \xrightarrow{\lambda} M^* \hat{\otimes}_K C^* \\
\downarrow \lambda & \downarrow 1 \otimes \gamma \\
M^* \hat{\otimes}_K C^* & \xrightarrow{\lambda \otimes 1} M^* \hat{\otimes}_K C^* \hat{\otimes}_K C^* \\
M^* \hat{\otimes}_K C^* & \xrightarrow{1 \otimes \varepsilon} M^* \hat{\otimes}_K K^*
\end{align*}
\]
Let \((A^*, \mu, \eta)\) be a \(K^*\)-algebra and \(M^*\) an object of \(\text{TopMod}_{K^*}\) satisfying the conditions (i), (ii), (iii) and “(iv) or (v)”.

(i) \(\text{Hom}^*(A^* \otimes_{K^*} A^*, K^*)\) is complete.

(ii) \(\hat{\phi}: \text{Hom}^*(A^*, K^*) \otimes_{K^*} \text{Hom}^*(A^*, K^*) \rightarrow \text{Hom}^*(A^* \otimes_{K^*} A^*, K^*)\) is an isomorphism.

(iii) \(M^*\) is Hausdorff.

(iv) \(A^*\) is supercofinite and \(M^*\) is subcofinite.

(v) \(A^*\) has a finitely generated open submodule.

Then, a morphism \(\alpha: A^* \otimes_{K^*} M^* \rightarrow M^*\) in \(\text{TopMod}_{K^*}\) gives a left \(A^*\)-module structure of \(M^*\) if and if the image of \(\alpha\) by

\[\Lambda_{A^*, A^*, M^*}: \text{Hom}_{K^*}(A^* \otimes_{K^*} M^*, M^*) \rightarrow \text{Hom}_{K^*}(M^*, M^* \hat{\otimes}_{K^*} A^{**})\]

gives a right \(A^{**}\)-comodule structure of \(M^*\).
Corollary 12.6

Let \((A^*, \mu, \eta)\) be a \(K^*\)-algebra and \(M^*\) an object of \(\text{TopMod}_{K^*}\). If the following conditions are satisfied, then

\[ \Lambda_{A^*, A^*, M^*} : \text{Hom}_{K^*}(A^* \otimes_{K^*} M^*, M^*) \to \text{Hom}_{K^*}(M^*, M^* \hat{\otimes}_{K^*} A^{**}) \]

maps the subset of \(\text{Hom}_{K^*}(A^* \otimes_{K^*} M^*, M^*)\) consisting of left \(A^*\)-module structures of \(M^*\) bijectively onto the subset of \(\text{Hom}_{K^*}(M^*, M^* \hat{\otimes}_{K^*} A^{**})\) consisting of right \(A^*\)-comodule structures of \(M^*\).

(i) \(\text{Hom}^*(A^* \otimes_{K^*} A^*, K^*)\) is complete.

(ii) \(\hat{\phi} : \text{Hom}^*(A^*, K^*) \hat{\otimes}_{K^*} \text{Hom}^*(A^*, K^*) \to \text{Hom}^*(A^* \otimes_{K^*} A^*, K^*)\) is an isomorphism.

(iii) \(A^*\) is supercofinite or \(A^*\) has a finitely generated open submodule.

(iv) \(M^*\) is profinite.

(v) \(A^* \otimes_{K^*} M^*\) is supercofinite.
§13. Quasi-topological category

We denote by $\textbf{Top}$ the category of topological spaces and continuous maps.

For $x \in X$, we denote by $\text{ev}_x : \textbf{Top}(X,Y) \to Y$ the map defined by $\text{ev}_x(f) = f(x)$. For $O \subset Y$, put $W(x,O) = \text{ev}_x^{-1}(O)$.

We give $\textbf{Top}(X,Y)$ the pointwise convergent topology generated by $\{W(x,O) \mid x \in X, O \text{ is an open set of } Y\}$. In other words, the pointwise convergent topology on $\textbf{Top}(X,Y)$ is the coarsest topology that $\text{ev}_x$ is continuous for every $x \in X$. 
Proposition 13.1

Let $X$, $Y$ and $Z$ be topological spaces.

(1) A map $\varphi : Z \to \text{Top}(X,Y)$ is continuous if and only if $\text{ev}_x \varphi : Z \to Y$ is continuous for any $x \in X$.

(2) For a continuous map $f : X \to Y$, the maps

$$f^* : \text{Top}(Y,Z) \to \text{Top}(X,Z)$$

and

$$f_* : \text{Top}(Z,X) \to \text{Top}(Z,Y)$$

induced by $f$ are continuous.
Definition 13.2

A category $T$ is called a quasi-topological category if the following conditions are satisfied.

1. For each $R, S \in \text{Ob } T$, $T(R, S)$ is a topological space.
2. For any morphism $f : R \to S$ in $T$ and $Z \in \text{Ob } T$, the maps $f_* : T(Z, R) \to T(Z, S)$ and $f^* : T(S, Z) \to T(R, Z)$ are continuous.

It follows from (2) of (13.1) that $\text{Top}$ is a quasi-topological category.
Condition 13.3
Let $T$ be a quasi-topological category and $D:D \to T$ a functor. For an object $X$ of $T$, define functors $D_X:D \to \text{Top}$ and $D^X:D^{op} \to \text{Top}$ by $D_X(i)=T(X,D(i))$, $D_X(\tau)=D(\tau)_*$ and $D^X(i)=T(D(i),X)$, $D^X(\tau)=D(\tau)^*$ for $i \in \text{Ob} D$ and $\tau \in \text{Mor} D$.

We consider the following conditions.

(L) If $(L \to D(i))_{i \in \text{Ob} D}$ is a limiting cone of $D$, 
$$(T(X,L) \xrightarrow{\pi_i} T(X,D(i)))_{i \in \text{Ob} D}$$

is a limiting cone of $D_X$.

(C) If $(D(i) \xrightarrow{l_i} C)_{i \in \text{Ob} D}$ is a colimiting cone of $D$, 
$$(T(C,X) \xrightarrow{l_i^*} T(D(i),X))_{i \in \text{Ob} D}$$

is a limiting cone of $D_X$.

Proposition 13.4
The conditions (L) and (C) of (13.3) are satisfied for any functor $D:D \to \text{Top}$ and topological space $X$. 
For categories $\mathbf{C}$ and $\mathbf{D}$, we denote by $\text{Funct}(\mathbf{C}, \mathbf{D})$ the category of functors from $\mathbf{C}$ to $\mathbf{D}$ and natural transformations between them.

**Definition 13.5**

Let $\mathbf{C}$ and $\mathbf{T}$ be quasi-topological categories. We say that a functor $F : \mathbf{C} \to \mathbf{T}$ is **continuous** if $F : C(R, S) \to T(F(R), F(S))$ is continuous for any $R, S \in \text{Ob} \mathbf{C}$.

We denote by $\text{Funct}_c(\mathbf{C}, \mathbf{T})$ the full subcategory of $\text{Funct}(\mathbf{C}, \mathbf{T})$ consisting of continuous functors.
Proposition 13.6
Let $T$ be a quasi-topological category and $R$ an object of $T$. Then, the functor $h_R : T \to \text{Top}$ represented by $R$ is continuous.

Proposition 13.7
Let $C$ be a quasi-topological category and $D : D \to \text{Funct}_c(C, \text{Top})$ a functor.

1. If $(L \xrightarrow{\pi_i} D(i))_{i \in \text{Ob}_D}$ is a limiting cone of $D$, $L$ is a continuous functor.
2. If $(D(i) \xrightarrow{\pi_i} C)_{i \in \text{Ob}_D}$ is a colimiting cone of $D$, $L$ is a continuous functor.
Let us denote by $\textbf{Set}$ the category of sets and maps and by $\Phi: \textbf{Top} \to \textbf{Set}$ the forgetful functor.

**Corollary 13.8**

For a quasi-topological category $\mathcal{C}$, the composition

$$\tilde{\Phi}: \text{Funct}_c(\mathcal{C}, \textbf{Top}) \to \text{Funct}(\mathcal{C}, \textbf{Set})$$

of the inclusion functor

$$\text{Funct}_c(\mathcal{C}, \textbf{Top}) \to \text{Funct}(\mathcal{C}, \textbf{Top})$$

and the functor

$$\Phi_*: \text{Funct}(\mathcal{C}, \textbf{Top}) \to \text{Funct}(\mathcal{C}, \textbf{Set})$$

induced by $\Phi$ creates limits and colimits.

Hence $\text{Funct}_c(\mathcal{C}, \textbf{Top})$ is complete and cocomplete.
Let $\mathbf{C}$ and $\mathbf{T}$ be categories. For $R \in \text{Ob}\mathbf{C}$, define an evaluation functor $E_R : \text{Funct}(\mathbf{C}, \mathbf{T}) \to \mathbf{T}$ at $R$ by $E_R(F) = F(R)$ and $E_R(\varnothing) = \varnothing_R$.

Let $\mathbf{T}$ be a quasi-topological category. For $F, G \in \text{Ob}\text{Funct}(\mathbf{C}, \mathbf{T})$, we give $\text{Funct}(\mathbf{C}, \mathbf{T})(F, G)$ the coarsest topology such that $E_R : \text{Funct}(\mathbf{C}, \mathbf{T})(F, G) \to \mathbf{T}(F(R), G(R))$ is continuous for any object $R$ of $\mathbf{C}$.

**Proposition 13.9**

Let $\mathbf{C}$ be a category and $\mathbf{T}$ a quasi-topological category. Then, $\text{Funct}(\mathbf{C}, \mathbf{T})$ is a quasi-topological category.
Proposition 13.10

Let $T$ be a quasi-topological category and $F : C \to T$ a functor.

(1) Suppose that $(L \xrightarrow{\pi_i} D(i))_{i \in \text{Ob} D}$ is a limiting cone of a functor $D : D \to \text{Funct}(C, T)$ and that, for any $R \in \text{Ob} C$, $E_R D : D \to T$ and $F(R) \in \text{Ob} T$ satisfy the condition $(L)$ of (13.3) ($T = \text{Top}$, for example). Then,

$$(\text{Funct}(C, T)(F, L) \xrightarrow{\pi_i} \text{Funct}(C, T)(F, D(i)))_{i \in \text{Ob} D}$$

is a limiting cone of a functor $D_F : D \to \text{Top}$ defined by $D_F(i) = \text{Funct}(C, T)(F, D(i))$ and $D_F(\tau) = D(\tau)_\ast$ for $i \in \text{Ob} D$, $\tau \in \text{Mor} D$. 
(2) Suppose that \((D(i) \xrightarrow{l_i} C)_{i \in \text{Ob} D}\) is a colimiting cone of a functor \(D : D \to \text{Funct}(C, T)\) and that, for any \(R \in \text{Ob} C\), \(E_R D : D \to T\) and \(F(R) \in \text{Ob} T\) satisfy the condition (C) of (13.3) \((T = \text{Top}, \text{for example})\). Then,

\[
(\text{Funct}(C, T)(C, L) \xrightarrow{l_i^*} \text{Funct}(C, T)(D(i), F))_{i \in \text{Ob} D}
\]

is a limiting cone of a functor \(D^F : D^{op} \to \text{Top} \) defined by \(D^F(i) = \text{Funct}(C, T)(D(i), F)\) and \(D^F(\tau) = D(\tau)^*\) for \(i \in \text{Ob} D\), \(\tau \in \text{Mor} D\).
Definition 13.11
If \( T = \text{Set} \) or \( \text{Top} \), for a functor \( F: C \to T \), we denote by \( C_F \) the category of \( F \)-models, that is, \( C_F \) is given by

\[
\text{Ob} C_F = \{(R, x) | R \in \text{Ob} C, x \in F(R)\},
\]

\[
C_F((R, x), (Y, y)) = \{f \in C(R, Y) | F(f)(x) = y\}.
\]

Remark 13.12
Since \( \{E_R^{-1}(W(x, O)) | (R, x) \in \text{Ob} C_F, O \text{ is an open set of } G(R)\} \) is a subbasis of the topology on \( \text{Funct}(C, \text{Top})(F, G) \), a map \( f: Z \to \text{Funct}(C, \text{Top})(F, G) \) is continuous if and only if \( \text{ev}_x E_R f: Z \to G(R) \) is continuous for any \( (R, x) \in \text{Ob} C_F \).
Lemma 13.13

Let \( \mathcal{C} \) be a quasi-topological category and \( F: \mathcal{C} \to \text{Top} \) a functor. For \( (R, x) \in \text{Ob} \mathcal{C} \) and \( S \in \text{Ob} \mathcal{C} \), define a map \( (\varphi(F)_{(R, x)})(S): h_R(S) \to F(S) \) by \( (\varphi(F)_{(R, x)})(S)(f) = F(f)(x) \) for \( f \in \mathcal{C}(R, S) = h_R(S) \). If \( F \) is continuous, \( (\varphi(F)_{(R, x)})(S) \) is continuous. Thus we have a morphism \( \varphi(F)_{(R, x)}: h_R \to F \) in \( \text{Funct}(\mathcal{C}, \text{Top}) \).

Let \( \mathcal{C} \) be a category and \( R \) an object of \( \mathcal{C} \). Define a map \( \theta_R(G): \text{Funct}(\mathcal{C}, \text{Top})(h_R, G) \to G(R) \) by \( \theta_R(G)(\varphi) = \varphi_R(\text{id}_R) \).
Proposition 13.14
For an object $R$ of $C$ and a functor $G : C \rightarrow \text{Top}$, the following topologies $\mathcal{O}, \mathcal{O}_1$ and $\mathcal{O}_2$ on $\text{Funct}(C, \text{Top})(h_R, G)$ are the same.

(i) $\mathcal{O}$ is the coarsest topology such that
\[ E_S : \text{Funct}(C, \text{Top})(h_R, G) \rightarrow \text{Top}(h_R(S), G(S)) \]
is continuous for any $S \in \text{Ob} C$.

(ii) $\mathcal{O}_1$ the coarsest topology such that
\[ \theta_R(G) : \text{Funct}(C, \text{Top})(h_R, G) \rightarrow G(R) \]
is continuous.

(iii) $\mathcal{O}_2$ is the coarsest topology such that
\[ E_R : \text{Funct}(C, \text{Top})(h_R, G) \rightarrow \text{Top}(h_R(R), G(R)) \]
is continuous.
Corollary 13.15 (Yoneda embedding)
A functor $h: C^{op} \to \text{Funct}_c(C, \text{Top})$ defined by $h(R) = h_R$ and $h(f) = h_f$ is continuous.

Proposition 13.16 (Yoneda’s lemma)
Let $C$ be a quasi-topological category and $F: C \to \text{Top}$ a continuous functor. Then, $\theta_R(F): \text{Funct}(C, \text{Top})(h_R, F) \to F(R)$ is a homeomorphism.

Proposition 13.17
Let $C$ be a quasi-topological category. The functor
$$\tilde{\Phi}: \text{Funct}_c(C, \text{Top}) \to \text{Funct}(C, \text{Set})$$
given in (13.8) has a left adjoint.
If $\mathbf{C}$ is a quasi-topological category, we denote by $D(F): \mathbf{C}_F^{\text{op}} \to \text{Funct}_c(\mathbf{C}, \text{Top})$ a functor given by $D(F)(R, x) = h_R$ and $D(F)(f) = h_f$.

**Proposition 13.18**

Let $\mathbf{C}$ be a quasi-topological category. If $F: \mathbf{C} \to \text{Top}$ is a colimit of representable functors, then

$$(D_F(R, x) \xrightarrow{\varphi(F)(R, x)} F)_{(R, x) \in \text{Ob } \mathbf{C}_F}$$

is a colimiting cone of the functor $D(F): \mathbf{C}_F^{\text{op}} \to \text{Funct}_c(\mathbf{C}, \text{Top})$. Hence $F$ is in the image of a left adjoint of the functor

\[
\tilde{\Phi}: \text{Funct}_c(\mathbf{C}, \text{Top}) \to \text{Funct}(\mathbf{C}, \text{Set})
\]

given in (13.8).
For a quasi-topological category $\mathcal{C}$ and a functor $D: \mathcal{D}^{\text{op}} \to \mathcal{C}$, we denote by $h_D: \mathcal{D}^{\text{op}} \to \operatorname{Funct}_c(\mathcal{C}, \text{Top})$ the composition of functors $\mathcal{D}^{\text{op}}: \mathcal{D} \to \mathcal{C}^{\text{op}}$ and $h: \mathcal{C}^{\text{op}} \to \operatorname{Funct}_c(\mathcal{C}, \text{Top})$ defined in (13.15).

**Proposition 13.19**

(1) If $F: \mathcal{C} \to \text{Top}$ is a colimit of representable functors and $(h_D(i) \xrightarrow{\iota_i} F)_{i \in \text{Ob} \mathcal{D}}$ is a cone of $h_D$ such that $(\tilde{\Phi}(h_D(i)) \xrightarrow{\tilde{\Phi}(\iota_i)} \tilde{\Phi}(F))_{i \in \text{Ob} \mathcal{D}}$ is a colimiting cone of $\tilde{\Phi} h_D$. Then, $(h_D(i) \xrightarrow{\iota_i} F)_{i \in \text{Ob} \mathcal{D}}$ is a colimiting cone of $h_D$.

(2) Suppose that $F$ is a colimit of $h_D: \mathcal{D}^{\text{op}} \to \operatorname{Funct}_c(\mathcal{C}, \text{Top})$ and that $L$ is a limit of $D$. Then, $L$ is a limit of the functor $\hat{D}(F): \mathcal{C}_F^{\text{op}} \to \mathcal{C}$ defined by $\hat{D}(F)(R, x) = R$ and $\hat{D}(F)(f) = f$. 
Proposition 13.20
Let $A$ be an object of a quasi-topological category $\mathcal{C}$ and $F: \mathcal{C} \to \text{Top}$ a functor. Suppose that a limit $L(F)$ of the functor $\hat{D}(F): \mathcal{C}^\text{op} \to \mathcal{C}$ defined in (13.19) exists.

If $F$ is a colimit of representable functors, there is a bijection $\Theta_{F,A}: \text{Funct}(\mathcal{C}, \text{Top})(F, h_A) \to \mathcal{C}(A, L(F)) = \mathcal{C}^\text{op}(L(F), A)$. Moreover, if the condition (L) of (13.3) is satisfied for $A$ and $(L(F) \xrightarrow{\pi(R,x)} D(F)(R,x))_{(R,x) \in \text{Ob} \mathcal{C}_F}$, $\Theta_{F,A}$ is a homeomorphism.

Proposition 13.21
For objects $F$ and $G$ of $\text{Funct}(\mathcal{C}, \text{Top})$, suppose that limits of $\hat{D}(F): \mathcal{C}^\text{op}_F \to \mathcal{C}$ and $\hat{D}(G): \mathcal{C}^\text{op}_G \to \mathcal{C}$ exist and that $F$ is a colimit of representable functors. If the condition (L) of (13.3) is satisfied for $L(G)$ and $(L(F) \xrightarrow{\pi(R,x)} D(F)(R,x))_{(R,x) \in \text{Ob} \mathcal{C}_F}$, then $L: \text{Funct}(\mathcal{C}, \text{Top})(F, G) \to \mathcal{C}(L(G), L(F))$ is continuous.
For a quasi-topological category $C$, we denote by $\text{Funct}_r(C, \text{Top})$ the full subcategory of $\text{Funct}(C, \text{Top})$ consisting of functors which are colimit of representable functors.

**Proposition 13.22**

For $F, G \in \text{Ob}\, \text{Funct}_r(C, \text{Top})$, there is a product of $F$ and $G$ in $\text{Funct}_r(C, \text{Top})$. 
Definition 13.23
Let $A$ and $B$ be objects of a quasi-topological category $C$. A topological coproduct of $A$ and $B$ is a coproduct $A \coprod B$ of $A$ and $B$ such that

$$
C(A,R) \xleftarrow{l_1^*} C(A \coprod B,R) \xrightarrow{l_2^*} C(B,R)
$$

is a product of $C(A,R)$ and $C(B,R)$ in Top for any $R \in \text{Ob } C$. If each pair of objects of $C$ has a topological coproduct, we say that $C$ is a category with finite topological coproducts.

Theorem 13.24
If $C$ is quasi-topological category with finite topological coproducts, $\text{Funct}_r(C,\text{Top})$ is a cartesian closed category.
§14. Topological affine scheme

For objects $A^*$ and $B^*$ of $\text{TopAlg}_{K^*}$, we give a topology on the set of morphisms $\text{TopAlg}_{K^*}(A^*, B^*)$ by giving a uniform structure as follows. For $p \in \text{TopAlg}_{K^*}(A^*, B^*)$, $S \subset A^*$ and $J \in I_{B^*}$, we put

$$U(S, J) = \{(f, g) \in \text{TopAlg}_{K^*}(A^*, B^*) \times \text{TopAlg}_{K^*}(A^*, B^*) | f(x) - g(x) \in J \text{ if } x \in S\},$$

$$U(p; S, J) = \{f \in \text{TopAlg}_{K^*}(A^*, B^*) | (f, p) \in U(S, J)\}.$$

Recall that $F_{A^*}$ denotes the set of finitely generated $K^*$-submodules of $A^*$. We also put

$$B = \{U(S^*, J) | S^* \in F_{A^*}, J \in I_{B^*}\},$$

$$B_p = \{U(p; S^*, J) | S^* \in F_{A^*}, J \in I_{B^*}\}.$$
\( \mathcal{B} \) is a basis of a uniform structure of \( \text{TopAlg}_{K^*}(A^*, B^*) \) and \( \mathcal{B}_p \) is a basis of the neighborhood of \( p \) with respect to the topology defined by the uniform structure of \( \text{TopAlg}_{K^*}(A^*, B^*) \).

If \( C \) is a subcategory of \( \text{TopAlg}_{K^*} \), we give \( C(A^*, B^*) \) the topology induced by \( \text{TopAlg}_{K^*}(A^*, B^*) \) for \( A^*, B^* \in \text{Ob} \, C \).

We remark that \( \text{TopAlg}_{K^*}(A^*, B^*) \) is a subspace of \( \text{Hom}^0(A^*, B^*) \) if we regard \( A^* \) and \( B^* \) as objects of \( \text{TopMod}_{K^*} \).
Definition 14.1

Let \textbf{Top} be the category of topological spaces and continuous maps. For an object $A^*$ of $\textbf{TopAlg}_{K^*}$ and a subcategory $C$ of $\textbf{TopAlg}_{K^*}$, we denote by $h_{A^*} : C \to \textbf{Top}$ the functor represented by $A^*$, that is, $h_{A^*}$ maps $B^* \in \text{Ob} \ C$ to $\textbf{TopAlg}_{K^*}(A^*, B^*)$.

We call $h_{A^*}$ a \textbf{topological affine $K^*$-scheme}. Thus we have a functor $h : C^\text{op} \to \text{Funct}(C, \textbf{Top})$ given by $h(A) = h_{A^*}$ and $h(f) = f^*$. Generally, we call a functor from a subcategory of $\textbf{TopAlg}_{K^*}$ to $\textbf{Top}$ a \textbf{topological $K^*$-functor}. 
To be continued.